CWI Quarterly Volume 9 (1 & 2) 1996, pp. 51 – 59

Some Indications that the Exceptional Groups Form a Series

Arjeh M. Cohen
Department of Mathematics Technical University Eindhoven
P.O. Box 513, 5600 MB Eindhoven The Netherlands

This is a written version of the talk delivered on Feb 6, 1996 at the SMC conference, Amsterdam.
Last Summer, P. Vogel noted that the dimensions of the exceptional Lie algebras are given by

$$-2\frac{(\lambda+5)(\lambda-6)}{\lambda(\lambda-1)}$$

for an appropriate value of λ, as well as other uniform behavior of the exceptional Lie algebras with regard to the parameter λ. Inspired by this, P. Deligne conjectured that there might be a tensor category explaining the behavior. Ronald de Man and I have provided additional computational evidence for his conjecture. In my talk I will explain the conjecture by elaborating on a tensor category for the general linear group, built up by means of braid-like combinatorial objects.

1. Introduction

The topic of this talk originates from joint work with Ronald de Man instigated by Deligne, with whom we have been brought into contact by Van der Kallen. The problem which I will address here is that of decomposing tensor powers of "natural modules" for a series of simple complex Lie groups into irreducible modules. The series I have in mind are indexed by a parameter n as follows:

n in GL_n
n in O_n
n in Sp_{2n}
? in G_2, F_4, E_6, E_7, E_8

Here the final line corresponds with the "series" of exceptional groups. There are some indications that, much like for the other series, there is a general pattern for the tensor power decompositions of the adjoint module for a series of 9 groups, including the five exceptional ones. The purpose of my talk is to put these indications into context.

2. Module decomposition

The general pattern starts from a duality principle, which can roughly be explained as follows. Let W be a G-module. We shall take our scalars to be the complex numbers $\mathbf{C}$. Since G is simple (more generally, reductive), W is a direct sum of G-irreducibles. How to find these irreducible constituents of W? Since it is not very useful to separate one irreducible submodule from another that is isomorphic to it, we usually restrict to the determination of the isotypical components, that is, the sum of all irreducible submodules isomorphic to a given irreducible, and the multiplicities of each isotypical component, that is, the number of copies of the irreducibles occurring in a direct sum decomposition of the component.

The duality principle concerns the centralizer algebra $A = \mathrm{End}_G(W)$, the algebra of all linear maps from W to itself which commute with every element of G. The important fact that makes A so useful is that the isotypical components of G in W are also the isotypical components of A. Moreover, if the dimension of the A isotypical component is ab where a is the dimension of the A irreducible involved and b its multiplicity, then b is the dimension of the corresponding G-irreducible and a is its multiplicity.

We shall apply this principle to the d-th tensor power $W = V^{\otimes d}$. For the series of groups G mentioned above it is easier by far to determine the isotypical components of A than to determine them as isotypical components of G. One reason is that there is a crisp description of A, another is that the dependence of A on n is minor, so that a great deal of the decomposition can be carried out simultaneously for all n by a proper study of A. What is needed to find the irreducibles for A and their multiplicities, is to find the central irreducible idempotents of A. If e is such an idempotent, the dimension of the isotypical component in W is $\mathrm{Tr}(e, W)$, its trace on the W, and the dimension of the irreducible A-module is the square root of $\mathrm{Tr}(L_e, A)$, where L_e stands for left multiplication by e.

3. The GL_n example

In order to make these words come to life, I will devote some attention to the case where $G = GL_n$ and V is the natural n-dimensional GL_n-module. So, V is an n-dimensional complex vector space, and GL_n can be viewed as the group of all invertible linear transformations of V.

It is a classic fact that the centralizer algebra $A = \mathrm{End}_G(V)$ is the group algebra of the symmetric group on d letters, Sym_d, which acts on $V^{\otimes d}$ by permuting the d components. Thus, the central irreducible idempotents of A are e_λ, certain elements indexed by partitions λ of d, and we can write

$$V^{\otimes d} = \bigoplus_{\lambda \dashv d} V_\lambda \otimes V^\lambda,$$

where V_λ is the GL_n-module corresponding to λ (uniquely determined, easy to express in terms of heighest weights), and V^λ is the unique irreducible Sym_d-module corresponding to λ.

Thus, to find V_λ we need to explicitize A, its idempotents e_λ, and compute traces.

4. Tensor category for GL_n

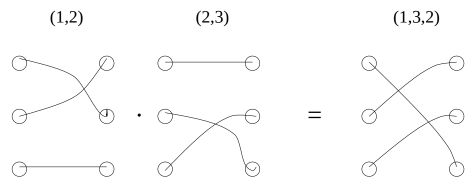
Now the labor that has to be done to describe A as well as the necessary traces of idempotents for finding the dimensions of the G-irreducibles and their multiplicities, can be succinctly formulated in a nice combinatorial way. The setting for this description is that of a tensor category. I will refrain from even attempting a formal definition of this notion, but just continue with a table connecting the notions in the language of tensor categories with those of the combinatorial/pictorial setting and with the representation theory just discussed.

tensor category	realisation	representation theory
object X	d-set X	G-module $V^{\otimes d}$
morphism $X \to Y$	linear combination of paths joining nodes $X \to Y$	linear combination of permutations of d letters, in $\mathrm{End}(V^{\otimes d})$
tensor $X \otimes Y$	disjoint union $X \sqcup Y$	tensor $V^{\otimes(d+e)}$
morphism composition	straightening out paths	morphism composition
End(X)	algebra generated by morphisms $X \to X$	A

Thus, the semantic interpretation of an object X in the tensor category is a G-module, but the object itself is nothing but a set of d points. These nodes are represented as in graph theory: $\circ \quad \circ \quad \circ$ stands for a 3-set. Actually, we shall recognize only one object consisting of d nodes, so we have depicted the only object that is a 3-set. In the tensor category, the tensor of two objects of this kind is just the disjoint sum of the corresponding sets.

A morphism from one object X to another object Y has the semantics of a G-equivariant linear map from the G-module X to the G-module Y, but is nothing else but a $\mathbf{C}$-linear combination of permutations, where a permutation is a bunch of paths connecting nodes of X with nodes of Y, turning all nodes into endpoints. In particular, morphisms only occur between objects having the same size (and hence being equal). This observation is in accordance with the well-known fact that, for n big with respect to d, a G-irreducible occurs in $V^{\otimes d}$ for only one d.

Composition of morphisms is easily explained in terms of pictures. For example, take the permutations $(1,2)$ and $(2,3)$. Their composition is seen to be $(1,3,2)$ by juxtaposition of the two individual pictures for $(1,2)$ and $(2,3)$ and then straightening out the resulting path from start to finish, forgetting the intermediate stage:

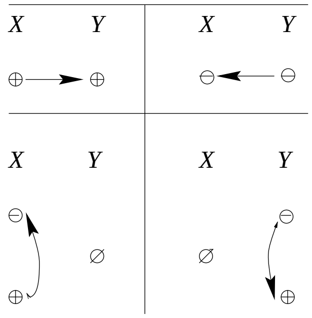


So far, we have only given a very simple well-known presentation of the basis of A consisting of the elements of Sym_d, and a pictorial description of the multiplication of basis elements, which, remarkably but not surprisingly enough, is monomial in the sense that the product of any two basis elements is a scalar multiple of another basis element. Here the scalar multiple is always 1 (as the basis is actually a group), but in future examples this will no longer be the case.

5. Adding duals

Rather than working out the idempotents of A in the classical case of GL_n on $V^{\otimes d}$, I will handle a slightly more general and more interesting case, extending the previous setting. We shall let the dual of V, denoted by V^*, enter the scene. In order to add it to the tensor category as an object, we charge the objects handled so far with a $+$, replacing $\circ$ in the pictures by $\oplus$, and introduce the object semantically known as V^*, to the category as object $\ominus$.

The tensor is now as easy as it was before: just the disjoint union of sets. A morphism between two objects is now slightly more subtle. Again it is a linear combination of "basic morphisms", which we can picture as oriented paths. The orientation for a path means that it has a start and an end point. Now a basic morphism from X to Y connects all the points with oriented paths in such a way that we always have one of four cases:



The semantics for the two morphisms at the bottom are evaluation and coevaluation. Indeed, fixing a basis v_i of V and its dual basis v_i^* of V (assuming all modules to be finite dimensional), one can think of the former as the map $\sum_i \alpha_i v_i^* \otimes v_i \mapsto \sum_i \alpha_i v_i^*(v_i)$ from $V^* \otimes V$ to $\mathbf{C} = V^0$ and of the latter as the map $a \mapsto \sum_i a v_i^* \otimes v_i$ from $\mathbf{C}$ to $V \otimes V^*$ for $a \in \mathbf{C} = V^0$. (Apparently, we also do not distinguish between $X \otimes Y$ and $Y \otimes X$.)

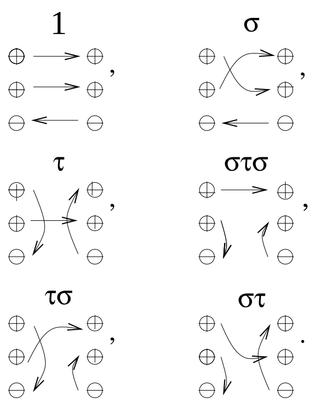
However, the description of $\mathrm{End}(X)$ for an object X is not yet complete. For, when composing maps, closed paths will occur. We convene that a closed path is equal to the scalar factor n.

Example: $V \otimes V \otimes V^*$ for GL_n

I shall take along an explicit example: $W = V \otimes V \otimes V^*$. The corresponding object is

$$X = \oplus \quad \oplus \quad \ominus$$

A basis for $A = \mathrm{End}(X) = \mathrm{End}_G(W)$ consists of the following 6 basic morphisms:



It is now straightforward to draw the full multiplication table of A from these pictures. It is in accordance with the names already given and is fully determined by the following relations:

$$\begin{aligned} \sigma^2 &= 1, \\ \tau^2 &= n\tau, \\ \tau\sigma\tau &= \tau, \\ (\sigma\tau)^2 &= \sigma\tau, \\ (\tau\sigma)^2 &= \tau\sigma. \end{aligned}$$

Note that $A = \mathrm{End}(X)$ has been defined in terms of the tensor category, that is, with fewer information than the full knowledge of G and its module W.

6. Idempotents

Since A as an algebra is completely known, we can determine its irreducible central idempotents.

First of all, from solving a set of linear equations, one finds that the center of A is the linear span of

$$1, \quad a = \sigma - \sigma\tau\sigma - \tau, \quad b = \sigma\tau + \tau\sigma - n\sigma\tau\sigma - n\tau.$$

Then, from a set of quadratic equations, the irreducible central idempotents are found to be:

$$e_1 = (1 + a - \frac{b}{n+1})/2, \quad e_2 = (1 - a + \frac{b}{n-1})/2, \quad e_3 = \frac{b}{1-n^2}.$$

The dimensions of the resulting A-components are:

$$\dim(Ae_1) = 1, \quad \dim(Ae_2) = 1, \quad \dim(Ae_3) = 4,$$

as follows from the identity $\dim(Ae_j) = \mathrm{Tr}(L_{e_j}, A)$, and so the dimensions of the corresponding A-irreducibles are 1, 1, 2, respectively.

7. Traces and Dimension

We still need to describe how to compute traces on W of elements of A in terms of the tensor category, and similarly for $\dim(W)$. To begin with the latter, we define

$$\dim(X) = \mathrm{Tr}(1_X),$$

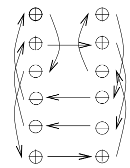
thus reducing everything to the computation of traces of morphisms $\phi \in \mathrm{End}(X)$. Here is how that is done:

$$\mathrm{Tr}(\phi, X) = \quad \emptyset \xrightarrow{\text{coeval}} X \otimes X^* \xrightarrow{\phi \otimes 1_{X^*}} X \otimes X^* \xrightarrow{\text{eval}} \emptyset.$$

Thus, for example, $\dim(V) = \mathrm{Tr}(1_V) =$ a single closed path $= n$.

But let us continue with $W = V \otimes V \otimes V^*$ and compute the traces of the three idempotents of A. Using the above combinatorial rules, this a matter of counting closed paths. For instance, for τ, we find

$$\mathrm{Tr}(\tau, W) = \mathrm{eval}\ \cdot\ \tau \otimes 1_{W^*}\ \cdot\ \mathrm{coeval}\ =$$



$$= \ n \cdot n = n^2.$$

By linearity, we then find the traces of a and b, and subsequently those of the idempotents e_1, e_2 and e_3 on W. We put the results in a table.

x	$\mathrm{Tr}(x, V \otimes V \otimes V^*)$	$\mathrm{Tr}(x, A)$
1	n^3	6
σ	n^2	0
τ	n^2	$2n$
$\sigma\tau\sigma$	n^2	$2n$
$\sigma\tau$	n	2
$\tau\sigma$	n	2
a	$-n^2$	$-4n$
b	$-2n^3 + 2n$	$4 - 4n^2$
e_1	$n(n^2 + n - 2)/2$	1
e_2	$n(n^2 - n - 2)/2$	1
e_3	$2n$	4

The result for the decomposition into GL_n-irreducibles for $W = V \otimes V \otimes V^*$ is

idempotent	dim of GL_n-irreducible	its multiplicity
e_1	$\dfrac{n(n^2+n-2)}{2}$	1
e_2	$\dfrac{n(n^2-n-2)}{2}$	1
e_3	n	2

A warning regarding small values of n with respect to d is in order. For example, if $n = 2$, then the e_2 component has dimension 0. To complete the procedure for a specific value of n, more work is needed than is described so far. This work can be fully described in the tensor category. Very roughly speaking, it comes down to modding out traceless morphisms (that, is morphisms $\phi : X \to Y$ with the property that the trace of any composition $\phi\psi$ with a morphism $\psi : Y \to X$ has value 0) and forming the Karoubian closure of the resulting tensor category (that amounts to introducing objects corresponding to images and kernels of idempotent morphisms).

In conclusion, we have a tensor category with a parameter n, which, for each specific positive value of the parameter, gives the "generic" description of the decomposition into irreducibles of each of the tensor powers formed from the natural modules and their duals. After the specialisation to the specific positive integer, however, adaption of the tensor category is possible (modding out traceless morphisms and forming the Karoubian closure) so as to obtain the complete picture.

8. The other two classical series

Similar results exist for $O(n)$ and $Sp(2n)$. Here an identification of $\oplus$ and $\ominus$ (or V and V^*) takes place, due to the existence of a G-invariant bilinear form on the natural module V, which identifies V and V^* as G-modules. Thus, eval gains an interpretation as the evaluation of the bilinear form in elements of $V \otimes V$. As a consequence, the nodes no longer have a sign, and morphisms are based on undirected paths. Closed paths may still occur (as there are paths joining two nodes of the same object) and are again identified with the scalar n (or $2n$ in case $G = Sp_{2n}$).

9. The exceptional series

Deligne's conjecture concerns the tensor powers of the adjoint module L of the following series of groups:

$$1,\ A_1,\ A_2.2,\ G_2,\ D_4.\mathrm{Sym}_3,\ F_4,\ E_6.2,\ E_7,\ E_8.$$

The full statement will appear in [2]. Here is a very rough description. Let

$$\lambda := -5, -3, -2, -\frac{3}{2}, -1, -\frac{2}{3}, -\frac{1}{2}, -\frac{1}{3}, -\frac{1}{5} \tag{$*$}$$

in the nine cases, ordered as above. Then, by an observation of Vogel,

$$\begin{aligned} \dim L &= -2\frac{(\lambda+5)(\lambda-6)}{\lambda(\lambda-1)} \qquad \text{and} \\ L^{\otimes 2} &= L^{2+} \oplus L^{2-} = (1 + Y_2 + Y_2^{\sharp}) \oplus (L + X_2), \quad \text{where} \\ \dim X_2 &= 5\frac{(\lambda+5)(\lambda-6)(\lambda+3)(\lambda-4)}{\lambda^2(\lambda-1)^2} \qquad \text{and} \\ \dim Y_2 &= -90\frac{(\lambda-4)(\lambda+5)}{\lambda^2(\lambda-1)(2\lambda-1)} \end{aligned}$$

Thus, we find an overall pattern that looks much the same as for GL_n. Like the tensor category for that series, one would like to find a fitting combinatorial description of the tensor power decompositions in which a map $X \mapsto X^{\sharp}$, the evaluation $L \otimes L \to \mathbf{C}$ using the Killing form, and a map $L \otimes L \to L$ representing the Lie bracket are all represented. The existence of such a combinatorial description, phrased in terms of tensor categories, is the content of Deligne's conjecture.

Since for the λ in (*) all dimensions of the objects of the tensor category will be nonnegative integers, one might speculate that the Karoubian closure of the tensor category modulo traceless morphisms for each of the corresponding specialisations, is Tannakian, and would thus lead to a "serial" definition of the "nine" exceptional groups. As for other values of λ, they usually will not lead to integral dimensions, and so there is no indication as to what their significance would be.

Let me conclude with three remarks concerning this phenomenon.

1. A very mysterious "semilinear" map $X \mapsto X^\sharp$ seems to emerge. We have used it in describing the other nontrivial irreducible in L^{2+} (distinct from Y_2). It fixes certain objects like L and X_2. It is semilinear in the sense that it fixes the constant scalars but maps the parameter λ onto $1-\lambda$, that is, $\lambda^\sharp = 1-\lambda$. With this convention, we have, for irreducibles X,

$$\dim(X^\sharp) = (\dim X)^\sharp.$$

2. The tensor powers continue to decompose according to such regular patterns for $d = 3, 4$, as has been accounted for in [1], and by De Man for $d = 5$ (even parts of $d = 6, 7$).

3. The computations regarding these decompositions were made by use of LiE, see [3]. Apart from the expected ingredients, such interpolations of the dimensions of the computed decompositions for the inidividual groups from the series, the values of the Casimir operators were used to find the appropriate descriptions of the dimensions as elements of $\mathbf{Q}(\lambda)$.

References

1. A. M. Cohen, R. de Man, *Computational evidence for Deligne's conjecture regarding exceptional Lie groups*, Comptes rendus (to appear).
2. P. Deligne, *La série exceptionelle de groupes de Lie*, Comptes rendus (to appear).
3. M. A. A. van Leeuwen, A. M. Cohen, B. Lisser (1992). *LiE, A Package for Lie Group Computations*, CAN, Amsterdam