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No-arbitrage, Change of Measure and Conditional Esscher Transforms

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1. Introduction

This paper grew out of a seminar at the Department of Mathematics at the ETH, Zürich during the Summer Semester of 1995 on the subject of mathematical finance and insurance mathematics. It should be viewed as a contribution towards bridging the existing methodological gap between both fields, especially in the area of pricing derivative instruments. Both insurance and finance are interested in the fair pricing of financial products. For instance, in the case of car insurance, depending on the various characteristics of the driver, a so-called net premium is calculated which should cover the ecpected losses over the period of the contract. To this net premium, various loading factors (for costs, fluctuations,...) are added. The resulting gross premium is also subject to market forces which imply that a market-conform premium is finally charged. The more an insurance market is liquid (many potential offers of insurance, deregulated markets), the more a " correct, fair" price may be expected to emerge. Very important in the process of determining the above premium is the attitude of both parties involved towards risk. Within the more economic literature this attitude towards risk can be described through the notion of utility. Utility theory enters as a tool to provide insight into decision making in the face of uncertainty. For a very readable introduction within the context of insurance, see Bowers et al. [2]. An alternative economic tool is equilibrium theory. Depending on the economic theory used, a multitude of possible premiums may result, one of which is the time–honoured Esscher principle. Rather than being based on the expected loss itself, the Esscher principle starts from the expectation of the loss under an exponentially transformed distribution, properly normalised. In Bühlmann [3], [4], the Esscher principle is discussed within the utility and equilibrium framework. Besides the pricing of individual risks (claims, say), more complicated insurance products involve time and hence are based on specific stochastic processes. The classical insurance risk processes are of the compound Poisson type or their generalisations like mixed and doubly stochastic compound Poisson processes. The main feature of such processes, making them distinct from the typical diffusion type models in finance, is their jump structure. Indeed, when we turn to fair pricing in finance, the standard reasoning uses the so-called no-arbitrage (or no free lunch) approach which says that there is no such thing as a riskless gain. The precise mathematical formulation of this economic principle brings in the by now fundamental notion of risk neutral martingale measure. In the case where the underlying stochastic process is " nice " (geometric Brownian motion, say), exactly one such measure exists and the fair price of a contingent claim is the expectation with respect to this measure, properly discounted. The latter, so-called complete case is rare in insurance. Due to the jump structure of standard risk processes, we are in the so-called incomplete case. As a consequence, risk cannot fully be hedged away and in most cases, there will be infinitely many such equivalent martingale measures so that pricing is directly linked to an attitude towards risk. Whereas in classical insurance, the question becomes "which premium principle to use", within the (incomplete) finance context it becomes "which equivalent martingale measure to use". This is exactly the point where the Esscher transform enters as one of the possible pricing candidates. Going back to a fundamental paper of ESSCHER [12], the Esscher transform is by now standard methodology in insurance, gradually however its appearance within mathematical finance is becoming more and more prominent: see for instance the beautiful paper by GERBER and SHIU [15] and the references and discussions therein. An interesting paper, coming more from the realm of mathematical finance is GRANDITS [16]. The present paper should be looked at in conjunction with BÜHLMANN ET. AL. [5] where special attention is given to discrete models. As explained above, typical insurance processes involve a jump component besides a possible diffusion term. It is therefore natural to present the necessary mathematical methodology needed for discussing pricing within both insurance and finance within the wider theory of semi-martingales. This is exactly what is done in the present paper. The classical notion of Esscher transform for distribution functions is generalised to stochastic processes. For a discussion of Esscher transform in a distributional context, see JENSEN [20]. In EMBRECHTS ET. AL. [11] an application to the approximation of the total claim amount distribution in the compound Poisson and negative binomial case is given.

1.1. Some notation

Suppose that a financial process (stock returns, spot rates, zero coupon bonds, value of a derivative instrument, $\cdots$) $S = (S_t)_{t\geq 0}$ is given on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ where $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ denotes the " flow of information". Mathematically the latter means that $\mathbb{F}$ consists of an increasing family of sub σ- algebras, i.e. for all $s \le t,\ \mathcal{F}_s \subset \mathcal{F}_t \subset \mathcal{F}$. Assume further that S is of "exponential form",

$$S_t = S_0\, e^{H_t}, \qquad H_0 = 0, \qquad t \ge 0, \tag{1}$$

where $H = (H_t)_{t \ge 0}$ is a semimartingale with respect to $\mathbb{F}$ and P. The latter will be denoted by $H \in$ Sem $(\mathbb{F}, P)$ or $H \in$ Sem (P). We remark that the notion of semimartingale does not depend on the measure P. More precisely, if Q $\sim$ P are two equivalent probability measures, then Sem (P) = Sem (Q). For a precise definition see for instance Jacod and Shiryaev [19] and Rogers and Williams [28]. Using Itô's formula for $f \in C^2$, one obtains:

$$\begin{aligned} f(H_t) = f(H_0) + \int_0^t f'(H_{s-}) dH_s + \frac{1}{2}\int_0^t f''(H_{s-}) d\langle H^c\rangle_s \\ + \sum_{0<s\le t} \big[f(H_s) - f(H_{s-}) - f'(H_{s-})\Delta H_s\big], \end{aligned} \tag{2}$$

where $\Delta H_s = H_s - H_{s-}$ and $\langle H^c\rangle$ is a quadratic characteristic of the continuous martingale part H^c of H. Hence for the case (1) above:

$$dS_t = S_{t-} d\hat{H}_t \tag{3}$$

with

$$\hat{H}_t = H_t + \frac{1}{2}\langle H^c\rangle_t \;+ \sum_{0<s\le t} (e^{\Delta H_s} - 1 - \Delta H_s). \tag{4}$$

In the class of semimartingales the linear equation (3) has a unique solution:

$$S_t = S_0\, \mathcal{E}(\hat{H})_t \tag{5}$$

where $\mathcal{E}(\hat{H})$ is called the *Doléans stochastic exponential*

$$\mathcal{E}(\hat{H})_t = \exp\left\{\hat{H}_t - \frac{1}{2}\langle \hat{H}^c\rangle_t\right\} \prod_{0<s\le t} (1 + \Delta\hat{H}_s) e^{-\Delta \hat{H}_s}. \tag{6}$$

It should be remarked that for every semimartingale $H = (H_t)$, with probability one,

$$\sum_{0<s\le t} |\Delta H_s|^2 \,<\, \infty, \quad \forall t > 0. \tag{7}$$

From (7) it immediately follows that for each $t > 0$, there are only finitely many time points $s \le t$ such that $|\Delta H_s| > \frac{1}{2}$. Consequently, the infinite sums and products in (4) and (6) are absolutely convergent and hence $\hat{H}$ and $\mathcal{E}(\hat{H})$ are well defined.

1.2. Discrete time

Consider the set-up (1) but now in discrete time,

$$S_n = S_0\, e^{H_n}, \quad H_0 = 0, \quad n = 0, 1, 2, \cdots \tag{8}$$

where $H = (H_n)_{n \geq 0}$ is a stochastic sequence defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_n)_{n \geq 0}, P)$. Clearly, (8) can formally be considered as a special case of (1) by defining

$$\mathcal{F}_t = \mathcal{F}_n \ , \ H_t = H_n \ , \ n \leq t < n+1.$$

Put

$$\hat{H}_n = \sum_{0 < k \leq n} (e^{\Delta H_k} - 1) \tag{9}$$

(to be compared with (4)), then we obtain

$$S_n = S_0 \prod_{0 < k \leq n} (1 + \Delta \hat{H}_k) = S_0 \mathcal{E}(\hat{H})_n. \tag{10}$$

The latter should be compared with (5) and (6). In the sequel we denote

$$h_k = \Delta H_k (= H_k - H_{k-1})$$

and

$$\hat{h}_k = \Delta \hat{H}_k (= \hat{H}_k - \hat{H}_{k-1}).$$

Recall that

$$h_k = ln \frac{S_k}{S_{k-1}}$$

and hence can be viewed as a *compound return*, whereas

$$\hat{h}_k = \frac{S_k}{S_{k-1}} - 1 = \frac{\Delta S_k}{S_{k-1}} = e^{h_k} - 1$$

stands for *simple return*. Using this terminology and the correspondances stated above, (1) can be viewed as a continuous model for compound return, whereas (5) is the continuous analogon of simple return. It is useful to remark that the representation (1) lends itself naturally for *statistical data analysis*. However, with respect to *probabilistic analysis*, the representation (5) turns out to be more advantageous. An example of the latter is the following: *$\mathcal{E}(\hat{H})$ is a local martingale if $\hat{H}$ is a local martingale.*

1.3. No-arbitrage and equivalent martingale measures.

The " equivalence " of the notions *no-arbitrage, no free lunch* and the *existence of equivalent martingale measures* belongs to the folklore of mathematical finance. The key underlying idea is the local equivalence of martingale measures, i.e. $\tilde{P} \overset{loc}{\sim} P$ on $(\Omega, \mathcal{F})$ meaning that for each $t > 0$, $\widetilde{P}_t \sim P_t$ (equivalence of probability measures) where $P_t = P|\mathcal{F}_t$, $\widetilde{P}_t = \widetilde{P}|\mathcal{F}_t$ and such that $S = (S_t)$ is a martingale or local martingale with respect to $\widetilde{P}$.

In discrete time, $n = 0, 1, \cdots, N,$ the precise formulation of the above is as follows.

Equivalent are

(a) *no-arbitrage, and*

(b) *there exists a probability measure* $\widetilde{P}$ *on* $(\Omega, \mathcal{F})$ *so that* $\widetilde{P}_N \sim P_N$ *and* $S = (S_n)_{n \leq N}$ *is a* $\widetilde{P}_N$ *-martingale.*

In the continuous time case, the situation is much more delicate. A solution is to be found in Delbaen and Schachermayer [7] and [8] and the references therein. Independent of the precise equivalence statements, the construction of *all* equivalent martingale measures in a particular situation is important. A slightly less ambitious goal would be the construction of certain subclasses. The main aim of our paper is exactly the solution of this technical problem. We shall also discover the so-called *conditional Esscher transform* as a special case of the change of measure paradigm in stochastic calculus.

2. Some facts about semimartingales

2.1. Definition

Below we summarise the basic definitions and results concerning semimartingale theory of relevance in insurance and finance. The *càdlàg* (right-continuous with left limits) stochastic process $H = (H_t)_{t \geq 0}$ defined on a filtered probability space $(\Omega, \mathcal{F}, (F_t)_{t \geq 0}, P)$ is a *semimartingale* if H admits a *canonical decomposition*

$$H_t = H_0 + A_t + M_t, \quad t \geq 0, \tag{11}$$

where $A = (A_t) \in \mathcal{V}$ (a process of *bounded variation*), $M = (M_t) \in \mathcal{M}_{loc}$ (a *local martingale*). Furthermore, we have that for each $t \geq 0, A_t$ and M_t are $\mathcal{F}_t$-measurable.

We recall that $M \in \mathcal{M}_{loc}$ if and only if there exists a sequence of $(\mathcal{F}_t)_{t \geq 0}$ -stopping times $(\tau_n)_{n \geq 1}$ such that $\tau_n \uparrow \infty$ $(P - a.s.)$ for $n \to \infty$ and for each $n \geq 1$, the *stopped process*

$$M^{\tau_n} = (M_t^{\tau_n}) \quad \text{with} \quad M_t^{\tau_n} = M_{t \wedge \tau_n}, \quad n \geq 1,$$

is a *martingale*:

$$E|M_t^{\tau_n}| < \infty, \quad E(M_t^{\tau_n}|\mathcal{F}_s) = M_s^{\tau_n} \ (P - a.s.), \quad s \leq t.$$

We would like to stress that local martingales are more than just martingale modulo boundedness conditions. Indeed, there exist local martingales possessing strong integrability properties which nonetheless are *not* martingales. See for instance REVUZ AND YOR [26], Chapter V, Exercise (2.13) where a local martingale is given, bounded in L^2, but which is not a martingale. In the case of discrete time, we have the following nice *characterisation of local martingales*; see for instance JACOD AND SHIRYAEV [19], Chapter 1, 1.64 or LIPTSER AND SHIRYAEV [23], Chapter VII,§1. Let $X = (X_n)_{n \geq 0}$ be a stochastic sequence defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_n)_{n \geq 0}, P)$. X is assumed adapted, i.e. X_n is $\mathcal{F}_n$ - measurable for all $n \geq 0$ and $E|X_0| < \infty$. Then the following conditions are equivalent:

(1) X is a *local martingale*,

(2) X is a *martingale transformation*, i.e. there exists a martingale $Y = (Y_n)$ and a *predictable* sequence $V = (V_n)$ (meaning that for each $n \geq 1$, V_n is $\mathcal{F}_{n-1}$ - measurable) such that for $n \geq 1$:

$$X_n = X_0 + \sum_{0 < k \leq n} V_k \Delta Y_k, \quad \Delta Y_k = Y_k - Y_{k-1},$$

(3) X is a *generalised martingale*, i.e.

$$E(|X_n| \, |\mathcal{F}_{n-1}) \quad < \infty, \quad n \geq 1,$$

and

$$E(X_n | \mathcal{F}_{n-1}) = X_{n-1}.$$

(The key point in the latter conditions is that we do not assume integrability of $X_n, n \geq 1$)

REMARK: The condition (2) above can be interpreted as X_n *is the value of a trading strategy* V *on an underlying asset* Y. This shows that the notion of local martingales lies at the heart of stochastic processes in finance and insurance. Unfortunately, the continuous time analogue of the above result is false.

2.2. Semimartingale representations

Denote by $\mu = \mu(\omega; ds, dx)$(or $d\mu$) the *measure describing the jump structure* of H:

$$\mu(\omega; (0, t,] \times A) = \sum_{0 < s \leq t} I(\Delta H_s(\omega) \in A), \ \ t > 0.$$

where $A \in \mathcal{B}(\mathbb{R} \setminus \{0\}), \Delta H_s = H_s - H_{s-}$ and $I(\cdot)$ stands for the indicator function. By $\nu = \nu(\omega; ds, dx)$ (or dv) we denote a *compensator* of μ, i.e. a predictable measure (see JACOD AND SHIRYAEV [19], Chapter II, 1.8) with the property that $\mu - \nu$ *is a local martingale measure.* This means that for each $A \in \mathcal{B}(\mathbb{R}\setminus\{0\})$:

$$\big(\mu(\omega;(0,t] \times A) - \nu(\omega;(0,t] \times A)\big)_{t>0}$$

is a local martingale with value 0 for $t = 0$. The latter property is *almost* equivalent to the local martingale property of the signed measure $\mu - \nu$. We shall not enter into the subtle difference here.

A semimartingale $H = (H_t)_{t \geq 0}$ is called *special* if there exists a decomposition (11) with a *precticable* process $A = (A_t)_{t \geq 0}$. See JACOD AND SHIRYAEV [19] where it is also shown that every semimartingale with *bounded jumps* $(|\Delta H_t(\omega)| \leq b < \infty,\ \omega \in \Omega,\ t > 0)$ is special.

Let φ be a truncation function, e.g. $\varphi(x) = xI(|x| \leq 1)$. Then $\Delta H_s - \varphi(\Delta H_s) \neq 0$ if and only if $|\Delta H_s| > b$ for some $b > 0$. Hence

$$\overset{\vee}{H}(\varphi)_t = \sum_{0<s\leq t} (\Delta H_s - \varphi(\Delta H_s))$$

denotes the jump part of H corresponding to *big jumps*. The number of the latter is still finite on $[0,t]$, for all $t > 0$, because for all semimartingales

$$\sum_{0<s\leq t} (\Delta H_s)^2 < \infty,\ P - a.s.$$

The process $H(\varphi) = H - \overset{\vee}{H}(\varphi)$ is a semimartingale with *bounded jumps* and hence it is special:

$$H(\varphi)_t = H_0 + B(\varphi)_t + M(\varphi)_t, \tag{12}$$

where $B(\varphi)$ is a predictable process and $M(\varphi)$ is a local martingale.

Every local martingale $M(\varphi)$ can be decomposed as:

$$M(\varphi) = M^c(\varphi) + M^d(\varphi), \tag{13}$$

where $M^c(\varphi)$ is a *continuous* (martingale) part and $M^d(\varphi)$ is a *purely discontinuous* (martingale) part,

$$M^d(\varphi)_t = \int_0^t \int \varphi(x) d(\mu - \nu). \tag{14}$$

More details, including a proof of (14), are to be found in JACOD AND SHIRYAEV [19], Chapter II, 2.34. It is clear that

$$\overset{\vee}{H}(\varphi)_t = \int_0^t \int (x - \varphi(x)) d\mu. \tag{15}$$

Consequently H has the following canonical representation:

$$H_t = H_0 + B(\varphi)_t + M^c(\varphi)_t + \int_0^t \int \varphi(x) d(\mu - \nu) + \int_0^t \int (x - \varphi(x)) d\mu, \quad (16)$$

a formula going back to Lévy and Khintchin.

The continuous martingale part $M^c(\varphi)$ does not depend on φ and will be denoted by H^c (the *continuous martingale* part of H). Consequently,

$$H_t = H_0 + B(\varphi)_t + H_t^c + \int_0^t \int \varphi(x) d(\mu - \nu) + \int_0^t \int (x - \varphi(x)) d\mu. \quad (17)$$

Denote by $\langle H^c \rangle$ a predictable quadratic characteristic of H^c, i.e. $(H^c)^2 - \langle H^c \rangle$ is a local martingale.

We finally arrive at the *triplet of predictable characteristics* of the semimartingale H:

$$T(\varphi) = (B(\varphi), \langle H^c \rangle, \nu).$$

In the case $\varphi(x) = xI(|x| \leq 1)$ we denote $B = B(\varphi)$. Then (17) takes on the form:

$$H_t = H_0 + B_t + H_t^c + \int_0^t \int_{|x| \leq 1} x d(\mu - \nu) + \int_0^t \int_{|x| > 1} x d\mu. \quad (18)$$

In Jacod and Shiryaev [19], Chapter II, 2 it is shown that if H is a semimartingale, then

$$\Delta B(\varphi)_t(\omega) = \int \varphi(x) \nu(\omega; \{t\} \times dx),$$

where

$$\nu(\omega; \{t\} \times dx) = \nu(\omega; (0, t] \times dx) - \nu(\omega; (0, t) \times dx)$$

and

$$(x^2 \wedge 1) * \nu \in \mathcal{A}_{loc},$$

i.e. the process $(\int_0^t \int (x^2 \wedge 1) d\nu)_{t \geq 0}$ is locally integrable in so far that there exist stopping times $\tau_n \uparrow \infty$ as $n \to \infty$, such that for $n \geq 1$

$$E\Big(\int_0^{\tau_n} \int (x^2 \wedge 1) d\nu \Big) < \infty.$$

Using this notation, H turns out to be a *special semimartingale* if and only if

$$(x^2 \wedge |x|) * \nu \in \mathcal{A}_{loc}.$$

Further, H is a *square integrable semimartingale* if and only if

$$x^2 * \nu \in \mathcal{A}_{loc}.$$

If H is a *special semimartingale*, then the canonical representation (17) is valid with $\varphi(x) = x$, i.e.

$$H_t = H_0 + B_t + H_t^c + \int_0^t \int x d(\mu - \nu), \quad t \geq 0, \tag{19}$$

with $B = B(\varphi)$.

There are various reasons why semimartingales play a fundamental role in insurance and finance (and indeed in many more applications):

(i) They form a wide class of processes including stochastic sequences in discrete time, martingales, super - and sub - martingales, diffusion processes, diffusions with jumps, processes with independent incremements (if for every $\lambda \in \mathbb{R}, (Ee^{i\lambda H_t})_{t \geq 0}$ has bounded variation). This is especially important in the intersection of insurance and finance where models involving both a diffusion component as well as a jump component are relevant.

(ii) They form the most general class of stochastic processes for which a stochastic integration theory can be worked out, the latter is a consequence of the famous Bichteler, Dellacherie, Kussmaul, Métivier and Pellaumail theorem (see Rogers and Willliams [28], Section IV. 16). A full stochastic calculus, including Itô's lemma for semimartingales exists.

(iii) The knowledge that a stochastic process is *not* a semimartingale may have important implications in finance in so far that then often explicit arbitrage strategies can be worked out. A typical example concerning so-called fractional Brownian motion is to be found in Rogers [27]. See also Delbaen and Schachermayer [7] where it is shown that a very weak form of the no-arbitrage property implies that the price process is already a semimartingale.

2.3. Examples

Discrete time In this case we don't really need the heavy semimartingale machinery, we only include this case for illustrative purposes. Consider the stochastic sequence $H = (H_n)_{n \geq 0}$ with $h_n = \Delta H_n = H_n - H_{n-1}$.

Hence,

$$\begin{aligned} H_n = H_0 \quad &+ \sum_{0<k\le n} h_k \\ = H_0 \quad &+ \sum_{0<k\le n} \varphi(h_k) + \sum_{0<k\le n} (h_k - \varphi(h_k)) \\ = H_0 \quad &+ \sum_{0<k\le n} E[\varphi(h_k)|\mathcal{F}_{k-1}] \\ &+ \sum_{0<k\le n} (\varphi(h_k) - E[\varphi(h_k)|\mathcal{F}_{k-1}]) + \sum_{0<k\le n} (h_k - \varphi(h_k)). \end{aligned} \tag{20}$$

Define for all $A \in \mathcal{B}(\mathbb{R}\setminus\{0\}),\ k \ge 0:$

$$\mu_k(A) = I(h_k \in A) = I(\Delta H_k \in A),$$
$$\nu_k(A) = E[I(h_k \in A)|\mathcal{F}_{k-1}] = P(h_k \in A|\mathcal{F}_{k-1}),$$

where conditional expectations are always taken as *regular versions*. Then

$$\mu(\omega; (0,n] \times A) = \sum_{0<k\le n} \mu_k(A),$$
$$\nu(\omega; (0,n] \times A) = \sum_{0<k\le n} \nu_k(A),$$

yielding the *canonical representation* (see (17))

$$\begin{aligned} H_n = H_0 + B(\varphi)_n &+ \sum_{0<k\le n} \int \varphi(x) d(\mu_k - \nu_k) \\ &+ \sum_{0<k\le n} \int (x - \varphi(x)) d\mu_k, \end{aligned} \tag{21}$$

where

$$B(\varphi)_n = \sum_{0<k\le n} \int \varphi(x) d\nu_k. \tag{22}$$

(We could have written $\nu_k(dx)$ for $d\nu_k$ etc. ...) Because there is *no* continuous part, the *characteristic triplet* reduces to

$$T(\varphi) = (B(\varphi), 0, \nu) \tag{23}$$

where

$$B(\varphi) = (B(\varphi)_n)_{n\ge 0}, = A6 \quad \nu = (\nu_n)_{n\ge 1}.$$

Processes with independent increments (I.I.) A process $H = (H_t)_{t \geq 0}$ with I.I. is a semimartingale if and only if for each $\lambda \in \mathbb{R}, (Ee^{i\,\lambda\,H_t})_{t \geq 0}$ is a function of bounded variation. For a proof, see JACOD AND SHIRYAEV [19], Chapter II,4.14. A remarkable fact for such processes is that their triplet of predictable characteristics only has *deterministic components*. If $H = (H_t)$ is continuous in probability, then $B(\varphi)_t, < H^c >_t$ and $\nu((0,t] \times dx)$ are continuous in t and the Lévy - Khintchin formula yields

$$E\ \exp\{i\lambda(H_t - H_0)\} = \exp\ \{i\lambda B(\varphi)_t - \frac{\lambda^2}{2}C_t + \int_0^t \int (e^{i\lambda x} - 1 - i\lambda\varphi(x))\nu(ds \times dx)\} \quad (24)$$

where $C_t =< H^c >_t$ is the variance of the continuous Gaussian part of H, and $B(\varphi)$ and ν are the first and third component in the triplet $T(\varphi) = (B(\varphi), < H^c >, \nu)$ of H written in semimartingale form. If the I.I. process is moreover homogeneous (stationary), also referred to as a *Lévy process*, then

$$\begin{aligned} B(\varphi)_t &= tb(\varphi) \\ C_t &= tC \\ \nu(dt \times dx) &= dt \times F(dx) \end{aligned} \quad (25)$$

where F is a distribution function on $\mathbb{R}$. For a textbook treatment of Lévy processes, see BERTOIN [1]. Hence in this case the triplet $T(\varphi)$ is reduced to $(b(\varphi), C, F(dx))$.

Brownian motion with drift and Poisson jumps Suppose that

$$H_t = bt + \sigma W_t + \sum_{k=1}^{N_t} \xi_k \quad (26)$$

where $\xi, \xi_1, \xi_2, \ldots$ are iid random variables with $F(x) = P(\xi \leq x), N = (N_t)_{t \geq 0}$ is a homogeneous Poisson process with intensity $\lambda > 0$, and $W = (W_t)_{t \geq 0}$ is standard Brownian motion. Suppose furthermore that the processes W, N and (ξ_i) are jointly independent. In this formulation, H in (26) in the recent literature either occurs as a classical risk process perturbed by Browian motion(see GERBER [14])) or as a model for catastrophic insurance futures (see for instance CUMMINS and GEMAN [6])

Then

$$\begin{aligned}
H_t &= bt + \sigma W_t + \sum_{k=1}^{N_t} \xi_k \qquad (27)\\
&= bt + \sigma W_t + \int_0^t \int x d\mu \\
&= (bt + \int_0^t \int \varphi(x) d\nu) + (\sigma W_t + \int_0^t \int \varphi(x) d(\mu - \nu)) \\
&\qquad + (\int_0^t \int (x - \varphi(x)) d\mu) \\
&= t(b + \lambda \int \varphi(x) F(dx)) + (\sigma W_t + \int_0^t \int \varphi(x) d(\mu - \nu)) \\
&\qquad + (\int_0^t \int (x - \varphi(x)) d\mu).
\end{aligned}$$

Consequently,

$$T(\varphi) = (B(\varphi), \langle H^c \rangle, \nu),$$

where

$$\begin{aligned}
B(\varphi)_t &= t(b + \lambda \int \varphi(x) F(dx)), \\
< H^c >_t &= \sigma^2 t, \qquad (28)\\
d\nu &= \lambda dt F(dx).
\end{aligned}$$

Diffusion processes with jumps These processes can be viewed as semimartingales with predictable characteristic triplet $T(\varphi) = (B(\varphi), C, \nu)$ where

$$\begin{aligned}
B(\varphi)_t &= \int_0^t b(s, H_s) ds, \quad (b = B_\varphi), \\
C_t &= \int_0^t C(s, H_s) ds \qquad (29)\\
\nu(\omega; dt \times dx) &= dt \times K_t(H_s(\omega), dx),
\end{aligned}$$

where $K_t(x, dy)$ is a Borel transition kernel from $\mathbb{R}_+ \times \mathbb{R}$in $\mathbb{R}$; see JACOD AND SHIRYAEV [19], Chapter III,2.

2.4. Conditional Esscher transforms

Consider a semimartingale $H = (H_t)_{t \geq 0}$ with triplet $T = (B, C, \nu)$ where we dropped for notational convenience the dependence on φ. Also for simplicity, we take $\varphi(x) = xI(|x| \leq 1)$. We first introduce the cummulant process $A(u) = (A(u)_t)_{t \geq 0}$ associated with H:

$$A(u)_t = iuB_t - \frac{1}{2}u^2 C_t + \int (e^{iux} - 1 - iu\varphi(x)) \nu\big((0, t] \times dx\big). \qquad (30)$$

Suppose that $\Delta A(u) \neq -1$, then the stochastic exponential $G(u) = \mathcal{E}(A(u))$ defined in (6) cannot take zero values. Now define the process

$$X_t(u) = \frac{e^{i\,u\,H_t}}{\mathcal{E}(A(u))_t}, \; t \geq 0. \tag{31}$$

An important property of semimartingales is the following characterisation:

H is a semimartingale with triplet (B, C, ν)

if and only if (32)

$X = (X_t(u))_{t \geq 0}$ is a local martingale for every $u \in \mathbb{R}$;

see Jacod and Shiryaev [19], Chapter II, 2.49. For discrete time processes, (30) reduces to

$$\Delta A(u)_n = \int (e^{i\,u\,x} - 1)\nu_n(dx) = E(e^{iuh_n} - 1|\mathcal{F}_{n-1}). \tag{33}$$

However, in this case (6) implies that

$$\begin{aligned} \mathcal{E}(A(u))_n &= \prod_{0<k\leq n} (1 + \Delta A(u)_k), \\ &= \prod_{0<k\leq n} E(e^{iuh_n}|\mathcal{F}_{n-1}). \end{aligned}$$

Hence in discrete time for a stochastic sequence $H = (H_n)$ with $\Delta H_n = h_n$ and so that $E(e^{iuh_n}|\mathcal{F}_{n-1}) \neq 0$, $n \geq 1$, the sequence

$$\left(\frac{e^{i\,u\,H_n}}{\prod_{0<k\leq n} E(e^{i\,u\,h_k}|\mathcal{F}_{k-1})} \right)_{n\geq 1} \tag{34}$$

is a local martingale. Of course we don't need the deep characterisation result (32) in order to prove (34), a more direct argument can be given in this case. Similarly, suppose $Ee^{a_k h_k} < \infty$, $k \geq 1$, for some constants $a_1, a_2, \ldots$, then the sequence $Z = (Z_n)_{n\geq 1}$ with $Z_0 = 1$ and

$$Z_n = \prod_{k\leq n} \frac{e^{a_k h_k}}{E(e^{a_k h_k}|\mathcal{F}_{k-1})}, \; n \geq 1, \tag{35}$$

is a martingale. The latter follows immediately from the adaptiveness of H and elementary properties of conditional expectation. Property (35) allows us to construct a family of measures $\{\widetilde{P}_N\}$ such that $d\widetilde{P}_N = Z_N dP_N$ and $\widetilde{P}_N = \widetilde{P}_{N+1}|\mathcal{F}_N$. The conditional distribution

$$\widetilde{P}_N(h_N \in A|\mathcal{F}_{N-1}) = E\left[I_A(h_N)\frac{e^{a_N h_N}}{E(e^{a_N h_N}|\mathcal{F}_{N-1})}|\mathcal{F}_{N-1}\right] \tag{36}$$

is called *the conditional Esscher transform.* In the traditional actuarial context, the h_i's are independent and hence (36) reduces to an unconditional expectation, *the Esscher transform*:

$$\widetilde{P}_N(h_N \in A) = E\left[I_A(h_N)\frac{e^{a_N h_N}}{Ee^{a_N h_N}}\right]. \tag{37}$$

3. Predictable Conditions for $S \in \mathcal{M}_{\text{LOC}}(P), S \in \mathcal{M}(P)$

3.1. One asset

In order to investigate whether $S \in \mathcal{M}(P)$ (i.e. S is a P-martingale) it may be more convenient to first look for conditions so that $S \in \mathcal{M}_{\text{loc}}(P)$ (i.e. S is a local P-martingale) and then use the result in Jacod and Shiryaev [19], Chapter I, 1.47 that *a local martingale S is a uniformly integrable martingale if and only if S belongs to the class* (D), that is the set of random variables $\{S_T : T \quad \text{finite stopping time}\}$ *is uniformly integrable.* We hence start with the representation (5), i.e.

$$S_t = S_0\mathcal{E}(\hat{H})_t$$

and use the property (see Section 1.2) that

$$S \in \mathcal{M}_{\text{loc}}(P) \quad \textit{if and only if} \quad \hat{H} \in \mathcal{M}_{\text{loc}}(P). \tag{38}$$

From (4) and (17) we obtain:

$$\begin{aligned}
\hat{H}_t &= H_t + \frac{1}{2}\langle H^c\rangle_t + \int_0^t\int(e^x - 1 - x)d\mu \\
&= H_0 + B(\varphi)_t + H_t^c + \frac{1}{2}\langle H^c\rangle_t + \int_0^t\int \varphi(x)d(\mu-\nu) \\
&\qquad + \int_0^t\int(x-\varphi(x))d\mu + \int_0^t\int(e^x-1-x)d\mu \\
&= H_0 + B(\varphi)_t + H_t^c + \frac{1}{2}\langle H^c\rangle_t + \int_0^t\int \varphi(x)d(\mu-\nu) \\
&\qquad + \int_0^t\int(e^x-1-\varphi(x))d\mu.
\end{aligned} \tag{39}$$

Suppose now that $|e^x - 1 - \varphi(x)| * \nu \in \mathcal{A}_{\text{loc}}$ (i.e. the process $(\int_0^t\int|e^x - 1 - \varphi(x)|d\nu)_{t\geq 0}$ is locally integrable), then

$$\begin{aligned}
\int_0^t\int(e^x-1-\varphi(x))d\mu &= \int_0^t\int(e^x-1-\varphi(x))d\nu \\
&\qquad + \int_0^t\int(e^x-1-\varphi(x))d(\mu-\nu),
\end{aligned} \tag{40}$$

where the last integral is a local martingale; see Jacod and Shiryaev [19], Chapter II, 1.28 and Liptser and Shiryaev [23], Chapter III,§ 5. Hence from

(39),

$$\hat{H}_t = H_0 + B(\varphi)_t + H_t^c + \frac{1}{2}\langle H^c\rangle_t + \int_0^t \int (e^x - 1 - \varphi(x))d\nu$$
$$+ \int_0^t \int (e^x - 1 - \varphi(x))d(\mu - \nu) + \int_0^t \int \varphi(x)d(\mu - \nu)$$
$$= H_0 + K_t + H_t^c + \int_0^t \int (e^x - 1)d(\mu - \nu),$$

where (see (30))

$$K_t = A(-i)_t = B(\varphi)_t + \frac{1}{2}\langle H^c\rangle_t + \int_0^t \int (e^x - 1 - \varphi(x))d\nu. \tag{41}$$

Therefore $\hat{H}_t = K_t +$ (local martingale)$_t$. Since $K = (K_t)$ is a predictable process, it follows that

$$\hat{H} \in \mathcal{M}_{\text{loc}}\ (P) \ \textit{ if and only if } \quad K \equiv 0.$$

See Jacod and Shiryaev [19], Chapter I, 3.16 and Liptser and Shiryaev [23], Chapter I,6, Theorem 4 for more details.

3.2. Two assets

Suppose that we now have a second asset $S^0 = (S_t^0)_{t\geq 0}$ with

$$S_t^0 = S_0^0\, e^{H_t^0}\ . \tag{42}$$

Similar to the discussions above (see (4)) we introduce

$$\hat{H}_t^0 = H_t^0 + \frac{1}{2}\langle H^{0c}\rangle_t + \sum_{0<s\leq t}(e^{\Delta H_s^0} - 1 - \Delta H_s^0) \tag{43}$$

and obtain

$$S_t^0 = S_0^0\, \mathcal{E}(\hat{H}^0)_t.$$

Therefore

$$\frac{S_t}{S_t^0} = \frac{S_0}{S_0^0}\frac{\mathcal{E}(\hat{H})_t}{\mathcal{E}(\hat{H}^0)_t}. \tag{44}$$

It is now easy to check by Itô's formula that

$$\mathcal{E}(\hat{H}^0)_t^{-1} = \mathcal{E}(-\hat{H}^*)_t, \tag{45}$$

where

$$\hat{H}_t^* = \hat{H}_t^0 - \langle \hat{H}^{0c} \rangle_t - \sum_{0 < s \leq t} \frac{(\Delta \hat{H}_s^0)^2}{1 + \Delta \hat{H}_s^0} \cdot \tag{46}$$

From (44) and (45) we obtain that

$$\frac{S_t}{S_t^0} = \frac{S_0}{S_0^0} \mathcal{E}(\hat{H}_t) \mathcal{E}(-\hat{H}^*)_t. \tag{47}$$

If in general $U, V \in Sem(P)$, then the so-called *Yor addition formula* (see for instance Rogers and Williams [28], Section IV. 19) yields

$$\mathcal{E}(U)\mathcal{E}(V) = \mathcal{E}(U + V + [U, V]) \tag{48}$$

with *the quadratic covariation process*

$$[U, V]_t = \langle U^c, V^c \rangle_t + \sum_{0 < s \leq t} \Delta U_s \Delta V_s. \tag{49}$$

So from (47), (48):

$$\frac{S_t}{S_t^0} = \frac{S_0}{S_0^0} \mathcal{E}(\hat{H} - \hat{H}^* + [\hat{H}, -\hat{H}^*])_t.$$

It is not difficult to check that

$$\hat{H} - \hat{H}^* + [\hat{H}, -\hat{H}^*] = \hat{H} - \hat{H}^0 + \langle \hat{H}^{0c} - \hat{H}^c, \hat{H}^{0c} \rangle + \sum \frac{\Delta \hat{H}^0 (\Delta \hat{H}^0 - \Delta \hat{H})}{1 + \Delta \hat{H}^0} \cdot$$

If S^0 stands for a riskless asset, i.e. H^0 is predictable, then $\hat{H}^{0c} = H^{0c} = 0$ and

$$\frac{S_t}{S_t^0} = \frac{S^0}{S_0^0} \mathcal{E}\big(\hat{H} - \hat{H}^0 + \sum \frac{\Delta \hat{H}^0 (\Delta \hat{H}^0 - \Delta \hat{H})}{1 + \Delta \hat{H}^0}\big). \tag{50}$$

Hence

$$\frac{S}{S^0} \in \mathcal{M}_{loc}\ (P)$$

if and only if (51)

$$\hat{H} - \hat{H}^0 + \sum \frac{\Delta \hat{H}^0 (\Delta \hat{H}^0 - \Delta \hat{H})}{1 + \Delta \hat{H}^0} \in \mathcal{M}_{loc}\ (P)$$

The result (51) can be very useful in finding sufficient conditions for S/S^0 to be a local $P-$ martingale. For instance, if $\Delta\hat{H}^0 = 0$, then $\hat{H}^0_t = H^0_t$ (we suppose that H^0 is predictable) and we obtain

$$K_t - H^0_t \equiv 0 \quad \text{implies} \quad \frac{S}{S^0} \in \mathcal{M}_{loc}\,(P). \tag{52}$$

Also, if $\Delta\hat{H} = \Delta\hat{H}^0$, then

$$K_t - H^0_t - \sum_{0<s\le t}(e^{\Delta H^0_t} - 1 - \Delta H^0_s) \equiv 0$$

$$\text{implies} \tag{53}$$

$$\frac{S}{S^0} \in \mathcal{M}_{loc}\,(P).$$

3.3. Examples

Discrete time In the case of discrete time

$$\hat{H} - \hat{H}^0 + \sum \frac{\Delta\hat{H}^0(\Delta\hat{H}^0 - \Delta\hat{H})}{1+\Delta\hat{H}^0} = \sum \frac{\Delta\hat{H} - \Delta\hat{H}^0}{1+\Delta\hat{H}^0},$$

so that because of (51),

$$\frac{S}{S^0} \in \mathcal{M}_{loc}\,(P)$$

mboxif and only if

$$\sum_{k\le n} \frac{\hat{h}_k - \hat{h}^0_k}{1+\hat{h}^0_k} \in \mathcal{M}_{loc}\,(P)\,\cdot \tag{54}$$

However $\hat{h}_k = e^{h_k} - 1, \hat{h}^0_k = e^{h^0_k} - 1$, so that by $\mathcal{F}_{k-1}$- measurability of h^0_k we obtain the following sufficient condition

$$E(e^{h_k}|\mathcal{F}_{k-1}) = e^{h^0_k},\ k \ge 1 \text{ implies } \frac{S}{S^0} \in \mathcal{M}_{loc}\,(P).$$

Processes with independent increments Suppose that $H = (H_t)_{t\ge 0}$ is a process with independent increments, the triplet $T(\varphi)$ given by (25) and let $\hat{H}^0_t = rt$. Then

$$K_t = t\Big(b(\varphi) + \frac{C}{2} + \int (e^x - 1 - \varphi(x))F(dx)\Big),$$

so that

$$b(\varphi) + \frac{C}{2} + \int (e^x - 1 - \varphi(x))F(dx) = r \text{ implies } \frac{S}{S^0} \in \mathcal{M}_{loc}\,(P).$$

Brownian motion with drift and Poisson jumps For the notation, see Section 2.3.3. In this case

$$K_t = t\big(b + \frac{\sigma^2}{2} + \lambda \int (e^x - 1)F(dx)\big),$$

whence

$$b + \frac{\sigma^2}{2} + \lambda E(e^\xi - 1) = \tau \text{ implies } \frac{S}{S^0} \in \mathcal{M}_{loc}(P).$$

4. Predictable conditions for the existence of a locally equivalent probability measure $\widetilde{P}$ such that $S \in \mathcal{M}_{loc}(\widetilde{P}),\ S \in \mathcal{M}(\widetilde{P})$

4.1. General results

If we have a measure $\widetilde{P} \overset{\text{loc}}{\sim} P$, then the likelihood (Radon-Nikodym derivative) process $Z = (Z_t)_{t \geq 0}$ with

$$Z_t = \frac{d\widetilde{P}_t}{dP_t} \tag{55}$$

is strictly positive ($Z_t > 0,\ P$ and $\widetilde{P} - a.s., t \geq 0$; see for instance Rogers and Williams 1987, Theorem IV,17.1. We therefore can define the process $M = (M_t)_{t \geq 0}$ as follows:

$$M_t = \int_0^t \frac{dZ_s}{Z_{s-}}, \tag{56}$$

which satisfies $M \in \mathcal{M}_{loc}\,(P)$. Since $dZ_t = Z_{t-}\, dM_t$, we have that

$$Z_t = Z_0\, \mathcal{E}(M)_t \tag{57}$$

where

$$\mathcal{E}(M)_t = \exp\{M_t - \frac{1}{2}\langle M^c \rangle_t\} \prod_{0 < s \leq t} (1 + \Delta M_s) e^{-\Delta M_s}. \tag{58}$$

The local martingale property of M implies that the following decomposition holds:

$$M_t = M_0 + \int_0^t \beta_s dH_s^c + \int_0^t \int W(\cdot, s, x) d(\mu - \nu) + \widetilde{M}_t,$$

where β and W satisfy some integrability conditions (see Jacod and Shiryaev [19], Chapter III, 4.24) and $\widetilde{M}$ is a residual martingale part which is orthogonal to $\int_0^\cdot \beta_s dH_s^c$ and $\int_0^\cdot \int W(\cdot, s, x) d(\mu - \nu)$. Unfortunately, we do not have sufficient tools in order to control the properties of $\widetilde{M}$. However, for many interesting cases, $\widetilde{M} \equiv 0$. The latter for instance holds if the triplet $T(\varphi) = (B, C, \nu)$ for H defines the distribution of H uniquely. The following are cases where this property holds:

(i) Processes with independent increments.

(ii) Strong solutions of stochastic differential equations with respect to Brownian motion.

(iii) In the case of Poisson random measure in discrete time where $\nu(\omega, \{n\} \times A) = P(\Delta H_n \in A|\mathcal{F}_{n-1}), n \geq 1$, which gives us the possibility to calculate the (unconditional) distribution of $(H_n)_{n\geq 0}$.

A possible approach consists of considering the structure of Z under the assumption that $\widetilde{P} \overset{\text{loc}}{\sim} P$ exists. Hence assume that $Z = (Z_t)_{t\geq 0}$ satisfies the representation $(57) - (58)$, where

$$M_t = M_0 + \int_0^t \beta_s dH_s^c + \int_0^t \int W(.,s,x)\, d(\mu - \nu). \tag{59}$$

Can we from this representation deduce the existence of $\widetilde{P}$? This approach may work if at least the characteristic triplet of H defines the measure P (*i.e.* the law of H) uniquely. We assume the finite horizon case $0 \leq t \leq T < \infty$ and normalise $E\, Z_T = 1$. In this case we can simply define

$$d\widetilde{P}_T = Z_T\, dP_T.$$

The difficult part in this plan de campagne is to find conditions on (β, W) and (B, C, ν) which imply that $Z = (Z_t)_{0\leq t\leq T}$ is a martingale with $E\, Z_T = 1$. A whole series of papers exists on this topic, see for instance Jacod and Memin [18], Liptser and Shiryaev [22], Novikov [24], [25], Lepingle and Memin [21] and Grigelionis [17]. (See Schachermayer [29] and Delbaen and Schachermayer [9] for a case where $\widetilde{M}$ cannot be taken to be zero!) So suppose that $M = (M_t)_{0\leq t\leq T}$ defined as in (59) is a positive martingale with $E\, Z_T = 1$. We now want to understand which conditions on (β, W) imply that $S \in \mathcal{M}_{loc}\,(\widetilde{P}_T)$. First observe that

$$SZ \in \mathcal{M}_{loc}\,(P_T) \text{ implies } S \in \mathcal{M}_{loc}\,(\widetilde{P}_T), \tag{60}$$

(see Jacod and Shiryaev, Chapter III, 3.8) so that it suffices to find conditions implying

$$\mathcal{E}(\hat{H})\mathcal{E}(M) \in \mathcal{M}_{loc}\,(P_T)\, \cdot \tag{61}$$

Also note that

$$\begin{aligned} \hat{H}Z \in \mathcal{M}_{loc}\,(P_T) \Rightarrow\ & \hat{H} \in \mathcal{M}_{loc}\,(\widetilde{P}_T) \\ \Leftrightarrow\ & \mathcal{E}(\hat{H}) \in \mathcal{M}_{loc}\,(\widetilde{P}_T) \\ \Leftrightarrow\ & S \in \mathcal{M}_{loc}\,(\widetilde{P}_T), \end{aligned}$$

so that instead of checking (61), one may look for conditions implying

$$\hat{H}\mathcal{E}(M) \in \mathcal{M}_{loc}(P_T). \tag{62}$$

One easily shows that (61) and (62) are equivalent.

From (61) and Yor's formula ((48)) one obtains:

$$\begin{aligned}\mathcal{E}(\hat{H})\mathcal{E}(M) &= \mathcal{E}(\hat{H} + M + [\hat{H}, M]) \\ &= \mathcal{E}(\hat{H} + M + \langle \hat{H}^c, M^c\rangle + \sum \Delta\hat{H}\Delta M) \cdot \end{aligned} \tag{63}$$

Moreover, (41) yields

$$\hat{H}_t = K_t + H_t^c + \int_0^t \int (e^x - 1) d(\mu - \nu) \cdot \tag{64}$$

From (59) and (64), assuming that the process $[\hat{H}, M]$ is locally integrable, we can find its *compensator* $\widetilde{[\hat{H}, M]}$. The latter is a predictable process with the property that $[\hat{H}, M] - \widetilde{[\hat{H}, M]} \in \mathcal{M}_{loc}\,(P)$. The following form results see also Jacod and Shiryaev [19], Chapter II, 2.17:

$$\begin{aligned}\widetilde{[\hat{H}, M]}_t = \int_0^t \beta_s d\langle H^c\rangle_s + \int_0^t \int W(e^x - 1) d\nu \\ - \sum_{s \le t} \int W(s, x)\nu(\{s\} \times dx) \int (e^x - 1)\nu(\{s\} \times dx).\end{aligned} \tag{65}$$

It turns out to be convenient to denote $W = Y - 1$. The main reason for this is the following. If H is a $P-$ semimartingale with triplet (B, C, ν) and $d\widetilde{P}_T = Z_T\, dP_T$, then H is also a $\widetilde{P}_T-$ semimartingale with triplet $(\widetilde{B}, \widetilde{C}, \widetilde{\nu})$ where $d\widetilde{\nu} = Y d\nu$, $Y(\omega, t, x)$ is positive and predictable and the process W in the definition of M (see (59)) has the following representation (Jacod and Shiryaev [19], Chapter III, 5.19)

$$\begin{aligned}W &= Y - 1 + \frac{\hat{Y} - a}{1 - a}\, I(a < 1), \\ a &= (a_t(\omega)) \text{ where } a_t(\omega) = \nu(\omega; \{t\} \times \mathbb{R}) \\ \hat{Y}_t &= \int Y(\omega, t, x)\nu(\omega; \{t\} \times dx) \cdot\end{aligned} \tag{66}$$

Both in the so-called quasi-left continuous case (*i.e.* $a_t \equiv 0$) as well as in the discrete-time case where $a_t(\omega) = P(\Delta H_t \in \mathbb{R} | \mathcal{F}_{t-1}) = 1$ we have that $W = Y - 1$. Therefore, as a corollary we obtain

$$\widetilde{[\hat{H}, N]}_t = \int_0^t \beta_s d\langle H^c\rangle_s + \int_0^t \int (Y - 1)(e^x - 1) d\nu \cdot \tag{67}$$

Together with (59), (63) and (64) we are led to the following result.
Suppose that $Z = (Z_t)_{t \le T}$ is a positive martingale with $dZ_t = Z_{t-} dM_t$, where $M = (M_t)_{t \le T}$ is given by (59) and $E \mid Z_T \mid = 1$. Then in the cases where

$$\nu(\omega; \{t\} \times \mathbb{R}) \in \{0, 1\},$$

the condition

$$K_t + \int_0^t \beta_s d\langle H^c \rangle_s + \int_0^t \int (Y-1)(e^x - 1) d\nu = 0, \quad t \le T,$$

implies that there exists a measure $\widetilde{P}_T$ constructed by (56), (59), (66) such that

$$\widetilde{P}_T \sim P_T \quad \textit{and} \quad S \in \mathcal{M}_{loc}(\widetilde{P}_T).$$

Suppose now that we are interested in the construction of probability measures $\widetilde{P}_T$ with

$$\widetilde{P}_T \sim P_T \quad \textit{and} \quad \frac{S}{S^0} \in \mathcal{M}_{loc}(\widetilde{P}_T).$$

In this case,

$$\frac{S_t}{S_t^0} Z_t = \frac{S_0}{S_0^0} Z_0 \; \mathcal{E}(\hat{H})_t \; \mathcal{E}^{-1}(\hat{H}^0)_t \; \mathcal{E}(M)_t \cdot \tag{68}$$

Because of (51), $\mathcal{E}(\hat{H})\mathcal{E}^{-1}(\hat{H}^0) = \mathcal{E}(\bar{H})$, where

$$\bar{H} = \hat{H} - \hat{H}^0 + \sum \frac{\Delta \hat{H}^0 (\Delta \hat{H}^0 - \Delta \hat{H})}{1 + \Delta \hat{H}^0} \cdot$$

Moreover, by (63),

$$\begin{aligned}
\mathcal{E}(\hat{H})\mathcal{E}^{-1}(\hat{H}^0)\mathcal{E}(M) &= \mathcal{E}(\bar{H})\mathcal{E}(M) \\
&= \mathcal{E}(\bar{H} + M + [\bar{H}, M]) \\
&= \mathcal{E}(\bar{H} + M + \langle \bar{H}^c, M^c \rangle + \sum \Delta \bar{H} \Delta M) \\
&= \mathcal{E}(\hat{H} - \hat{H}^0 + M + \langle \hat{H}^c - \hat{H}^{0c}, M^c \rangle \\
&\qquad + \sum \frac{\Delta \hat{H} - \Delta \hat{H}^0}{1 + \Delta \hat{H}^0} \Delta M \\
&\qquad + \sum \frac{\Delta \hat{H}^0 (\Delta \hat{H}^0 - \Delta \hat{H})}{1 + \Delta \hat{H}^0}) \\
&\equiv \mathcal{E}(I), \text{ say.}
\end{aligned} \tag{69}$$

Again we assume that H^0 is predictable so that $H^{0c} = 0$ and consequently

$$\begin{aligned}
I = \hat{H} - \hat{H}^0 + M + \langle \hat{H}^c, M^c \rangle \\
+ \sum \frac{(\Delta M - \Delta \hat{H}^0)(\Delta \hat{H} - \Delta \hat{H}^0)}{1 + \Delta \hat{H}^0} \cdot
\end{aligned} \tag{70}$$

Compare this expression with (51) when $M \equiv 0$ and (63) when $\hat{H}^0 \equiv 0$. In order to find now a predictable condition ensuring that $S/S^0 \in \mathcal{M}_{loc}\,(\widetilde{P})$, as before, we search for a decomposition

$$I = \textit{Predictable process} + \textit{local martingale}\cdot$$

Another way of putting this is: if $\widetilde{I}$ is the compensator (i.e. predictable part) of I, then

$$\widetilde{I} = 0 \quad \text{implies} \quad \frac{S}{S^0} \in \mathcal{M}_{loc}\,(\widetilde{P})\cdot \tag{71}$$

Returning to (70), observe the following facts.

$$\begin{aligned}
(a) \qquad \hat{H}_t - \hat{H}_t^0 + M_t &= (K_t - \hat{H}_t^0) + \hat{H}_t^c + \int_0^t \int (e^x - 1) d(\mu - \nu) \\
&\quad + \int_0^t \beta_s dH_s^c + \int_0^t \int W d(\mu - \nu) \\
&= (K_t - \hat{H}_t^0) + (\textit{local martingale})_t,
\end{aligned}$$

(b) $\quad \langle \hat{H}^c, \hat{M}^c \rangle_t = \int_0^t \beta_s d\langle \hat{H}^c \rangle_s \cdot$

The calculation of the compensator of the last (i.e. $\sum -$) term in (70) is in general involved. However, for many interesting special cases (including those already discussed in previous sections) the compensator $\widetilde{I}$ can be obtained in explicit form. Rather than pursuing the general case as outlined above, in the next section we shall look at some examples.

4.2. Some examples

Discrete time In this case,

$$\begin{aligned}
\Delta I &= \Delta M + \frac{(1 + \Delta M)(\Delta \hat{H} - \Delta \hat{H}^0)}{1 + \Delta \hat{H}^0} \\
&= \frac{\Delta M (1 + \Delta \hat{H}) + (\Delta \hat{H} - \Delta \hat{H}^0)}{1 + \Delta \hat{H}^0} \cdot
\end{aligned} \tag{72}$$

Again denote $h_k = \Delta H_k$, whence $\Delta \hat{H}_k = e^{\Delta H_k} - 1 = e^{h_k} - 1$; we use the same notation for H^0. Hence, with

$$\begin{aligned}
\Delta M_n &= \int (Y_n(x;\omega) - 1)(\mu_n(dx) - \nu_n(dx)) \\
&= Y_n(h_n;\omega) - E(Y_n(h_n;\omega)|\mathcal{F}_{n-1}),
\end{aligned}$$

we obtain from (72) that

$$\begin{aligned}
\Delta I_n &= \Delta M_n e^{h_n - h_n^0} + e^{h_n - h_n^0} - 1 \\
&= e^{h_n - h^0{}_n}(\Delta M_n + 1) - 1 \cdot
\end{aligned} \tag{73}$$

Together with the assumed predictability of (h_n^0), we obtain from (73) the following key result:

$$E\big[e^{h_n}\big(Y_n(h_n;\omega) - E\big(Y_n(h_n;\omega)|\mathcal{F}_{n-1}\big) + 1\big)|\mathcal{F}_{n-1}\big] = e^{h_n^0},\ n \geq 1$$

$$\text{implies} \tag{74}$$

$$\frac{S}{S^0} \in \mathcal{M}_{loc}\,(\widetilde{P})\cdot$$

Therefore, the existence problem of a (local) matingale measure $\widetilde{P}$ is reduced to finding (Y_n) which satisfy (74). This task may still seem to be formidable in the stated generality. It is exactly at this point that the *conditional Esscher transform* defined in (36) enters naturally. Indeed, we *assume* that

$$Y_n(h_n;\omega) = \frac{e^{a_n h_n}}{E(e^{a_n h_n}|\mathcal{F}_{n-1})}, \tag{75}$$

where the *unknown* functions a_n are $\mathcal{F}_{n-1}$ - measurable. Our aim is to determine the a_n's in the special case of (75). With (74) we arrive at the following equation:

$$E\big[e^{(a_n+1)h_n}|\mathcal{F}_{n-1}\big] = e^{h_n^0} E\big[e^{a_n h_n}|\mathcal{F}_{n-1}\big]\cdot \tag{76}$$

If the increment sequence (h_n) is *iid* and $h_n^0 \equiv h_1^0$ say, then for $a \equiv a_n$ we obtain the equation:

$$Ee^{(a+1)h_1} = e^{h_1^0} Ee^{ah_1}\cdot \tag{77}$$

Hence in this case, the Esscher transform allows for a special construction of (Y_n) by reducing the problem to finding constants (a_n) or predictable functions $(a_n(\omega))$ satisfying (76). In Gerber and Shiu [15], the construction (77) is applied in a finance context. See also the references in the latter paper for further reading on the subject. Embrechts [10] discusses the Esscher transform in the light of financial versus actuarial pricing systems.

Processes with stationary, independent increments (S.I.I.) Let $H = (H_t)$ be a process with S.I.I., continuous in probability, and triplet

$$B(\varphi)_t = tb(\varphi)$$
$$C_t = tC$$
$$\nu(dt \times dx) = dtF(dx)\cdot$$

Moreover, $H_t^0 = rt$, say. Then K_t is defined in Section 3.3.2. In this case,

$$I = \hat{H} - \hat{H}^0 + M + [\hat{H}, M],$$

and from (59), (64), (67) we obtain:

$$K_t + \int_0^t \beta_s d\langle H^c\rangle_s + \int_0^t \int (Y-1)(e^x-1)d\nu = rt \tag{78}$$

implies

$$\frac{S}{S^0} \in \mathcal{M}_{loc}\,(\widetilde{P})\cdot$$

The sufficient condition (78) can be rewritten as:

$$t\big(b(\varphi) + \frac{C}{2} + \int (e^x - 1 - \varphi(x))F(dx)\big) + C\int_0^t \beta_s ds$$
$$+ \int_0^t \int (Y-1)(e^x-1)dsF(dx) = rt\,\cdot \tag{79}$$

Because of the homogeneity (i.e. incremental stationarity) of the process, it seems reasonable to take $\beta_s(\omega) \equiv \beta, Y(s.x.\omega) \equiv Y(x)$. Then for unknown β and $Y(x)$, (79) reduces to:

$$C(\frac{1}{2}+\beta) + b(\varphi) + \int (e^x - 1 - \varphi(x))F(dx)$$
$$+ \int (Y(x)-1)(e^x-1)F(dx) = r\,\cdot \tag{80}$$

Take as particular case the standard Black - Scholes set-up in finance, i.e.

$$S_t = e^{\mu t + \sigma W_t}\ ,\ S_t^0 = e^{rt}\cdot$$

In this case, $b(\varphi) = \mu$, $C = \sigma^2$, $\nu \equiv 0$ so that the condition (80) reduces to the well-known equation

$$\mu + \sigma^2(\frac{1}{2}+\beta) = r \tag{81}$$

and $M_t = \beta H_t^c = \beta\sigma W_t$. It should be stressed that (81) can be obtained much more easily directly, i.e. without using the general theory introduced above. Indeed

$$Z_t = \exp\left\{\beta\sigma W_t - \frac{(\beta\sigma)^2}{2}t\right\}$$

and

$$\begin{aligned}\frac{S_t}{S_t^0}Z_t &= \exp\Big\{(\mu - r)t + \sigma W_t\Big\}\exp\left\{\beta\sigma W_t - \frac{(\beta\sigma)^2}{2}t\right\}\\ &= \exp\left\{\sigma(1+\beta)W_t - \frac{\sigma^2(1+\beta)^2}{2}t\right\}\exp\left\{(\sigma^2(\frac{1}{2}+\beta)+\mu-r)t\right\}\cdot\end{aligned}$$

Since

$$\left(\exp\left\{\sigma(1+\beta)W_t - \frac{\sigma^2(1+\beta)^2}{2}t\right\}\right)_{t\geq 0}$$

is a P-martingale, condition (81) immediately implies that $S/S^0 \in \mathcal{M}\,(P)$ and so $S/S^0 \in \mathcal{M}\,(\widetilde{P})$ where $d\widetilde{P}_t = Z_t dP_t$.

Brownian motion with drift and Poisson jumps Consider the model (26) with triplet representation (28). Hence from (80) we obtain the following condition for $(\beta, Y(x))$:

$$\sigma^2(\frac{1}{2}+\beta)+b+\lambda\int Y(x)(e^x-1)F(dx)=r \tag{82}$$

or equivalently,

$$\sigma^2(\frac{1}{2}+\beta)+b+\lambda E(e^\xi-1)Y(\xi)=r\cdot$$

Compare this condition with the condition in Section 3.3.3., where $\beta=0$, $Y\equiv 1$ and $\widetilde{P}_t=P_t$. If we consider a solution

$$Y(x)=\frac{e^{\alpha x}}{Ee^{\alpha\xi}}$$

for suitable α, then we get for (α,β):

$$\sigma^2(\frac{1}{2}+\beta)+b+\lambda\frac{E(e^\xi-1)e^{\alpha\xi}}{Ee^{\alpha\xi}}=r,$$

or with $\psi(\alpha)=Ee^{\alpha\xi}$,

$$\beta\sigma^2+\lambda\frac{\psi(\alpha+1)}{\psi(\alpha)}=r-b-\frac{\sigma^2}{2}-\lambda\cdot$$

5. Conclusion

In order to price and hedge derivative instruments in insurance and finance, a no-arbitrage approach leads to the construction of equivalent (local) martingale measures of specific semimartingales. For a general class of such processes, including discrete models, processes with stationary and independent increments and certain diffusion models with jumps, a general construction toward obtaining such measures is outlined. Though these methods are well known in the literature on general stochastic processes, we found it useful to summarise the main results and applications of this theory to the context of insurance and finance. In doing so, we hope to contribute to closing the methodological gap currently existing between both fields. The main common tool concerns the so-called Esscher transform, a time-honoured tool in insurance risk theory. Its construction is generalised to the so-called conditional Esscher transform which may serve a similar purpose within more general pricing models.

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