Hyperfunctions and Analytic Functionals*

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Dedicated to the respectful memory of Dr. Johannes Willem de Roever

In this introductory paper, dedicated to the respectful memory of Dr. Johannes Willem (Jowi) de Roever, I discuss some ideas and motivations lying behind the deep and intriguing theory of hyperfunctions and close classes of analytic functionals, a domain of mathematics where Jowi de Roever made a worthy contribution.

Generalized functions

In the early fifties it became clear that the development of modern mathematical physics essentially needs a generalization of the concept of a function. For example, it was proved that fundamental physical axioms imply that quantum fields cannot be described by usual functions, having values in all points, and that some singularities should be present. As a matter of fact, quantum physics provided mathematicians with a natural way of this generalization. Namely, one of the basic principles of quantum theory is the concept of *measurement*. To get the idea, suppose that some physical quantity is described by a function $y = f(x)$, and we want to evaluate it at a point x_0. To do this we use some physical instrument, characterized by an aperture function $y = \phi(x)$. The process of measurement gives us the mean value (see Figure 1)

$$\int f(x)\phi(x)dx \simeq f(x_0)\int \phi(x)dx = f(x_0) \quad \text{if} \quad \int \phi(x)dx = 1.$$

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FIGURE 1.

In particular, different instruments and different tunings of the same instrument produce different results! Hence, our physical quantity is in fact described by a *linear functional* $\phi \mapsto f(\phi) = \int f(x)\phi(x)dx$, where ϕ belongs to some space of smooth functions. Now, it is quite natural to admit linear functionals of a more general form to do this, and to define *generalized functions* as continuous linear functionals over an appropriate space of *test functions*. This idea was implicitly used in the thirties by P. A. M. Dirac in his work on quantum mechanics (the famous δ-function). On the other hand, also in the thirties, S. L. Sobolev systematically used generalized derivatives to solve the Cauchy problem for hyperbolic equations. In fact, S. L. Sobolev has laid the foundations of the theory of generalized functions [1]. The full exposition of the theory was given by L. SCHWARTZ [2] in 1950-51, who introduced the space of *distributions*. After that, generalized functions became rather popular, and were extensively used in pure and applied mathematics, especially in the theory of partial differential equations (PDEs) and mathematical physics.

Thus, generalized functions are elements of the dual space to a test function space. The concrete choice of the test functions space depends on a problem under study, but usually it consists of smooth functions subject to some growth or support conditions. Operations over generalized functions are defined as the dual ones to corresponding operations over test functions. For example, the partial derivative $D_\mu f$, where f is a generalized function, and $D_\mu = \frac{\partial}{\partial x^\mu}$, is defined by

$$(D_\mu f)(\phi) = -f\,(D_\mu \phi) \qquad \text{for all test functions } \phi,$$

while the Fourier transform $F[f]$ is defined by

$$F[f](\phi) = f\left(F[\phi]\right) \qquad \text{for all test functions } \phi,$$

where $F[\phi]$ is the conventional Fourier transform of ϕ. It is said that a generalized function f is vanishing on an open set $\mathcal{O}$ if $f(\phi) = 0$ for all test functions ϕ having their supports in $\mathcal{O}$, and the support $\operatorname{supp} f$ of f is then defined in the standard fashion.

It is clear that the set of solutions of a given PDE depends on a choice of a space where these solutions are searched for. So when we pass from smooth functions and classical solutions to generalized functions and generalized solutions this set increases. It is remarkable that often the choice of a function space must be included in the axioms of a *physical* theory. Thus, admissible singularities of a quantum field in the configuration space determine its possible high-energy asymptotics in the momentum space. Such considerations motivated the search and the study of new functions and new spaces of generalized functions. The basic model is the Schwartz distributions. Here the test functions space $\mathcal{D}(\mathcal{O})$ over an open domain $\mathcal{O}$ consists of all $\mathcal{C}^\infty$-functions having their supports in $\mathcal{O}$. Distributions $f \in \mathcal{D}'(\mathcal{O})$ locally have a *finite order*, i.e., $f = P(D)g$ on every compact subset K of $\mathcal{O}$, where $P(D)$ is a partial differential operator with constant coefficients, g is a continuous function on K, and the order of $P(D)$ depends on K. To get generalized functions of *infinite order* or *ultradistributions* one should pass to more smooth test functions, i.e., to test functions with controlled growth of high order derivatives, like Gevrey classes or analytic functions. But then we encounter the problem of defining the support of an ultradistribution, because in the quasianalytic case there are no functions with compact supports. In order to overcome these contradictions M. Sato [3] proposed a quite new concept of *hyperfunction*.

Hyperfunctions of one variable

We start with the case of one independent variable. Let ω be an open interval in $\mathbb{R}$, Ω be its complex neighborhood in $\mathbb{C}$, so $\omega = \Omega \cap \mathbb{R}$. Let

$$\Omega^\pm = \{z = x + iy \in \Omega : y \gtrless 0\}$$

be the intersections of Ω with the upper and lower half-plane (see Figure 2).

Figure 2.

Let $f(z) \in \mathcal{H}(\Omega^+)$ be a function analytic in Ω^+. Suppose that for every $\phi(x) \in \mathcal{T}(\omega)$ with compact support in ω, where $\mathcal{T}(\omega)$ is some test function space, there exist a limit

$$\mathrm{bv}f(\phi) = \lim_{y \to +0} \int_\omega f(x+iy)\phi(x)dx,$$

and this limit uniquely defines a generalized function $\mathrm{bv}f \in \mathcal{T}'(\omega)$, then $\mathrm{bv}f$ is called the *generalized boundary value* of the analytic function $f(z)$. Usually, the boundary value $\mathrm{bv}f$ exists in a given space $\mathcal{T}'(\omega)$ if $f(x+iy)$ satisfies some estimate when $y \to +0$. For example, $\mathrm{bv}f$ exists in $\mathcal{D}'(\omega)$ if for every compact $K \subset \omega$ there exist $M, p, \epsilon > 0$, such that

$$|f(x+iy)| \le My^{-p} \quad \text{for all } x \in K,\ 0 < y < \epsilon.$$

As a rule, generalized functions have *analytic representations*. Namely, if $\mathcal{T}'(\omega)$ is a typical space of generalized functions, and $g \in \mathcal{T}'(\omega)$, then there exist a complex neighborhood Ω of ω and analytic functions $f^\pm \in \mathcal{H}(\Omega^\pm)$, such that

$$g = \mathrm{bv}f^+ + \mathrm{bv}f^- \quad \text{in} \quad \mathcal{T}'(\omega).$$

EXAMPLE A nice and familiar example is the Dirac δ−function, which can be written as

$$\delta(x) = \frac{1}{2\pi i}\left(\frac{1}{x-i0} - \frac{1}{x+i0}\right).$$

Moreover, two pairs $\{f_1^+, f_1^-\}$ and $\{f_2^+, f_2^-\}$ represent the same generalized function g iff there exists an analytic function $f \in \mathcal{H}(\Omega)$, such that

$$f_1^\pm - f_2^\pm = \pm f|_{\Omega^\pm}.$$

Now, the bright idea of Sato was *to give up any growth conditions*! Namely, for an open interval $\omega \subset \mathbb{R}$ he considers pairs $\{f^+, f^-\}$ of arbitrary analytic functions $f^\pm \in \mathcal{H}(\Omega^\pm)$, where the complex neighborhood Ω depends on a pair, he introduced the equivalence relation

$$\{f^+, f^-\} \sim 0 \text{ iff } f^\pm = \pm f|_{\Omega^\pm} \text{ for some } f \in \mathcal{H}(\Omega),$$

and defined a *hyperfunction* g over ω as the equivalence class $[f^+, f^-]$ of a pair $\{f^+, f^-\}$. The linear space of all hyperfunctions over ω he denoted by $\mathcal{B}(\omega)$.

At first glance, this definition is rather abstract. To see its essence I present some simple properties of hyperfunctions.

1. Embedding of generalized functions into hyperfunctions.
Every space $\mathcal{T}'(\omega)$ of generalized functions having analytic representations is embedded into $\mathcal{B}(\omega)$ via this analytic representation. So, the concept of hyperfunction is indeed a wide extension of the concept of generalized function.

2. Differentiation.
The derivative Dg of a hyperfunction $g = [f^+, f^-] \in \mathcal{B}(\omega)$ is defined as the equivalence class

$$Dg = [f_z^+, f_z^-] \in \mathcal{B}(\omega),$$

where $f_z^\pm = \partial f^\pm/\partial z$. This definition agrees with the above embedding of generalized functions.

3. Multiplication by real analytic functions.
The product ϕg of a real analytic function $\phi \in \mathcal{A}(\omega)$ and a hyperfunction $g = [f^+, f^-] \in \mathcal{B}(\omega)$ is defined as the equivalence class

$$\phi g = [\varphi f^+, \varphi f^-] \in \mathcal{B}(\omega),$$

where $\varphi(z) \in \mathcal{H}(\Omega)$ is an analytic extension of $\phi(x)$ into some complex neighborhood Ω of ω.

By these rules the action of a linear differential operator

$$P(D) = \sum_{i=0}^{m} a_i(x) D^i, \qquad a_i(x) \in \mathcal{A}(\omega),$$

is defined over $\mathcal{B}(\omega)$, and we can look for hyperfunction solutions of a differential equation

$$P(D)g = h, \qquad g, h \in \mathcal{B}(\omega).$$

If h is a generalized function, then any generalized solution g of the above equation will be its hyperfunction solution, also, but even in this case new solutions may appear. Moreover, in hyperfunction classes uniqueness and existence theorems take their natural closed form.

4. Hyperfunction boundary values.
For every analytic function $f \in \mathcal{H}(\Omega^+)$ ($f \in \mathcal{H}(\Omega^-)$) its *hyperfunction boundary value* $\mathrm{bv} f \in \mathcal{B}(\omega)$ is defined as the equivalence class $[f, 0]$ ($[0, f]$). Notice that

$$\mathrm{bv} f = [f|_{\Omega^+}, 0] = [0, f|_{\Omega^-}] \qquad \text{for all } f \in \mathcal{H}(\Omega).$$

5. Restrictions.
Let $\omega \supset \omega_0$ be open intervals in $\mathbb{R}$. For every hyperfunction $g = [f^+, f^-] \in \mathcal{B}(\omega)$ its *restriction* $g|_{\omega_0} \in \mathcal{B}(\omega_0)$ is defined by the rule

$$g|_{\omega_0} = [f^+|_{\Omega_0^+}, f^-|_{\Omega_0^-}],$$

where $\Omega_0 = \Omega \cap \{z = x + iy : x \in \omega_0\}$ (see Figure 3).

Figure 3.

6. Sheaf of hyperfunctions.

Using the Mittag-Leffler theorem one can prove that if a hyperfunction $g \in \mathcal{B}(\omega)$, and $g|_{\omega_\alpha} = 0$ for all $\alpha \in A$, where a family of open intervals $\omega_\alpha, \alpha \in A$, covers ω, then $g = 0$ on the whole interval ω. In particular, the *support* supp$g$ of a hyperfunction g is defined in the usual way. Moreover, by the same theorem, a hyperfunction can be defined *locally*. Namely, let again $\omega_\alpha, \alpha \in A$, be an open covering of ω, and let $g_\alpha \in \mathcal{B}(\omega_\alpha), \alpha \in A$, be a family of hyperfunctions with the property: $g_\alpha = g_\beta$ on $\omega_\alpha \cap \omega_\beta$ for all $\alpha, \beta \in A$. Then there exists an unique hyperfunction $g \in \mathcal{B}(\omega)$ such that $g|_{\omega_\alpha} = g_\alpha$ for all $\alpha \in A$. This means that hyperfunctions on $\mathbb{R}$ form a *sheaf*, denoted by $\mathcal{B}$. Notice that many classes of generalized functions (for example Schwartz distributions) also form sheaves. It is specific for hyperfunctions that the sheaf $\mathcal{B}$ is *flabby.* In more detail,

$$\textit{the restriction mapping} \quad \mathcal{B}(\omega) \to \mathcal{B}(\theta) \quad \textit{is surjective}$$

for any open sets $\theta \subset \omega \subset \mathbb{R}$. In particular, every hyperfunction $g \in \mathcal{B}(\omega)$ can be extended to some hyperfunction $h \in \mathcal{B}(\mathbb{R})$, $h|_\omega = g$. Moreover, there exists an analytic function $f \in \mathcal{H}(\Omega)$, $\Omega = \mathbb{C} \setminus \overline{\omega}$, such that $g = [f|_{\Omega^+}, -f|_{\Omega^-}]$, where $\overline{\omega}$ is the closure of ω.

7. Hyperfunctions with compact supports.

Let a hyperfunction $g \in \mathcal{B}(\omega)$ have a compact support $K \subset \omega$. Then there exists an unique analytic function $f \in \mathcal{H}(\Omega)$, $\Omega = \mathbb{C} \setminus K$, such that

$$f(z) \to 0, \text{ when } |z| \to \infty, \text{ and } g = [f|_{\Omega^+}, -f|_{\Omega^-}].$$

Consider the linear space $\mathcal{A}(K)$ of all real analytic functions on K with its natural locally convex topology, and let $\mathcal{A}'(K)$ be the dual space of real analytic functionals with their supports in K. The above representation defines an element $\hat{g} \in \mathcal{A}'(K)$ by the formula

$$\mathcal{A}(K) \ni \phi \mapsto \hat{g}(\phi) = \oint_\gamma f(z)\varphi(z)dz,$$

where $\varphi(z) \in \mathcal{H}(\Omega)$ is an analytic extension of $\phi(x)$ into some complex neighborhood Ω of K, and a closed curve γ in Ω encircles K once (see Figure 4; for simplicity we assume K to be a closed interval).

FIGURE 4.

This construction establishes an isomorphism of the linear spaces

$$\mathcal{B}_K(\omega) \xrightarrow{\sim} \mathcal{A}'(K),$$

where $\mathcal{B}_K(\omega)$ is a linear space of all hyperfunctions with their supports in K (notice that $\mathcal{B}_K(\omega)$ does not depend on an open set $\omega \supset K$).

HYPERFUNCTIONS OF SEVERAL VARIABLES

In essence, hyperfunctions are a branch of complex analysis. No wonder that the many–dimensional case is much more complicated, like it is for analytic functions. To understand it, one should be intimately acquainted with algebraic topology and cohomology theory, as well as with many–dimensional complex analysis itself. I will try to give the definitions and elementary properties of hyperfunctions using only elementary methods.

In one dimension every complex neighborhood Ω of an interval ω splits naturally in two parts Ω^+ and Ω^-. Their counterparts in several dimensions are "wedges" and more general complex domains "tuboids" (J. BROS, D. IAGOLNITZER [4]).

To describe them we need some preliminaries. Remind that a set $C \subset \mathbb{R}^n$ is called a *cone* if $\lambda x \in C$ for all $\lambda > 0$ and $x \in C$. We say that a cone C_1 is *compact* in a cone C_2, write $C_1 \subset C_2$, if $\overline{C_1} \subset \{0\} \cup \mathrm{int} C_2$, where $\overline{M}$ is the *closure* and $\mathrm{int} M$ is the *interior* of a set M.

Let $\pi : \mathbb{C}^n \to \mathbb{R}^n$ be the natural projection

$$\mathbb{C}^n \ni z = x + iy \mapsto \pi(z) = x \in \mathbb{R}^n$$

of the complex space $\mathbb{C}^n$ onto the real space $\mathbb{R}^n$. Thus, for every complex set M its real projection πM is defined, and for every $x \in \pi M$ the fiber $M_x = \{y \in \mathbb{R}^n : x + iy \in M\}$ over x is defined. We shall speak of *fiberwise* properties, such as fiberwise convexity, fiberwise compactness, etc., understanding by this that the corresponding property is satisfied for each fiber.

An open complex set $V \subset \mathbb{C}^n$ is called a *profile* over an open real set $\omega \subset \mathbb{R}^n$ if $\pi V = \omega$, and every fiber V_x, $x \in \omega$, is a cone. A simple example of a profile is a *wedge profile*

$$V = \omega + iC, \qquad \omega \text{ is a domain in } \mathbb{R}^n, \quad C \text{ is an open cone in } \mathbb{R}^n.$$

In one dimension there are exactly three profiles

$$\omega + i\mathbb{R}, \quad \omega + i\mathbb{R}_+, \quad \omega + i\mathbb{R}_-$$

over an open interval ω, where $\mathbb{R}_\pm = \{x \gtrless 0\}$.

An open complex set $T \subset \mathbb{C}^n$ is called a *tuboid* with a profile V over an open real set $\omega \subset \mathbb{R}^n$ if $T \subset V$, $\pi T = \pi V = \omega$, and for every point $a \in \omega$ and every open cone $C \subset V_a$ there exists a number $R > 0$, such that the *wedge*

$$\begin{aligned} W &= \{|x-a| < R\} + i\{y \in C : |y| < R\} \\ &= B_R(a) + iC_R \subset T. \end{aligned}$$

A simple example of a tuboid is an "orange section"

$$T = \{z = x + iy \in \mathbb{C}^n : |z - a| < R, y \in C\},$$

where $a \in \mathbb{R}^n$, C is an open cone, and $R > 0$. In general, a tuboid is something like a twisted orange section.

Now, let T be a tuboid with a fiber connected profile V over ω. Suppose an analytic function $f(z) \in \mathcal{H}(T)$ has *locally slow growth* when $y \to 0$ in T, i.e., for every $a \in \omega$ and every cone $C \subset V_a$ there exist numbers $R, M, p > 0$, such that the wedge $W = B_R(a) + iC_R \subset T$, and

$$|f(x+iy)| \le My^{-p} \quad \text{for all} \quad z = x + iy \in W.$$

Then the function f has the *distribution boundary value* $\text{bv}f \in \mathcal{D}'(\omega)$, locally defined by the limits

$$\text{bv}f(\phi) = \lim_{y \to 0 \text{in} C} \int_{B_R(a)} f(x+iy)\phi(x)dx,$$

for all $\phi \in \mathcal{D}(B_R(a))$. Now, it was proved by A. MARTINEAU [5] that every distribution $g \in \mathcal{D}'(\omega)$ has an analytic representation

$$g = \sum_{\alpha=1}^{p} \text{bv} f_\alpha, \qquad f_\alpha \in \mathcal{H}(T_\alpha),$$

where T_α are some tuboids over ω, and analytic functions $f_\alpha(z)$ have locally slow growth when $y \to 0$ in T_α, $\alpha = 1, \ldots, p$ (notice that p depends on g). Further, according to Martineau's version of the "edge of the wedge" theorem

$$\sum_{\alpha=1}^{p} \text{bv} f_\alpha = 0 \quad \text{in} \quad \mathcal{D}'(\omega),$$

where analytic functions $f_\alpha \in \mathcal{H}(T_\alpha)$ have locally slow growth when $y \to 0$ in tuboids T_α with profiles V_α, iff there exist analytic functions $f_{\alpha\beta} = -f_{\beta\alpha} \in \mathcal{H}(T_{\alpha\beta})$ of locally slow growth when $y \to 0$ in tuboids $T_{\alpha\beta}$ with profiles

$$V_{\alpha\beta} = \mathrm{ch}_\omega(V_\alpha \cup V_\beta), \quad \alpha, \beta = 1, \ldots, p,$$

such that

$$f_\alpha = \sum_{\beta=1}^{p} f_{\alpha\beta} \quad \text{in} \quad T_\alpha \cap T_{\alpha 1} \ldots \cap T_{\alpha p} \quad \text{for} \quad \alpha = 1, \ldots, p.$$

Here $\mathrm{ch}_\omega(V_\alpha \cup V_\beta)$ denotes fiberwise convex hull of profiles V_α and V_β over ω (see Figure 5). Similar results are valid in many classes of generalized functions.

FIGURE 5.

Again, to define hyperfunctions over an open set $\omega \subset \mathbb{R}^n$ one should give up growth conditions. Namely, in the linear space of all finite unordered families

$$\{f_1, \ldots, f_p\}, \qquad f_\alpha \in \mathcal{H}(T_\alpha),$$

where T_α are tuboids with profiles V_α over ω, $\alpha = 1, \ldots, p$, the natural number p depends on the family, let us introduce an equivalence relation by the rule

$$\{f_1, \ldots, f_p\} \sim 0$$

iff there exist analytic functions $f_{\alpha\beta} = -f_{\beta\alpha} \in \mathcal{H}(T_{\alpha\beta})$, $\alpha, \beta = 1, \ldots, p$, such that

$$f_\alpha = \sum_{\beta=1}^{p} f_{\alpha\beta} \quad \text{in} \quad T_\alpha \cap T_{\alpha 1} \ldots \cap T_{\alpha p} \quad \text{for} \quad \alpha = 1, \ldots, p,$$

where tuboids $T_{\alpha\beta}$ have profiles $V_{\alpha\beta} = \mathrm{ch}_\omega(V_\alpha \cup V_\beta)$. By definition, a hyperfunction $g = [f_1, \ldots, f_p]$ over an open set $\omega \subset \mathbb{R}^n$ is the equivalence class of a family $\{f_1, \ldots, f_p\}$. The linear space of all hyperfunctions over ω is denoted by $\mathcal{B}(\omega)$. Notice that for a hyperfunction $g = [f_1, \ldots, f_p] \in \mathcal{B}(\omega)$ analytic functions $f_\alpha \in \mathcal{H}(T_\alpha)$, tuboids T_α and even the number p are far from unique. In particular, every hyperfunction $g \in \mathcal{B}(\omega)$ has a representation

$$g = [f_\varepsilon \ ; \ \varepsilon = (\varepsilon_1, \ldots, \varepsilon_n) \in \{+1, -1\}^n], \qquad (p = 2^n)$$

where analytic functions $f_\varepsilon \in \mathcal{H}(\Omega^\varepsilon)$, tuboids $\Omega^\varepsilon = \Omega \cap \mathbb{C}^n_\varepsilon$, $\Omega \subset \mathbb{C}^n$ is a complex neighborhood of ω,

$$\mathbb{C}^n_\varepsilon = \{z = x + iy \in \mathbb{C}^n : y \in \mathbb{R}^n_\varepsilon\},$$

n-tants

$$\mathbb{R}^n_\varepsilon = \{y = (y^1, \ldots, y^n) \in \mathbb{R}^n : \varepsilon_1 y^1 > 0, \ldots, \varepsilon_n y^n > 0\}.$$

In his original definition Sato used precisely this representation, and gave it a cohomological interpretation

$$\mathcal{B}(\omega) = H^n_\omega(\Omega, \mathcal{H}),$$

where $H^n_\omega(\Omega, \mathcal{H})$ is the linear space of n-th relative cohomologies of an open set $\omega \subset \mathbb{R}^n$ with coefficients in the sheaf $\mathcal{H}$ of germs of analytic functions in $\mathbb{C}^n$. By the excision theorem, the space $H^n_\omega(\Omega, \mathcal{H})$ does not depend on an open complex neighborhood Ω of ω. This definition allowed Sato to establish the main properties of hyperfunctions, using powerful methods of algebraic topology and theory of sheaves. Moreover, Sato's definition is coordinate free, so it is valid for any real analytic manifold. Also, taking different locally free analytic sheaves, instead of the sheaf $\mathcal{H}$, one gets hyperfunctions of different types. The definition I presented here is close to the microlocal point of view, developed in seventies by M. Sato, T. Kawai, M. Kashiwara, J. Bros, D. Iagolnitzer, J. M. Bony, P. Schapira, F. Treves and others (see, for example, [6], [7]).

We see that the definition of hyperfunctions (especially the original definition of Sato) is rather abstract, and to apply hyperfunctions to concrete problems one should know the necessary algebraic tools. Thus, the application of hyperfunctions to linear partial differential equations with real analytic coefficients produced remarkable results, but many of these results are known only to a narrow circle of specialists in microlocal analysis. On the other side, the absence of a natural physical interpretation of hyperfunctions and the unusual mathematical apparatus, essentially restricted an application of hyperfunctions to mathematical physics, the traditional source of new mathematical ideas. No wonder that from the start there were made attempts to give a functional description of hyperfunctions.

Analytic functionals

The first such description was presented by A. Martineau [8], who proved that hyperfunctions are locally represented by real analytic functionals.

Let us look at Martineau's construction in some detail. First, for any compact $K \in \mathbb{R}^n$ there is an isomorphism of the linear spaces

$$\mathcal{B}_K(\omega) \xrightarrow{\sim} \mathcal{A}'(K),$$

where $\mathcal{B}_K(\omega)$ is a linear space of all hyperfunctions with their supports in K (cf. the one dimensional case above, the explicit formula, realizing the isomorphism, is no longer valid, of course), $\mathcal{A}'(K)$ is the linear space of real analytic functionals with their supports in K. Further, when an open set ω is bounded, every hyperfunction $g \in \mathcal{B}(\omega)$ has an extension $G \in \mathcal{B}(\mathbb{R}^n)$, $G|_\omega = g$, with a compact support supp$G \subset \overline{\omega}$, hence the isomorphism

$$\mathcal{B}(\omega) \overset{\sim}{\longrightarrow} \mathcal{A}'(\overline{\omega})/\mathcal{A}'(\partial\omega)$$

is defined, where $\overline{\omega}$ is the closure and $\partial\omega$ is the boundary of ω. At last, every hyperfunction g over an (unbounded or bounded) open set ω is completely defined by its restrictions $g_\alpha = g|_{\omega_\alpha} \in \mathcal{B}(\omega_\alpha)$, $\alpha \in A$, where ω_α are some bounded open sets covering ω (i.e., $\omega = \cup_{\alpha \in A}\omega_\alpha$), A is a finite or infinite set of indices. Thus, for any open set ω we get the local correspondence

$$\mathcal{B}(\omega) \ni g \mapsto h = \quad g|_\theta \longleftrightarrow [\hat{h}] \in \mathcal{A}'(\overline{\theta})/\mathcal{A}'(\partial\theta), \quad \text{for any bounded open subset } \theta \text{ of } \omega.$$

These considerations, as well as the studies of new and new classes of ultradistributions, including quasianalytic ones, drew interest to *analytic functionals*, i.e., continuous functionals over linear spaces of analytic functions, equipped with natural topologies. The main problem here is the absence of a concept of support of an analytic functional, because there are no analytic functions with compact supports. The only reasonable substitute is the *carrier*, defined in terms of seminorms, assigning the topology of the test space. For example, the locally convex topology of the space $\mathcal{H}(\Omega)$ of all analytic functions in an open complex set $\Omega \subset \mathbb{C}^n$ is given by seminorms

$$|\varphi|_K = \sup_{z \in K} |\varphi(z)|, \qquad \varphi \in \mathcal{H}(\Omega),$$

where K runs through all compact sets in Ω. Hence, a compact set K is called a carrier of an analytic functional $h \in \mathcal{H}'(\Omega)$, I shall write $K = \text{carr}h$, if for every complex neighborhood Θ of K, such that the closure $\overline{\Theta}$ is compact in Ω, there exists a positive number M, such that

$$|h(\varphi)| \le M|\varphi|_{\overline{\Theta}}, \quad \text{for all } \varphi \in \mathcal{H}(\Omega).$$

The major defect of the carrier is that

$$K_1 = \text{carr}h \text{ and } K_2 = \text{carr}h \quad \text{generally doesn't imply} \quad K_1 \cap K_2 = \text{carr}h,$$

so the least carrier is not obliged to exist. Notice that the least carrier exists, and is called the support, in the real analytic case, i.e., for every real analytic functional $h \in \mathcal{A}'(\mathbb{R}^n)$, where

$$\mathcal{A}(\omega) = \lim_{\Omega \supset \omega} \text{ind } \mathcal{H}(\Omega),$$

where Ω runs all complex neighborhoods of a real open set $\omega \subseteq \mathbb{R}^n$. This property was essentially used in Martineau's definition of hyperfunctions.

A special interest to the concept of support is connected with the fact that it plays an important role in mathematical physics, describing local properties of physical objects. In particular, such concept is necessary to formulate the causality principle in the configuration space and the spectral property in the momentum space. Further, the Fourier transform relates properties of a physical object in the configuration space and its spectral function in the momentum space. A splendid example of results in this field is the Jost-Lehmann-Dyson representation, providing an explicit description of functions satisfying certain support conditions both in configuration and momentum spaces (causality and spectrum).

These considerations stimulated the study of different classes of analytic functionals and their Fourier (more precisely, Laplace) transforms. The starting example is the space $\mathcal{H}'(\Omega)$, where $\Omega \subset \mathbb{C}^n$ is a complex domain. Here, the test function space $\mathcal{H}(\Omega)$ contains all linear exponents $\mathrm{e}^{i\zeta}$, where $\mathrm{e}^{i\zeta}(z) = \mathrm{e}^{i\zeta z}$, with parameter $\zeta \in \mathbb{C}^n$, independent variable $z \in \Omega$, and $\zeta z = \zeta^1 z^1 + \ldots + \zeta^n z^n$. The *Laplace transform* $L[g]$ of an analytic functional $g \in \mathcal{H}'(\Omega)$ is defined by the formula

$$L[g](\zeta) = g(\mathrm{e}^{i\zeta}), \qquad \zeta \in \mathbb{C}^n.$$

One can easily check that $L[g]$ is an entire function (i.e., analytic in $\mathbb{C}^n$) of exponential growth. Namely, if $\mathrm{carr} g = K$ then for every $\epsilon > 0$ there exist a positive M such that

$$|L[g](\zeta)| \leq M \mathrm{e}^{\mathrm{s}_K(\zeta) + \epsilon|\zeta|}, \quad \text{for all } \zeta \in \mathbb{C}^n,$$

where

$$\mathrm{s}_K(\zeta) = \sup_{z \in K} \Re(i\zeta z), \qquad \zeta \in \mathbb{C}^n,$$

is the *support function* of the compact $K \subset \Omega$, $\Re(i\zeta z)$ is the real part of $i\zeta z$. Thus, we have the linear mapping

$$L : \mathcal{H}'(\Omega) \longrightarrow \mathrm{Exp}(\mathbb{C}^n),$$

where $\mathrm{Exp}(\mathbb{C}^n)$ is the linear space of all entire functions of exponential growth. If Ω is a Runge domain then linear combinations of exponents $\mathrm{e}^{i\zeta}$ are dense in $\mathcal{H}(\Omega)$, and the mapping L is injective. Moreover, if the domain Ω is convex then the image $L[\mathcal{H}'(\Omega)]$ has a simple description [9]

$$L[\mathcal{H}'(\Omega)] = \Big\{ \quad f(\zeta) \in \mathcal{H}(\mathbb{C}^n) : |f(\zeta)| \leq M \mathrm{e}^{\mathrm{s}_K(\zeta)} \text{ for all } \zeta \in \mathbb{C}^n; \\ \text{for some compact } K = K(f) \subset \Omega \text{ and } M = M(f) > 0 \Big\}.$$

To get new classes of analytic functionals it is enough to replace seminorms $|\cdot|_K$ with weight seminorms

$$|\varphi|_\rho = \sup_{z\in\Omega}\{e^{-\rho(z)}|\varphi(z)|\}, \qquad \varphi \in \mathcal{H}(\Omega),$$

where $\rho(z)$, $z \in \Omega$, is some real weight function (in the original case of the seminorms $|\cdot|_K$ the function $\rho(z)$ is the *indicator function* of a compact K, i.e., $\rho(z) = 0$ if $z \in K$, and $\rho(z) = +\infty$ if $z \in \Omega \setminus K$). In particular,

$$|e^{i\zeta}|_\rho = \sup_{z\in\Omega}\{e^{-\rho(z)+\Re(i\zeta z)}\} = e^{\rho^*(\zeta)},$$

where

$$\rho^*(\zeta) = \sup_{z\in\Omega}\{\Re(i\zeta z) - \rho(z)\}, \qquad \zeta \in \mathbb{C}^n,$$

is the *Legendre transform* of the weight function ρ. Now, if an analytic functional $g \in \mathcal{H}'(\Omega)$ is bounded by a seminorm $|\cdot|_\rho$ then its Laplace transform

$$L[g](\zeta) = g(e^{i\zeta}), \qquad \zeta \in \mathrm{dom}\rho^*,$$

is defined, where

$$\mathrm{dom}\rho^* = \{\zeta \in \mathbb{C}^n : \rho^*(\zeta) < \infty\}.$$

If the domain Ω and the weight function ρ are good enough (for example, Ω and $\mathrm{dom}\rho^*$ are convex complex domains), then the Laplace transform has nice properties, like in the original case with seminorms $|\cdot|_K$.

Notice that the Fourier transform of a functional as the dual operation, defined above, usually appears as a *limit case* of the Laplace transform. Many researchers don't separate these operations and use the joint term *Fourier-Laplace transform* or simply *Fourier transform*.

A profound study of different classes of analytic functionals with real and complex carriers and their Fourier-Laplace transforms was done by Jowi de Roever in his thesis [10].

There is a special class of analytic functionals closely connected with the Fourier-Laplace transform, when test function spaces consist of functions analytic in tube domains

$$\Omega = T^B = \mathbb{R}^n + iB = \{z = x + iy : y \in B\}, \qquad B \subset \mathbb{R}^n \text{ is a base },$$

with controlled growth in *real* directions, i.e., weight functions $\rho = \rho(x)$ do not depend on y.

A simple but important space of the last type is the space

$$\Phi(\mathbb{R}^n) = \Big\{\varphi \in \mathcal{H}(T^B) : \sup_{z\in T^B}\{e^{\epsilon|z|}|\varphi(z)|\} < \infty, \quad B = \{|y| < \epsilon\}, \\ \text{for some } \epsilon = \epsilon(\varphi) > 0\Big\}.$$

This space has a natural topology of the inductive limit of Banach spaces. For every function $\varphi \in \Phi(\mathbb{R}^n)$ the Fourier transform

$$F[\varphi](\zeta) = \int e^{i\zeta x}\varphi(x)dx$$

is defined. It is easy to check that $F[\varphi] \in \Phi(\mathbb{R}^n)$. Moreover, there is the isomorphism

$$F : \Phi(\mathbb{R}^n) \xrightarrow{\sim} \Phi(\mathbb{R}^n),$$

and the dual isomorphism of the dual spaces. Elements of the dual space $\Phi(\mathbb{R}^n)'$ are called *Fourier hyperfunctions*, and below I shall explain why.

FOURIER HYPERFUNCTIONS

First, the one dimensional case. Let $g \in \mathcal{B}(\omega)$ be a hyperfunction over an open interval ω. Then there exists an analytic function $f \in \mathcal{H}(\mathbb{C} \setminus \overline{\omega})$ representing g, i.e., $g = [f^+, f^-]$, where $\overline{\omega}$ is the closure of ω,

$$f^{\pm} = \pm f|_{\mathbb{C}^{\pm}}, \qquad \mathbb{C}^{\pm} = \{z = x + iy : y \gtrless 0\}.$$

Moreover, such representing function f is not unique, and one can choose a representing function *bounded or even decreasing fast in real directions*! In particular, there exists a representing function f of g of an *infra-exponential growth* in real directions, i.e., satisfying the estimate: for every $\epsilon > 0$ and $0 < \alpha < \beta < \infty$ there exists $M = M(\epsilon, \alpha, \beta) > 0$, such that

$$|f(x + iy)| \leq M e^{\epsilon|x|} \quad \text{for all } -\infty < x < +\infty \text{ and } \alpha < |y| < \beta.$$

Let f be a representing function of g of an infra-exponential growth, define an analytic functional $\hat{g} \in \Phi'(\mathbb{R})$ by the formula

$$\Phi(\mathbb{R}) \ni \varphi \mapsto \hat{g}(\varphi) = \int f(x + iy_0)\varphi(x + iy_0)dx - \int f(x - iy_0)\varphi(x - iy_0)dx,$$

where $y_0 = y_0(\varphi) > 0$ is small enough. The described procedure is called an *extension* of a hyperfunction g to a functional $\hat{g}$. In general, a bounded representing function f of g is not unique, so the extension $\hat{g}$ is not unique, also. (Notice that if $\overline{\omega} = K$ is a compact then

$$\hat{g}(\varphi) = \oint_{\gamma} f(z)\varphi(z)dz,$$

where a closed curve γ is shown in Figure 4.)

On the other side, for every analytic functional $h \in \Phi'(\mathbb{R})$ there exist functions $f^{\pm}(z)$ analytic in the half-planes $\mathbb{C}^{\pm} = \{z = x + iy : y \gtrless 0\}$ and of an infra-exponential growth in real directions, such that

$$h(\varphi) = \int f^+(x + iy_0)\varphi(x + iy_0)dx + \int f^-(x - iy_0)\varphi(x - iy_0)dx, \quad \varphi \in \Phi(\mathbb{R}),$$

where $y_0 > 0$ is small enough, again. Clear, the pair $\{f^+, f^-\}$ defines a hyperfunction $h|_{\mathbb{R}} = [f^+, f^-] \in \mathcal{B}(\mathbb{R})$. This hyperfunction is called the *restriction* of a functional h.

Thus, elements of the space $\Phi'(\mathbb{R})$ are, in fact, extensions of hyperfunctions. Taking into account that it is invariant under the Fourier transform, one can easily understand why elements of $\Phi'(\mathbb{R})$ are called *Fourier hyperfunctions*.

The theory of Fourier hyperfunctions was developed by T. KAWAI [11] using the above mentioned Sato's general cohomological approach of SATO [3].

Namely, T. Kawai considered the compactification $\mathbb{D}^n = \mathbb{R}^n \cup S_\infty^{n-1}$ of $\mathbb{R}^n$ by means of the "sphere of infinite radius" S_∞^{n-1} and for every open subset Ω of the complex space $\mathbb{D}^n + i\mathbb{R}^n$ he introduced the linear space $\widetilde{\mathcal{H}}(\Omega)$ of all functions $f(z)$ analytic in the restriction $\Omega' = \Omega \cap \mathbb{C}^n$, $\mathbb{C}^n = \mathbb{R}^n + i\mathbb{R}^n$, and having infra-exponential growth, i.e., satisfying the estimate: for every $\epsilon > 0$ there exists $M = M(\epsilon) > 0$, such that

$$|f(z)| \leq M e^{\epsilon |z|} \qquad \text{for all } z = x + iy \in \Omega'.$$

The collection of all spaces $\widetilde{\mathcal{H}}(\Omega)$ forms a presheaf over $\mathbb{D}^n + i\mathbb{R}^n$. The associated sheaf is denoted by $\widetilde{\mathcal{H}}$. Notice that the restriction $\widetilde{\mathcal{H}}|_{\mathbb{C}^n}$ coincides with the sheaf $\mathcal{H}$ of germs of analytic functions in $\mathbb{C}^n$.

Then, following the Sato's general approach, for every open subset ω of $\mathbb{D}^n$ Kawai defined the linear space

$$\mathcal{R}(\omega) = H_\omega^n(\Omega, \widetilde{\mathcal{H}}),$$

of n-th relative cohomologies of ω with coefficients in the sheaf $\widetilde{\mathcal{H}}$. By the excision theorem, the space $H_\omega^n(\Omega, \widetilde{\mathcal{H}})$ does not depend on an open complex neighborhood $\Omega \subset \mathbb{D}^n + i\mathbb{R}^n$ of ω. Elements of the spaces $\mathcal{R}(\omega)$ are called Fourier hyperfunctions.

Kawai proved the following statements.

(a) The collection of all spaces $\mathcal{R}(\omega)$ forms a flabby sheaf $\mathcal{R}$ over $\mathbb{D}^n$. In particular, every Fourier hyperfunction g over some open $\omega \subset \mathbb{D}^n$ has a standard representation

$$g = [f_\varepsilon \ ; \ \varepsilon = (\varepsilon_1, \ldots, \varepsilon_n) \in \{+1, -1\}^n],$$

where representing functions $f_\varepsilon \in \mathcal{H}(\mathbb{C}_\varepsilon^n)$ have infra-exponential growth in real directions, i.e., satisfy the estimate: for every $\epsilon > 0$ and $0 < r < R < \infty$ there exists $M = M(\epsilon, r, R)$ such that

$$|f_\varepsilon| \leq M e^{\epsilon |z|} \qquad \text{for all } z = x + iy \in \mathbb{C}_\varepsilon^n \cap \{r < |y| < R\}.$$

Remind that $\mathbb{C}_\varepsilon^n = \mathbb{R}^n + i\mathbb{R}_\varepsilon^n$, with n-tants

$$\mathbb{R}_\varepsilon^n = \{y = (y^1, \ldots, y^n) \in \mathbb{R}^n : \varepsilon_1 y^1 > 0, \ldots, \varepsilon_n y^n > 0\}.$$

(b) The restriction $\mathcal{R}|_{\mathbb{R}^n}$ of the sheaf $\mathcal{R}$ to $\mathbb{R}^n$ coincides with the sheaf $\mathcal{B}$ of hyperfunctions. In particular, every hyperfunction $g \in \mathcal{B}(\omega)$, ω is an open subset of $\mathbb{R}^n$, can be extended as a Fourier hyperfunction to $\mathbb{D}^n$, i.e., there exists a Fourier hyperfunction $h \in \mathcal{R}(\mathbb{D}^n)$, such that the restriction $h|_\omega = g$.

(c) There is the isomorphism

$$\mathcal{R}(\mathbb{D}^n) \xrightarrow{\sim} \Phi'(\mathbb{R}^n),$$

given by the following procedure. For a Fourier hyperfunction $g \in \mathcal{R}(\mathbb{D}^n)$ with a standard infra-exponential representation $g = [f_\varepsilon]$ the analytic functional $\hat{g} \in \Phi'(\mathbb{R}^n)$ is defined by the formula

$$\hat{g}(\varphi) = \sum_{\varepsilon} \int f_\varepsilon(x + iy_\varepsilon)\varphi(x + iy_\varepsilon)dx, \qquad \varphi \in \Phi(\mathbb{R}^n),$$

where $y_\varepsilon \in R^n_\varepsilon$, $|y_\varepsilon|$ is small enough, the sum is taken over all $\varepsilon \in \{+1, -1\}^n$. Notice that, in fact, Kawai has proved a more refined version of this statement.

Thus, every hyperfunction (with noncompact support, also) can be realized (nonuniquely) as an analytic functional on $\Phi(\mathbb{R}^n)$. This provides a possibility to use tools of functional analysis (including Fourier-Laplace transform) while solving hyperfunction problems. In his original paper [11] Kawai applied his theory to the study of hyperfunction solutions of linear partial differential equations with constant coefficients. There are other applications of Fourier hyperfunctions to differential equations. In mathematical physics Fourier hyperfunctions were also used as a basis for the formulation of a hyperfunction version of axiomatic quantum field theory. But, at that time interests and main trends of quantum physics had changed, and this work had a small impact.

In the seventies, hyperfunction theory adopted the microlocal point of view, mentioned above. An important concept here is the *singular spectrum*, the direct counterpart of the *analytic wave front* introduced by L. Hörmander in distribution theory. This concept relates possible analytic representations of a hyperfunction (or a distribution) and directions in which its Fourier transform is decreasing. A clear introductory exposition of microlocal theory without any algebra can be found in L. HÖRMANDER [12], but I finish my talk at this point.

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