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Choice Sequences: a Retrospect

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1. Introduction

The topic of this talk will be the lasting interest of L.E.J. Brouwer's notion of choice sequence for the philosophy of mathematics. In the past here has been done a good deal of work on choice sequences, but in the last decade the subject is a bit out of fashion, for several reasons, which I shall not go into here. In this retrospective I want to take a look with you at a special aspect of choice sequences, namely their interest as an important "case–study" in the philosophy of mathematics.

How does mathematics arrive at its concepts, and discover the principles holding for those concepts? This is a typically philosophical question, more easily posed than answered. A procedure which certainly has played a role and still plays a role might be described as

informally rigorous analysis of a concept

That is to say,

- given an informally described, but intuitively clear concept,
- one analyzes the concept as carefully as possible, and attempts to formulate formally precise principles characterizing the concept.

This is full of vague words such as "intuitively clear", "informal", "rigorous" etc. Instead of plunging myself deeper into the morass of vagueness and imprecision by trying to explain in general terms what I mean by these words, I shall give some examples below.

But before doing this, let me point out that usually an informal concept serving as basis for such an analysis is not found in nature but suggests itself on the basis of a good deal of *mathematical* experience only.

Also it is good to point out that many mathematicians in the course of their work are never called upon to make such a concept analysis, since there is always a vast amount of work to be done based on sharply delimited, well–understood mathematical notions. Nevertheless, the study of mathematical–concept analysis is of obvious interest for the philosophy and history of mathematics.

2. EXAMPLES

- *Example 1.* The notion of *natural number*, formally characterized by induction. Natural numbers are historically at the basis of mathematics, the analysis of the concept culminated in the work of Dedekind and Peano, formally characterizing them by the validity of induction in full generality.
- *Example 2.* The ε–δ–definition of *continuity of functions* may be obtained by reflecting on the idea of a smooth curve, a curve which makes no "jumps".
- *Example 3.* The notion of *area* for a wide class of pointsets in the plane. A few intuitively obvious properties for the notion of area lead to a complete and unique characterization. ("Area " should conform to the standard euclidean definition of the area of a rectangle, should be additive, monotone and take only positive values.)
- *Example 4.* The notion of *humanly computable function*, or *algorithm*, has been analyzed by Turing and leads to the mathematically precise notion of a recursive function (= Turing–computable function, λ–definable function, etc.). A similar analyis is possible of the notion of *mechanically computable function*, leading to the same formal concept.
- *Example 5.* The notions of a *random sequence* and *independency* of random sequences. An analysis in our sense (not completely satisfactory) of this concept has been carried out by the probability theorist R. von Mises; in recent years M. van Lambalgen has published on this([6, 7, 8]. I shall not discuss any details here, since this would certainly carry us too far. Van Lambalgen argues that even if we can develop probability theory on an axiomatic basis, starting from the notion of a probability measure, instead of starting from the notion of random sequence, we still have to face the question "what is a random sequence" when confronted with the applications of probability theory.

  This example shows some similarities with our next and final example:
- *Example 6.* Choice sequences. The details of this example will follow later.

For contrast, I shall also briefly discuss a "non–example": the notion of "set", as it figures in the modern practice of axiomatic set theory. Basis for my discussion are two papers by PENELOPE MADDY ([9, 10]) and MICHAEL HALLETT's book ([4]).

The justifications for axioms of set theory are of two kinds, in Maddy's terminology "intrinsic" and "extrinsic" ones. The intrinsic arguments argue from the intuitive understanding of the concept of set; the extrinsic (pragmatic, heuristic) arguments are based on, e.g., the desirability of certain mathematical consequences, intertheoretic consequences, explanatory power. In the use of extrinsic justifications there is an analogy with the natural sciences: experimenting with the consequences helps us to choose the "right" axioms. Extrinsic arguments are brought forward, not only for fancy axioms stating the existence of certain very large cardinals, but also for the basic axioms of the well–known Zermelo–Fraenkel axiomatization. This happens even for an axiom looking so uncontroversial as the extensionality axiom; most authors regard it as intrinsically justified, but some give also extrinsic reasons: the extensional notion of set is simpler, clearer and more convenient than its non–extensional ("intensional") counterpart (sets as properties).

Several authors have attempted to justify the Zermelo–Fraenkel axioms by analyzing the intuitive picture of the cumulative hierarchy. As shown convincingly by Hallett (in his book "Cantorian set theory and limitiation of size") and Maddy, many difficulties are glossed over in these "intrinsic" justifications, and all in all, these attempts at concept analysis remain thoroughly unconvincing.

From the examples, and the non–example above, it will be clear that on the one hand concept analysis has played, and may be expected to continue playing, an important role in the development of mathematics, but on the other hand that it is by no means the only way of introducing new concepts.

3. Choice sequences

Choice sequences were introduced by L.E.J. Brouwer around 1916, in order to give an intuitively more satisfactory account (than hitherto) of the continuum in the context of his intuitionistic mathematics ([12]). Before that time, Brouwer, just like the french "empiristes" such as E. Borel, had viewed the continuum as a primitive concept which could not be understood as the "set of its elements". The introduction of choice sequences restored this possibility (of having an "arithmetic" theory of the continuum, in contemporary terminology).

A very interesting aspect of Brouwer's notion was that choice sequences, when taken seriously as mathematical objects, *enforced* the use of intuitionistic logic, since some of the principles valid for choice sequences contradicted classical logic. Thus they provided an alternative mathematical universe with a deviant logic. Nowadays this is commonplace, in particular in topos theory, but it was certainly not commonplace in Brouwer's time.

A choice sequence in Brouwer's sense is a sequence of natural numbers (to keep it simple), which is not a priori given by a law or recipe for the elements, but which is created step by step by the ideal mathematician (anyone of us may serve as an approximation of the ideal mathematician); the process will go on indefinitely, but is never finished. Brouwer's discovery was that one could use such "unfinished objects" in a meaningful way in (intuitionistic) mathematics. It is not the individual choice sequence which matters, but the principles which hold for all of them. Since continuous operations on sequences can be carried out by operating on initial segments, choice sequences are closed under continuous operations.

The subjective element in the construction of choice sequences, the conflict with classical logic, and Brouwer's little–understood argument for the so–called "bar theorem" all tended to make choice sequences look mysterious or dubious (Errett Bishop even talked of "mumbo–jumbo").

People reacted in different ways. Some rejected intuitionism wholesale. Others preferred developing constructive mathematics without choice sequences

(A.A. Markov, E. Bishop). Logicians such as S.C. KLEENE ([5]) encoded the mathematical practice involving choice sequences, and gave a relative consistency proof (relative to a system in the intersection of intuitionistic and classical mathematics).

Over the years Brouwer repeatedly indicated how one should think of the individual choice sequence, but his detailed views seemed to oscillate.

Some intuitionists are of the opinion that further analysis of the individual choice sequence is unneccessary, nay, even impossible — there is a single concept, and no room for meaningful "subuniverses" ([3, 2]). I follow here another, "analytical" approach, which highlights the "intensional" aspects of choice sequences.

A property of sequences, or an operation on sequences is said to be *extensional*, if it only depends on the "course–of–values" of the sequence, that is to say, for operations Φ and properties P:

$$\forall x(\alpha x = \beta x) \to \Phi\alpha = \Phi\beta, \text{ and } \forall x(\alpha x = \beta x) \to (P(\alpha) \leftrightarrow P(\beta)).$$

In this talk, intensional properties are simply non–extensional properties.

For example, suppose we consider sequences given by algorithms. The operation which assigns to a sequence its algorithm is non–extensional. Usually this introduction of non–extensionality is avoided by talking about the algorithms themselves, instead of the sequences. But this way of avoiding non–extensionality is not feasible in the case of choice sequences.

On the analytical approach, we try to analyze what it means to say that a choice sequence is "given", and what it means to assert that something holds for all choice sequences. An algorithmic sequence is obviously "given" by describing the algorithm, but how is a choice sequence "given"?

I shall endeavour to illustrate the analytical approach by one example, for a rather extreme variant of choice sequence, the so-called *lawless sequences* ([16, chapter 12],).

Informally, we think of a lawless sequence of natural numbers as a process of choosing values in $\mathbb{N}$, started by the ideal mathematician (in the mind of which, in Brouwer's view, all mathematics is constructed); the process is started under the a priori restriction that

- at any stage of the construction never more than an initial segment has been determined, and that *no* restrictions have been imposed on future choices;
- there is a commitment to determine more and more values (so the sequence is infinite).

If we reflect on these stipulations, the following principle seems to be evident.

THE PRINCIPLE OF OPEN DATA

If P is a property of lawless sequences (α, β will be used for lawless sequences), and $P(\alpha)$, then there is an initial segment $\overline{\alpha}n = \langle \alpha(0), \ldots, \alpha(n-1)\rangle$ such that

$$\forall\beta(\overline{\alpha}n = \overline{\beta}n \to P(\beta)) \text{ or } \forall\beta(\forall m{<}n(\alpha m = \beta m) \to P\beta)).$$

The reason is that if we can assert at some stage in our activity $P(\alpha)$ for a lawless sequence α, then the basis for our assertion can only be an initial segment of α, and therefore any lawless sequence starting with the same initial segment ought to have the same property.

A lawless sequence may be compared to the sequence of casts of a die. There too, at any given stage in the generation of the sequence never more than an initial segment is known. But in discussing random sequences, we are usually interested in another notion of truth: instead of absolute truth, we have "true with probability 1". A random sequence of zero's and one's contains a zero with probability one, but there is no absolute certainty that we will encounter a one before this actually happens.

All this is very nice, but what can we say about the existence of lawless sequences? Certainly we can assume that we can start creating some lawless sequences, even as many as we like, but some experimenting soon reveals that hat we need to get a mathematical theory at all, is that all possible initial segments occur:

Density principle

For each initial segment there is a lawless sequence starting with that segment.

In order to guarantee this, we have to modify our notion of lawless sequence a little: at the start, when generating a lawless sequence, we permit the stipulation of a finite initial segment in advance. After that, things go on as before. In the "model" of the casts of a die, this would amount to permitting initially a number of deliberate placings of the die.

However, this modification of our notion weakens the justification for the principle of open data. After this modification it may happen that, for two lawless sequences α, β, at some stage the same initial segment may have been determined, while nevertheless there is a piece of information about these sequences which cannot be read off from these initial segments alone, namely *which* parts of these initial segments had been fixed at the start. So we may initially stipulate that α begins with 2,1 and β with 2,1,4, but in a later stage for both sequences we know the initial segment 2,1,4,2,3 and nothing more. One may well feel that we do not want to distinguish mathematically between these two situations at later stages, or in other words, these differences should not enter in the properties of sequences we are interested in. In short, we want to *abstract* from these distinctions. If we do, we expect "open data" to hold. In this way several mor eprinciples may be formulated which yield a formally consistent non–classical theory, which, up to a point, completely characterizes lawless sequences.

In this example, mathematical desiderata led us to introduce a complication into the intuitive picture, and to weaken the link between the informal concept and the formal principles obtained by analyzing the concept. We have weakened the intuitive basis in order to gain in mathematical manageability.

Of course this happens elsewhere too. Time and again objections have been raised against unrestricted induction over the natural numbers, since natural numbers as usually treated are not "conceptually uniform": our grasp of a small number like 9 is vastly different from our understanding of, say, 9^{9^9}. The latter presupposes the understanding of exponentiation as an everywhere defined operation on the natural numbers. In practice, we treat 9 and 9^{9^9} as "essentially the same", i.e. we forget about (abstract from) the intuitive differences. This is an abstraction with powerful consequences; the difficulties encountered in the various attempts to take these differences seriously show how essential the role of this kind of abstraction is. (To some extent the distinction is taken into account in *bounded arithmetic*; but this undeniably interesting theory is mathematically much more complicated than ordinary arithmetic.)

Similarly, certain intuitionists adopt a "holistic" view of the continuum, and maintain that is neither desirable nor possible even to distinguish subdomains such as the lawless sequences ([3, 2]).

However, if one does distinguish subdomains, according to how precisely a sequence is given, there is a wealth of possibilities giving rise to consistent and interesting formalisms. From a mathematical point of view, the main problem is to discover which notions lead to a nice theory.

We mention two possibilities, more or less at random.

(A) Instead of stipulating, as for lawless sequences, that at every stage future choices are completely unrestricted, we now permit at any stage either to leave future choices completely free, or to restrict henceforth choices to a law (definite recipe), at least if this option has not already earlier been chosen.

(B) We permit at any stage restriction of the continuation of the sequence to lie with a definite (i.e., not depending on choice parameters) tree of possibilities; later restrictions should be compatible with the restrictions chosen before. As an extreme case the choices may be narrowed down to a law, as a single–branch tree.

Lack of time prevents me from discussing the properties of the sequences in these examples — not in themselves important — therefore I only mention the result of the analysis: (A) and (B) have quite distinct properties, which distinguish them from each other and from the lawless sequences.

But I hope this suffices to show that choice sequences provides very interesting examples for experimentation with concept analysis, perhaps precisely because these examples are far less familiar to us than say, "area" or "continuous function".

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