Zeros at Infinity for Affine Nonlinear Control Systems

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Abstract—A definition of zeros at infinity for affine nonlinear control systems is proposed. The definition is local, which means that we exclude certain singularities. We argue the reasonableness of our definition by showing its relevance to the problem of nonlinear decoupling. In particular, we give a necessary and sufficient condition for the solvability of the general regular decoupling problem for affine systems in terms of the zeros at infinity.

I. INTRODUCTION

THE purpose of the present paper is to study the decoupling problem and its connection to zeros at infinity for the class of affine nonlinear systems. The connection between the two subjects has been well established in the context of linear systems (cf. [1], [2]), and it turns out that it is possible to establish quite similar results for nonlinear systems—as long as one restricts oneself, as we do in this paper, to a “local” point of view, i.e., one allows the introduction of assumptions that will hold on open parts of the state manifold but possibly not on the entire manifold as such. Our main result (Theorem 4.1) gives a necessary and sufficient condition for the solvability of the regular static-state feedback noninteracting control problem for affine systems (the problem is defined in Section IV). It is shown in Theorem 3.1 how this necessary and sufficient condition can be interpreted in terms of zeros at infinity. The decoupling results of the present work extend those of [24], where the treatment was restricted to situations in which the number of scalar inputs equals the number of vector outputs. Of course, the development sketched above would not be possible without having available a definition of “zeros at infinity” for the class of affine systems. For more restricted classes of nonlinear systems, indexes which could serve to define zeros at infinity have been introduced by Hirschorn [6] and Isidori [8]. We consider it a point of major interest of the present paper that here, for the first time, the notion of “zeros at infinity” is defined for the full class of affine systems. It is shown in [26] that our definition encompasses those given by Hirschorn and Isidori.

It is perhaps worthwhile to expand on what the concept of “zeros at infinity” means (see also [3] for linear systems, [23] for nonlinear systems). Basically, the zeros at infinity are numbers that indicate the orders of integration in a (multivariable) system. Consider first a linear single-input single-output system \( x = Ax + Bu, y = c^T x \). The “order of integration” in such a system can be defined, for instance, as the lowest number \( k \) for which the input function \( u \) appears explicitly in the expression for the \( k \)th derivative of \( y \). Since \( y = c^T A^k x + c^T A^{k-1} u + \cdots \), it is clear that this order of integration could also be expressed algebraically as the lowest value of \( k \) for which the number \( c^T A^{k-1}B \) is unequal to zero.

Because the development around infinity of the transfer function \( g(s) = c^T (sI - A)^{-1} B \) is \( g(s) = c^T B s^{-1} + c^T A B s^{-2} + \cdots, \) yet another way of expressing the order of integration would be that it is the unique value of \( k \) for which \( s^k g(s) \) has a finite and nonzero value at infinity. Following the standard terminology of function theory, this number is also called the order of the zero at infinity of \( g(s) \). Note that the first definition that we gave for “order of integration” would also apply to nonlinear systems. The situation is more complicated if we turn to multivariable systems. For decoupled scalar systems (with a diagonal transfer matrix), it is clear that the proper definition of the zeros at infinity for the system as a whole would be to take the zeros at infinity of each channel separately. In general, however, one has to reorganize the input- and output-channels in such a way that the integration structure is displayed by a set of numbers. In the linear case, this can be done by using the concept of a “bicausal matrix,” i.e., a proper rational matrix which also has a proper rational inverse, so that it has, in this sense, neither poles nor zeros at infinity. The idea is that multiplication of a transfer matrix by a bicausal matrix does not “essentially” change the integration structure. One then proves (see [3], [14]) that for every strictly proper rational matrix \( G(s) \) there exist bicausal matrices \( B_1(s) \) and \( B_2(s) \) such that

\[
B_1(s)G(s)B_2(s) = \begin{bmatrix} \Delta(s) & 0 \\ 0 & 0 \end{bmatrix}
\]

Moreover, the numbers \( d_1, \ldots, d_\ell \) are determined uniquely by \( G(s) \). It is then natural to call these numbers the (orders of the) zeros at infinity of the system described by \( G(s) \).

The above definition is not easily extended to nonlinear systems since it is given in terms of the transfer matrix. Fortunately, there are also characterizations available directly in state-space terms. Such a characterization was already given in [14], but a recent and slightly different version due to Malabre [13] turns out to be more useful for our purposes. Let a system \( \Sigma(A, B, C) \) be given, with state-space \( X \), and consider the “\( V^* \)-algorithm” [37]

\[
V^0 = X
\]

\[
V^{k+1} = \{ x \in V^k | Ax \in V^k + \text{Im } B \}.
\]

In a finite number of steps, this sequence of subspaces tends to a limit, which is denoted by \( V^* \). It can then be shown that the number

\[
p^k = \dim (\text{Im } B \cap V^{k+1}) - \dim (\text{Im } B \cap V^k)
\]

is equal to the number of zeros at infinity of order \( \geq k \), as defined above. So, the zeros at infinity can be recovered from the numbers \( p^k \) as defined by (1.5). Malabre’s [13] proof of this is rather indirect; for a short proof, see [26]. It is this characterization of the integration structure that will be generalized to nonlinear systems in the next section.

II. DIFFERENTIAL GEOMETRIC STRUCTURE THEORY

We consider an affine nonlinear control system

\[
\dot{x}(t) = A(x(t)) + \sum_{i=1}^{m} B_i(x(t))u_i(t)
\]

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where \( x \) are local coordinates of a smooth \( n \)-dimensional manifold \( M, A, B_1, \ldots, B_m \) are smooth vector fields on \( M \) and \( u_i: \mathbb{R}_+ \rightarrow \mathbb{R} \) is a piecewise smooth input function, \( i \in m \). Together with the dynamics (2.1), we consider the output functions

\[
z(t) = C(x(t)), \quad i \in k
\]  

(2.2)

where \( C_i: M \rightarrow N_i \) is a smooth map from \( M \) to a smooth \( p_i \)-dimensional manifold \( N_i, p_i \geq 1, i \in k \). We assume that each \( C_i \), \( i \in k \), is a surjective submersion. Throughout the paper we will make the following standard assumptions for the systems (2.1), (2.2):

1. \( \dim A_0 := \dim \text{span} \{B_1, \ldots, B_m\} = m \)  
2. \( \text{rank of the map } \mathcal{C} := (C_1, \ldots, C_d) \)  
3. \( M \rightarrow N_1 \times \cdots \times N_k \) equals \( p_1 + \cdots + p_k \)  
4. system (2.1) satisfies the strong accessibility-rank condition (see [33], [18]).

We allow here static-state feedback, i.e., an admissible control law has the form

\[
u = \alpha(x) + \beta(x)u
\]

(2.6)

where \( \alpha: M \rightarrow \mathbb{R}^m, \beta: M \rightarrow \mathbb{R}^{m \times m} \) are smooth functions. To keep as much open-loop control as possible, we assume that \( \beta(x) = (\beta_i(x))_{i=1}^m \) is nonsingular for all \( x \in M; v = (v_1, \ldots, v_m)' \in \mathbb{R}^m \) represents a new input. By applying the feedback law (2.6) to (2.1) we obtain as new dynamics

\[
x(t) = \tilde{A}(x(t)) + \sum_{i=1}^m \tilde{B}_i(x(t))u(t)
\]

(2.7)

where

\[
\tilde{A}(x) = A(x) + \sum_{i=1}^m B_i(x)\alpha_i(x),
\]

(2.8a)

\[
\tilde{B}_i(x) = \sum_{j=1}^m B_j(x)\beta_{ij}(x).
\]

(2.8b)

Next we come to one of the basic concepts in the "differential geometric approach" to nonlinear system theory. For detailed accounts we refer to [4], [8]-[12], [16]-[28] and to [37] for the linear counterpart.

**Definition 2.1:** A fixed-dimensional involutive distribution \( D \) on \( M \) is locally controlled invariant if, locally around each point \( x_0 \in M \) there exists a control law (2.6) such that the modified dynamics (2.7) satisfies

\[
[A, D] \subset D,
\]

(2.9a)

\[
[B_i, D] \subset D, \quad i \in m.
\]

(2.9b)

There also exists a definition of global controlled invariance [8], [9], but the advantage of the local concept above the global one is that the following test is available to determine whether or not a distribution is locally controlled invariant.

**Theorem 2.2:** Let \( D \) be an involutive distribution on \( M \) of fixed dimension and assume that \( D \cap \Delta_0 \) has fixed dimension. Then, \( D \) is locally controlled invariant if and only if

\[
[A, D] \subset D + \Delta_0,
\]

(2.10a)

\[
[B_i, D] \subset D + \Delta_0, \quad i \in m.
\]

(2.10b)

An important class of controlled invariant distributions is given by the following.

**Definition 2.3:** A fixed-dimensional involutive distribution \( D \) on \( M \) is a regular controllability distribution if, locally around each point \( x_0 \in M \) there exists a control law (2.6) such that

\[
[A, D] \subset D,
\]

(2.9a)

\[
[B_i, D] \subset D, \quad i \in m
\]

(2.9b)

and

\[
D = \text{involutive closure of } \{ad^*_k\Delta_0 \cap D, ad^*_k\Delta_0 \cap D \cap \{k \in \mathbb{Z}_+, i \in m\} \}
\]

(2.11)

Or equivalently (see [18]) \( D = \text{involutive closure of } \{ad^*_k\Delta_0, ad^*_k\Delta_0 \cap \{k \in \mathbb{Z}_+, i \in m \} \} \) for a certain subset \( I \subset m \).

As in the linear geometric theory (see [37]) locally controlled invariant distributions and regular local controllability distributions play an important role in the (local) solution of synthesis problems like the disturbance decoupling problem and the noninteracting control problem (see [4], [8]-[12], [16]-[28]). In this context one is especially interested in supremal elements satisfying Definition 2.2 or (2.5), which are contained in a given fixed-dimensional involutive distribution \( K \) on \( M \). However, in general, these supremal elements may not exist. In order to overcome this problem we consider the following algorithm:

\[
\begin{align*}
V^0 &= TM \\
V^{n+1} &= K \cap \Delta^{-1}(\Delta_0 + V^n)
\end{align*}
\]

(2.12)

where

\[
\Delta^{-1}(V) = \{X \in V(M) | [\Delta, X] \subset V\}
\]

(2.13)

and \( \Delta \) is the affine distribution associated with (2.1)

\[
\Delta(x) = A(x) + \Delta_0(x).
\]

(2.14)

It is straightforward to show that the algorithm (2.12) converges in at most \( \dim K \) steps to a limit, which will be denoted as \( V^\infty \), so \( V^\infty = V^{\dim k} \).

Now, in general, the (involutive) distributions \( V^\mu, \mu \geq 0 \), appearing in (2.12) will not have constant dimension. However, for analytic systems the \( V^\mu \), \( \mu \geq 0 \), are of constant dimension on an open and dense submanifold \( M^\infty \) of \( M \). Now, if we exclude all possible singularities in the dimensions of the \( V^\mu \), \( \mu \geq 0 \) and \( V^\nu \cap \Delta_0, \mu \geq 0 \), then we know (see, e.g., [4], [9], [16]) that \( V^\mu \) is the maximal element in the family of all controlled invariant distributions contained in \( K \). Therefore, we will make the following basic assumption (valid on open parts of \( M \)).

**Assumption 2.4:** For each \( \mu \geq 0 \), the distributions \( V^\mu \) and \( V^\nu \cap \Delta_0 \) will have fixed dimension, where \( V^\nu \) is defined in (3.12).

The (nonlinear) algorithm (2.12) contains structural information about a control system, as shown in [37] for the linear case. In what follows we will mimic the linear theory on infinite zeros as far as possible. For linear references see, e.g., [35] and [26]. Consider a smooth nonlinear control system (2.1) together with one output function \( C \) as in (2.2). By assumption, the function \( C \) being a surjective submersion induces a fixed-dimensional involutive distribution \( \ker C_\infty \) on \( M \). Therefore, we may apply algorithm (2.12) to \( \ker C_\infty \) and assume that Assumption 2.4 holds in the case. Then supremal locally controlled invariant distribution contained in \( \ker C_\infty \) is denoted as \( V^\infty \) and satisfies \( V^\infty = V^{k+1} = V^\infty \) for all \( k \geq n - p \) where \( p = \text{rank } C \). Now we define a set of integers by the following.

**Definition 2.5:**

\[
p^\mu := d(\Delta_0 \cap V^{n-1}) - d(\Delta_0 \cap V^\mu), \quad \mu > 0.
\]

(2.15)

Associated with the sequence \( \{p^\mu\}_{\mu=0}^{\infty} \) we define another list by the following.
\textbf{Definition 2.6:}

\[ n^* := \text{number of } p^* \text{'s which are greater than or equal to } \mu. \]  
\[ (2.16) \]

There is a one-to-one correspondence between the sequences \( \{ p^* \}_{j=1}^{\infty} \) and \( \{ n^* \}_{j=1}^{\infty} \) given by (2.16) and \( p^* = \text{number of } n^* \text{'s which are greater than or equal to } \mu. \)

\[ \text{(2.17)} \]

As in the linear case (see [13], [26]) we will say that the nonlinear system (2.1), with output

\[ z = C(x) \]  
\[ (2.18) \]

has \( p^i \) zeros at infinity of orders \( \{ n^* \}. \) As we have seen in Section I, these integers play an important role in the linear theory (as, for example, in Silverman’s structure algorithm), but also in the solution of the noninteracting control problem; see [1], [2]. In the next sections it will be shown that in the general nonlinear noninteracting control problem, the integers \( \{ p^* \} \) (or \( \{ n^* \} \) play the same role as in the linear theory of [1], [2]. It is for this reason that we have chosen to call the \( n^* \)’s the orders of the \( p^i \) infinite zeros. Further explanation is given in [26].

\textbf{Remark: For general nonlinear systems of the form (locally)}

\[ \begin{cases} \dot{x} = f(x, u) \\ \dot{y} = h(x, u) \end{cases} \]  
\[ (2.19) \]

one can define zeros at infinity in the following way. We form an “extended system” (cf. [27]) by introducing a new input function \( v \) as follows:

\[ u = v. \]  
\[ (2.20) \]

The extended system (2.19), (2.20) with \((v)\) as state and \( u \) as input is affine, and we can apply the above definition (see also [28]). From the orders of zero at infinity so obtained, we subtract one in order to compensate for the integration we have added. Note that, in this way, one may find zeros at infinity of order zero; this is in agreement with the linear situation. Of course, one has to show that this definition is consistent in the sense that if (2.19) happens to be affine, then the definition given above agrees with the direct definition given earlier. This has been done in [25]. In the rest of this paper, we will limit ourselves to affine systems.

Let us finally say a few words on controllability distributions. Again it can be shown (see [12], [18]) that there exists a supremal regular local controllability distribution \( R^*_g \) contained in a given fixed-dimensional involutive distribution \( K \) on \( M. \) Notice, however, that \( R^*_g \) is not necessarily of constant dimension. As in the linear theory there is no direct algorithm for computing \( R^*_g \).

The easiest way of computing \( R^*_g \) is with the aid of \( V^g. \) This can be summarized in the following procedure.

\textbf{Step 1: Compute } \( V^g \) \text{ (assume } \( V^g \text{ has constant dimension).} \)

\textbf{Step 2: Compute appropriate } \( A_1, B_1, \ldots, B_n \) which leave \( V^g \) invariant.

\textbf{Step 3: Compute } \( \Delta_0 \cap V^g. \)

\textbf{Step 4: } \( R^*_g = \text{involutive closure of } \{ ad^k \Delta_0 \cap V^g, ad^k \Delta_0 \cap V^g \mid k \in \mathbb{Z}_+, \ i \in m \}. \)

Notice that, almost by construction, the following identity holds (cf. [18]):

\[ \Delta_0 \cap R^*_g = \Delta_0 \cap V^g \]  
\[ (2.21) \]

which will be used in the sequel.

\textbf{III. STRUCTURE AT INFINITY FOR MULTIPLE OUTPUTS}

We now consider the system (2.1), (2.2) under the standard assumptions (2.3)–(2.5). While (2.4) holds, we have that for each \( I \subseteq k \) the involutive distribution \( \bigcap_{j \in I} \text{Ker } C_{j^*} \) is of constant dimension, and therefore we may apply the algorithm (2.12) for each of them. Assuming that Assumption 2.4 holds for each sequence of distributions, we obtain the corresponding supremal local controlled invariant elements. We will list them as follows:

\[ V^*_g = \text{supremal locally controlled invariant distribution in } \text{Ker } C_{g^*}. \]  
\[ (3.1a) \]

\[ V^*_j = \text{supremal locally controlled invariant distribution in } \bigcap_{j \in I} \text{Ker } C_{j^*}, \ I \subseteq k. \]  
\[ (3.1b) \]

We also write

\[ D_j = V^g \setminus I, \ I \subseteq k \]  
\[ (3.1c) \]

and

\[ R^*_j = \text{supremal regular local controllability distribution in } \bigcap_{j \in k} \text{Ker } C_{j^*}, \ I \subseteq k. \]  
\[ (3.1d) \]

The corresponding lists of orders of the zeros at infinity will be denoted as follows:

\[ p^j = d(\Delta_0 \cap V^j), \ i \in k, \ \mu > 0. \]  
\[ (3.2a) \]

\[ q^j = d(\Delta_0 \cap V^j), \ i \in k, \ \mu > 0. \]  
\[ (3.2b) \]

\[ q^* = d(\Delta_0 \cap V^*), \ i \in k, \ \mu > 0. \]  
\[ (3.2c) \]

\[ q^* = d(\Delta_0 \cap V^*), \ i \in k, \ \mu > 0. \]  
\[ (3.2d) \]

\[ q^* = d(\Delta_0 \cap V^*), \ i \in k, \ \mu > 0. \]  
\[ (3.2e) \]

It is convenient in this notation that we set

\[ V^g = TM \text{ and } D^*_g = V^*_g. \]

The following relations are immediate:

\[ V^*_j = D^*_j \setminus I, \ I \subseteq k, \ \mu > 0. \]  
\[ (3.3) \]

Through these preliminaries we come to the main theme of this section. Consider the indentity

\[ \Delta_0 = \sum_{i \in k} \Delta_0 \cap D^*_i. \]  
\[ (3.7) \]

Relation (3.7), to which we will refer as the noninteraction condition (this terminology will be fully justified in Section IV),
is equivalent to certain relations among the indexes $p^p_\mu$, $p^p_\mu$, and $p^p_s$. Notice that for linear systems it is known that (3.7) is equivalent to, cf. [1]

$$p^p_s = \sum_{\mu \in k} p^p_\mu, \quad \text{for all } \mu > 0. \quad (3.8)$$

Here we will give an extension of this result.

**Theorem 3.1:** Assume that for all $I \subset k, V^p_s$ satisfies Assumption 2.4. Then the following are equivalent:

a) $\Delta_0 = \sum_{\mu \in k} \Delta_0 \cap D^p_\mu,$ \hspace{1cm} (3.7)

b) $p^p_s = \sum_{\mu \in k} p^p_\mu,$ for all $\mu > 0,$ \hspace{1cm} (3.8)

c) $p^p_s : p(k) \to \mathbb{R}.$ is a weight function, for all $\mu > 0.$

For the proof of this theorem we need some preliminary results.

**Lemma 3.2:** Suppose Assumption 2.4 holds for all $D^p_\mu, I \subset k.$ Then, for all $\mu, \psi_1, \psi_2, \ldots \in 0,$

$$\Delta_0 = \sum_{\psi \in k} \Delta_0 \cap D^p_\psi, \quad (3.10)$$

and thus, for all $\mu \geq 0,$

$$\Delta_0 = \Delta_0 \cap D^p_\psi + \Delta_0 \cap D^p_\psi. \quad (3.11)$$

Induction and Lemma 3.2 lead to the desired result (3.9). Now for arbitrary $I, J \subset k$ we have

$$D^p_\psi \cap D^p_\psi \subset D^p_\psi \cap D^p_{\psi \cup \psi \setminus \psi} \quad (3.12)$$

and because $I \cup J \cup k \setminus I = k,$ we have that

$$D^p_\psi \cap D^p_{\psi \cup \psi \setminus \psi} = D^p_{\psi \cup \psi \setminus \psi \cup \psi \setminus \psi} = D^p_{\psi \cup \psi \setminus \psi \cup \psi \setminus \psi} \quad (3.13)$$

On the other hand

$$D^p_{\psi \cup \psi \setminus \psi \cup \psi \setminus \psi} = D^p_{\psi \cup \psi \setminus \psi \cup \psi \setminus \psi} \quad (3.14)$$

so by (3.12), (3.13), and (3.14) we obtain

$$D^p_{\psi \cup \psi \setminus \psi \cup \psi \setminus \psi} = D^p_{\psi \cup \psi \setminus \psi \cup \psi \setminus \psi} \quad (3.15)$$

if and only if

$$\forall I, J \subset k \Delta_0 \cap D^p_\psi + \Delta_0 \cap D^p_\psi = \Delta_0 \cap D^p_{\psi \cup \psi \setminus \psi \cup \psi \setminus \psi} \quad (3.16)$$

**Proof:**

\[ \Delta_0 = \Delta_0 \cap D^p_\psi + \Delta_0 \cap D^p_\psi + \Delta_0 \cap D^p_\psi + \Delta_0 \cap D^p_\psi \]

We are now able to prove the main theorem of this section.

**Proof (of Theorem 3.1):**

\[ a = c \] We have by Lemmas 3.3 and 3.4, for all $I, J \subset k$ and $\mu > 0,$ that

$$\Delta_0 \cap D^p_\psi + \Delta_0 \cap D^p_\psi + \Delta_0 \cap D^p_\psi + \Delta_0 \cap D^p_\psi \quad (3.17)$$

Thus, for all $\mu > 0,$

$$p^p(\theta) = d(\Delta_0 \cap V^p_\psi) - d(\Delta_0 \cap V^p_\psi) = m - m = 0 \quad (3.18)$$

$$p^p(I \cup J) = d(\Delta_0 \cap V^p_{\psi \cup \psi \setminus \psi \cup \psi \setminus \psi}) - d(\Delta_0 \cap V^p_{\psi \cup \psi \setminus \psi \cup \psi \setminus \psi}) \quad (3.19)$$

$$p^p(I \cup J) = d(\Delta_0 \cap V^p_{\psi \cup \psi \setminus \psi \cup \psi \setminus \psi}) - d(\Delta_0 \cap V^p_{\psi \cup \psi \setminus \psi \cup \psi \setminus \psi}) \quad (3.20)$$
Using (3.17) we have
\[ d(\Delta_0 \cap V_i^{+1} + \Delta_0 \cap V_i^{-1}) = d(\Delta_0 \cap V_i^{+1} - \Delta_0 \cap V_i^{-1}) \]
\[ = d(\Delta_0 \cap V_i^{+1}) + d(\Delta_0 \cap V_i^{-1}) - d(\Delta_0 \cap V_i^{+1} - \Delta_0 \cap V_i^{-1}) \]
\[ = d(\Delta_0 \cap V_i^{+1}) + d(\Delta_0 \cap V_i^{-1}) - d(\Delta_0 \cap V_i^{+1} - \Delta_0 \cap V_i^{-1}) \] (3.21)
and by (3.18)
\[ d(\Delta_0 \cap V_i^{+1} + \Delta_0 \cap V_i^{-1}) = d(\Delta_0 \cap V_i^{+1} - \Delta_0 \cap V_i^{-1}) \] (3.22)
Furthermore, (3.21) and (3.22) hold true if we replace \( \mu \) by \( * \), i.e., by taking \( \mu \) sufficiently large. Combination of these expressions, together with (3.20), (3.21), and (3.22) leads to
\[ p_{*}^{(I \cup J)} = p_{*}^{(I \cup J)} - p_{*}^{(I \cap J)} \] (3.23)
Using \( p_{*}^{(I \cup J)} = \sum_{i \in k} p_{*}^{(I \cup J)} \), we obtain the following identities
\[ m - d(\Delta_0 \cap V_i^{+1} + \Delta_0 \cap V_i^{-1}) \]
\[ = \sum_{i \in k} m - d(\Delta_0 \cap V_i^{+1} + \Delta_0 \cap V_i^{-1}) \]
\[ = \sum_{i \in k} [m - d(\Delta_0 \cap V_i^{+1} + \Delta_0 \cap V_i^{-1})] \]
\[ = \sum_{i \in k} [m - d(\Delta_0 \cap V_i^{+1} + \Delta_0 \cap V_i^{-1})] \] (3.24)

Clearly the statement is true for \( \mu = 0 \). Assume (3.24)-(3.26) hold for a certain \( \mu > 0 \), then by repeated application of Lemma 3.2 (3.25) and (3.26) hold true for \( \mu + 1 \). Furthermore, we have [see (3.4c)] for all \( i \in k \)
\[ \Delta_0 = \Delta_0 \cap D_i^{*+1} + \Delta_0 \cap V_i^{*+1} \] (3.27)
Next we compute \( d(\Delta_0 \cap D_i^{*+1} + \Delta_0 \cap V_i^{*+1}) \).
\[ d(\Delta_0 \cap D_i^{*+1} + \Delta_0 \cap V_i^{*+1}) \]
\[ = d(\Delta_0 \cap D_i^{*+1} + \Delta_0 \cap V_i^{*+1}) \]
\[ = \sum_{j \in l} d(\Delta_0 \cap V_j^{*+1}) - (k - 2)m + d(\Delta_0 \cap V_j^{*+1}) \]
\[ = \sum_{j \in l} d(\Delta_0 \cap V_j^{*+1}) - (k - 2)m - d(\Delta_0 \cap V_j^{*+1}) \] (3.28)
Therefore, (3.24) is established for all \( \mu \geq 0 \) and (3.7) readily follows by taking \( \mu \) sufficiently large.

While the numbers \( p_{*}^{(I \cap J)} \) and \( p_{*}^{(I \cup J)} \) are in one-to-one correspondence to the orders of the infinite zeros (see Definition 2.6), condition (3.8) can also be established by using them. Let
\[ n_{*}^{(I \cup J)} \]
\[ n_{*}^{(I \cap J)} \]
where \( \mathcal{U} \) denotes the set theoretic union (with repeated common elements).

**Remarks:** 1) In case the number of scalar inputs (= \( m \)) equals the number of vector outputs (= \( k \)), the noninteracting condition
(3.7) reduces to a direct sum
\[ \Delta_0 = \bigoplus_{i \in k} \Delta_0 \cap D_i^* \] (3.34)

and
\[ \Delta_0 \cap V^* = 0 \] (3.35)

(see [24] for details). ii) As already noted in (2.21) we can replace (3.7) by
\[ \Delta_0 = \sum_{i \in k} \Delta_0 \cap R_i^* \] (3.36)

which will be the starting point of the next section.

IV. The General Noninteracting Control Problem

We now come to the generalization of the linear regular block-decoupling problem (here regular means that one uses full control in the decoupling state feedback): see [15, 38, 1]. Let us briefly outline the input-output decoupling problem under consideration. For a more complete discussion of this topic, we refer to [24], where the same problem has been solved in case the number of scalar \( V \) inputs equals the number of vector outputs. Consider the system (2.1), (2.2) under the assumptions (2.3)-(2.5). Suppose that, after applying a feedback law (2.6) the new input \( v_i \) does not affect the output \( z_i, j \in k, j \neq i \), and moreover the input \( v_j \) "controls" the output \( z_j, i \in k \). Here \( (v_1, \ldots, v_k) = (v_1, \ldots, v_n) \), but some \( v_j, i \in m \), may appear in various vector inputs \( v_j, j \in k \). That is, there is a partitioning
\[ m = I_1 U \cdots U_l k \] (4.1)

with the property that \( j \in I_1 \) \( \Rightarrow v_j \) belongs to \( v_n, l \in k \).

Clearly, if \( v_j \in v_n, v_g \) for some \( i \neq \beta \in k \), then neither \( z_j, j \neq i \), nor \( z_j, j = i \), is affected by \( v_j \); so all outputs \( z_j, j \in k \), are independent of the input \( v_i \). Therefore, excluding overlappings in the various input vectors \( v_i, i \in k \) leads to a partitioning \( v = (v^0, v^1, \ldots, v^k) \) such that \( v^k \) does not affect \( z_j, j \in k \) and \( v^0 \) does not affect \( z_j, j \neq i \), and "controls" \( z_i \). This allows us to rewrite the partitioning (4.1) as
\[ m = I^0 \oplus I^1 \oplus \cdots \oplus I^k \] (4.2)

with the property \( j \in I^0 \Rightarrow v_j \in v^0, l = 0, 1, \ldots, k \).

Consider the regular (local) controllability distributions
\[ R_j \triangleq \text{span (involutive closure of } \{ad_{\xi}^k \tilde{E}_{ij}, ad_{\eta}^k \tilde{E}_{ij} \} \text{)} \quad k \in \mathbb{Z}_+, i \in I, j \in k \] (4.3)

The noninteraction conditions can be nicely expressed by means of the distributions \( R_1, \ldots, R_k \), namely the input \( v_i \) does not affect \( z_j, j \neq i \), if and only if
\[ R_j \subseteq \bigcap_{i \in j} \text{Ker } C_{ij^*}, \quad j \in k \] (4.4)

while \( v_j \) "controls" \( z_j, i \in k \) is equivalent to (see [20], [22], [24] for the definition of output controllability)
\[ R_j + \text{Ker } C_{ij^*} = TM, \quad j \in k \] (4.5)

or equivalently
\[ C_{ij^*}(R_j) = TM, \quad j \in k \] (4.6)

The static-state feedback noninteracting control problem can now be formulated as follows.

Given the system (2.1), (2.2) find, if possible, a feedback law (2.6) such that (4.4) and (4.5) hold for each \( i = 1, \ldots, k \), defined by (4.3). This problem will be solved here in a local fashion. Given an arbitrary initial point \( x_0 \in M \) we are interested in finding a local feedback law (2.6), i.e., \( \alpha \) and \( \beta \) are possibly only well-defined in a neighborhood of \( x_0 \) (compare to Definition 2.1 and Theorem 2.2 on local controlled invariance).

Without any further requirements we cannot get global solutions of the above problem. The solution of the nonlinear noninteracting control problem is similar to the linear (geometric) version of this problem (see [15, 38]) so the differential geometric approach again provides a good framework for such a synthesis problem. Recall the definition (3.1d) of \( R_i^* \), \( I \subseteq k \).

The theorem we are after is as follows.

Theorem 4.1: Consider the system (2.1), (2.2) and assume that for all \( I \subseteq k \), \( V_I \) and \( V_I \cap \Delta_0 \) all have fixed dimension. Then the static-state feedback noninteracting control problem is locally solvable around each point \( x_0 \in M \) if and only if
\[ \Delta_0 = \bigoplus_{i \in k} \Delta_0 \cap D_i^* \] (3.37)

Furthermore, if these conditions hold, then \( \{ R_i^* \}_{i=1}^k \) is the only solution satisfying (4.4) and (4.5).

We will prove this theorem by using the following result of [24].

Theorem 4.2: Consider the system (2.1), (2.2) and assume that for all \( I \subseteq k \) the distribution \( \Sigma_{i \in I} R_i^* \) is involutive. This is the basic observation of [24] for the construction of a decoupling feedback law.

The idea to use Theorem 4.2 for proving the sufficient part of Theorem 4.1 is that we first "factor out" the maximal unobservability distribution in \( \text{Ker } dC, \) i.e., \( V^* \) [see (3.1a)], and then we show that the reduced system on the quotient manifold \( M(\text{mod } V^*) \), exactly satisfies the sufficient condition (4.7). Note that the quotient system will have \( m \setminus l^0 \) inputs [see (4.2)]. In formalizing this we need the following results.

Lemma 4.3: If (3.7) holds, then
\[ \Delta_0 \cap V^* = \sum_{i \in k} (\Delta_0 \cap D_i^*)/(\Delta_0 \cap V^*). \] (4.8)

Proof: By definition, we have \( V^* \subseteq D_i^* \) for \( i = 1, \ldots, k \).

Therefore,
\[ \Delta_0 \cap V^* = \left( \sum_{i \in k} (\Delta_0 \cap D_i^*)/\Delta_0 \cap V^* \right) = \sum_{i \in k} (\Delta_0 \cap D_i^*)/(\Delta_0 \cap V^*). \]

Lemma 4.4: If
\[ \Delta_0 \cap V^* = 0 \] (4.9)

then (3.7) is equivalent to
\[ \Delta_0 = \bigoplus_{i \in k} \Delta_0 \cap D_i^* \] (4.7)

that is the distributions \( \{ \Delta_0 \cap D_i^* \}_{i=1}^k \) are independent.

Proof: As a result of the previous section we know that (3.7) is equivalent to \( p^*: p(k) \to \mathbb{Z}_+ \) being a weight function for all \( \mu >...
0. Therefore,
\[ p'(k \setminus \{1\}) + p'(k \setminus \{2\}) + \cdots + p^2(k \setminus \{k\}) \]
\[ = p'(k) + p'(k \setminus \{1, 2\}) + p'(k \setminus \{3\}) + \cdots + p'(k \setminus \{k\}) \]
\[ = p'(k) + p'(k) + p'(k \setminus \{1, 2, 3\}) + p'(k \setminus \{4\}) \]
\[ + \cdots + p'(k \setminus \{k\}) = \cdots \]
\[ = (k-1)p'(k) + p'(0) = (k-1)(m-0) = (k-1)m. \]
So \( m - d(\Delta_0 \cap D^*) + m - d(\Delta_0 \cap D^*) + \cdots + m - d(\Delta_0 \cap D^*) = (k-1)m \).

Clearly, (4.10) is equivalent to (4.7).

Now we proceed with the proof of the main theorem.

**Proof (of Theorem 4.1):** For sufficiency, we assume that (3.7) or the equivalent (3.38) holds. The proof now proceeds in two steps. Let \( x_0 \in M \). Then we first construct a local feedback law
\[ u = \alpha(x) + \beta(x)\dot{x} \]  
(4.11)
such that the modified dynamics leaves \( V^* \) invariant, i.e.,
\[ \{\tilde{A}, V^*\} \subset V^* \]
\[ \{\tilde{B}_i, V^*\} \subset V^*, \quad i \in m \]  
(4.12)
(see also [8], [9] for a thorough explanation of this “factoring out”—procedure in connection with controlled invariance. For our control system this projection amounts to a quotient system on \( 0(x_0) \mod V^* \) given by
\[ x_2 = \tilde{A}(x_2) + \tilde{B}(x_2)\dot{x}_2, \quad i = 1, \ldots, k \]  
(4.13)
(4.14)

Because \( V^* \subset D^* \), \( i \in k \), the distributions \( \pi_i(D^*) \) are well defined on \( 0(x_0) \mod V^* \) and each of them is involutive (Math. 34). Setting \( D^* = \pi_i(D^*) \), \( i \in k \), and \( \Delta_0 = \{ \tilde{B}_{i+1}, \ldots, \tilde{B}_m \} \) we see by Lemma 4.3 that (3.7) implies
\[ \Delta_0 = \{ \tilde{B}_0 \cap \tilde{D}_k \} \]  
(4.15)

Moreover, the supremal controlled invariant distribution of (4.14) contained in \( \text{Ker} C_\sigma \), respectively, \( \cap_{\sigma_{ij}} \text{Ker} C_j \), \( i \in k \), equals \( \pi_\sigma^*(V^*) = 0 \), respectively, \( \pi_\sigma(D^*) = D^* \), \( i \in k \). Therefore, we may apply Lemma 4.4 to conclude that (4.7) holds. So by Theorem 4.2 there exists a feedback
\[ \tilde{u}^2 = \alpha(x) + \beta(x)\dot{x}_2 \]  
(4.16)

where \( \tilde{u}^2 = (\tilde{u}_1, \ldots, \tilde{u}_k)' \) for the system (4.14) which solves the static-state feedback noninteracting control problem for this system. Getting
\[ u_i = \tilde{u}_i, \quad i = 1, \ldots, k \]  
(4.17)

(4.11), (4.16), and (4.17) together locally define a state feedback which solves the noninteracting control problem for the original system. To show that (3.7) is necessary, let \( \{R_I\}_{I \in E} \) be a set of regular local controllability distributions that gives a solution of the decoupling problem, see (4.3)—(4.5) (cf. [24]). Since
\[ \Delta_0 \subset \{ \bigcup_{I \in E} \Delta_I \cap \bigcup_{I \in E} R_I \} \]
we see immediately that (3.7) must hold.

Remark: The proof given here is completely different from the corresponding “linear proof” of [15]. In fact, after the tedious calculations of Section III, our proof becomes in the linear case much simpler than in [15].

V. Conclusions

We have proposed a definition of “zeros at infinity” for affine nonlinear control systems, and we demonstrated the usefulness of our definition in the solution of the general decoupling problem. It seems that we have here a promising area of further research. For instance, we expect that the problem of (left and right) invertibility [6], [7], [19], [32] can be studied profitably using the concepts of this paper (see also [23]). Further study can be made of the algebraic aspects of the decoupling problem [21], and of canonical forms in the context [24]. The nonregular input–output decoupling problem remains open to further investigation. An important issue is the existence of global solutions to the decoupling problem; in this connection, we mention the recent work of Byrnes on global controlled invariance. Finally, several aspects of the \( V^* \)-algorithm (2.12) need to be investigated further: among these are the computational side of the algorithm and the study of the consequences of nonconstant dimensions of the distributions \( V^* \).

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