Special Functions and Group Theory

Oberwolfach Meeting from 13th to 19th March 1983

by Bob Hoogenboom

The purpose of this meeting was to discuss recent developments in the theory of special functions, with emphasis on the connections between special functions and group theory.

First of all, what is a special function? Many (ridiculous) definitions have been given, most of which exclude some very important examples of special functions. The one and only good definition I know is by Richard Askey, in [1]: A function is a special function if it occurs often enough that it gets a name. Important examples of special functions are the exponential, the gamma function, the Riemann zeta function, theta functions, Bessel functions, Jacobi polynomials, etc. The first three examples were not discussed at the meeting; the last three, and many others, were.

Secondly, how do special functions occur? The answer is: in many ways. Let me give three examples. Theta functions occur in physics, for instance as periodic solutions of Korteweg - de Vries type equations, and in connection with completely integrable systems, cf. [2]. Other special functions occur as solutions of certain second order differential equations, for instance the hypergeometric function ${}_2F_1(\alpha,\beta;\gamma;x)$, which is defined by

$$ {}_2F_1(\alpha,\beta;\gamma;x) := \sum_{n=0}^{\infty} \frac{(\alpha)_n(\beta)_n}{(\gamma)_n n!} x^n, \tag{1} $$

where the shifted factorial $(a)_n$ is defined by $(a)_n := a(a+1)...(a+n-1)$ for $n = 1,2,...$. The hypergeometric function ${}_2F_1$ is a solution of the equation

$$ x(1-x)\frac{d^2}{dx^2}f + [\gamma-(\alpha+\beta+1)x]\frac{d}{dx}f = \alpha\beta f, \tag{2} $$

cf. [3, §2.1.1]. Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ are a special case of (1), namely

$$ P_n^{(\alpha,\beta)}(x) := \binom{n+\alpha}{n} {}_2F_1(-n, n+\alpha+\beta+1; \alpha+1; \tfrac{1}{2}(1-x)). \tag{3} $$

As a third example, I would like to mention representation theory. Some special functions appear in the study of linear representations of certain groups, for instance as matrix entries in irreducible representations of Lie groups. An example is the *nth* Bessel function

$$ J_n(x) := \frac{1}{2\pi}\int_0^{2\pi} e^{ix\sin\theta - in\theta}\, d\theta, \tag{4} $$

which arises as a matrix entry of the irreducible representations of the group of motions of the plane $\mathbf{R}^2$, cf. [5, Ch. IV]. I shall discuss some of the talks grouped around four different themes.

1. Semisimple Lie Groups

Under this category, functions on groups were studied, mostly spherical functions on semisimple Lie groups and generalizations. A spherical function on a semisimple Lie group G with respect to a maximal compact subgroup K is a K-invariant function on G/K which is a joint eigenfunction of all G-invariant differential operators on G/K.

New interpretations for addition formulas for functions of the second kind (that is, a second solution of a differential equation of type (2) which is regular at ∞) were given in terms of the pair $(G, K) = (SO(n,1), SO(n-1,1))$. Observe that the subgroup K is not compact. Here the space G/K can be interpreted as the hyperboloid of one sheet in $\mathbf{R}^n$ (Talks by Durand, Mizony). Matrix elements of the representations of $S0(n,1)$ were studied by a global approach, without using Lie algebra theory (Koornwinder : $n=3$, Takahashi: partial results for $n=4$). Other talks in this category were by Reimann, Terras and Hoogenboom.

2. Special functions and q-analogues

Under this category special functions were studied by analytic methods, without the use of group theory. The emphasis was put on the generalization of classical results to the so-called q-analogues of special functions. As someone told me lately, q-analogues are just the ordinary special functions, only with all the 1's replaced by a q. There remains only one problem: where are the 1's ? To give an example, the binominal theorem

$$(1-x)^{-\alpha} = \sum_{n=0}^{\infty} \frac{(\alpha)_n}{n!} x^n \tag{5}$$

has the following q-analogue

$$\frac{(\alpha x;q)_\infty}{(x;q)_\infty} = \sum_{n=0}^{\infty} \frac{(\alpha;q)_n}{(q;q)_n} x^n \tag{6}$$

$(|x|<1, |q|<1)$, where

$$(\alpha;q)_\infty = \prod_{n=0}^{\infty}(1-\alpha q^n),\ (\alpha;q)_n = (1-\alpha)(1-\alpha q) \cdots (1-\alpha q^{n-1}).$$

By writing $\alpha = q^a$, where a is a nonnegative integer, (6) formally leads to (5) if $q \uparrow 1$.

This phenomenon is inspired by the study of (parameters of) permutation representations of the Chevalley groups (i.e, the finite analogues of Lie groups), the theory of partitions, etc. (Talks by Gasper, Askey, Rahman).

Other talks in this section dealt with orthogonal polynomials (Dunkl), and solutions of differential equations (Sprinkhuizen-Kuyper, Trime'che)

3. Selberg integrals and Dyson conjectures

The classical Beta-integral reads as follows

$$\int_0^1 u^{\alpha-1}(1-u)^{\beta-1}du = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}, \tag{7}$$

where Γ is the classical gamma function. As I learned at this meeting, this equation has been generalized in 1944 to the following equation

$$\int_0^1 \ldots \int_0^1 \prod_{i=1}^{p} u_i^{\alpha-1}(1-u_i)^{\beta-1}\prod_{i<j}(u_i-u_j)^{2\gamma}du_1 \cdots du_p = \tag{8}$$

$$= \prod_{n=1}^{p} \frac{\Gamma(1+n\gamma)\Gamma(\alpha+(n-1)\gamma)\Gamma(\beta+(n-1)\gamma)}{\Gamma(1+\gamma)\Gamma(\alpha+\beta+(p+n-2)\gamma)},$$

see Selberg [4]. Unfortunately, Selberg published his result in an obscure Norwegian journal, so that it remained unnoticed for some time. Formula (8) was used to prove some conjectures by Dyson about the constant term in a certain Laurent series expansion. The simplest instance of Dyson conjectures is as follows. Here *C.T.* denotes the constant term in the Laurent series expansion.

$$C.T. \prod_{i \neq j}^{n}(1-x_i x_j^{-1})^k = \frac{(nk)!}{(k!)^n}. \tag{9}$$

The talks in this category were devoted to generalizations of these Dyson conjectures, some of which are still open. (Talks by Macdonald, Stanton, Metha, Kadell).

4. Signal processing and the Heisenberg group.

The Radon transform plays an important role in the theory of computerized tomography. Also, in radar signal processing the Heisenberg group arises. Talks in this section were on radar tomography (Grünbaum, Schempp, Louis), and on the Heisenberg group (Greiner, Auslander).

The other talks (among a total of 37) dealt with Lie groups and physics (Louck, Milne, Kramer, Onofri), Combinatorics and Special functions (Seidel, Bannai, Foata), Gelfand pairs and hypergroups (Lasser, Letac), Seperation of variables (Miller), Theta functions (Hazewinkel), and various subjects (Ronveaux, Delvos, Calogero, Hermann).

The meeting was organized by R.A. Askey (Madison), T.H. Koornwinder (Amsterdam) and W. Schempp (Siegen). The proceedings will be published by Reidel, Dordrecht, the provisional title being: *'Special functions: group* *theoretic aspects and applications'* in the series 'Mathematics and its applications.'

References

[1] Askey, R.A., *Foreword*, in: *'Symmetry and separation of variables'* (W. Miller, jr.), Addison - Wesley, Reading, Mass., 1977.

[2] Dubrovin, D.A., *Theta functions and non-linear equations,* Russian Math. Surveys, **36**:2 (1982) 11-92.

[3] Erdélyi, A., W. Magnus, F. Oberhettinger & F. Tricomi, *Higher Trancedental Functions,* Vol. 1, McGraw-Hill, New York 1953.

[4] Selberg, A., *Bemerkninger om et multipelt integral,* Norsk Matematisk Tidskrift, **26** (1944) 71-78.

[5] Vilenkin, N. Ja., *Special functions and the theory of group representations,* Am. Math. Soc. Transl. of Math. Monographs, Vol. 22, Providence, R.I., 1968.

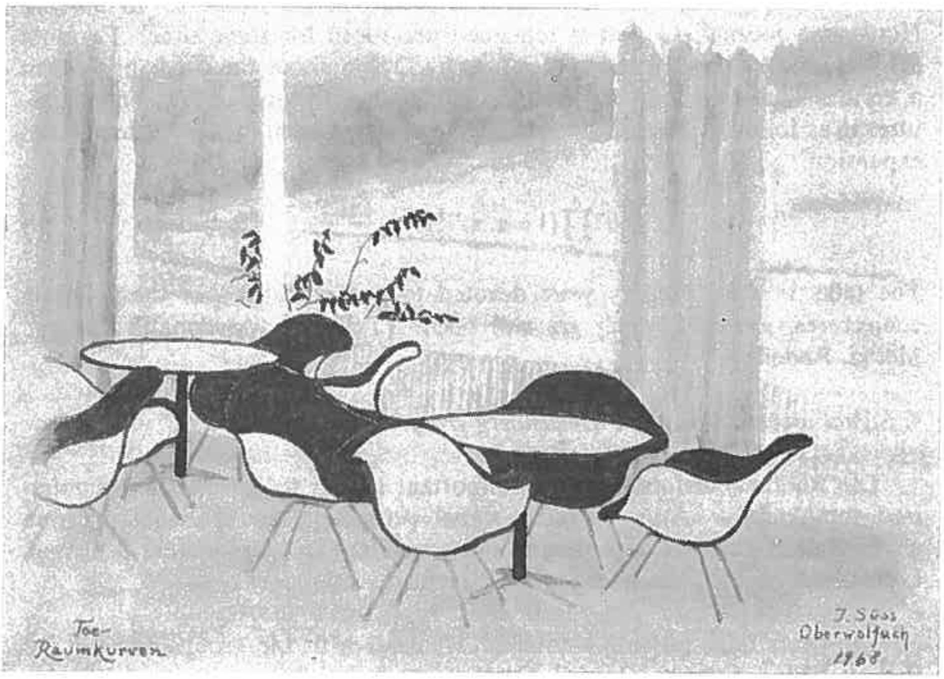

Illustration of 'Tee-Raumkurven' at Oberwolfach
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