CWI Quarterly Volume 10 (3&4) 1997, pp. 239 – 251

Numerical Solution of Elliptic Convection-Diffusion Problems on Fitted Meshes

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The problem of the transport of a quantity (heat, matter or momentum) by advection-diffusion is considered for arbitrarily large Peclet number in a two-dimensional domain. In some neighbourhoods of the boundary of the domain, boundary layers may appear for large Peclet number. It is well known that classical finite difference methods do not yield approximate solutions with a guaranteed accuracy if the Peclet number can be arbitrarily large. For such problems, numerical methods which use monotone finite difference operators on appropriately fitted piecewise-uniform meshes have been shown to yield approximate solutions with a guaranteed accuracy independent of the Peclet number. Numerical examples are presented which verify this property under various boundary conditions, and comparisons are made with approximate solutions obtained using alternatively fitted meshes.

1. Introduction

We consider methods for solving the advection-diffusion problem

$$\varepsilon\Delta u + \mathbf{a}.\nabla u = f$$

in a domain in $\mathbb{R}^2$ under various boundary conditions. In the case where $\varepsilon << 1$, which is equivalent to large Peclet number, the problem is said to be advection-dominated. Here u may be thought of as a concentration of some quantity (e.g., heat, a pollutant etc.) that is driven by a known advective velocity field $\mathbf{a} = (a_1, a_2)$ where ε is the diffusivity. We presume here that the differential operator is approximated by an upwinded finite difference operator, as defined later, and focus attention on the nature of the computational mesh, which is presumed to be refined in the boundary layer regions.

The computational meshes, which are referred to in this paper as *fitted meshes*, were introduced and analysed in [8] and [7]. There, it was shown, for various singular perturbation problems, that when appropriate stable finite difference operators are used in conjunction with these meshes, numerical solutions of guaranteed accuracy (for all values of the diffusion coefficient ε) are obtained. The numerical methods are then said to be ε-uniform. The attraction of these meshes, apart from their reliability, is their extreme simplicity: they are piecewise-uniform meshes, which are condensed in the "layer regions", and rely on the identification of a transition point $\sigma_{N,\varepsilon}$, which is the interface between the coarser and finer uniform meshes. (Here N is the order of the mesh.)

The precise definition of $\sigma_{N,\varepsilon}$ varies with the nature of the layer; for example, in the case of a regular exponential layer, $\sigma_{N,\varepsilon} = \min\{1/2, K\varepsilon \ln N\}$, where K is a constant; and, in the case of a parabolic layer, $\sigma_{N,\varepsilon} = \min\{1/4, \sqrt{\varepsilon} \ln N\}$. It is not surprising that σ depends on ε; however, it is the dependence on N that is the key to the success of the mesh. The effect of this N-dependence is that the mesh becomes uniform when N is sufficiently large. The guaranteed accuracy predicted theoretically in [8] has been verified computationally in, *inter alia*, [2], [5],[6].

There are certainly other families of computational meshes which will also produce ε-uniform methods, such as those introduced by Bakhvalov [1]; however, these meshes are much more complicated in structure than those of Shishkin. It is thus perhaps not surprising that other piecewise-uniform meshes have also been suggested. For a layer of order ε^μ it is relatively easy to see that a transition point dependent on ε^μ alone will not suffice [6]; however, recently, various authors [3], [4], [9] have proposed piecewise-uniform meshes where the transition point σ_ε is independent of N but dependent on $\varepsilon \ln(1/\varepsilon)$[1]. One immediately evident disadvantage of such a mesh is that, no matter how large N is, the mesh will not become uniform. On the other hand, the choice of a dependence on $\varepsilon \ln(1/\varepsilon)$ is based on an awareness of the boundary layer thickness; for any fixed ε and relatively small N, the difference between $\sigma_{N,\varepsilon}$ and σ_ε is small. There is no suggestion that these meshes can produce a uniformly in ε convergent solution. Nevertheless it is worthwhile to consider whether there is a significant difference between solutions on one mesh and on the other. The main purpose of this paper is to show that there is indeed a significant difference.

2. A problem with a Neumann boundary condition.

Numerical results are obtained for the specific model problem

$$\varepsilon \Delta u + \frac{\partial u}{\partial x} + 2\frac{\partial u}{\partial y} = x^2(1-x)^2(1-y)^2 y^2 \quad \text{on} \quad \Omega = (0,1)\times(0,1) \qquad (1a)$$

[1] In the case of [3], [4] a Schwarz procedure is also used

with the Dirichlet boundary conditions

$$u(1,y) = 1 \quad u(x,1) = 1 \tag{1b}$$

and the Neumann boundary conditions

$$\frac{\partial u}{\partial x}(0,y) = 0; \quad \varepsilon\frac{\partial u}{\partial y}(x,0) = -1; \tag{1c}$$

The fitted mesh for this problem is now defined as

$$\overline{\Omega}^N = \bar{\omega}_1 \times \bar{\omega}_2^*$$

where $\bar{\omega}_1$ is a uniform mesh of order N in the x-direction. The layer at $x = 0$ is weak enough not to require a fitted mesh in this direction. In the y-direction, $\bar{\omega}_2^*$ is defined as follows. The interval $[0,1]$ is divided into two parts

$$[0,\sigma_y], [\sigma_y, 1]$$

The transition point σ_y is defined by

$$\sigma_y \equiv \min\{1/2, \varepsilon \ln N\},$$

The intervals $[0,\sigma_y], [\sigma_y,1]$ are then divided into $N/2$ equal parts.

The computed solution for $\varepsilon = .01$ is shown in Figure 1 with $N = 64$.

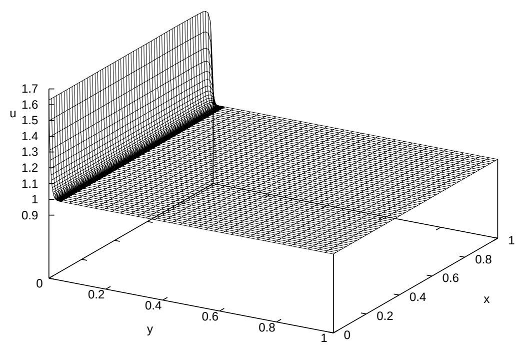

FIGURE 1. Solution of Problem (1) on fitted mesh Ω^{64} with $\varepsilon = 0.01$

We examine the performance of standard upwinding on this mesh. That is,

$$L_u^N U_\varepsilon = \varepsilon\delta_x^2 U_\varepsilon + \varepsilon\delta_y^2 U_\varepsilon + D_x^+ U_\varepsilon + 2D_y^+ U_\varepsilon = x^2(1-x)^2(1-y)^2 y^2 \quad \text{on } \Omega^N$$

where

$$h_i = x_i - x_{i-1}, \quad \bar{h}_i = (h_i + h_{i+1})/2$$
$$D_x^+ Z(x_i,y_j) = (Z(x_{i+1},y_j) - Z(x_i,y_j))/h_{i+1}$$
$$D_x^- Z(x_i,y_j) = (Z(x_i,y_j) - Z(x_{i-1},y_j))/h_i$$

and

$$\delta_x^2 Z(x_i,y_j) \equiv (D_x^+ Z(x_i,y_j) - D_x^- Z(x_i,y_j))/\bar{h}_i$$

with analogous definitions of D_y^+, D_y^- and δ_y^2. The Neumann boundary conditions are discretised in the obvious way

$$D_x^+ u_N(x_0, y_j) = 0 \quad \varepsilon D_y^+ u_N(x_i, y_0) = -1$$

The errors are given in Table 1 and the estimated orders of convergence in Table 2. The errors and rates are estimated as described in [2]

ε	Number of Intervals N					
	8	16	32	64	128	256
1	.123D+00	.605D-01	.293D-01	.137D-01	.586D-02	.195D-02
2^{-2}	.246D+00	.121D+00	.586D-01	.273D-01	.117D-01	.391D-02
2^{-4}	.492D+00	.242D+00	.117D+00	.547D-01	.234D-01	.781D-02
2^{-6}	.514D+00	.331D+00	.201D+00	.109D+00	.469D-01	.156D-01
2^{-8}	.527D+00	.327D+00	.193D+00	.106D+00	.515D-01	.190D-01
2^{-10}	.539D+00	.331D+00	.193D+00	.106D+00	.515D-01	.190D-01
2^{-12}	.546D+00	.333D+00	.194D+00	.106D+00	.515D-01	.190D-01
2^{-14}	.550D+00	.334D+00	.194D+00	.106D+00	.515D-01	.190D-01
2^{-16}	.552D+00	.334D+00	.194D+00	.106D+00	.515D-01	.190D-01
2^{-18}	.553D+00	.335D+00	.195D+00	.106D+00	.515D-01	.190D-01
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
2^{-28}	.554D+00	.335D+00	.195D+00	.106D+00	.515D-01	.190D-01

TABLE 1. Errors in approximating u using fitted mesh method

ε	Number of Intervals N				
	8	16	32	64	128
1	1.00	1.00	1.00	1.00	1.00
2^{-2}	1.00	1.00	1.00	1.00	1.00
2^{-4}	1.00	1.00	1.00	1.00	1.00
2^{-6}	0.49	0.51	0.55	1.00	1.00
2^{-8}	0.56	0.63	0.69	0.74	0.78
2^{-10}	0.61	0.64	0.69	0.74	0.78
2^{-12}	0.62	0.65	0.70	0.74	0.78
2^{-14}	0.63	0.66	0.70	0.74	0.78
2^{-16}	0.64	0.66	0.70	0.74	0.78
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
2^{-28}	0.64	0.66	0.70	0.74	0.78
Uniform:	0.64	0.66	0.70	0.74	0.78

TABLE 2. Convergence rates in approximating u using fitted mesh method

An alternative mesh for this problem is now defined as

$$\overline{\Omega}_A^N = \bar{\omega}_1 \times \bar{\omega}_2^A$$

where $\bar{\omega}_1$ is again a uniform mesh of order N, and $\bar{\omega}_2^A$ is defined as follows. The interval $[0,1]$ is divided into two parts

$$[0, \sigma_y^A], [\sigma_y^A, 1]$$

The transition point σ_y^A is defined by

$$\sigma_y^A \equiv \min\{1/2, \varepsilon \ln(1/\varepsilon)\},$$

The intervals $[0, \sigma_y^A], [\sigma_y^A, 1]$ are then divided into $N/2$ equal parts.

We now examine the performance of standard upwinding on this mesh. If we again produce a table of estimated orders of convergence for the alternative mesh we obtain the results in Table 3

ε	Number of Intervals N				
	8	16	32	64	128
1	1.00	1.00	1.00	1.00	1.00
2^{-2}	1.00	1.00	1.00	1.00	1.00
2^{-4}	1.00	1.00	1.00	1.00	1.00
2^{-6}	1.00	1.00	1.00	1.00	1.00
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
2^{-46}	1.00	1.00	1.00	1.00	1.00

TABLE 3. Convergence rates in approximating u using alternative mesh method

The rates look very good, in the sense that the method is clearly convergent with order 1 for any fixed ε; but when we examine the errors as given in Table 4, we see that, although the errors are decreasing for each fixed ε, the error is *increasing* as $\varepsilon \to 0$ for each fixed N. Thus it is impossible to calculate a uniform convergence rate for this mesh.

A cross-section of the solution on each mesh for $x = 1/2$ is shown in Figures 2 and 3 for $\varepsilon = 10^{-2}$, 10^{-4}, 10^{-6} and 10^{-8}.

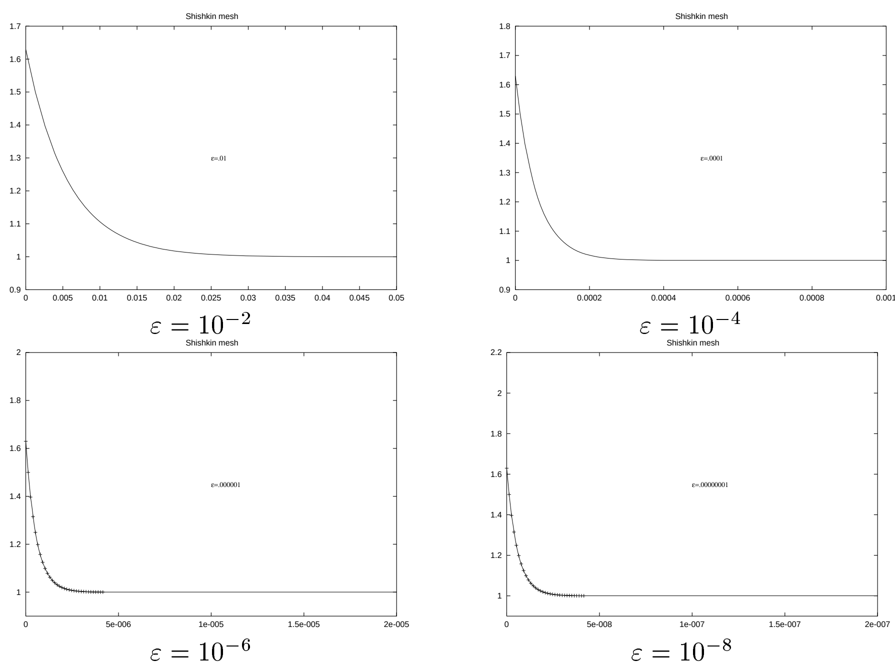

FIGURE 2. Zoomed cross-sections of solutions on fitted mesh at $x = 1/2$

ε	Number of Intervals N 8	16	32	64	128	256
1	.123D+00	.605D-01	.293D-01	.137D-01	.586D-02	.195D-02
2^{-2}	.171D+00	.839D-01	.406D-01	.190D-01	.812D-02	.271D-02
2^{-4}	.341D+00	.168D+00	.812D-01	.379D-01	.162D-01	.542D-02
2^{-6}	.521D+00	.256D+00	.124D+00	.576D-01	.247D-01	.822D-02
2^{-8}	.699D+00	.341D+00	.164D+00	.765D-01	.328D-01	.109D-01
2^{-10}	.869D+00	.424D+00	.204D+00	.952D-01	.408D-01	.136D-01
2^{-12}	.104D+01	.506D+00	.244D+00	.114D+00	.488D-01	.163D-01
2^{-14}	.121D+01	.589D+00	.285D+00	.133D+00	.569D-01	.190D-01
2^{-16}	.137D+01	.673D+00	.325D+00	.152D+00	.650D-01	.217D-01
2^{-18}	.154D+01	.756D+00	.366D+00	.171D+00	.731D-01	.244D-01
2^{-20}	.171D+01	.840D+00	.406D+00	.190D+00	.813D-01	.271D-01
2^{-22}	.188D+01	.924D+00	.447D+00	.209D+00	.894D-01	.298D-01
2^{-24}	.205D+01	.101D+01	.487D+00	.227D+00	.975D-01	.325D-01
2^{-26}	.222D+01	.109D+01	.528D+00	.246D+00	.106D+00	.352D-01
2^{-28}	.239D+01	.118D+01	.569D+00	.265D+00	.114D+00	.379D-01
2^{-30}	.256D+01	.126D+01	.609D+00	.284D+00	.122D+00	.406D-01
2^{-32}	.273D+01	.134D+01	.650D+00	.303D+00	.130D+00	.433D-01
2^{-34}	.290D+01	.143D+01	.691D+00	.322D+00	.138D+00	.460D-01
2^{-36}	.307D+01	.151D+01	.731D+00	.341D+00	.146D+00	.487D-01
2^{-38}	.324D+01	.159D+01	.772D+00	.360D+00	.154D+00	.515D-01
2^{-40}	.341D+01	.168D+01	.812D+00	.379D+00	.162D+00	.542D-01
2^{-42}	.358D+01	.176D+01	.853D+00	.398D+00	.171D+00	.569D-01
2^{-44}	.375D+01	.185D+01	.894D+00	.417D+00	.179D+00	.596D-01
2^{-46}	.392D+01	.193D+01	.934D+00	.436D+00	.187D+00	.623D-01
2^{-48}	.409D+01	.201D+01	.975D+00	.455D+00	.195D+00	.650D-01
2^{-50}	.427D+01	.210D+01	.102D+01	.474D+00	.203D+00	.677D-01
2^{-52}	.444D+01	.218D+01	.106D+01	.493D+00	.211D+00	.704D-01
2^{-54}	.461D+01	.227D+01	.110D+01	.512D+00	.219D+00	.731D-01
2^{-56}	.478D+01	.235D+01	.114D+01	.531D+00	.227D+00	.758D-01
2^{-58}	.495D+01	.243D+01	.118D+01	.550D+00	.236D+00	.785D-01
2^{-60}	.512D+01	.252D+01	.122D+01	.569D+00	.244D+00	.812D-01
2^{-62}	.529D+01	.260D+01	.126D+01	.588D+00	.252D+00	.839D-01
2^{-64}	.546D+01	.269D+01	.130D+01	.607D+00	.260D+00	.867D-01
2^{-68}	.563D+01	.277D+01	.134D+01	.626D+00	.268D+00	.894D-01
2^{-70}	.580D+01	.285D+01	.138D+01	.644D+00	.276D+00	.921D-01

TABLE 4. Maximum errors in approximating u using alternative mesh method

3. One-dimensional model problems

3.1. A Neumann problem

To try to explain what is happening in the example above, we consider the following model one-dimensional problem.

$$\varepsilon u_\varepsilon''(x) + a u_\varepsilon'(x) = 0, \quad x \in \Omega = (0,1)$$

$$\varepsilon u_\varepsilon'(0) = -a, \quad u_\varepsilon(1) = 0$$

where a is a positive constant. The exact solution is

$$u_\varepsilon(x) = \exp(-ax/\varepsilon) - \exp(-a/\varepsilon)$$

Note that

$$\lim_{\varepsilon \to 0} u_\varepsilon(0) = 1$$

and that the boundary layer width is $O(\varepsilon \ln(1/\varepsilon))$ since

$$\frac{d^k u_\varepsilon}{dx^k} \le C, \quad x > k\varepsilon \ln(1/\varepsilon)$$

We will examine the numerical performance of standard upwinding

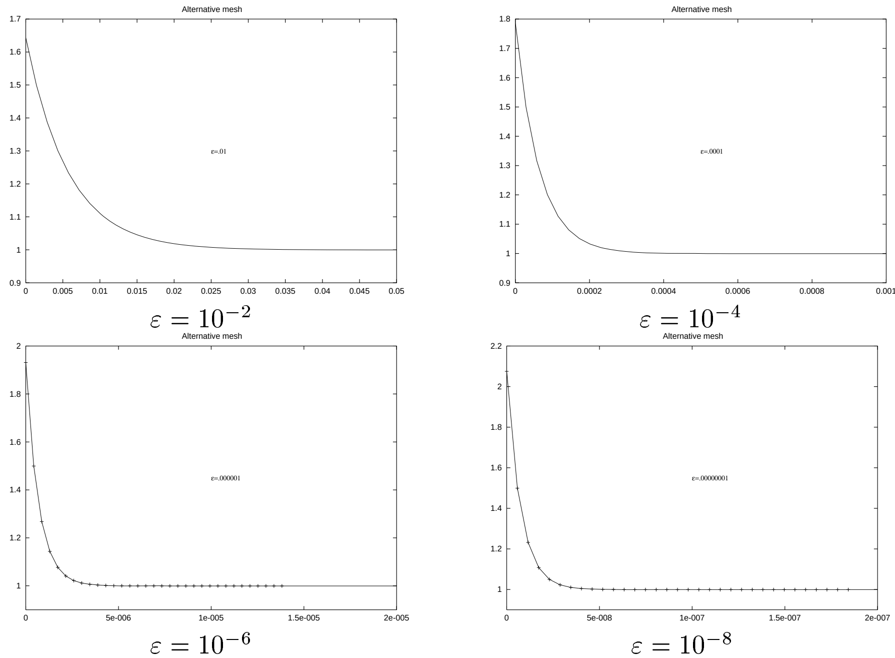

FIGURE 3. Zoomed cross-sections of solutions on alternative mesh at $x = 1/2$

$$\varepsilon\delta^2 U^N + aD^+U^N = 0, \quad x_i \in \Omega^N_\sigma$$
$$\varepsilon D^+ U^N_0 = -a, \quad U^N_N = 0$$

on piecewise-uniform meshes of the form

$$\Omega^N_\sigma = \{x_i | x_i = x_{i-1} + h, \quad 0 \le i \le N/2; \quad x_i = x_{i-1} + H, \quad N/2 < i \le N\}.$$

where the fine and coarse mesh widths are

$$h = 2\sigma/N, \quad H = 2(1-\sigma)/N.$$

A natural choice for the transition parameter is

$$\sigma_\varepsilon = \min\{0.5, (1/a)\varepsilon \ln(1/\varepsilon)\},$$

The exact solution of the difference scheme is

$$U^N(x_i) = \frac{\lambda_1^i - \lambda_1^{N/2}}{\lambda_1} + \frac{2\lambda_1^{N/2}(1-\lambda_2^{N/2})}{(\lambda_1+\lambda_2)}, \quad i \le N/2$$

$$U^N(x_i) = \frac{2\lambda_1^{N/2}(\lambda_2^{i-N/2} - \lambda_2^{N/2})}{(\lambda_1+\lambda_2)}, \quad i \ge N/2,$$

where

$$\rho_1 = \frac{2a\sigma}{N\varepsilon} \quad \lambda_1 = \frac{1}{1+\rho_1} \quad \rho_2 = \frac{2a(1-\sigma)}{N\varepsilon} \quad \lambda_2 = \frac{1}{1+\rho_2}.$$

If the asymptotically natural choice is taken for the transition parameter (i.e., $\sigma = \sigma_\varepsilon$) then for each fixed N

$$\lim_{\varepsilon \to 0} \lambda_1 = 0$$

and thus

$$\lim_{\varepsilon\to 0} U^N(0) \to \infty$$

and so the error in the numerical approximation becomes unbounded. More precisely

$$|U^N(0) - u_\varepsilon(0)| = O(\ln(1/\varepsilon)/N)$$

Note that this logarithmic rate of growth of the error as $\varepsilon \to 0$ may be observed in any of the columns of Table 4, i.e., for any fixed N.

3.2. A Dirichlet problem

Consider now the following problem

$$\varepsilon u''_\varepsilon(x) + a u'_\varepsilon(x) = 0, \quad x \in \Omega = (0,1)$$

$$u_\varepsilon(0) = 1, \quad u_\varepsilon(1) = 0$$

where the coefficient a is again a positive constant. Note that

$$\lim_{\varepsilon\to 0} \varepsilon u'_\varepsilon(0) = -a$$

We will examine the numerical performance of standard upwinding

$$\varepsilon\delta^2 U^N + aD^+U^N = 0, \quad x_i \in \Omega^N_\sigma$$

$$U^N_0 = 1, \quad U^N_N = 0$$

on piecewise-uniform meshes of the form

$$\Omega^N_\sigma = \{x_i | x_i = x_{i-1} + h, \quad 0 \le i \le N/2; \quad x_i = x_{i-1} + H, \quad N/2 < i \le N\}.$$

The exact solution of this difference scheme is

$$U^N(x_i) = A(\lambda_1^i - 1) + 1, \quad i \le N/2$$

$$U^N(x_i) = A\left(\frac{2\lambda_1^{N/2+1}(\lambda_2^{i-N/2} - \lambda_2^{N/2})}{\lambda_1 + \lambda_2}\right), \quad i \ge N/2,$$

where

$$A = \left(\frac{2\lambda_1^{N/2+1}(1 - \lambda_2^{N/2})}{\lambda_1 + \lambda_2} + 1 - \lambda_1^{N/2}\right)^{-1}$$

Note that

$$\lim_{\varepsilon\to 0} \varepsilon D^+U^N(0) = \lim_{\varepsilon\to 0} \frac{-aA}{1 + 2\frac{a\sigma}{N\varepsilon}}$$

If the natural transition point is taken then this limit tends to zero. Thus, for this choice of transition parameter, and for any fixed choice of N,

$$\lim_{\varepsilon\to 0} |\varepsilon D^+U^N(0) - \varepsilon u'_\varepsilon(0)| = a$$

We will observe this behaviour of the error in the next section.

4. A problem with parabolic boundary layers.

It was verified in [6] that methods using fitted meshes yield ε-uniformly convergent solutions of singular-perturbation problems which contain both regular and parabolic layers. Here we are concerned with the approximation of the normalised derivatives εu_x and $\sqrt{\varepsilon} u_y$. Note the normalising is required as the derivatives themselves become unbounded as $\varepsilon \to 0$; however, from the known bounds on the first order derivatives we know that the normalised derivatives remain bounded as $\varepsilon \to 0$, and we approximate them by $\varepsilon D_x^+ U_\varepsilon^N$ and $\sqrt{\varepsilon} D_y^+ U_\varepsilon^N$.

We consider the sample convection-diffusion problem

$$\varepsilon \Delta u_\varepsilon + (1 + x^2 + y^2)\frac{\partial u_\varepsilon}{\partial x} = 0, \quad (x, y) \in \Omega = (0,1) \times (0,1) \tag{2a}$$

with Dirichlet boundary conditions

$$u_\varepsilon(x, 0) = 64x^3(1-x)^3 \quad u_\varepsilon(1, y) = 64y^3(1-y)^3 \tag{2b}$$

and

$$u_\varepsilon(x, 1) = u_\varepsilon(0, y) = 0 \tag{2c}$$

Both regular and parabolic layers are present in its solution.

Now we define the fitted mesh for this problem.

$$\overline{\Omega}^N = \bar{\omega}_1^* \times \bar{\omega}_2^*$$

where $\bar{\omega}_1^*$ is defined as follows. The interval $[0, 1]$ is divided into two parts

$$[0, \sigma_x], [\sigma_x, 1]$$

As before, the transition point σ_x is defined by

$$\sigma_x \equiv \min\{1/2, \varepsilon \ln N\},$$

The intervals $[0, \sigma_x], [\sigma_x, 1]$ are divided into $N/2$ equal parts.

The mesh $\bar{\omega}_2$ is also a fitted mesh constructed as follows. The interval $[0, 1]$ is divided into three parts

$$[0, \sigma_y], [\sigma_y, 1 - \sigma_y], [1 - \sigma_y, 1]$$

where $\sigma_y \in (0, 1/4]$ depends on ε and N and is given by

$$\sigma_y = \min[1/4, \sqrt{\varepsilon} \ln N]$$

The intervals $[0, \sigma_y], [1-\sigma_y, 1]$ are divided into $N/4$ equal parts and the interval $[\sigma_y, 1 - \sigma_y]$ is divided into $N/2$ equal parts.

Let us first look at the performance of standard upwinding on this mesh. That is,

$$L_u^N U_\varepsilon = \varepsilon \delta_x^2 U_\varepsilon + \varepsilon \delta_y^2 U_\varepsilon + (1 + x^2 + y^2) D_x^+ U_\varepsilon = 0 \quad \text{on } \Omega^N$$

Figure 4 shows the numerical solution of (2) on a fitted mesh with $\varepsilon = 10^{-3}$ and $N = 64$

The following tables give the maximum pointwise errors and the convergence rates for the numerical approximations of the normalised derivatives on the fitted meshes. They show that the errors stabilise as $\varepsilon \to 0$ and indicate uniform in ε convergence of the method.

As an alternative mesh for this problem, we consider the following, proposed in [9]

$$\overline{\Omega}^N = \bar{\omega}_1^A \times \bar{\omega}_2^A$$

ε	Number of Intervals N 8	16	32	64	128	256
1	0.444D+00	0.204D+00	0.989D-01	0.473D-01	0.204D-01	0.681D-02
2^{-2}	0.375D+00	0.219D+00	0.104D+00	0.481D-01	0.205D-01	0.684D-02
2^{-4}	0.142D+00	0.101D+00	0.536D-01	0.284D-01	0.134D-01	0.480D-02
2^{-6}	0.120D+00	0.805D-01	0.488D-01	0.266D-01	0.128D-01	0.465D-02
2^{-8}	0.101D+00	0.767D-01	0.463D-01	0.254D-01	0.123D-01	0.452D-02
2^{-10}	0.942D-01	0.755D-01	0.459D-01	0.251D-01	0.122D-01	0.448D-02
2^{-12}	0.934D-01	0.748D-01	0.457D-01	0.250D-01	0.121D-01	0.446D-02
2^{-14}	0.932D-01	0.744D-01	0.455D-01	0.250D-01	0.121D-01	0.446D-02
2^{-16}	0.931D-01	0.742D-01	0.453D-01	0.250D-01	0.121D-01	0.446D-02
2^{-18}	0.931D-01	0.741D-01	0.453D-01	0.250D-01	0.121D-01	0.445D-02
2^{-20}	0.931D-01	0.741D-01	0.453D-01	0.250D-01	0.121D-01	0.445D-02
⋮	⋮	⋮	⋮	⋮	⋮	⋮
2^{-34}	0.931D-01	0.740D-01	0.453D-01	0.250D-01	0.121D-01	0.445D-02

TABLE 5. Maximum pointwise errors in approximating εu_x using fitted mesh method

ε	Number of Intervals N 8	16	32	64	128
1	0.73	0.79	0.85	0.92	0.96
2^{-2}	0.52	0.74	0.87	0.93	0.97
2^{-4}	0.35	0.79	0.83	0.66	0.74
2^{-6}	0.71	0.46	0.37	0.63	0.70
2^{-8}	0.60	0.09	0.36	0.60	0.69
2^{-10}	0.85	0.05	0.35	0.61	0.69
2^{-12}	0.94	0.04	0.34	0.61	0.69
2^{-14}	0.98	0.02	0.34	0.60	0.69
2^{-16}	0.99	0.02	0.34	0.60	0.69
2^{-18}	1.00	0.01	0.34	0.60	0.69
⋮	⋮	⋮	⋮	⋮	⋮
2^{-34}	1.00	0.01	0.34	0.60	0.69
Uniform:	0.73	0.79	0.85	0.92	0.96

TABLE 6. Orders of convergence in approximating εu_x using fitted mesh method.

ε	Number of Intervals N 8	16	32	64	128	256
1	0.430D+00	0.210D+00	0.102D+00	0.469D-01	0.205D-01	0.690D-02
2^{-2}	0.430D+00	0.230D+00	0.129D+00	0.627D-01	0.275D-01	0.925D-02
2^{-4}	0.650D+00	0.305D+00	0.154D+00	0.742D-01	0.325D-01	0.110D-01
2^{-6}	0.934D+00	0.482D+00	0.230D+00	0.113D+00	0.498D-01	0.169D-01
2^{-8}	0.736D+00	0.587D+00	0.370D+00	0.219D+00	0.999D-01	0.346D-01
2^{-10}	0.680D+00	0.562D+00	0.347D+00	0.208D+00	0.108D+00	0.411D-01
2^{-12}	0.672D+00	0.560D+00	0.347D+00	0.208D+00	0.108D+00	0.410D-01
⋮	⋮	⋮	⋮	⋮	⋮	⋮
2^{-34}	0.671D+00	0.560D+00	0.347D+00	0.208D+00	0.108D+00	0.410D-01

TABLE 7. Maximum pointwise errors in approximating $\sqrt{\varepsilon}u_y$ using fitted mesh method.

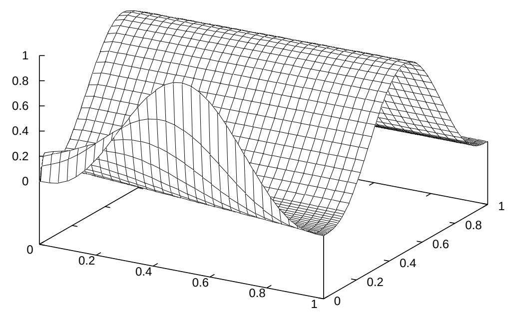

FIGURE 4. Solution of Problem (2) on fitted mesh Ω^{64} with $\varepsilon = 0.001$

ε	Number of Intervals N				
	8	16	32	64	128
1	0.88	0.97	0.95	0.94	0.98
2^{-2}	0.63	0.80	0.90	0.95	0.98
2^{-4}	0.65	0.83	0.87	0.93	0.97
2^{-6}	0.25	0.65	0.77	0.88	0.94
2^{-8}	-0.49	-0.25	0.01	0.77	0.88
2^{-10}	-0.55	-0.11	0.22	0.44	0.60
2^{-12}	-0.56	-0.11	0.22	0.44	0.60
⋮	⋮	⋮	⋮	⋮	⋮
2^{-34}	-0.56	-0.11	0.22	0.44	0.60
Uniform:	0.88	0.75	0.01	0.77	0.65

TABLE 8. Orders of convergence in approximating $\sqrt{\varepsilon} u_y$ using fitted mesh method.

where $\bar{\omega}_1^A$ is defined as follows. The interval $[0,1]$ is divided into two parts

$$[0, \sigma_x^A], [\sigma_x^A, 1]$$

and the transition point σ_x^A is defined by

$$\sigma_x^A \equiv \min\{1/2, \varepsilon \ln 1/\varepsilon\},$$

The intervals $[0, \sigma_x^A], [\sigma_x^A, 1]$ are divided into $N/2$ equal parts.

The mesh $\bar{\omega}_2^A$ is constructed as follows. The interval $[0,1]$ is divided into three parts

$$[0, \sigma_y^A], [\sigma_y^A, 1 - \sigma_y^A], [1 - \sigma_y^A, 1]$$

where $\sigma_y^A \in (0, 1/4]$ depends on ε and is given by

$$\sigma_y^A = \min[1/4, \sqrt{\varepsilon \ln 1/\varepsilon}]$$

The intervals $[0, \sigma_y^A], [1 - \sigma_y^A, 1]$ are divided into $N/4$ equal parts and the interval $[\sigma_y^A, 1 - \sigma_y^A]$ is divided into $N/2$ equal parts.

ε	Number of Intervals N					
	8	16	32	64	128	256
1	.444D+00	.204D+00	.989D-01	.473D-01	.204D-01	.681D-02
2^{-2}	.452D+00	.276D+00	.138D+00	.649D-01	.289D-01	.110D-01
2^{-4}	.222D+00	.101D+00	.494D-01	.234D-01	.104D-01	.108D+00
2^{-6}	.362D+00	.152D+00	.643D-01	.266D-01	.964D-02	.186D-02
2^{-8}	.497D+00	.214D+00	.909D-01	.390D-01	.155D-01	.452D-02
2^{-10}	.622D+00	.296D+00	.123D+00	.521D-01	.214D-01	.736D-02
2^{-12}	.720D+00	.369D+00	.153D+00	.655D-01	.274D-01	.102D-01
2^{-14}	.796D+00	.435D+00	.180D+00	.790D-01	.334D-01	.129D-01
2^{-16}	.853D+00	.504D+00	.217D+00	.919D-01	.395D-01	.158D-01
2^{-18}	.895D+00	.568D+00	.257D+00	.108D+00	.459D-01	.187D-01
2^{-20}	.925D+00	.625D+00	.297D+00	.124D+00	.522D-01	.215D-01
2^{-22}	.947D+00	.677D+00	.336D+00	.139D+00	.584D-01	.244D-01
2^{-24}	.963D+00	.722D+00	.375D+00	.153D+00	.656D-01	.274D-01
2^{-26}	.974D+00	.762D+00	.412D+00	.166D+00	.724D-01	.303D-01
2^{-28}	.982D+00	.797D+00	.449D+00	.179D+00	.791D-01	.334D-01
2^{-30}	.988D+00	.827D+00	.483D+00	.197D+00	.855D-01	.365D-01

TABLE 9. Maximum pointwise errors in approximating εu_x using alternative mesh method.

ε	Number of Intervals N					
	8	16	32	64	128	256
1	.430D+00	.210D+00	.102D+00	.469D-01	.205D-01	.690D-02
2^{-2}	.430D+00	.231D+00	.134D+00	.658D-01	.293D-01	.102D-01
2^{-4}	.691D+00	.305D+00	.152D+00	.740D-01	.326D-01	.551D-01
2^{-6}	.965D+00	.478D+00	.231D+00	.113D+00	.497D-01	.168D-01
2^{-8}	.923D+00	.512D+00	.244D+00	.111D+00	.326D-01	.573D-01
2^{-10}	.850D+00	.534D+00	.254D+00	.109D+00	.294D-01	.161D+00
2^{-12}	.885D+00	.580D+00	.283D+00	.126D+00	.387D-01	.777D-01
2^{-14}	.917D+00	.619D+00	.310D+00	.142D+00	.476D-01	.463D-02
2^{-16}	.940D+00	.652D+00	.332D+00	.155D+00	.548D-01	.324D-02
2^{-18}	.957D+00	.679D+00	.354D+00	.168D+00	.616D-01	.480D-02
2^{-20}	.970D+00	.702D+00	.374D+00	.180D+00	.681D-01	.768D-02
2^{-22}	.978D+00	.722D+00	.395D+00	.193D+00	.745D-01	.107D-01
2^{-24}	.984D+00	.739D+00	.414D+00	.203D+00	.807D-01	.138D-01
2^{-26}	.989D+00	.753D+00	.434D+00	.213D+00	.867D-01	.167D-01
2^{-28}	.992D+00	.765D+00	.451D+00	.223D+00	.925D-01	.196D-01
2^{-30}	.994D+00	.775D+00	.468D+00	.233D+00	.983D-01	.225D-01
2^{-32}	.996D+00	.785D+00	.484D+00	.242D+00	.104D+00	.252D-01
2^{-34}	.997D+00	.792D+00	.499D+00	.250D+00	.108D+00	.279D-01

TABLE 10. Maximum pointwise errors in approximating $\sqrt{\varepsilon}u_y$ using alternative mesh method.

We consider the performance of standard upwinding on this mesh.

$$L_u^N U_\varepsilon = \varepsilon\delta^2 U_\varepsilon + D_x^+ U_\varepsilon = 0 \quad \text{on } \Omega^N$$

The final two tables give the errors in the numerical approximations of the normalised derivatives using the alternative mesh. As predicted by the analysis of the one dimensional model problem, the errors tend to 1 as $\varepsilon \to 0$.

5. Summary

We have demonstrated in this paper, that a numerical method which uses upwind finite difference operators on a fitted mesh where the transition points depend on both ε and N is ε-uniform as predicted in the theory of Shishkin.

We also illustrated some deficiencies of a mesh which has transition points dependent on $\varepsilon \ln 1/\varepsilon$.

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