CWI Quarterly Volume 9 (1 & 2) 1996, pp. 97 – 101

Foundation of the Cumulative Hierarchy of Sets

A.J.C. Hurkens
University of Nijmegen, Faculty of Philosophy
P.O.Box 9103, 6500 HD Nijmegen, The Netherlands
TNO, Section Knowledge Based Systems
P.O.Box 49, 2600 AA Delft, The Netherlands
thurkens@vms.uci.kun.nl

The axioms of set theory are sometimes motivated as follows:
(1) A collection is a *set* iff at some stage all of its members exist.
(2) A set *exists* at some stage iff at some earlier stage all of its members exist.
In order to justify the Axiom of Foundation, one often adds:
(3) The stages are well-ordered by "earlier than".
This is a circular "reduction" of the foundation of sets to the well-ordering of stages. We present a simple definition of "stage" such that only (1) needs to be assumed: (2) follows by definition and (3) can be derived from (1) and (2).

1. Foundations: sets, mathematics, and the cumulative hierarchy

The foundation SMC was founded 50 years ago. Mathematicians know that 50 is just a number. But what is a number? This is a question about the foundations of mathematics.

Set theorists will answer that a *number* is the *set* of all smaller numbers[1]. So 50 is the set $\{0, 1, \ldots, 49\}$. Its fifty members are also sets, and so on: It is a *pure* set. So we can choose sets $x_1 \in 50$, $x_2 \in x_1$, and so on until we arrive at the empty set. In fact, this *will* happen (after at most fifty steps): The set is *well-founded*. Well-founded sets can be "formed" from their members step by step, in such a way that each set is formed *after* its members. In case of pure sets, we start this process with nothing at all, forming the empty set. Fifty stages later we can form the set 50, after an infinite number of stages the set

[1] One might think it would be more natural to identify e.g. 50 with the set $\{1, 2, \ldots, 50\}$. This is possible if one accepts the Anti Foundation Axiom (Aczel, 1988). But then all positive natural numbers would be equal to *the* set x whose only member is x.

$\omega = \{0, 1, \ldots\}$ of all natural numbers, and so on. Boolos mentioned 25 years ago that "authors of set-theory texts either omit [this iterative conception of set] or relegate it to back pages". One usually indexes the "stages" by ordinal numbers and defines the *partial universe* V_α of all sets that are formed before (or "exist at") stage α by transfinite induction as $\bigcup_{\beta<\alpha} \mathcal{P}\mathrm{V}_\beta$. Forget this complicated definition of the *cumulative hierarchy*! We will present an elementary one that can play a role in the foundations of set theory.

2. The founders: Scott, Mirimanoff, and Cantor

By reducing mathematical objects to sets, set theory provides mathematics with foundations. But what is a set? The founder of set theory gave an informal answer a century ago:[2] A *set* is a collection of objects into a whole. Such a collection may be small (like 50) or big (like ω). But the collection $\Omega = \{0, 1, \ldots, \omega, \ldots\}$ of all ordinal numbers, which was studied by Burali-Forti (1897), turned out to be "too big": It is not a *set*. Russell (1903) found a much simpler example: the collection of alle sets that are not members of themselves. Mirimanoff (1917) found an example in between: the collection of all well-founded sets. He formulated the fundamental problem of set theory:[3] *Which* collections of objects are sets?

Departing from some postulats, Mirimanoff solved this problem for well-founded sets by using ordinals to measure the well-foundedness: The *rank* of a set is the least ordinal above the ranks of its members. He showed that a collection of well-founded sets is a *set* if and only if there is an upperbound for the ranks of its members. In particular, the collection V_α of all sets of rank smaller than α is a set.

About 75 years ago, Zermelo and Fraenkel gave set theory its current *axiomatic* foundations. These axioms do not mention the cumulative hierarchy. From a logical point of view, they just describe a directed graph $(\mathrm{V}, \in)$ whose vertices are called "sets" and whose edges stand for "is a member of". For example, the Axiom of Foundation (added in 1925 by Von Neumann) implies that this graph has no cycles.

But why these particular axioms? Why do we assume that all sets are well-founded? Avoidance of contradictions cannot be the only motivation. About 25 years ago, Scott was one of the authors who tried to justify the axioms by reformulating them in terms of the "stages" or "levels" of the cumulative hierarchy.

3. The axiom of foundation: stages, motivation and circularity

The Axiom of Foundation is usually motivated as folows: The members of a set should be given *before* the set itself (in a *logical*, not *temporal* sense).

[2] Cantor (1895–1897): "Unter einer "Menge" verstehen wir jede Zusammenfassung M von bestimmten wohlunterschiedenen Objekten m unsrer Anschauung oder unseres Denkens (welche die "Elemente" von M genannt werden) zu einem Ganzen."

[3] "Quelles sont les conditions nécessaires et suffisantes pour qu'un ensemble d'individues existe?"

By assuming that each set x is formed at some stage s and that each member of x is formed at an earlier stage than s, Boolos (1971) reduced the well-foundedness of the relation "is a member of" to the well-foundedness of "is earlier than". In fact he even *assumed* that this relation between stages is a well-ordering. But is this so evident? This seems to be just as circular as proving that the partial universes form a *well-ordered* hierarchy after having *defined* them by means of ordinals.

According to Levy (1973), the idea of "forming" sets stage by stage cannot be viewed as fundamental for the notion of set:[4] "The weakest part of this point of view is that the reasoning leading to the concept of a well-founded set uses the *well-ordering* of the layers."

However, Scott's axiomatization *avoids* the concept of well-ordering: He takes the notion of "is a partial universe" as primitive, assumes some elementary properties of partial universes and (other) sets, and then *proves* that the cumulative hierarchy is well-ordered by "is a member of". In fact, some of these properties can be *proved* after *defining* the notion of "partial universe".

4 A simple, minimal characterization

In the first-order language of set theory, the membership relation $\in$ is the only primitive relation. The inclusion relation $\subseteq$ can be defined as usual, and if the Axiom of Extensionalty is assumed, then $x = y$ can be *defined* as $x \subseteq y \wedge y \subseteq x$.

If the members of some set x can be characterized by $\forall y(y \in x \leftrightarrow \varphi(y))$, then we write $x = \{y|\varphi(y)\}$ and say that *the class* $\{y|\varphi(y)\}$ *exists.* (By Russell's paradox, the class $\{y|y \notin y\}$ does not exist.)

In order to define partial universes, we first define partial hierarchies:

- A set h is a *partial hierarchy* if and only if for each member u of h, $u = \{y|\exists v(v \in h \wedge v \in u \wedge y \subseteq v)\}$.

Note that if $y \in u$ and $x \subseteq y$, then also $x \in u$.

- A set u is a *partial universe* if and only if for some partial hierarchy h, $u = \{y|\exists v(v \in h \wedge v \in u \wedge y \subseteq v)\}$.

Note that each member of the partial hierarchy h is a partial universe too.

Now one easily gets the following characterization of partial universes, which is a simplification of Scott's Accumulation Axiom:

For each partial universe u and set y:

$$y \in u \leftrightarrow \exists v(v \text{ is a partial universe } \wedge v \in u \wedge y \subseteq v)$$

This truth *by definition* can be translated in the language of "stage" theory directly:

[4] *Studies in Logic and the Foundations of Mathematics*, Volume 67: *Foundations of Set Theory*, Chapter 2: *Axiomatic Foundations of Set Theory*, Section 5: *The Axiom of Foundation*, p. 89

– A set y *exists* at stage u if and only if at some earlier stage v each member of y exists.

We can now state an *axiom* (for each formula φ) that expresses that the universal class V is, in some sense, a "big" partial universe. This axiom scheme combines Scott's Comprehension and Restriction Axiom.

$$\text{The class } \{y|\varphi(y)\} \text{ exists } \leftrightarrow \exists v(v \text{ is a partial universe } \wedge \forall y(\varphi(y) \rightarrow y \in v)$$

In particular, for each set x, since $x = \{y|y \in x\}$, there is some partial universe v such that $x \subseteq v$. A translation of our axiom is:

– A collection is a *set* if and only if at some stage v each of its members exists.

We can now prove that the class $\{u|u$ is a partial universe$\}$ (i.e., the cumulative hierarchy) does not exist. For suppose it were a set h. Then for some partial universe v, $h \subseteq v$. But also $v \in h$, so $v \in v$: "v is earlier than itself". This implies that each subset r of v is a member of v. Now take $r = \{y|y \in v \wedge y \notin y\}$. Then $r \in r \leftrightarrow r \notin r$.

In a similar way we can *prove* that each set is well-founded:
If there were partial universes v with non-well-founded elements, then the intersection $\{y|\forall v(v$ is a partial universe with a non-well-founded set $\rightarrow y \in v)\}$ of all such partial universes would be a set x such that each subset of x is a member of x.

We can now prove theorems by transfinite induction, like:

– Each partial universe v is *transitive*: each member of v is a subset of v.
– For all partial universes u and v, either $u \in v$ or $v \subseteq u$.

By extensionality, this last theorem (whose proof requires classical logic) implies that either $u \in v$, $v \in u$ or $u = v$. This shows that the cumulative hierarchy is well-ordered.

References

1. P. Aczel (1988). *Non-Well-Founded Sets.* CSLI Lecture Notes **14**. CSLI, Stanford, CA.
2. G. Boolos (1971). The iterative conception of set. *The Journal of Philosophy* **68** 215–231.
3. C. Burali-Forti (1897). Una questione sui numeri transfiniti. *Rendiconti del Circolo matematico di Palermo* **11** 154–164.
4. G. Cantor (1895–1897). Beiträge zur Begründung der transfiniten Mengenlehre. *Mathematische Annalen* **46** 481–512, **49** 207–246.
5. A.A. Fraenkel, Y. Bar-Hillel and A. Levy (1973). *Foundations of Set Theory.* Second revised edition. Studies in Logic and the Foundations of Mathematics **67**. North-Holland Publishing Company, Amsterdam.

6. D. MIRIMANOFF (1917). Les antinomies de Russell et de Burali-Forti et le problème fondamental de la théorie des ensembles. *L'Enseignement Mathématique* **19** 37–52.
7. B. RUSSELL (1903). *The Principles of Mathematics.* Cambridge University Press, Cambridge, G.B.
8. D. SCOTT (1974). Axiomatizing set theory. In: T.J. JECH (editor). *Axiomatic Set Theory.* Proceedings of Symposia in Pure Mathematics **13**, Part 2. American Mathematical Society, Providence, R.I. 207–214.