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A Retrospective View on Sampled-Data – Control Systems

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A personal and retrospective view on sampled-data control systems is given. Emphasis is placed on the course of the discovery of a function space model, currently known as the lifted model, for this class of systems. Its impact on the frequency domain characteristics is discussed. A design example is given to illustrate the contents.

1. Prelude

In this brief article, I will attempt to give a fairly personal and retrospective view on the recent developments in the theory of sampled-data control systems. The emphasis is upon how I came to discover a new function space framework, currently called lifting, for sampled-data systems, and how this notion helped to clarify certain issues in the study of such systems.

It was a fairly cold day in January 1971. I was sitting in an old classroom of Kyoto University equipped with a charcoal-type heater already obsolete even in those days, attending an introductory course on control theory for junior students. The professor was teaching us a basic treatment of sampled-data systems Figure 1. Here $P(s)$ describes the correspondence between the incoming and outgoing signals $u(s)$ and $y(s)$ of the controlled plant as $y(s) = P(s)u(s)$ in terms of Laplace transforms. Similarly, $C(z)$ describes those of the controller via z-transform to specify its operation in discrete-time. To interface these two kinds of systems, sampling/hold operations are introduced. The former is

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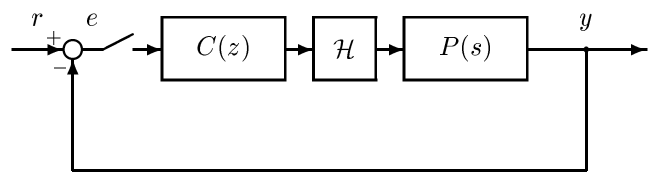

Figure 1. Unity Feedback Sampled-Data System

denoted by the slanted line segment while the latter is designated by the box $\mathcal{H}$. The objective here is to analyze and design this control system.

The lecture started out with the introduction of z-transform, description of how we can compute the z-transform from a given Laplace transform, and then proceeded to the z-domain representation etc. I can still recall with a rather vivid image that I got quite puzzled as the course proceeded. In a word, my bewilderment may be summarized to the fact that Figure 1 was really never discussed. What was grasped is the behavior of the system at sampled points only. In other words, instead of studying Figure 1, one used to resort to Figure 2, by

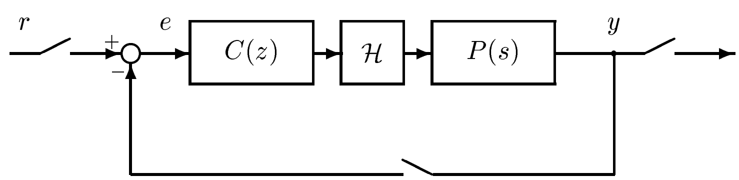

Figure 2. Modified Unity Feedback Sampled-Data System

introducing fictitious samplers. Of course we know the reasons why we do this. Figure 1 is not a time-invariant system. If we try to regard it as a discrete-time system, the continuous-time plant $P(s)$ gives us a trouble. On the other hand, if we attempt to view Figure 1 as a continuous-time system, the discrete-time controller and sample/hold devices cannot be regarded as time-invariant continuous-time components. By introducing fictitious samplers as above, we can bypass these problems and consider Figure 2 a time-invariant discrete-time system. This also made it possible to introduce such time-invariant notions as transfer functions.

There is, however, a price to be paid. The intersample behavior is lost in this discrete-time model of Figure 2. One may recover it by the modified z-transform, but it is possible only after we specified the controller. It is not taken into account in the design when the model is Figure 2.

Then why call this theory a sampled-data control? If we disregard the continuous-time behavior of the plant and convert the plant to Figure 2, it is almost merely a theory for discrete-time systems. To make it worse, the conversion of Figure 1 $\mapsto$ Figure 2 was often implicitly done. Many textbooks simply stated, after discussing the methods of computing z-transforms from given continuous-time transfer functions, "the problem is thus *equivalently* transformed into the design of a discrete-time control problem," often without exhibiting Figure 2. On the other hand, if we insist upon maintaining the intersample behavior, the mixture of continuous and discrete-time systems makes it impossible to treat the sampled-data system as a time-invariant object. Of course, it is not precise to say that I, junior student at the time, thought of the problem precisely in this way. This only represents the mood I felt, but it was somehow there. At any rate, this dichotomy disturbed me, but I did not, of course, see any easy way out.

2. A Step Forward

From time to time, sampled-data systems came back to my mind. I was subconsciously looking for a framework that has the advantage of incorporating the intersample behavior while maintaining the advantage of transfer functions. But whenever I tried to think about the problem relying upon the sampled points, it led to the loss of intersample information. Or else, the continuous-time plant introduced a time-varying behavior in the intersampling periods. In a word, the dilemma left unresolved.

In the meantime, my main research interest was centered around the realization and control of distributed parameter systems, especially delay systems. In due course and in collaboration with Shinji Hara, I encountered a new-type servo scheme called *repetitive control*, see Figure 3 [25]. This is a control scheme which aims at tracking every periodic signal of a fixed period L. To generate all such signals, one needs a pure delay element of length L.

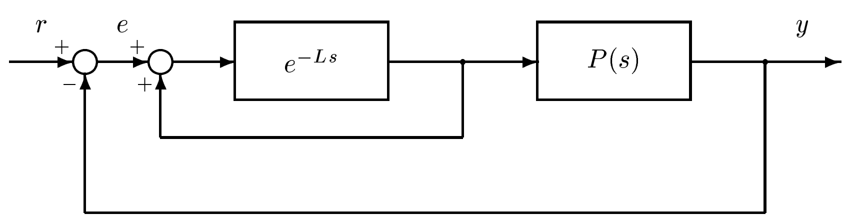

Figure 3. Repetitive Control System

The idea of using an infinite-dimensional element (i.e., delay) to control a finite dimensional plant intrigued me. Needless to say, to model such systems one needs a function space on an interval of length L (typically $L^2[0, L]$). Although this did not have a direct connection with sampled-data systems, I started thinking about combining this idea with sampled-data systems.

3. Lifting–Function Space Model

With the advance of micro-processors, the need for digital control systems became more and more prevalent in the 80's. Such a trend first started with a discrete-time treatment. However, the need for handling the intersample behavior in such control systems became more seriously recognized. In particular, Bruce Francis and his co-workers started a new pioneering approach toward H^∞ and H^2 control of sampled-data systems with built-in intersample behavior [8, 9, 10, 19]. During the summer of 1989, he visited Japan as a JSPS fellow, and gave lectures on his new approach.

Meanwhile, I was struggling to reconcile the traditional approach with intersampling behavior. Let us take the well-known model by Kalman and Bertram [28]:

$$x_{k+1} = e^{A_c h} x_k + \int_0^h e^{A_c \tau} B_c u_k d\tau, \quad y_k = C_c x_k \tag{1}$$

for

$$\dot{x}(t) = A_c x(t) + B_c u(t), \quad y(t) = C_c x(t). \tag{2}$$

If we employ the model (1), it appears that we inevitably have to introduce sampling in the input term, and hence this results in ignoring the intersampling behavior. I had been stuck at this point for some time, but suddenly around the end of summer of 1989, I came to realize that *if we do not want to lose the intersample information, we should simply keep it as a function vector.* This almost idiotically straightforward idea led me to the following: Let $f(t)$ be a function of t, defined on $[0, \infty)$, and h a fixed sampling period. If we sample $f(t)$ at each h interval period, we get a sequence $\{f(kh)\}_{k=0}^\infty$. This surely results in loss of intersampling information $f(kh+\theta)$, $0 < \theta < h$. If we try to argue that we can recover intersampling information in Figure 2 via z-transform, it is misleading because intersampling information in the reference input $r(t)$ surely cannot be recovered. If it cannot be recovered, we should keep it. It is an almost trivial mathematical idea to set up a mapping

$$\mathcal{L} : f \mapsto \{f(kh+\theta)\}_{k=0}^\infty, \quad 0 \le \theta < h \tag{3}$$

where each term on the right-hand side is regarded as a function of θ. See Figure 4.

This maps a continuous-time signal to a *sequence* of functions

$$\{f(kh+\theta)\}_{k=0}^\infty,$$

and is now called *lifting*. An important feature is that with this correspondence, periodic continuous-time plants can be transformed into time-invariant discrete-time systems. This may be explained as follows: Suppose we are at the k-th sampling step, i.e., $t = kh$. According to the lifting definition (3), from $t = kh$ to $t = (k+1)h$, input $u_k(\theta)$ is applied. We regard this as an infinite-dimensional input vector that drives the system to the next step $t = (k+1)h$. Accordingly, the state moves in a trajectory.

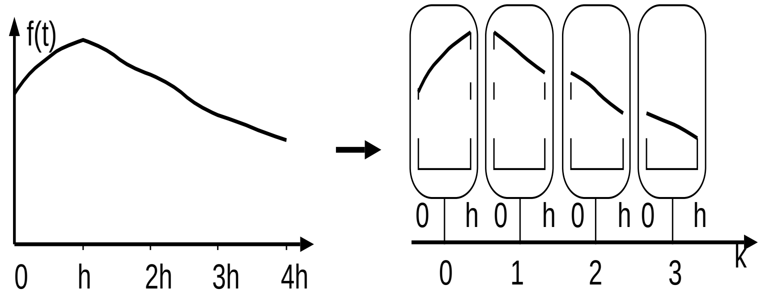

FIGURE 4. Lifting

To see such a state transition, one can replace h by θ in the Kalman-Bertram model (1), and replace u_k by $u_k(\theta)$. This leads to the lifted model I first obtained [39, 40]:

$$\begin{aligned} x_k(\theta) &= e^{A_c h} x_{k-1}(h) + \int_0^\theta e^{A_c(\theta-\tau)} B_c u_{k+1}(\tau) d\tau \\ y_k(\theta) &= C_c x_k(\theta) \end{aligned} \tag{4}$$

Actually, incorporating the full state history $x_k(\theta)$ into the model is a bit superfluous, and one can well model it by using their values at kh, provided we allow a direct feedthrough term in the model. This model was independently proposed and used by TOIVONEN [38], and BAMIEH ET AL. [5, 7][2] and exhibits the finite-dimensionality of various optimization problems more explicitly.

$$\begin{aligned} x_{k+1} &= e^{A_c h} x_k + \int_0^h e^{A_c(h-\tau)} B_c u_k(\tau) d\tau \\ y_k(\theta) &= C_c e^{A_c(\theta)} x_k + \int_0^\theta C_c e^{A_c(\theta-\tau)} B_c u_k(\tau) d\tau \end{aligned} \tag{5}$$

From here on we assume this model.

The important point here is that the operators

$$\begin{aligned} \mathbf{A}: &\quad \mathbf{C}^n \to \mathbf{C}^n : x \mapsto e^{A_c h} x \\ \mathbf{B}: &\quad L^2[0,h] \to \mathbf{C}^n : u \mapsto \int_0^h e^{A_c(h-\tau)} B_c u(\tau) d\tau \\ \mathbf{C}: &\quad \mathbf{C}^n \to L^2[0,h] : x \mapsto C_c e^{A_c(\theta)} x \\ \mathbf{D}: &\quad L^2[0,h] \to L^2[0,h] : u \mapsto \int_0^\theta C_c e^{A_c(\theta-\tau)} B_c u(\tau) d\tau \end{aligned}$$

[2] The idea of *discrete-time* lifting was known earlier in the signal processing literature, e.g., DAVIS [14]. However, this was not known to the control system community until these developments were made. See also the work by KHARGONEKAR, POOLLA and TANNENBAUM [29], where lifting was introduced for linear periodically time-varying discrete-time systems, and application to continuous-time systems was indicated.

do not depend on k ($\mathbf{C}^n$ denotes the n-dimensional complex Euclidean space). Hence (5) can be written as a time-invariant equation

$$\begin{aligned} x_{k+1} &= \mathbf{A}x_k + \mathbf{B}u_k \\ y_k &= \mathbf{C}x_k + \mathbf{D}u_k. \end{aligned}$$

Thus the continuous-time plant (2) can be described by a *time-invariant* discrete-time model. Once this is done, it is entirely routine to connect this expression with a discrete-time controller, and hence sampled-data systems can be fully described by time-invariant discrete-time equations, this time without sacrificing the intersampling information.

It is now easy to write down the combined equation for sampled-data systems, for example, for Figure 1. We denote it abstractly as

$$\begin{aligned} x_{k+1} &= \mathcal{A}x_k + \mathcal{B}u_k \\ y_k &= \mathcal{C}x_k + \mathcal{D}u_k. \end{aligned} \tag{6}$$

Let us introduce the notion of *transfer functions.* Let $f := \{f_k(\cdot)\}_{k=0}^{\infty}$ be a sequence of functions, each f_k belonging to $L^2[0,h]$. We can define its z-transform as

$$\mathcal{Z}[f](z) := \sum_{k=0}^{\infty} f_k z^{-k}. \tag{7}$$

This is just a formal power series. Similarly, the z-transform of a sequence of operators $G = \{G_k\}_{k=0}^{\infty}$, $G_k \in \mathcal{L}(L^2[0,h])$ can be defined by

$$\mathcal{Z}[G](z) := \sum_{k=0}^{\infty} G_k z^{-k}. \tag{8}$$

When $G_k = \mathcal{C}\mathcal{A}^{k-1}\mathcal{B}$ and $G_0 = \mathcal{D}$, the sequence describes the input/output correspondence of system (6), and its z-transform is equal to

$$G(z) := \mathcal{D} + \mathcal{C}(zI - \mathcal{A})^{-1}\mathcal{B}. \tag{9}$$

This is the *transfer function* of system (6). When an input $u(z)$ as in (7) is applied to system (6), its zero-initial state output is given by $G(z)u(z)$.

While such an expression primarily makes sense as a formal power series, substitution of certain complex values for z often makes sense. Observe that $\mathcal{A}$ is a matrix. Then (9) makes sense for $z = \lambda$ with λ not being in the spectrum of $\mathcal{A}$. The resulting operator becomes a bounded operator in $L^2[0,h]$.

4. Steady State and Frequency Response

Because of the time-invariance it induces, lifted model (5) is very effective in characterizing steady state response.

Let $G(z)$ be a stable transfer function as defined above. Here stability means that the spectrum of $\mathcal{A}$ is contained in the open unit disc of the complex plane.

Let us pose the question: *What is the steady state response against sinusoidal inputs?*

Let us take a sinusoid $e^{j\omega t}$. Its lifted image is

$$\{e^{j\omega kh}e^{j\omega\theta}\}_{k=0}^{\infty}.$$

This has the form $\{\lambda^k v(\theta)\}$, i.e., power function with infinite-dimensional initial vector. So we generalize the question above, and look for a steady state response against inputs of type $\{\lambda^k v(\theta)\}$ for $|\lambda| \geq 1$.

By definition (7), the z-transform of $\lambda^k v(\theta)$ is easily seen to be

$$\frac{zv(\cdot)}{z-\lambda}.$$

Since $G(z)$ is stable, the initial state response decays to zero, so the response against this input asymptotically approaches $G(z)\,(zv(\cdot)/(z-\lambda))$. Since $G(z)$ is analytic in a neighborhood of $z = \lambda$, it admits the expansion:

$$G(z) = G(\lambda) + (z-\lambda)\tilde{G}(z) \tag{10}$$

with some $\tilde{G}(z)$ that is also analytic in $|z| \geq 1$ (by the stability of G). It follows that

$$G(z)\frac{zv(\theta)}{z-\lambda} = \frac{zG(\lambda)v(\theta)}{z-\lambda} + z\tilde{G}(z)v(\theta).$$

The second term on the right goes to zero as $k \to \infty$ by the analyticity of $\tilde{G}$, so that the output approaches $zG(\lambda)v(\theta)/(z-\lambda)$. Therefore, the output y asymptotically approaches

$$y(kh+\theta) = \lambda^k (G(\lambda)v)(\theta) \tag{11}$$

as $k \to \infty$.

This result leads to the following observations.

1. When $\lambda = 1$ and $v(\theta) \equiv 1$, (11) gives $G(1)v$. This is the steady state response against the step input. If $G(z)$ is the transfer function from the reference signal to the error output, then asymptotic tracking is achieved only when $[G(1)v](\theta) \equiv 0$.
2. When $|\lambda| = 1$, $\lambda^k G(\lambda)v(\theta)$ is really not in "steady-state" unless $\lambda = 1$. However, its *modulus* $|G(\lambda)v(\theta)|$ remains the same at each sampling time. The change at each step is a phase shift induced by the multiplication by $e^{j\omega h}$.

The latter observation motivates the following definition:

DEFINITION 4.1. Let $G(z)$ be the transfer function of the lifted system as above. Let $\omega_s := 2\pi/h$. The *frequency response operator* is the operator

$$G(e^{j\omega h}) : L^2[0,h] \to L^2[0,h] \tag{12}$$

regarded as a function of $\omega \in [0, \omega_s)$. Its *gain* at ω is defined to be

$$\|G(e^{j\omega h})\| = \sup_{v \in L^2[0,h]} \frac{\|G(e^{j\omega h})v\|}{\|v\|}. \tag{13}$$

The maximum $\|G(e^{j\omega h})\|$ over $[0, \omega_s)$ is the H^∞ norm of $G(z)$.

The above definition of gain takes all the aliasing effects into account. To see this, observe that the lifting of $e^{j(\omega + n\omega_s)t}$ can be expressed also as

$$\{e^{j\omega kh} e^{j(\omega+n\omega_s)\theta}\}_{k=0}^{\infty}$$

because $e^{j(\omega+n\omega_s)h} = e^{j\omega h}$. Therefore, z-transforms of type

$$\frac{zv(\theta)}{z - e^{j\omega h}}$$

can express all aliased signals of $e^{j\omega t}$, and the gain defined above gives the magnitude of the worst case response against those signals.

5. Frequency Response via Sequence Spaces

Independently, Araki and co-workers proposed a different definition of frequency response for sampled-data systems [1, 2, 3]. The idea is the following. Fix a fundamental frequency $0 \le \omega < \omega_s$, and write $\omega_n := \omega + n\omega_s$. What they found through the impulse modulation formula (e.g., [32]) is that the steady state response against inputs of type $\sum_{n=-\infty}^{\infty} u_n e^{j\omega_n t}$ is again expressible in the same form. In other words, sample and hold operations introduce aliased waveforms of type $e^{j\omega_n t}$, $n = 0, \pm 1, \pm 2, \ldots$, but that completely describes the system and no further waveforms are necessary. They introduced a function space

$$X_\omega := \{x(t) = \sum_{n=-\infty}^{\infty} x_n e^{j\omega_n t}; \sum_n |x_n|^2 < \infty\}, \tag{14}$$

isomorphic to ℓ^2 for each fixed ω, and showed that a stable sampled-data system induces a bounded linear operator $\mathcal{G}_\omega : X_\omega \to X_\omega$ in the form

$$\mathcal{G}_\omega(\sum_{\ell=-\infty}^{\infty} u_\ell e^{j\omega_\ell t}) = \sum_{n,\ell=-\infty}^{\infty} g_n^\ell u_\ell e^{j\omega_n t}$$

They call this $\mathcal{G}_\omega$ (or the matrix (g_n^ℓ)) an *FR operator*. The same concept was also used effectively by Dullerud and Glover [16] for the study of robust stability.

This definition looks quite different from the frequency response introduced in the previous section. They are, however, actually equivalent. To see this, the following lemma is fundamental.

LEMMA 5.1. *Fix any $\omega \in [0, \omega_s)$. Then every $\varphi \in L^2[0,h]$ can be expanded in terms of $e^{j\omega_n \theta}$ as*

$$\varphi(\theta) = \sum_{n=-\infty}^{\infty} a_n e^{j\omega_n \theta} \tag{15}$$

with

$$a_n = \frac{1}{h}\int_0^h e^{-j\omega_n \tau}\varphi(\tau)d\tau = \frac{1}{h}\hat{\varphi}(j\omega_n) \tag{16}$$

where $\varphi \in L^2[0,h]$ is embedded in $L^2[0,\infty)$ as a function having support contained in $[0,\infty)$. Furthermore, the L^2 norm $\|\varphi\|$ is given by

$$\|\varphi\|^2 = h \sum_{n=-\infty}^{\infty} |a_n|^2. \tag{17}$$

In short, the family $\{e^{j\omega_n \theta}/\sqrt{h}\}_{n=-\infty}^{\infty}$ forms an orthonormal basis, so that in the lifting expression

$$\{e^{j\omega kh} v(\theta)\}_{k=0}^{\infty} \tag{18}$$

the initial function $v(\theta)$ can be expanded into the sum as above: $v(\theta) = \sum_{\ell=-\infty}^{\infty} v_\ell e^{j\omega_\ell \theta}$.

We give a lifting interpretation of $\mathcal{G}_\omega$. Suppose that (18) is applied to a stable transfer function $G(z)$. By what we have seen in the previous section, its steady state response at the k-th step is

$$\begin{aligned} e^{j\omega kh}G(e^{j\omega h})[v] &= e^{j\omega kh}G(e^{j\omega h})[\sum_{\ell=-\infty}^{\infty} v_\ell e^{j\omega_\ell \theta}] \\ &= \sum_{\ell=-\infty}^{\infty} e^{j\omega kh} G(e^{j\omega h})[e^{j\omega_\ell \theta}]v_\ell \end{aligned} \tag{19}$$

Expand $G(e^{j\omega h})[e^{j\omega_\ell \theta}]$ in terms of $e^{j\omega_n \theta}$ to get

$$G(e^{j\omega h})[e^{j\omega_\ell \theta}] = \sum_{n=-\infty}^{\infty} g_n^\ell e^{j\omega_n \theta}.$$

Substituting this to (19), we obtain

$$e^{j\omega kh}G(e^{j\omega h})[v] = e^{j\omega kh} \sum_{\ell=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} g_n^\ell e^{j\omega_n \theta} v_\ell.$$

Since $e^{j(\omega + n\omega_s)h} = e^{j\omega h}$, this is the k-th step response of

$$\sum_{\ell=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} g_n^\ell e^{j\omega_n t} v_\ell,$$

where $t = kh + \theta$. Interchanging the order of summation, this is equal to

$$\sum_{n=-\infty}^{\infty} \left(\sum_{\ell=-\infty}^{\infty} g_n^\ell(\omega) v_\ell \right) e^{j\omega_n t}. \tag{20}$$

This means that the sequence space definition given here is precisely the matrix expansion with respect to the basis $\{e^{j\omega_n \theta}\}$ of $G(e^{j\omega h})$. Since this is an orthonormal basis, this correspondence gives an isometric isomorphism, so that all of their singular values and norms coincide.

6. Gain Computation

The gain function $G(e^{j\omega h})$ is given as the operator norm at each frequency, and this is not so easy to compute. Fortunately, for most of the practical purposes, it can be computed as the maximal singular value [43]. To be more precise, the operator $G(e^{j\omega h})$ is at most a compact perturbation of a multiplication operator, and except the singular case in which the norm is equal to the norm of this multiplication operator, the norm of $G(e^{j\omega h})$ is given as the maximal singular value.

Our problem is thus reduced to that of solving the singular value equation

$$(\gamma^2 I - G^* G(e^{j\omega h}))w = 0. \tag{21}$$

This is still an infinite-dimensional equation. However, we can convert this into state space forms, by invoking the state space realization of $G(z)$ and its adjoint $G^*(z)$. Note that by lifting a realization of $G(z)$ can be written in the form

$$\begin{aligned} x_{k+1} &= \mathcal{A} x_k + \mathcal{B} w_k \\ y_k &= \mathcal{C} x_k + \mathcal{D} w_k. \end{aligned}$$

Its adjoint can then be easily derived as

$$\begin{aligned} p_k &= \mathcal{A}^* p_{k+1} + \mathcal{C}^* v_k \\ e_k &= \mathcal{B}^* p_{k+1} + \mathcal{D}^* v_k. \end{aligned}$$

Taking the z-transforms of both sides, setting $z = e^{j\omega h}$, and substituting $v = y$ and $e = \gamma^2 w$, we get

$$\begin{aligned} e^{j\omega h} x &= \mathcal{A} x + \mathcal{B} w \\ p &= e^{j\omega h} \mathcal{A}^* p + \mathcal{C}^* (\mathcal{C} x + \mathcal{D} w) \\ (\gamma^2 - \mathcal{D}^* \mathcal{D}) w &= e^{j\omega h} \mathcal{B}^* p + \mathcal{D}^* \mathcal{C} x \end{aligned}$$

Solving these, we obtain

$$\left(e^{j\omega h} \begin{bmatrix} I & \mathcal{B} R_\gamma^{-1} \mathcal{B}^* \\ 0 & \mathcal{A}^* + \mathcal{C}^* \mathcal{D} R_\gamma^{-1} \mathcal{B}^* \end{bmatrix} - \begin{bmatrix} \mathcal{A} + \mathcal{B} R_\gamma^{-1} \mathcal{D}^* \mathcal{C} & 0 \\ \mathcal{C}^* (I + \mathcal{D} R_\gamma^{-1} \mathcal{D}^*) \mathcal{C} & I \end{bmatrix} \right) \begin{bmatrix} x \\ p \end{bmatrix} = 0 \tag{22}$$

where $R_\gamma = (\gamma I - \mathcal{D}^*\mathcal{D})$. The important point to be noted here is that all the operators appearing here are actually matrices. For example, by checking the domain and range spaces, we easily see that $\mathcal{B}R_\gamma^{-1}\mathcal{B}^*$ is a linear operator from $\mathbf{C}^n$ into itself, i.e., a matrix. Therefore, in principle, one can solve the singular value equation (21) by finding a nontrivial solution for (22) (provided R_γ is invertible) [41, 43].

REMARK 6.1. In actual computation, it is often more convenient to employ approximation approach [23, 31]. A bisection search algorithm is also given in [23].

7. EXAMPLE

We give a simple H^2 design example that exhibits the difference of the modern and classical design methods.

Consider the following unstable continuous-time generalized plant G:

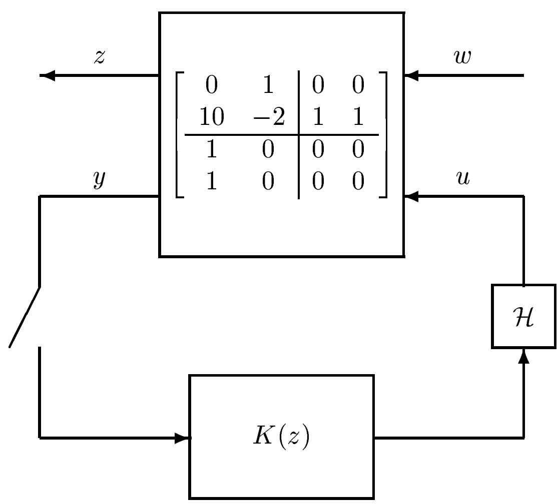

FIGURE 5. Sampled Feedback System

Here $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ is a shorthand for $D + C(zI - A)^{-1}B$.

Suppose we discretize the plant G at sampled points, get a discrete-time generalized plant G_d, and then execute the H^2 design for G_d. Naturally G_d loses not only the intersample information but also information on how long the sampling period is. We then pose the question: *What kind of performance does this design sacrifice?* To make a comparison, we take a direct, continuous-time based, sampled-data H^2 design [10, 30, 6]. Here the latter H^2 problem minimizes the average of responses against impulses applied over the first time interval $[0, h)$.

We set the sampling period $h = 0.2$, and execute the

- new sampled-data H^2 design, get a controller K_s,
- discretize the plant (representing the sample point behavior only), and execute the discrete-time H^2 design, and get a controller K_d.

The computation is executed over the function routines $\mathcal{H}$SYS Module program developed by HARA, YAMAMOTO, FUJIOKA and TAKEDA [24]. The results are as follows:
K_s (sampled-data design):

$$\left[\begin{array}{c|c} A_s & B_s \\ \hline C_s & D_s \end{array}\right] = \left[\begin{array}{c|c} \begin{bmatrix} -3.72509 & -20.8438 \\ 0.259501 & 0.999332 \end{bmatrix} & \begin{bmatrix} -16.6414 \\ 0.243988 \end{bmatrix} \\ \hline \begin{bmatrix} 11.0285 & 16.5185 \end{bmatrix} & 0 \end{array}\right]$$

K_d (discrete-time H^2 design):

$$\left[\begin{array}{c|c} A_d & B_d \\ \hline C_d & D_d \end{array}\right] = \left[\begin{array}{c|c} \begin{bmatrix} -5.12471 & -28.1772 \\ 0.337204 & 1.35992 \end{bmatrix} & \begin{bmatrix} -22.268 \\ 0.46908 \end{bmatrix} \\ \hline \begin{bmatrix} 13.1959 & 23.2278 \end{bmatrix} & 0 \end{array}\right]$$

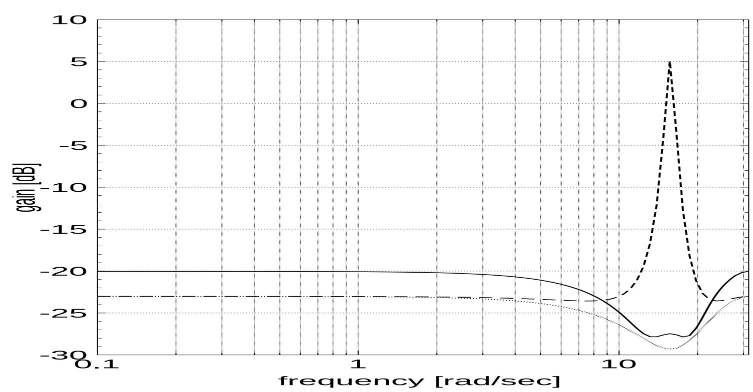

FIGURE 6. Frequency Responses

Figure 6 shows the frequency responses of the closed-loop systems. The solid curve shows that of the sampled-design, whereas the dotted curve shows the discrete-time frequency response when K_d is connected with the discretized plant G_d (i.e., purely discrete-time frequency response). Although it appears that the usual discretized design performs better, it is actually worse when we compute the real (continuous-time) frequency response of G connected with K_d. The dash curve shows this frequency response. It is similar to the discrete-time frequency response in the low frequency range, but exhibits a very sharp peak around the Nyquist frequency ($\pi/h \sim 15.2$ rad/sec).

This can also be observed in the time-response Figure 7 (impulse response). The solid curve shows the sampled-data design, and the dash curve the discrete-

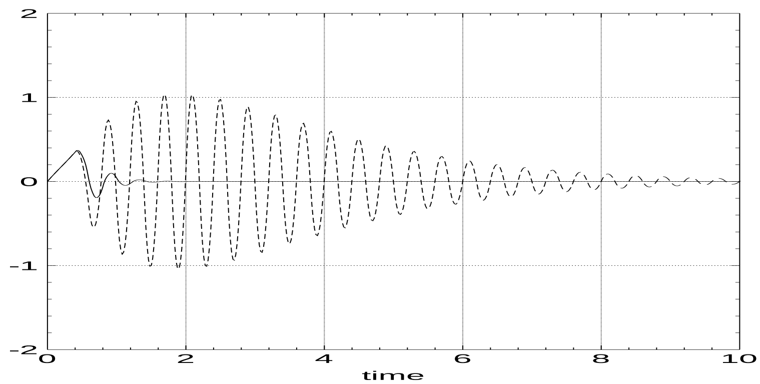

FIGURE 7. Impulse Responses

time design. It is easily seen that the latter shows a very oscillatory behavior. Also, we see that both responses decay to zero very rapidly *at sampled instants*. The difference is that the latter exhibits very large ripples, with period approximately 0.4 sec. This corresponds to 1/0.4Hz, which is the same as $(2\pi)/0.4 = \pi/h$ rad/sec, i.e., Nyquist frequency. This is precisely the phenomenon captured in the (lifted) frequency response in Figure 6.

Summarizing, we see that

- the modern sampled-data design effectively attenuates undesirable intersample ripples, and
- the introduced sampled-data frequency response successfully captures continuous-time behavior which the purely discrete-time notion fails to describe.

8. NOTES ON RELATED WORK

I have given a fairly personal and retrospective view on the recent developments of sampled-data systems, focused around the themes I was interested in. As I noted in the beginning, the views are fairly biased, and there are lots of other related, but independent work. To circumvent the overall perspective, let us give a few notes on some related work. Naturally, the list cannot be complete, but I have benefitted greatly from the recent account by CHEN and FRANCIS [12], which gives far more complete discussions; interested readers are referred to this monograph.

Varied design methods have been obtained, all taking account of intersample behavior, and differ from the classical design methods. The computation and optimization of H^∞-norm have been studied by many authors: CHEN and FRANCIS [9], KABAMBA and HARA [27], TOIVONEN [38], BAMIEH and PEARSON [5], TADMOR [37], SIVASHANKAR and KHARGONEKAR [35], HAYAKAWA

ET AL. [26], SUN ET AL. [36], YAMAMOTO [41], and YAMAMOTO and KHARGONEKAR [43], etc. Among them, CHEN and FRANCIS [9] gave the first attempt. KABAMBA and HARA [27] gave a first state space solution for the general case. BAMIEH and PEARSON [5] gave a solution in the lifting framework. SIVASHANKAR and KHARGONEKAR [35] gave a direct state space approach that leads to the Riccati equations.

H^2 control problems have also been studied fairly extensively. For example, see CHEN and FRANCIS [10, 11], KHARGONEKAR and SIVASHANKAR [30], BAMIEH and PEARSON [6], HARA, FUJIOKA and KABAMBA [22] for the case with discrete-time inputs, and HAGIWARA and ARAKI [21] for FR-operator type approach.

It is to be noted that if we allow any periodically time-varying perturbations, the small gain condition is necessary and sufficient for robust stability [34]. On the other hand, if we restrict perturbations to linear time-invariant ones it can be fairly conservative (DULLERUD and GLOVER [16]; an infinite-dimensional μ condition results.) Computation using "D-scaling" is discussed in [17].

L^1-norm problems are studied by DULLERUD and FRANCIS [15], SIVASHANKAR and KHARGONEKAR [33], and BAMIEH, DAHLEH and PEARSON [4].

For frequency domain approaches not dealt with here, especially that related to generalized hold functions or Bode type integral design constraints, see [18, 20].

There is also a new attempt to design digital filters with these new H^∞ sampled-data techniques. See [13].

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