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Actuarial Applications of Financial Models

M. J. Goovaerts
Katholieke Universiteit Leuven
Ministerbroedersstraat 5
B-3000 Leuven, België en
Universiteit van Amsterdam.
Roetersstraat 11
1018 WB Amsterdam, The Nederlands

J. Dhaene
Katholieke Universiteit Leuven
Ministerbroedersstraat 5
B-3000 Leuven, België en
Universitaire Instelling Antwerpen
Universiteitsplein 1
B-2610 Antwerpen, Belgium

In the present contribution we indicate the type of situations, seen from an insurance point of view, in which financial models serve as a basis for providing solutions to practical problems. In addition, some of the essential differences in the basic assumptions underlying financial models and actuarial applications are given.

1. Introduction

Many insurance problems, both for premium calculation as well as for the determination of provisions, can be cast into the form of the evaluation of the distribution of the quantity

$$A_t = \int_0^t \gamma(\tau) e^{-x(\tau)} d\tau, \tag{1}$$

where $x(\tau)$ denotes a stochastic process and $y(\tau)$ is a deterministic function.

This important quantity has been studied by several authors in the actuarial literature as well as in the theory of stochastic processes (see [1-7]).

In case $x(\tau)$ denotes a stochastic process starting in zero, hence $x(0) = 0$, the interpretation of (1) is clear. Indeed, (1) can be written as

$$A_t = \int_0^t \gamma_0(\tau)\exp\{-\int_0^\tau (\delta d\tau_0 + dx(\tau_0))\}d\tau. \tag{2}$$

From (2) it follows that A_t is the discounted value of a continuous stream of payments $\gamma_0(\tau)$ through $(0,t]$, discounted at an interest intensity $I(\tau)$ defined by

$$dI(\tau) = \delta\tau + dx(\tau) \tag{3}$$

In many applications, one assumes that $x(\tau)$ denotes a Wiener process, starting in zero, with variance $\sigma^2\tau$. Remark that Wiener processes are always stationary and have independent increments. Hence, in this case we have that for each $0 \leq \tau_a \leq \tau_b$, $x(\tau_b) - x(\tau_a)$ is normally distributed with expectation 0 and variance $\sigma_2(\tau_b - \tau_a)$. Further, for each positive integer n and for all $0 \leq \tau_0 \leq \tau_1 \leq \ldots \leq \tau_n$ the random variables $x(\tau_1), x(\tau_2) - x(\tau_1), \ldots, x(\tau_n) - x(\tau_{n-1})$ are independent.

The generating function of A_t can then be written down explicitely by considering a discretization of the Brownian motion with respect to the time variable, by dividing the interval in n subintervals of length ε and subsequently letting n tend to infinity. Following Vanneste et al. [16] or De Schepper [10], one obtains

$$\begin{aligned} E(e^{-uA_t}) = \; & \int_{-\infty}^{+\infty} dx_n \lim_{n\to\infty}\left(\frac{1}{\sqrt{2\pi\sigma^2\varepsilon}}\right)^n \\ & \int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty}\prod_{j=1}^{n-1} dx_j \exp\left\{-\frac{1}{\sigma^2}\sum_{j=0}^{n-1}\frac{(x_{j+1}-x_j)^2}{2\varepsilon} - \right. \\ & \left. \sum_{j=0}^{n-1} u\varepsilon\gamma_j e^{-x_j}\right\}, \end{aligned} \tag{4}$$

where $x_j = x(j\varepsilon)$, $\gamma_j = \gamma(j\varepsilon)$.

Unfortunately, an analytical form of this generating function is known only in a limited number of cases.

First, an analytic form is known for the generating function of a zero coupon bond, where $\gamma(\tau) = \delta(t-\tau)e^{-\delta\tau}$.

A second case with known analytic results follows from chosing $\gamma(\tau) = e^{-\delta\tau}$. In this case, one has that $\gamma_0(\tau) \equiv 1$ and $A_t = \int_0^t e^{-\delta\tau - x(\tau)}d\tau$ can be interpreted as a continuous annuity.

The following expression can then be derived for the probability density f_t of the annuity:

$$\begin{aligned} f_t(x) = \; & \frac{4\sqrt{2}}{\pi\sqrt{\pi}}\frac{1}{\sigma^3}\frac{1}{\sqrt{t}x^2} e^{-\frac{\delta^2 t}{2\sigma^2}} e^{\frac{2\pi^2}{\sigma^2 t}} e^{-\frac{2}{\sigma^2 x}} \int_0^\infty dz\, z^{-\frac{2\delta}{\sigma^2}} e^{-\frac{2z^2}{\sigma^2 x}} \\ & \int_0^{+\infty} dy\, e^{-\frac{2y^2}{\sigma^2 t}} \exp\left\{-\frac{4z}{\sigma^2 x}\cosh(y)\right\}\sinh(y)\sin\left(\frac{4\pi y}{\sigma^2 t}\right) \end{aligned} \tag{5}$$

(see GEMAN et al. [9] and DE SCHEPPER [10]). From this expression we can obtain the probability density of the perpetuity with random interest rate:

$$f_\infty(x) = \frac{1}{\Gamma\left(\frac{2\delta}{\sigma^2}\right)} \left(\frac{2}{\sigma^2}\right)^{\frac{2\delta}{\sigma^2}} \left(\frac{1}{x}\right)^{1+\frac{2\delta}{\sigma^2}} e^{-\frac{2}{\sigma^2 x}} \tag{6}$$

which is the density function of an inverse gamma distributed random variable, see DUFRESNE [1] and DE SCHEPPER ET AL. [4].

GEMAN and YOR [8, 9] also obtained expressions for the moments of the distribution of $A_t(\gamma(\tau) = e^{-\delta\tau})$, see also DE SCHEPPER [10]. These results can be extended to the case where $x(\tau)$ denotes an Ornstein-Uhlenbeck process starting at zero, determined by the parameters σ^2 and α. In this case, an expression for the moment generating function is given by

$$\begin{aligned} E(e^{-uA(t)}) = \sum_{n=0}^{\infty} (-\frac{u}{2\alpha})^n e^{\frac{\sigma^2 n}{2}} \int_{e^{-2\alpha t}}^{1} dt_n \, t_n^{\frac{\delta}{2\alpha}-1} \int_{t_n}^{1} dt_{n-1} \, t_{n-1}^{\frac{\delta}{2\alpha}-1} \cdots \\ \int_{t_2}^{1} dt_1 \, t_1^{\frac{\delta}{2\alpha}-1} \exp\left\{ -\frac{\sigma^2}{2}(t_1 + t_2 + \ldots + t_n) - \sigma^2(\sqrt{t_1 t_2} + \right. \\ \left. \sqrt{t_2 t_3} + \ldots + \sqrt{t_{n-1} t_n}) + \sigma^2 \left(\sqrt{\frac{t_2}{t_1}} + \sqrt{\frac{t_3}{t_1}} + \ldots + \sqrt{\frac{t_n}{t_{n-1}}} \right) \right\} \end{aligned} \tag{7}$$

(see DE SCHEPPER [10]).
In order to obtain the moments in an analytical form, the n integrations have to be worked out. It is not clear yet how this can be done.

As mentioned above, the general problem of the evaluation of the distribution function of A_t in case $x(\tau)$ is a Wiener process has not yet been solved and it seems to be a problem for which the corresponding case in mathematical physics, namely the quantum theoretical approach to the time-dependent potential $V(x,t) = \gamma(t)e^{-x}$, and the calculation of the corresponding kernel has not yet been solved either. Therefore, the question arises if there are stochastic processes for which the distribution of A_t can be evaluated analytically and which are still acceptable (or even more acceptable) for practical actuarial applications. Instead of considering the stochastic differential equation $dI(\tau) = \delta d\tau + \sigma^2 dx(\tau)$ (where x is a standard Wiener process), we consider a modified version, making the variance depend on I, e.g. by considering $dI(\tau) = (\alpha + \beta e^{I(\tau)})d\tau + 2\sigma e^{I(\tau)/2} dx(\tau)$. Considering $I = -2\ln J$, application of Itô calculus results in

$$dJ(\tau) = \left(-\frac{\alpha}{2} J(\tau) + \frac{1}{2J(\tau)}(\sigma^2 - \beta) \right) d\tau - \sigma dx(\tau).$$

We have to consider $\int_0^t \gamma(\tau) J^2(\tau) d\tau$, hence in case the stochastic term $\sigma dx(\tau)$ is put equal to zero one obtains for the deterministic path

$$\int_0^t \gamma(\tau) \left[e^{-\alpha\tau} \left(1 - \frac{\sigma^2 - \beta}{\alpha} \right) + \frac{\sigma^2 - \beta}{\alpha} \right] d\tau.$$

Let $z = \frac{\sigma^2-\beta}{\alpha}$, then in the deterministic case one obtains

$$(1-z)\int_0^t \gamma(\tau)e^{-\alpha\tau}d\tau + z\int_0^t \gamma(\tau)d\tau,$$

which is a credibility average between the nominal and discounted value of the payments $\gamma(\tau)$.

Consequently we have to evaluate

$$\begin{aligned} E(e^{-uA_t}) = \quad & \int_0^{+\infty} dx_n \lim_{n\to\infty}\left(\frac{1}{\sqrt{2\pi\sigma^2\varepsilon}}\right)^n \frac{f(x_n)}{f(u)} \\ & \int_0^{+\infty}\cdots\int_0^{+\infty}\prod_{j=1}^{n-1} dx_j \exp\left\{-\frac{1}{\sigma^2}\sum_{j=0}^{n-1}\frac{(x_{j+1}-x_j)^2}{2\varepsilon}\right. \\ & \left. -\alpha\sum_{j=0}^{n-1}x_j^2\varepsilon - \beta\sum_{j=0}^{n-1}\frac{\varepsilon}{x_j^2} - \varepsilon u\sum_{j=0}^{n-1}\gamma_j x_j^2\right\} \end{aligned}$$

for a suitable choice of f. The $(n-1)$-fold integrations can be worked out to give

$$E(e^{-uA_t}) = c\int_{-\infty}^{+\infty} dx \frac{f(x)}{f(u)} \exp\left(-\frac{\zeta(t)}{2\eta(t)}u^2 - \frac{\dot\eta(t)}{2\eta(t)}x^2\right)\sqrt{xu}\, I_{\sqrt{2\beta+1/4}}\left(\frac{ux}{\eta(t)}\right)$$

where $\eta(t)$ and $\zeta(t)$ are solutions of $\ddot\nu(t) = (\alpha + u\gamma(\tau))\nu(\tau)$ such that $\eta(0) = 0, \dot\eta(0) = 1, \zeta(0) = 1, \dot\zeta(0) = 0$, see Vanneste et al. [16].

An inversion with respect to u leads to an expression for the density of the quantity A_t. Although the known analytical results for the density of A_t are limited, this problem deserves a lot more attention because it can be applied in many situations arising in actuarial practice. In the subsequent sections of the paper we aim at presenting some of the situations in which the distribution of A_t is important.

2. A pure financial application

In an Asian option, the payoff is defined in terms of the average value of the underlying asset during a certain time period rather than in terms of its final value.

The exact pricing of Asian options is clearly a difficult exercise because the distribution of an arithmetic average of stock prices is unknown (and in particular not lognormal) when prices themselves are lognormally distributed.

Options based on average prices is an attractive feature for thinly traded assets and commodities where price manipulations near the option expiration date are possible. Some options on domestic interest rates exhibit the Asian feature when the base rate is an arithmetic average of spot rates.

Let $S(u)$ be the stock price at time u. The payoff of the Asian call option at maturity t is then given by

$$\left(\frac{1}{t}\int_0^t S(u)du - k\right)_+$$

It can be proven that under certain assumptions (see e.g. GEMAN ET AL. [9]), the Asian option pricing problem is solved if the following quantity can be determined:

$$E(\int_0^t e^{2v\tau+2x(\tau)}d\tau - q)_+ \tag{8}$$

where $x(\tau)$ is a standard Brownian motion starting from 0, and q and ν are real numbers, depending on the parameters involved. In order to cast this expectation into an analytical form, the distribution of $\int_0^t e^{2x(\tau)}d\tau$ has to be evaluated. This evaluation can be done by using the result displayed in formula (5).

3. An application for life insurance premiums

A frequently used law for modelling the mortality intensity μ_x in life insurance is the so called Makeham law which states that

$$\mu_x = \beta + \gamma c^x \tag{9}$$

where the probabilistic interpretation is immediate. This model is valid in case mortality is a static characteristic. In case mortality changes in time, an extension of the model could be obtained by modelling the probability that a person is alive at $x+t$ given that he is alive at x by

$$_tp_x = e^{-\beta t + x(t)} g^{c^x(c^t-1)}. \tag{10}$$

A continuous life annuity can then be cast into the form

$$A_t(\gamma(\tau) = e^{-\beta\tau} g^{c^x(c^\tau-1)}) = \int_0^t e^{-\beta\tau+x(\tau)} g^{c^x(c^\tau-1)} e^{-\delta\tau - y(\tau)} d\tau. \tag{11}$$

It is clear that the stochastic effect is now the result of $x(\tau)-y(\tau)$. While it can be argued that $y(\tau)$ is a pure interest stochastic process and should therefore satisfy the usual assumptions on the term structure of interest as e.g. explained in COX, INGERSOLL and ROSS [11], it is clear that the combined effect of $x(\tau)$ and $y(\tau)$ satisfies other basic assumptions.

We notice that in practice, c is close to 1. Consequently, the right hand side of (11) could be approximated by means of

$$\int_0^t e^{-(\beta+\delta+c^x \ln g \ln c)\tau} e^{x(\tau)-y(\tau)} d\tau. \tag{12}$$

Following BEEKMAN ET AL. [6], the expectation and the variance can be obtained in the case that

$$F(\tau) = e^{x(\tau)-y(\tau)},$$

with

$$dF(\tau) = \frac{1}{2}F(\tau)d\tau + F(\tau)dw(\tau),$$

where $w(\tau)$ is a Brownian motion. Taking a drift term proportional to $1/2F(\tau)$ ensures us that the average interest intensity is equal to zero. In our present model the same can be obtained by changing δ in (12) into $\delta + 1/2$ and considering

$$dF(\tau) = F(\tau)dw(\tau). \tag{13}$$

A pure actuarial motivation for changing δ in $\delta + 1/2$ (in case $\sigma^2 = 1$) is found e.g. in GERBER and SHIU, see ref. 13–15. They introduced the Esscher transform in order to obtain a risk neutral valuation principle (or what is the same, they used an equivalent martingale measure defined by the Esscher transform).

Let $S(t)$ denote the price of a stock at time t. Let the interest intensity be described by a process $\{x(t)\}_{t\geq 0}$ with stationary and independent increments with $x(0) = 0$, such that $S(0) = S(t)e^{-x(t)}$. As an example they consider $x(t)$ a Wiener process with mean μ and variance σ^2 per unit time. Then $F(x,t) = N(x;\mu t;\sigma^2 t)$ with generating function $M(z,t,) = \exp\left\{\left(-\mu z + \frac{\sigma^2 z^2}{2}\right)t\right\}$. The Esscher transform has as generating function $M(z,t,h) = M(z+h,t)/M(h,t)$. Hence the distribution of the Esscher transform reads

$$F(x,t,h) = N(x, \mu t + h\sigma^2 t, \sigma^2 t).$$

In order to obtain a risk neutral valuation principle, h (the Esscher parameter) is selected to have $S(0) = E^*(e^{-\delta t}S(t))$ where * indicates that the expectation is done with respect to the Esscher measure.

4. AN APPLICATION TO PENSION FUND CALCULATIONS

In a pension fund there are two essential streams of cash flows, namely pensions have to be paid out (eventually capitals at the retirement date) and contributions come in. This payment streams can be visualised on the following time axis

pensions:	$-P(0)$	$-P(1)$	$-P(t)$
contributions:	$+C(0)$	$+C(1)$	$+C(t)$
time:	0	1	t

where we consider a time horizon t.

In the sequel, we assume that pensions and contributions are paid continuously through time. The total amount of pensions paid at time t is denoted by $P(\tau)$, while the the total amount of contributions at time τ is given by $C(\tau)$.

We assume that the total number of active persons (= the contribution paying persons) and the total number of retired persons (= the persons who receive pensions) at time τ are both proportional to $e^{k(\tau)}$ where $k(\tau)$ denotes some stochastic process to be specified. Further, it is assumed that the salary income at time τ is proportional to $e^{s(\tau)}$, where $s(\tau)$ denotes some stochastic process also to be specified. We also assume that pensions increase with the same intensity $s(\tau)$ as the salaries. We assume that the contributions at any time are defined as a fixed percentage of the salary income at that time. Hence, the contributions at time τ are also proportional to $e^{s(\tau)}$.

The total contribution function $C(\tau)$ and the total pension function $P(\tau)$ are then modelled by

$$C(\tau) = c(\tau)e^{k(\tau)+s(\tau)},$$

$$P(\tau) = p(\tau)e^{k(\tau)+s(\tau)},$$

where $c(\tau)$ and $p(\tau)$ are given deterministic functions of time.

The value at the evaluation date 0 of the net-income of the pension fund at time t is then given by

$$c(\tau) - p(\tau))e^{k(\tau)+s(\tau)-x(\tau)}$$

where $x(\tau)$ is the stochastic process describing the interest intensity. The total value of the fund, evaluated prospectively as the discounted future income minus the discounted value of the pensions to be paid on a time horizon (o, t), is then given by

$$\int_0^t (c(\tau) - p(\tau))e^{k(\tau)+s(\tau)-x(\tau)}d\tau. \tag{14}$$

It is clear that, although the same type of expression is obtained as in (1) with $x(\tau)$ replaced by the combined effect of $k(\tau)+s(\tau)-x(\tau)$, the stochastic process to be assumed for this resulting intensity not necessarily satisfies the same basic hypotheses of pure financial models. The main difference is that while in financial theory interest is supposed not to become negative the resulting intensity certainly can become negative. It is sufficient to have a firm with a decreasing number of new entrants to the fund to have a negative k, so that it is possible that a negative global intensity occurs.

In order to cope with this situation we could consider a stochastic process $z(\tau)$ derived from the Lagrangian

$$\frac{1}{2}\left(e^{-2z}(\dot{z})^2 + ae^{-2z} + \beta e^{2z}\right), \tag{15}$$

as explained in DE VIJLDER ET AL. [15]. In that paper, the distribution of $A_t = \int_0^t \gamma(\tau)e^{-2z(t)}d\tau$ is evaluated by means of its Laplace transform

$$E(e^{-uA_t}) = c\int_{-\infty}^{+\infty} dx \frac{1}{\eta(t)} \exp\left(-\frac{\zeta(t)}{2\eta(t)} - \frac{\dot{\eta}(t)}{\eta(t)}e^{-2x}\right) e^{-x/2} I_{\sqrt{2\beta+1/4}}(\frac{e^{-x}}{\eta(t)})$$

where η and ζ are the solutions of the differential equation $\ddot{v}(\delta) = (\alpha + u\gamma(\delta))v(\delta)$ subject to some boundary conditions, see VANNESTE ET AL. [16]. Performing an inversion with respect to u leads to an expression for the density of A_t.

5. Reserving in Liability Insurance

For the unpaid losses, an insurance company will create a loss reserve, often called IBNR reserve. To estimate this reserve correctly is very important. Therefore, there have been developed several methods to estimate the IBNR reserves. Consider a run-off triangle with average loss figures X_{js}, representing the size of claims incurred in year of origin j and to be paid in development year s, divided by the estimated number of claims. One of the methods used to estimate future values of these losses X_{js} is the separation method where

$$\hat{X}_{js} = \hat{\lambda}_{j+s-1}\hat{r}_s$$

with λ_s describing the evolution per calendar year (e.g. allowing for inflation) and where r_s can be interpreted as fractions of claims finalized in their s-th development year, where j and s are integers. In fact, we will consider the following run-off triangle which is modeled by the parameters r and λ:

Year of origin	**development Year**				
	1	**2**	...	**t − 1**	**t**
1	$r_1\lambda_1$	$r_2\lambda_2$	...	$r_{t-1}\lambda_{t-1}$	$r_t\lambda_t$
2	$r_1\lambda_2$	$r_2\lambda_3$	...	$r_{t-1}\lambda_t$	
⋮	⋮	⋮	...		
t − 1	$r_1\lambda_{t-1}$	$r_2\lambda_t$			
t	$r_1\lambda_t$				

We assume that λ_s and r_s are estimated by the separation method or by some lognormal model for $s \le t$. For expectations of future losses X_{js} we need estimates for the future values of λ_s. In classical methods these future values have been determined by extrapolation or by exogenous estimates for inflation year by year. This means that j and s are integers (considering a discrete basis) such that the loss reserves are considered to be constant during a year.

Now, we will indicate how this methodology can be extended in a continuous way. Therefore, we assume that these values can be described by

$$\lambda_{\tau,\tau'}e^{-2x(\tau+\tau'-t)},$$

where $x(\tau + t' - t)$ is a stochastic process denoting the fluctuations of λ and where

$$\lambda_{\tau,\tau'} = r_{[t']+1}\lambda_{[\delta+\delta']}e^{-\delta(\delta+\delta'-t)},$$

where as usual $[k]$ denotes the largest integer smaller than k. The subindex δ is related to the time of origin (somewhere between j and $j+1$) and δ' denotes the development time (between s and $s+1$). It is clear that for $0 \leq \tau \leq t$ and $0 \leq \tau' \leq t-\tau$ the losses $X_{\tau,\delta'}$ are known where for $0 \leq \tau \leq t$ and $t-\tau \leq \tau' \leq t$, the realizations of $X_{\delta,\delta'}$ or their discounted values have to be estimated.

Our aim is the evaluation of the distribution of the reserve V_t given by

$$V_t = \int_0^t d\tau \int_{t-\tau}^t d\tau' \lambda_{\tau,\tau'} e^{-2x(\tau+\tau'-t)}.$$

By transformation of time, V_t can be cast into the following form:

$$V_t = \int_0^t d\tau \int_0^\tau d\tau'' \lambda_{\tau,\tau''-\tau+t}\; e^{-2x(\tau'')}d\tau'' = \int_0^t R(\tau'')e^{-2x(\tau'')}d\tau'',$$

having interchanged the order of integration, and where

$$R(\tau'') = \int_{\tau''}^t \lambda_{\tau,\tau''-\tau+t} d\tau$$

denotes a deterministic function of τ'', for which we can use the separation method to estimate the amount by:

$$\sum_{k=[\tau'']}^t \tau_{[\tau'']+t-k}\hat{\lambda}_{[\tau'']+t-1}e^{-\delta[\tau'']}.$$

As said before, we need a stochastic process $x(\tau)$ measuring the randomness of the rate of return, for which we may assume that the following properties are fulfilled:

1. a negative rate of return is allowed although exceptional because the rate of return is the result of inflation like effects and other stochastic growth factors;
2. the variance of the rate of return increases when the rate of return itself increases;
3. extreme values of the rate of return are unlikely.

It is clear that the stochastic correction can also be introduced along the other dimensions of the problem (e.g. development year, year of origin or calender year but taking into account other factors than inflation).

Again the process derived from the Lagrangian given in (15) can be used to obtain the distribution of the reserve.

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