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Spatial Patterns in Higher Order Phase Transitions

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1. Introduction

In this lecture we shall be concerned with stationary spatial patterns in bi-stable systems described by the *Extended Fisher-Kolmogorov (EFK)* equation

$$\frac{\partial u}{\partial t} = -\gamma \frac{\partial^4 u}{\partial x^4} + \frac{\partial^2 u}{\partial x^2} + u - u^3, \qquad \gamma > 0. \tag{1}$$

This equation was proposed in 1987 by Coullet, Elphick & Repaux [8], and in 1988 by Dee & van Saarloos [11] as a generalization of the classical Fisher-Kolmogorov (FK) equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u - u^3, \tag{2}$$

which has played an important role in the studies of pattern formation in bi-stable systems (cf. [2], [16], [20], [26], [27]). The term "bi-stable" refers here to the fact that the uniform states $u = \pm 1$ are stable as solutions of the equation

$$\frac{du}{dt} = u - u^3.$$

The EFK equation arises in the study of singular points (so called *Lifshitz points* [14]) in phase transitions, and as the evolution equation in gradient systems described by the energy functional

$$I(u) = \int \left\{ \frac{\gamma}{2}(u'')^2 + \frac{\beta}{2}(u')^2 + F(u) \right\} dx, \qquad \gamma > 0, \quad \beta \in \mathbb{R}, \tag{3}$$

where F denotes the double-well potential

$$F(u) = \frac{1}{4}(1 - u^2)^2. \tag{4}$$

We then obtain the EFK equation when we choose $\beta = 1$. Another important application of this equation is found in the theory of instabilities in nematic liquid crystals [5], [28].

In studies of second order materials [10], [18], [19], one also finds the functional $I(u)$. Here $\beta < 0$. The stationary points of $I(u)$ are then equivalent to the equilibrium solutions of the Swift-Hohenberg equation

$$\frac{\partial u}{\partial t} = -\Big(1 + \frac{\partial^2}{\partial x^2}\Big)^2 u + \alpha\, u - u^3, \qquad \alpha > 0 \tag{5}$$

when $\alpha > 1$ through a simple scaling of $x,\ t$ and u (see for instance [7], [9], and the references therein).

In this paper we are interested in stationary spatial patterns which can be described by the EFK equation. Thus, we are concerned with bounded solutions $u(x)$ of the equation

$$\gamma u^{iv} = u'' + u - u^3 \quad \text{on} \quad \mathbb{R}. \tag{6}$$

Related equations arise in the propagation of solitons in nonlinear optical fibres with *negative 4th order dispersion*, which are described by the nonlinear Schrödinger equation

$$i\frac{\partial u}{\partial t} = -\frac{\partial^4 u}{\partial x^4} + \frac{\partial^2 u}{\partial x^2} - |u|^2 u, \qquad \gamma > 0 \qquad u \in \mathbb{C}. \tag{7}$$

If one sets

$$u(x,t) = v(x)e^{i\omega t}, \qquad v \in \mathbb{R}, \quad \omega \in \mathbb{R},$$

then we obtain for v:

$$v^{iv} - v'' - \omega^2 v + v^3 = 0. \tag{8}$$

The deflection of an asymmetrically supported strutt leads to the equation

$$u^{iv} + Pu'' + u - u^2 = 0, \qquad P \in \mathbb{R}, \tag{9}$$

which in recent years has been studied in a series of papers [1], [6], [4] by Hamiltonian methods with a focus on homoclinic orbits (solitons).

Finally we mention an equation which has been studied in connection with observed travelling waves observed in suspension bridges, such as the Golden Gate bridge [17]:

$$u^{iv} + c^2 u'' + (1+u)_+ - 1 = 0. \tag{10}$$

The types of spatial patterns for the EFK equation we shall discuss here will be

- *Kinks*, that is heteroclinic orbits connecting the stable uniform states $u = -1$ and $u = +1$.
- *Periodic patterns.*
- *Solitons*, that is, homoclinic orbits to one of the uniform states.
- *Chaotic patterns*, as conjectured in [8].

2. Stationary solutions

The spatial patterns which arise in the context of the FK equation ($\gamma = 0$) are bounded solutions of the equation

$$u'' + u - u^3 = 0 \quad \text{on} \quad \mathbb{R}.$$

The spatial patterns which arise in the context of the FK equation are well known and consist of kinks, which are *odd*, *monotonone* and given explicitly by

$$u(x) = \pm \tanh\Big(\frac{x}{\sqrt{2}}\Big),$$

and periodic solutions: for each $\mu \in (0,1)$ there exists a unique periodic solution (modulo sign and translations) such that

$$\max\{|u(x)| : x \in \mathbb{R}\} = \sqrt{1-\sqrt{\mu}} < 1.$$

Here μ is the value of the first integral:

$$-(u')^2 + \frac{1}{2}(1-u^2)^2 = \frac{\mu}{2}. \tag{11}$$

Note that these periodic solutions have a single bump between zeros. Plainly, $\mu = 0$ for kinks; for $\mu \notin [0,1)$ there are no nontrivial bounded solutions of (6).

Turning to the EFK equation, we consider bounded odd solutions of the equation

$$\gamma u^{iv} = u'' + u - u^3 \quad \text{on} \quad \mathbb{R} \tag{12}$$

in which $\gamma > 0$. The first integral now becomes

$$2\gamma u'u''' - \gamma(u'')^2 - (u')^2 + \frac{1}{2}(1-u^2)^2 = \frac{\mu}{2}. \tag{13}$$

We note that at $\gamma = \frac{1}{8}$ the nature of the critical points $(u, u', u'', u''') = (\pm 1, 0, 0, 0)$ changes:

$$\left.\begin{array}{lll} \gamma \le \frac{1}{8} & \Rightarrow & \mathrm{Im}\lambda_k = 0, \\ \gamma > \frac{1}{8} & \Rightarrow & \mathrm{Im}\lambda_k \ne 0, \end{array}\right\} \qquad k = 1,2,3,4.$$

This critical value of γ has a profound effect on the structure of the family of stationary patterns.

About *Kinks* we find that if $\gamma \le \frac{1}{8}$ there exists a unique monotone kink, whilst if $\gamma > \frac{1}{8}$ there exist many kinks. Specifically, for any integer $N \ge 0$ there exists a kink with $2N+1$ zeros. These kinks all satisfy the universal bound:

$$|u(x)| < \sqrt{2} \quad \text{for} \quad x \in \mathbb{R}. \tag{14}$$

About *Periodic solutions*, we find that for any $\gamma > 0$ and any $\mu \in (0,1)$, there exists a single bump periodic solution which, as with the FK equation, has maxima below $u = 1$. However, for $\mu = 0$ new periodic solutions appear when $\gamma > \frac{1}{8}$: two branches of single bump periodic solutions $u_\pm(\gamma)$ bifurcate from the unique odd monotone kink U at $\gamma = \frac{1}{8}$ and continue to exist for all $\gamma > \frac{1}{8}$. They have the properties

$$0 < |u_-(\gamma)|_\infty < 1 < |u_+(\gamma)|_\infty < \sqrt{2}.$$

In addition to these single bump periodic solutions, many multi bump periodic solutions bifurcate from U at $\gamma = \frac{1}{8}$.

With the methods of [22], single as well as multi bump *Solitons* tending to $u = 1$ or $u = -1$ as $x \to \pm\infty$, were recently shown to exist when $\gamma > \frac{1}{8}$.

Finally, it was found that the EFK equation has odd *chaotic* stationary patterns when $\gamma > \frac{1}{8}$ and $\mu = 0$. The graph of these stationary solutions has an infinite sequence of local maxima tending to infinity and is characterized by the position of the local maxima with respect to the level $u = 1$: above or below. It is easily seen that maxima are always local and that they can never lie on the line $u = 1$.

Let us denote the locations of the consecutive maxima on $\mathbb{R}^+$ by ξ_k, $k = 1, 2, \ldots$, and let them be ordered so that $\xi_k < \xi_{k+1}$. With this sequence we associate a symbol sequence $\sigma = (\sigma_1, \sigma_2, \ldots)$ in the following manner:

$$\begin{array}{lcl} u(\xi_k) > 1 & \Longleftrightarrow & \sigma_k = 1 \\ u(\xi_k) < 1 & \Longleftrightarrow & \sigma_k = 0 \end{array} \qquad \text{for} \quad k \geq 1. \tag{15}$$

It is shown in [23] that for γ suffiently large and $\mu = 0$, there exists for every symbol sequence (σ_k) an odd solution u_σ of equation (6) on $\mathbb{R}$ of which the maxima satisfy (15). These solutions u_σ all have the properties

$$u_\sigma(\xi_k) > 0 \quad \text{and} \quad u_\sigma(\eta_k) < -1 \qquad \text{for every} \quad k \geq 1, \tag{16}$$

where (η_k) denotes the sequence of consecutive minima on $\mathbb{R}^+$, and so these solutions oscillate around $u = 0$. Furthermore, if a and b are two consecutive extrema, then

$$|u_\sigma(a) - u_\sigma(b)| > 1. \tag{17}$$

Recently is was shown in [15] that chaotic solutions, identified with the symbol sequences defined above, even exist for *all* $\gamma > \frac{1}{8}$. However, in this case, the solutions do not have zeros between extrema and do not satisfy (16) and (17).

3. Outline of the method

In this section we sketch the method used in [21], [22], [23], [24] to prove many of the results outlined in Section 2. For simplicity we shall set $\mu = 0$. The method is based on a shooting argument. Looking for odd solutions, we consider the initial value problem

$$\begin{cases} \gamma u^{iv} = u'' + u - u^3, \ x > 0 \\ u(0) = 0, \ u'(0) = \alpha, \ u''(0) = 0, \ u'''(0) = \beta. \end{cases} \tag{18}$$

Without loss of generality we may set $\alpha > 0$, and in view of the energy identity (13) we find that

$$\beta = \beta(\alpha) = \frac{1}{2\gamma\alpha}\Big(\alpha^2 - \frac{1}{2}\Big), \tag{19}$$

where we have set $\mu = 0$.

Plainly, Problem (18) has for every $\alpha > 0$ a unique local solution $u(x, \alpha)$. When looking for single bump periodic solutions, we seek a positive value of α such that $u(x, \alpha)$ has the properties

$$\begin{aligned} &u'(x,\alpha) > 0 \quad \text{for} \quad 0 \le x < \xi \\ &u'(\xi,\alpha) = 0 \quad \text{and} \quad u'''(\xi,\alpha) = 0 \end{aligned} \tag{20}$$

for some finite $\xi = \xi(\alpha) > 0$. It is easily verified that a solution defined on $[0, \xi]$, which satisfies (18–20) can be extended to yield a periodic solution of period 4ξ. Thus, we define

$$\xi(\alpha) = \sup\{x > 0 : u'(\cdot, \alpha) > 0 \ \text{ on } \ [0, x)\}.$$

This function has the following properties:

PROPOSITION 3.1. *Let $\gamma > \frac{1}{8}$. Then for every $\alpha > 0$, we have*

$$\xi(\alpha) < \infty \quad \textit{and} \quad u'(\xi(\alpha), \alpha) = 0.$$

REMARK. If $\gamma \le \frac{1}{8}$, then $\xi = \infty$ is possible, but only for the kink.

PROPOSITION 3.2. $\xi \in C(\mathbb{R}^+)$.

When $u'' \neq 0$, the continuity of ξ follows from the Implicit Function Theorem applied to the equation

$$u'(\xi(\alpha), \alpha) = 0.$$

The points at which $u'' = 0$ are delicate. At these points, the energy identity becomes

$$\frac{1}{2}(1 - u^2)^2 = 0,$$

and so we must have $u = \pm 1$. A careful analysis shows that at these points ξ is also continuous in α.

To prove the existence of periodic solutions we now study the function

$$\phi(\alpha) \stackrel{\text{def}}{=} u'''(\xi(\alpha), \alpha) \quad \text{on} \quad \mathbb{R}^+.$$

By Propositions 3.1 and 3.2, the function ϕ is well defined and continuous.

LEMMA 3.3. *We have*

(a) $\phi(\alpha) < 0$ *for* α *small*,

(b) $\phi(\alpha) < 0$ *if* $\alpha \ge \dfrac{1}{\sqrt{2}}$, PROOF. The first assertion follows at once from (19).

To prove the second assertion, we set $\alpha \geq \frac{1}{\sqrt{2}}$. Then we see from the second line in (18) that $u'''(0) \geq 0$, and that if $u'''(0) = 0$, then $u^v(0) = 1$. This means that $u'' > 0$ for x small and in fact, in view of the equation, that $u'' > 0$ as long as $u \leq 1$. Therefore

$$u(\xi(\alpha), \alpha) > 1 \quad \text{and} \quad u''(\xi(\alpha), \alpha) < 0 \quad \text{if} \quad \alpha \geq \frac{1}{\sqrt{2}}. \tag{21}$$

Let x_1 be the point on $[0, \xi]$, where $u = 1$. Then $u''(x_1) > 0$ whilst $u''(\xi) < 0$, so that there must be a point $x_2 \in (x_1, \xi)$ where $u'' = 0$ and $u''' \leq 0$. Since $u(x_2) > 1$, it follows from the differential equation that $u^{iv} < 0$. Another inspection of the differential equation shows that $u'' < 0$, $u''' < 0$ and $u^{iv} < 0$ on $(x_2, \xi]$. This proves Part (b). □

We are now ready to prove that there exist periodic solutions with maxima below $u = 1$ as well as periodic solutions with maxima above $u = 1$. Write

$$\alpha_1 = \sup\{\alpha^* > 0 : u(\xi(\alpha), \alpha) < 1 \;\text{ for }\; 0 < \alpha < \alpha^*\} \quad \text{and} \quad u_1 = u(\cdot, \alpha_1).$$

It follows from the continuity of $\xi(\alpha)$ that α_1 is well defined, and that

$$u_1(\xi) = 1 \quad \text{and} \quad \phi(\alpha_1) = u_1'''(\xi) > 0. \tag{22}$$

The last inequality follows from the fact that $u_1''' \geq 0$ and that if $u_1''' = 0$, then by uniqueness, $u_1 \equiv 1$, which is not the case.

Since by Lemma 3.3(a), $\phi(\alpha) = u'''(0, \alpha) < 0$ for small values of α, it follows from (22) and the continuity of $\xi(\alpha)$ that there exists an $\alpha_- \in (0, \alpha_1)$ such that

$$\phi(\alpha_-) = u'''(\xi(\alpha_-), \alpha_-) = 0. \tag{23}$$

Thus $u(x, \alpha_-)$ satisfies all of (18–20) and so is a periodic solution of equation (6).

However, by Lemma 3.3(b) there also exists an $\alpha_+ \in (\alpha_1, \frac{1}{\sqrt{2}})$ such that

$$\phi(\alpha_+) = u'''(\xi(\alpha_+), \alpha_+) = 0, \tag{24}$$

so that the function $u(x, \alpha_+)$ is also a periodic solution of equation of (6). Observe that our construction implies that

$$\max\{u_-(x) : x \in \mathbb{R}\} < 1 < \max\{u_+(x) : x \in \mathbb{R}\}.$$

More complicated periodic solutions can be constructed by following not only the first critical point but also the second, the third and so on. For instance, let

$$\eta = \sup\{x > \xi : u'(\cdot, \alpha) < 0 \;\text{ on }\; (\xi, x)\}.$$

Then we follow the function

$$\psi(\alpha) \stackrel{\text{def}}{=} u'''(\eta(\alpha), \alpha).$$

Since $u'''(0,\alpha) \to -\infty$ as $\alpha \to 0$ it is easily seen that $u(\eta) < -1$ for α suffiently small. As α increases, $u(\eta(\alpha),\alpha)$ passes through the line $x = -1$, say at α_{-1}. There we have

$$u(\eta(\alpha_{-1}),\alpha_{-1}) = -1, \quad u''(\eta(\alpha_{-1}),\alpha_{-1}) = 0 \quad \text{and} \quad \psi(\alpha_{-1}) < 0.$$

At $\alpha = \alpha_1$ we have $u(x,\alpha_1) = 1$ and $u''(x,\alpha_1) = 0$. Here

$$\eta(\alpha_1) = \xi(\alpha_1) \quad \text{and so} \quad \psi(\alpha_1) > 0.$$

Thus, $\psi(\alpha_{-1}) < 0$ and $\psi(\alpha_1) > 0$. Therefore, there must exist a point $\alpha_0 \in (\alpha_{-1},\alpha_1)$ such that $\psi(\alpha_0) = 0$ and hence $u(x,\alpha_0)$ is a periodic solution which is symmetric with respect to its *second* critical point. Note that

$$-1 < u(\eta(\alpha_0),\alpha_0) < 0 < u(\xi(\alpha_0),\alpha_0) < 1.$$

All these inequalities are obvious from the construction, except the second one, which requires some more detailed analysis. A numerical analysis of such solutions was carried out in [3].

By applying this analysis in an iterative manner, using ideas of [12], [13], it is possible to prove the existence of multi bump kinks and chaotic patterns, as indicated in Section 2.

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