The axioms of set theory are sometimes motivated as follows:
(1) A collection is a set iff at some stage all of its members exist.
(2) A set exists at some stage iff at some earlier stage all of its members exist.
In order to justify the Axiom of Foundation, one often adds:
(3) The stages are well-ordered by “earlier than”.
This is a circular “reduction” of the foundation of sets to the well-ordering of stages. We present a simple definition of “stage” such that only (1) needs to be assumed: (2) follows by definition and (3) can be derived from (1) and (2).

1. Foundations: sets, mathematics, and the cumulative hierarchy
The foundation SMC was founded 50 years ago. Mathematicians know that 50 is just a number. But what is a number? This is a question about the foundations of mathematics.
Set theorists will answer that a number is the set of all smaller numbers\(^1\). So 50 is the set \{0, 1, \ldots, 49\}. Its fifty members are also sets, and so on: It is a pure set. So we can choose sets \(x_1 \in 50, x_2 \in x_1\), and so on until we arrive at the empty set. In fact, this will happen (after at most fifty steps): The set is well-founded. Well-founded sets can be “formed” from their members step by step, in such a way that each set is formed after its members. In case of pure sets, we start this process with nothing at all, forming the empty set. Fifty stages later we can form the set 50, after an infinite number of stages the set

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\(^1\) One might think it would be more natural to identify e.g. 50 with the set \{1, 2, \ldots, 50\}. This is possible if one accepts the Anti Foundation Axiom (Aczel, 1988). But then all positive natural numbers would be equal to the set \(x\) whose only member is \(x\).
\[ \omega = \{0, 1, \ldots \} \] of all natural numbers, and so on. Boolos mentioned 25 years ago that “authors of set-theory texts either omit [this iterative conception of set] or relegate it to back pages”. One usually indexes the “stages” by ordinal numbers and defines the partial universe \( V_\alpha \) of all sets that are formed before (or “exist at”) stage \( \alpha \) by transfinite induction as \( \bigcup_{\beta < \alpha} \mathcal{P}V_\beta \). Forget this complicated definition of the cumulative hierarchy! We will present an elementary one that can play a role in the foundations of set theory.

2. **The Founders: Scott, Mirimanoff, and Cantor**

By reducing mathematical objects to sets, set theory provides mathematics with foundations. But what is a set? The founder of set theory gave an informal answer a century ago.\(^2\) A set is a collection of objects into a whole. Such a collection may be small (like 50) or big (like \( \omega \)). But the collection \( \Omega = \{0, 1, \omega, \ldots \} \) of all ordinal numbers, which was studied by Burali-Forti (1897), turned out to be “too big”: It is not a set. Russell (1903) found a much simpler example: the collection of all sets that are not members of themselves. Mirimanoff (1917) found an example in between: the collection of all well-founded sets. He formulated the fundamental problem of set theory:哪 Which collections of objects are sets?

Departing from some postulates, Mirimanoff solved this problem for well-founded sets by using ordinals to measure the well-foundedness: The rank of a set is the least ordinal above the ranks of its members. He showed that a collection of well-founded sets is a set if and only if there is an upperbound for the ranks of its members. In particular, the collection \( V_\alpha \) of all sets of rank smaller than \( \alpha \) is a set.

About 75 years ago, Zermelo and Fraenkel gave set theory its current axiomatic foundations. These axioms do not mention the cumulative hierarchy. From a logical point of view, they just describe a directed graph \( (V, \in) \) whose vertices are called “sets” and whose edges stand for “is a member of”. For example, the Axiom of Foundation (added in 1925 by Von Neumann) implies that this graph has no cycles.

But why these particular axioms? Why do we assume that all sets are well-founded? Avoidance of contradictions cannot be the only motivation. About 25 years ago, Scott was one of the authors who tried to justify the axioms by reformulating them in terms of the “stages” or “levels” of the cumulative hierarchy.

3. **The Axiom of Foundation: Stages, Motivation and Circularity**

The Axiom of Foundation is usually motivated as follows: The members of a set should be given before the set itself (in a logical, not temporal sense).

\(^2\) Cantor (1895–1897): “Unter einer “Menge” verstehen wir jede Zusammenfassung \( M \) von bestimmten wohlunterschiedenen Objekten \( m \) unserer Anschauung oder unseres Denkens (welche die “Elemente” von \( M \) genannt werden) zu einem Ganzen.”

\(^3\) “Quelles sont les conditions nécessaires et suffisantes pour qu’un ensemble d’individus existe?”
By assuming that each set $x$ is formed at some stage $s$ and that each member of $x$ is formed at an earlier stage than $s$, Boolos (1971) reduced the well-foundedness of the relation “is a member of” to the well-foundedness of “is earlier than”. In fact he even assumed that this relation between stages is a well-ordering. But is this so evident? This seems to be just as circular as proving that the partial universes form a well-ordered hierarchy after having defined them by means of ordinals.

According to Levy (1973), the idea of “forming” sets stage by stage cannot be viewed as fundamental for the notion of set. The weakest part of this point of view is that the reasoning leading to the concept of a well-founded set uses the well-ordering of the layers.”

However, Scott’s axiomatization avoids the concept of well-ordering: He takes the notion of “is a partial universe” as primitive, assumes some elementary properties of partial universes and (other) sets, and then proves that the cumulative hierarchy is well-ordered by “is a member of”. In fact, some of these properties can be proved after defining the notion of “partial universe”.

4 **A simple, minimal characterization**

In the first-order language of set theory, the membership relation $\in$ is the only primitive relation. The inclusion relation $\subseteq$ can be defined as usual, and if the Axiom of Extensionality is assumed, then $x = y$ can be defined as $x \subseteq y \land y \subseteq x$.

If the members of some set $x$ can be characterized by $\forall y(y \in x \leftrightarrow \varphi(y))$, then we write $x = \{y|\varphi(y)\}$ and say that the class $\{y|\varphi(y)\}$ exists. (By Russell’s paradox, the class $\{y|y \notin y\}$ does not exist.)

In order to define partial universes, we first define partial hierarchies:

- A set $h$ is a **partial hierarchy** if and only if for each member $u$ of $h$, $u = \{y|\exists v(v \in h \land v \in u \land y \subseteq v)\}$.

Note that if $y \in u$ and $x \subseteq y$, then also $x \in u$.

- A set $u$ is a **partial universe** if and only if for some partial hierarchy $h$, $u = \{y|\exists v(v \in h \land v \in u \land y \subseteq v)\}$.

Note that each member of the partial hierarchy $h$ is a partial universe too.

Now one easily gets the following characterization of partial universes, which is a simplification of Scott’s Accumulation Axiom:

For each partial universe $u$ and set $y$:

$$y \in u \leftrightarrow \exists v(v \text{ is a partial universe } \land v \in u \land y \subseteq v)$$

This truth by definition can be translated in the language of “stage” theory directly:

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A set $y$ exists at stage $u$ if and only if at some earlier stage $v$ each member of $y$ exists.

We can now state an axiom (for each formula $\varphi$) that expresses that the universal class $V$ is, in some sense, a "big" partial universe. This axiom scheme combines Scott's Comprehension and Restriction Axiom.

The class $\{y|\varphi(y)\}$ exists $\iff \exists v(v$ is a partial universe $\land \forall y(\varphi(y) \to y \in v)$

In particular, for each set $x$, since $x = \{y|y \in x\}$, there is some partial universe $v$ such that $x \subseteq v$. A translation of our axiom is:

- A collection is a set if and only if at some stage $v$ each of its members exists.

We can now prove that the class $\{u|u$ is a partial universe\} (i.e., the cumulative hierarchy) does not exist. For suppose it were a set $h$. Then for some partial universe $v$, $h \subseteq v$. But also $v \in h$, so $v \in v$: "$v$ is earlier than itself". This implies that each subset $r$ of $v$ is a member of $v$. Now take $r = \{y|y \in v \land y \notin v\}$. Then $r \in r \iff r \notin r$.

In a similar way we can prove that each set is well-founded:

If there were partial universes $v$ with non-well-founded elements, then the intersection $\{y|\forall v(v$ is a partial universe with a non-well-founded set $\rightarrow y \in v)\}$ of all such partial universes would be a set $x$ such that each subset of $x$ is a member of $x$.

We can now prove theorems by transfinite induction, like:

- Each partial universe $v$ is transitive each member of $v$ is a subset of $v$.
- For all partial universes $u$ and $v$, either $u \in v$ or $v \subseteq u$.

By extensionality, this last theorem (whose proof requires classical logic) implies that either $u \in v$, $v \in u$ or $u = v$. This shows that the cumulative hierarchy is well-ordered.

References

