

Multicommodity Flows and Polyhedra

A.M.H. Gerards
CWI
P.O. Box 94079
1090 GB Amsterdam
The Netherlands
email: bgerards@cwi.nl

Seymour's conjecture on binary clutters with the so-called weak (or \mathbb{Q}_+ -) max-flow min-cut property implies — if true — a wide variety of results in combinatorial optimization about objects ranging from matchings to (multicommodity) flows and disjoint paths. In this paper we review in particular the relation between classes of multicommodity flow problems for which the so-called cut-condition is sufficient and classes of polyhedra for which Seymour's conjecture is true.

1. INTRODUCTION

Polyhedral Combinatorics studies combinatorial problems using the theory of linear inequalities. One of its open questions is: for which 0,1-matrices $A \in \mathbb{R}^{m \times n}$ is

$$P(A) := \{x \in \mathbb{R}^n \mid Ax \geq \mathbf{1}; x \geq \mathbf{0}\}$$

an *integral* polyhedron, which means that it has integral extreme points only. For some subclass of 0,1-matrices this comes down to a question about disjoint paths and multicommodity flows. Consequently, many partial answers to the polyhedral question come with one or more multicommodity flow theorems. Over the years the research on this issue in combinatorial optimization has been driven by the relevance of disjoint paths for applications varying from transportation problems to VLSI layout on one side and the study of the integrality of polyhedra on the other side. In this paper we will review old and new pairs of results for polyhedra and for multicommodity flows. The research to the more recent of these results were particularly motivated by a conjecture by Paul Seymour, which — if true — contains a wide variety of polyhedral results on combinatorial objects, ranging from matchings to disjoint paths.

2. ODD CIRCUITS AND POLYHEDRA

A *signed graph* is a pair (G, Σ) consisting of an undirected graph G and a collection Σ of its edges. The edge set of G will be denoted by $E(G)$; the

node set by $V(G)$. A collection F of edges in G is called *odd in* (G, Σ) if $|F \cap \Sigma|$ is odd, otherwise F is called *even*. So we speak of odd and even edges, circuits, cycles, etc. A *circuit* is a connected (sub)graph with all degrees two. A *cycle* is a (sub)graph with all degrees even. The collection of cycles in G is denoted by \mathcal{G} . For the collection of odd circuits in (G, Σ) we write $\Omega(G, \Sigma)$. As we are mainly concerned with edge sets we will identify a subgraph of G with the edge set of that subgraph. So “ C is a circuit in G ” may also mean: “ C is the edge set of a circuit in G ”. A *cut* is a set of edges of the form $\delta(U) := \{uv \in E(G) | u \in U, v \notin U\}$ for some $U \subseteq V(G)$. If S is a finite set (typically $E(G)$ or $\Omega(G, \Sigma)$), $y \in \mathbb{R}^S$ and $T \subseteq S$, we write $y(T) := \sum_{t \in T} y_t$.

We call a signed graph (G, Σ) *weakly bipartite* if

- (1) $P(G, \Sigma) := \{x \in \mathbb{R}_+^{E(G)} | x(C) \geq \mathbf{1} \ (C \in \Omega(G, \Sigma))\}$ is an integral polyhedron.

Clearly, weak bipartiteness only depends on the collection of odd circuits in (G, Σ) . So it is invariant under *re-signing*, that is replacing Σ by the symmetric difference $\Sigma \Delta \delta(U)$ of Σ with some cut $\delta(U)$. We call (G, Σ) and (H, Θ) *isomorphic* if they are related through re-signing and graph-isomorphism. We call (G, Σ) *bipartite* if $\Sigma = \delta(U)$ for some $U \subseteq V(G)$. Clearly, (G, Σ) is bipartite if there are no odd circuits, or equivalently if (G, Σ) is isomorphic to (G, \emptyset) . This concurs with the terminology for ordinary graphs when we consider those as signed graphs with all edges odd. We will denote $(G, E(G))$ by \tilde{G} .

It should be noted that specializing any of the results in this paper to ordinary graphs (with all edges odd) does not really yield weaker statements. Still, we consider signed graphs because they enable a more natural presentation of results and arguments than when we confine ourselves to ordinary graphs. One reason for that is that the class of polyhedra $P(G, \Sigma)$ is closed under intersection with coordinate hyperplanes, whereas the class of polyhedra $P(\tilde{G})$ is not.

Minors — intersection with and projection on supporting hyperplanes

Obviously, the orthogonal projection of an integral polyhedron in $\mathbb{R}^{E(G)}$ on the hyperplane defined by $x_e = 0$ for any $e \in E(G)$ is integral too. In case of $P(G, \Sigma)$ this projection is $P((G, \Sigma) \setminus e)$. Here $(G, \Sigma) \setminus e := (G \setminus e, \Sigma \setminus \{e\})$, where $G \setminus e$ is obtained from G by deleting edge e . Similarly, when $P(G, \Sigma)$ is integral, then so is its intersection with the (supporting!) hyperplane $x_e = 0$. In (G, Σ) this corresponds to *contracting* e in (G, Σ) . This operation is defined as follows: first re-sign (G, Σ) , if necessary, such that e becomes even, next contract e in G (that is remove it and identify its end nodes). The resulting signed graph will be denoted by $(G, \Sigma)/e$. Clearly, the definition of $(G, \Sigma)/e$ is only defined up to re-signing. But, as this does not affect weak bipartiteness, this is specific enough for our purposes.

A *minor* of (G, Σ) is the result of a series of re-signings, deletions and contractions. We say that (G, Σ) has a (H, Θ) -*minor* if it has a minor isomorphic to (H, Θ) . So we have:

- (2) *The class of weakly bipartite signed graphs is closed under taking minors.*

Non-weakly bipartite signed graphs — Seymour’s conjecture

It is an easy exercise to show that the signed graph \widetilde{K}_5 consisting of the complete graph on 5 nodes with all edges odd is not weakly bipartite. Indeed, the vector $[\frac{1}{3}, \dots, \frac{1}{3}]$ is contained in $P(\widetilde{K}_5)$, but cannot be written as a convex combination of integral vectors in that polyhedron, as each such integral vector x satisfies $x(E(K_5)) \geq 4 > 10 \times \frac{1}{3}$. A conjecture by Seymour prophesies that \widetilde{K}_5 is the only minimal non-weakly bipartite signed graph.

- (3) CONJECTURE: (SEYMOUR [29, 31]) *(G, Σ) is weakly bipartite if and only if it has no \widetilde{K}_5 -minor.*

Intermezzo: Binary clutters

In fact, Seymour states his conjecture in a more general context. Let \mathcal{C} be a binary space on a finite ground set E , i.e. a linear subspace of the linear space $GF(2)^E$ over the two element field $GF(2)$. Think of \mathcal{C} as a collection of subsets of E closed under taking symmetric differences. Let Σ be a subset of E . Then we denote by (\mathcal{C}, Σ) the collection of elements $C \in \mathcal{C}$ with $|C \cap \Sigma|$ odd. So (\mathcal{C}, Σ) is an affine subspace of \mathcal{C} of co-dimension 1 (unless it is empty). The collection of inclusion-wise minimal elements of (\mathcal{C}, Σ) is denoted by $\Omega(\mathcal{C}, \Sigma)$. Any such set systems is called a *binary clutter*. We call (\mathcal{C}, Σ) , and also $\Omega(\mathcal{C}, \Sigma)$, *weakly bipartite* if $P(\mathcal{C}, \Sigma) := \{x \in \mathbb{R}_+^E \mid x(C) \geq 1 \ (C \in (\mathcal{C}, \Sigma))\}$ is an integral polyhedron. Conjecture (3) is a special case of:

- (4) CONJECTURE: (SEYMOUR [29, 31]) *$\Omega(\mathcal{C}, \Sigma)$ is weakly bipartite if and only if it has no minor isomorphic to one of the following three binary clutters: $\Omega(\widetilde{K}_5)$; the complements of cuts in K_5 ; the lines of the Fano plane.*

We leave it to the reader to find out how *minor* should be appropriately defined. (It should correspond to projection on and intersection with hyperplanes $x_e = 0$.) Conjecture (3) is (4) restricted to *graphic spaces*, i.e. when, for some graph G , $\mathcal{C} = \mathcal{G}$ (which is a binary space). Of the three configurations in (4) only the first one can occur when \mathcal{C} is graphic. Actually, none of the three configurations in (4) can occur when \mathcal{C} is *cographic*, that is the collection \mathcal{G}^* of cuts of a graph G . So, according to Seymour’s conjecture, (\mathcal{G}^*, Σ) should be weakly bipartite. This is in fact a theorem. It amounts to the polyhedral characterization of T -joins due to EDMONDS and JOHNSON [7] which is equivalent to Edmonds’ well-known polyhedral characterization of (perfect) matchings (EDMONDS [5]).

Seymour calls weakly bipartite binary clutters, *binary clutters with the \mathbb{Q}_+ -max-flow min-cut property*. The reason for his terminology will be obvious after reading this paper. Our terminology originates from GRÖTSCHEL and PULLEYBLANK [17]. They introduce the notion of weak bipartiteness for ordinary graphs, so for \widetilde{G} , in relation with the max-cut problem. The *max-cut problem* asks for a maximum weight cut in a graph. In general it is *NP*-hard

(KARP [20]), but for weakly bipartite graphs it is polynomially solvable. The reason is that, if \tilde{G} is weakly bipartite, minimizing a weight-function over $P(\tilde{G})$ — which is solving a linear programme — yields the complement of a maximum weight cut.

In the context of general 0,1-matrices, also the term *ideal* is often used, referring to the similarity with *perfect matrices* (= perfect graphs) where systems of the form $Ax \leq \mathbf{1}$ are considered. Ideal matrices are even less understood than weakly bipartite graphs. Not even a conjecture is available — at least not just in terms of forbidden minors. In fact, there are over a thousand “minor-minimal non-ideal matrices” known with just 10 columns (CORNUÉJOLS and NOVICK [2]). But there are hopeful results as well, in particular by LEHMAN [21] (cf. PADBERG [26], SEYMOUR [34]).

3. ODD CIRCUITS, PATHS AND FLOWS

We relate weak bipartiteness with multicommodity flows.

THE MULTICOMMODITY FLOW PROBLEM:

Given an undirected graph G , a collection $D \subseteq E(G)$ of *demand* edges $d = s_d t_d$, and a *capacity/demand-function* $c \in \mathbb{Z}_+^{E(G)}$. Do there exist paths P_{di} in $G \setminus D$, going from s_d to t_d , and flow values $f_{di} \geq 0$ (with $i = 1, \dots, n_d; d \in D$), such that:

$$\sum_{i=1}^{n_d} f_{di} = c_d \quad (d \in D) \quad \text{“For each } d \in D, \text{ there goes a total amount } c_d \text{ of flow in } G \setminus D \text{ from } s_d \text{ to } t_d, \dots \text{”}$$

$$\sum_{d \in D} \sum_{i=1, P_{di} \ni e}^{n_d} f_{di} \leq c_e \quad (e \in E(G) \setminus D) \quad \dots \text{ such that no edge } e \text{ carries more flow than its capacity } c_e \text{.”}$$

End nodes of demand edges are called *terminals*. Note that, against custom, we included the demands as edges in the graph. We denote an instance of the multicommodity flow problem by $\text{flow}(G, D; c)$. If additionally we require the flows f_{di} to take integral values, we write $\text{path}(G, D; c)$. The *disjoint paths problem* is the collection of instances $\text{path}(G, D; \mathbf{1})$, which ask for edge-disjoint paths P_d ($d \in D$) where P_d runs from s_d to t_d . When the answer to $\text{flow}(G, D; c)$ (resp. $\text{path}(G, D; c)$) is affirmative we say that *flows* (resp. *paths*) *exist for* $(G, D; c)$.

A natural necessary condition for the multicommodity problem to have a solution is the

$$(5) \quad \text{CUT-CONDITION: If } U \subseteq V(K), \text{ then } c(\delta(U) \cap D) \leq c(\delta(U) \setminus D).$$

The cut-condition is not always sufficient for flows to exist (see Figure 1). The relation between multicommodity flows and weakly bipartite graphs lies in the question of the sufficiency of the cut-condition.

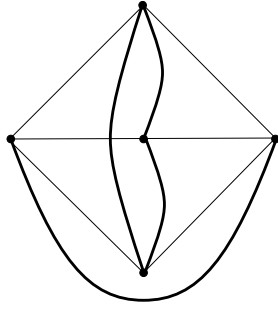


FIGURE 1. *Bold edges are in D ; $c := \mathbf{1}$.*

- (6) *If (G, D) is weakly bipartite and $c \in \mathbb{Z}_+^{E(G)}$ then the cut-condition is sufficient for the existence of flows for $(G, D; c)$.*

The converse is not true, but:

- (7) *(G, Σ) is weakly bipartite if and only if for each (G, D) isomorphic to (G, Σ) and for each $c \in \mathbb{Z}_+^{E(G)}$ the cut-condition is sufficient for the existence of flows for $(G, D; c)$.*

To see that (6) is true indeed, observe that flows exist if and only if:

$$(8) \quad c(D) = \max\{y(\Omega) \mid \sum_{C \in \Omega, C \ni e} y_C \leq c_e \ (e \in E(G)); \ y \in \mathbb{R}_+^\Omega\},$$

where $\Omega := \Omega(G, D)$. The reason is that each path from s_d to t_d closes with the edge $s_d t_d$ a circuit in G meeting D exactly once; so that is an odd circuit. Clearly, not each odd circuit yields a path, but if (8) holds, then each C for which y_C takes a positive value in an optimal solution y of (8) must meet D exactly once, hence corresponds to a path in $G \setminus D$.

If (G, D) is weakly-bipartite then, by *LP*-duality, (8) is equivalent to:

$$(9) \quad c(D) = \min\{c^\top x \mid x(C) \geq 1 \ (C \in \Omega); \ x \in \mathbb{Z}_+^{E(G)}\}.$$

However, each optimal solution of (9) can be proved to be the characteristic vector of a set $\delta(U) \Delta D$ for some $U \subseteq V(D)$. Moreover, all such vectors are feasible for (9). Hence (8) is equivalent to:

$$(10) \quad c(\delta(U) \setminus D) - c(\delta(U) \cap D) = c(\delta(U) \Delta D) - C(D) \geq 0 \text{ for all } U \subseteq V(K),$$

which is the cut-condition. So, (6) follows. As an illustration, observe that (G, D) in Figure 1 is isomorphic to \widetilde{K}_5 .

Intermezzo: Sufficient conditions for the existence of paths — dual integrality
When dealing with weak bipartiteness, one is more concerned with existence

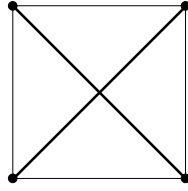


FIGURE 2. *Bold edges are in D ; $c := \mathbf{1}$.*

of flows than with existence of paths. But, also the existence of paths has polyhedral aspects. Even when the cut-condition is sufficient for flows to exist, it might not be good enough for paths (see Figure 2). There are two typical conditions which can help us out here. First, consider the case that (8) has an integral optimal solution for all integral non-negative c (call such signed graphs *strongly bipartite*). It is well-known from polyhedral theory (EDMONDS and GILES [6]) that a strongly bipartite signed graph is also weakly bipartite. So, in that case, the cut-condition is sufficient for paths to exist. Strongly bipartite graphs are known in terms of forbidden minors. (Note the relation of (11) with Figure 2.)

(11) THEOREM (SEYMOUR [30]). *(G, Σ) is strongly bipartite if and only if it has no \widetilde{K}_4 -minor.*

In fact, the theorem holds for all binary clutters, with the same forbidden minor (SEYMOUR [30]).

Another condition which together with the cut-condition is in some cases sufficient for the existence of paths is the so-called *parity condition*:

(12) PARITY CONDITION. *c is integral and $c(\delta(v))$ is even for all $v \in V(G)$.*

Let us call a signed graph *evenly bipartite* if the parity condition is sufficient for the maximization problem in (8) to take integral optimal solutions. It can be proved that even bipartiteness implies weak bipartiteness. So, if (G, D) is evenly bipartite, then the cut-condition and the parity condition together are sufficient for the existence of paths for $(G, D; c)$. Informally: the parity condition sometimes allows to re-route fractional flows into paths.

Let us spend a few lines on "strong bipartiteness implies weak bipartiteness" and "even bipartiteness implies weak bipartiteness". Let $Ax \leq b$ be a system of inequalities with $A \in \mathbb{Z}^{m \times n}$ and $b \in \mathbb{Z}^m$. Let Λ be a finitely generated lattice in the linear space generated by the rows of A and containing the lattice generated by the rows of A . What Edmonds and Giles essentially proved is the following: if $\min\{y^\top b \mid y^\top A = c^\top, y \geq 0\}$ has an integral optimal solution for all $c \in \Lambda$ for which the minimum exists, then for each c , $\max\{c^\top x \mid Ax \leq b\}$ has an optimal solution which is contained in $\Lambda^* := \{x \in \mathbb{Q}^n \mid y^\top x \in \mathbb{Z} \text{ if } y \in \Lambda\}$. The most

quoted version of this result is the case that $\Lambda = \Lambda^* = \mathbb{Z}^n$. For signed graphs this means that strong bipartiteness implies weak bipartiteness. In case Λ is the collection of all c satisfying the parity condition, Λ^* is the set of all half-integral vectors x with $x(C) \in \mathbb{Z}$ for each $C \in \mathcal{C}(G)$. Combining this with the fact that $\Omega(G, \Sigma)$ is a binary clutter, implies that even bipartiteness implies weak bipartiteness. So, "strong bipartiteness implies weak bipartiteness" is a property that the system of inequalities defining $P(G, \Sigma)$ shares with all systems of linear inequalities; whereas "even bipartiteness implies weak bipartiteness" also relies on the way the system relates to a binary space.

4. WEAKLY BIPARTITE GRAPHS AND MULTICOMMODITY FLOW THEOREMS

In the table below we list the best-known classes of weakly bipartite signed graphs and the corresponding multicommodity flow problems where the cut-condition is sufficient for the existence of flows. In all cases the signed graphs are in fact evenly bipartite, so the parity condition plus the cut-condition is sufficient for the existence of paths in the corresponding multicommodity flow problems.

We use the following compact (bordered) surfaces:

The *disk* $\mathcal{D} := \{z \in \mathbb{C} \mid |z| \leq 1\}$, with boundary $B(\mathcal{D}) := \{z \in \mathbb{C} \mid |z| = 1\}$.

The *annulus* $\mathcal{A} := \{z \in \mathbb{C} \mid 1 \leq |z| \leq 2\}$, with *inner boundary* $I(\mathcal{A}) := \{z \in \mathbb{C} \mid |z| = 1\}$ and *outer boundary* $O(\mathcal{A}) := \{z \in \mathbb{C} \mid |z| = 2\}$.

The *projective plane*, which can be obtained from \mathcal{D} by identifying opposite points on $B(\mathcal{D})$ (i.e. $e^{i\phi}$ with $-e^{i\phi}$).

The *Klein bottle*, which can be obtained from \mathcal{A} by identifying opposite points on $I(\mathcal{A})$ (i.e. $e^{i\phi}$ with $-e^{i\phi}$) and opposite points on $O(\mathcal{A})$ (i.e. $2e^{i\phi}$ with $-2e^{i\phi}$). The Klein bottle can also be obtained by identifying $I(\mathcal{A})$ in "reverse cyclic order" with $O(\mathcal{A})$, meaning: identifying $e^{i\phi}$ with $2e^{-i\phi}$.

The *torus*, which can be obtained from \mathcal{A} by identifying $I(\mathcal{A})$ in "cyclic order" with $O(\mathcal{A})$, meaning: identifying $e^{i\phi}$ with $2e^{i\phi}$.

Both the projective plane and the Klein bottle are *non-orientable* surfaces: they contain curves with the property that when you walk exactly once entirely along them, the "left-hand side" and "right-hand side" of the curve are interchanged. We call those curves *one-sided* (or *orientation-reversing*). Curves with the property that the sides are distinguishable are called *two-sided* (or *orientation-preserving*). $(G, \Sigma) \setminus U$, with $U \subseteq V(G)$, denotes the signed graph obtained from (G, Σ) by deleting all nodes in U and all edges with end nodes in U . If $F \subseteq E(G)$, then $V(F)$ denotes the collection of end points of edges in F .

The best-known classes of weakly bipartite signed graphs and the corresponding multicommodity flow problems where the cut-condition is sufficient for the existence of flows are:

	(G, Σ) is weakly bipartite:	Cut-condition is sufficient for $(G, D; c)$:
1	There exists a node $u \in V(G)$ such that $(G, \Sigma) \setminus \{u\}$ is bipartite (special case of (11))	$ D = 1$ (Menger [23], Ford and Fulkerson [9])
2	There exist two nodes $u, v \in V(G)$ such that $(G, \Sigma) \setminus \{u, v\}$ is bipartite (Barahona [1])	$ D = 2$ (Hu [19], Rothschild and Winston [27])
3	G is planar (Edmonds and Johnson [7], Hadlock [18])	G is planar (Seymour [33])
4	(G, Σ) can be embedded in the projective plane such that $\Omega(G, \Sigma)$ consists of the one-sided circuits in G (Lins [22])	$G \setminus D$ can be embedded in \mathcal{D} such that $V(D) \subseteq B(\mathcal{D})$ (Okamura and Seymour [25])
5	(G, Σ) can be embedded in the Klein bottle such that $\Omega(G, \Sigma)$ consists of the one-sided circuits in G (Schrijver [29])	$D = D_I \cup D_O$ and $G \setminus D$ can be embedded in \mathcal{A} such that $V(D_I) \subseteq I(\mathcal{A})$ and $V(D_O) \subseteq O(\mathcal{A})$ (Okamura [24])
		$D = \{\{s_1, t_1\}, \dots, \{s_k, t_k\}\}$ and $G \setminus D$ can be embedded in \mathcal{A} such that s_1, \dots, s_k lie on $I(\mathcal{A})$ in clockwise order and t_1, \dots, t_k lie on $O(\mathcal{A})$ in anticlockwise order (Schrijver [29])

We refer to the different results in the table by: Case 1, 2, etc. Note that in Case 5 only the multicommodity flow theorems together are equivalent to the result on weakly bipartite signed graphs in Case 5. The reason is that weak bipartiteness is invariant under re-signing, but sufficiency of the cut-condition not. Moreover, note that Case 1 is contained in Case 2 (although signed graphs as in Case 2 need not be strongly bipartite). Also: Case 4 is contained in Case 5.

REMARK. The careful reader might notice a little bit of cheating in the table. If one tries to derive the weak bipartiteness results in Cases 1 and 2 from the related flow theorems, one will end up with a multicommodity flow problems different from the ones indicated in the table. However a simple additional construction yields the proper ones. And, if one wants to derive the results by Okamura and Seymour in Cases 4 and 5 of the table from the related multicommodity flow problems one will have problems to embed (G, D) in the projective plane or the Klein bottle. The reason is that in these multicommodity flow problem we did not impose a special order in which the terminals lie on the boundary of the disk or annulus. However, there exists a simple construction (cf. Frank [10]) which transforms each multicommodity flow problem as in the results of Okamura and Seymour to one which has the terminals ordered

around the boundaries such that the desired embedding of (G, D) is possible.

We shall now try to weaken the conditions for weak bipartiteness given in the table.

GENERALIZATION OF CASE 3. What are the graphs which — like planar graphs — have the property that (G, Σ) is weakly bipartite irrespective of Σ ? If (3) is true: all the graphs with no K_5 -minor. (A minor for an undirected graph is taken as minor for a signed graph ignoring parity of edges). SEYMOUR [32] (cf. FONLUPT, MAHJOUR and UHRY [8]) proved that this is indeed the case, using a result of WAGNER's [35] saying that all graphs with no K_5 -minor are — with one easy-to-settle exception — either planar, or can be decomposed into smaller such graphs (allowing an inductive argument). In fact, Seymour characterized all binary spaces \mathcal{C} with (\mathcal{C}, Σ) weakly bipartite irrespective of Σ , using an extension of Wagner's result from SEYMOUR [31].

GENERALIZATION OF CASE 2. What is an obvious generalization of the case that $(G, \Sigma) \setminus \{u, v\}$ is bipartite? That $(G, \Sigma) \setminus \{u, v, w\}$ is bipartite? No, \widetilde{K}_5 satisfies that property! However, there is a correct extension possible:

(13) THEOREM (GERARDS [12]). *Let (G, Σ) be a signed graph. If there exists a node $u \in V(G)$ with the property that $(G, \Sigma) \setminus \{u\}$ has no \widetilde{K}_4 -minor then (G, Σ) is weakly bipartite.*

(Note that this is as far as you can get in terms of $(G, \Sigma) \setminus \{u\}$.) A special case of (13) is that $(G, \Sigma) \setminus \{u\}$ can be embedded in the plane such that exactly two faces are bounded by an odd cycle (GERARDS [13]).

COMMON GENERALIZATION OF CASES 1 AND 4. Signed graphs as in Cases 1 and 4 have the property that each two odd circuits have a node in common. Although this is also the case for \widetilde{K}_5 , the following can be proved:

(14) THEOREM. *Let (G, Σ) be a signed graph with $V(C_1) \cap V(C_2) \neq \emptyset$ for all $C_1, C_2 \in \Omega(G, \Sigma)$. If (G, Σ) has no \widetilde{K}_5 -minor, then it is weakly bipartite.*

This is true because: if in a signed graph each two odd circuits intersect, then it is either as in Case 1 or 4, or it is \widetilde{K}_5 , or it can be decomposed into smaller signed graphs without disjoint odd circuits — allowing an inductive argument (GERARDS, LOVÁSZ, SCHRIJVER, SHIH, SEYMOUR, TRUEMPER [14]).

GENERALIZATION OF CASES 4 AND 5. Observe that the condition in Case 4 can be equivalently formulated as “ (G, Σ) can be embedded in the projective plane such that all faces of that embedding are bounded by even cycles in (G, Σ) ”. Call an embedding of a signed graph in a surface such that all faces are bounded by even cycles an *even face embedding*. Which other compact surfaces have the property that all even face embedded signed graphs are weakly bipartite? An

embedding on the Klein bottle as in Case 5 is an even face embedding, but not all even face embeddable graphs on the Klein bottle arise in that way. But, as we will see in the next section, they still are weakly bipartite. On the other hand, as we see in Figure 3, \widetilde{K}_5 has an even face embedding in the torus. As

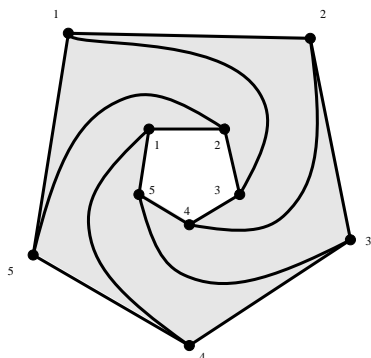


FIGURE 3. *The shaded area is the annulus. To obtain an even-face embedding of \widetilde{K}_5 in the torus, identify nodes and edges on the outer boundary with nodes and edges on the inner boundary (so identify 1 with 1, 2 with 2 etc.).*

all compact surfaces other than the sphere, the projective plane and the Klein bottle can be obtained from the torus by adding “handles” and “cross-caps”, it follows that the sphere, the projective plane and the Klein bottle are the only “weakly bipartite surfaces”. (Note that the case of the sphere is virtually empty: even-face embedded signed graphs in the sphere are bipartite.)

We will make one other attempt to find more weakly bipartite graphs. A *pinched surface* is a compact surface or a topological space obtained from a compact surface by identifying (possibly several times) two or more points to one *pinch point*. If Π is a (pinched) surface we define, (recursively) a *k-pinched* Π by identifying k non-pinch points of Π to one pinch point. It can be shown that all pinched surfaces contain an even face embedding of \widetilde{K}_5 , except for: the 2-pinched projective plane; the k -pinched spheres (with $k = 2, 3, \dots$) and the k - l -pinched spheres with $2 \leq k, l \leq 3$. And indeed, all these exceptions have the property that each even face embedded bipartite graph is weakly bipartite. The case of the k -pinched spheres is contained in Cases 1 and 3 together (as one easily observes) and the case of the k - l -pinched spheres is essentially a special case of (13). So these give no new classes of weakly bipartite graphs. But the 2-pinched projective plane does. Both the case of the Klein bottle and of the 2-pinched projective plane give rise to new multicommodity flow/disjoint paths theorems, namely on the Möbius strip.

5. DISJOINT PATHS ON THE MÖBIUS STRIP

(15) THEOREM. *Let G be an undirected graph embedded in the Möbius strip \mathcal{M} . Moreover, let $D = \{s_1 t_1, \dots, s_k t_k\} \subseteq E(G)$ with $s_1, \dots, s_k, t_1, \dots, t_k$ on the boundary of \mathcal{M} , and $c \in \mathbb{Z}_+^{E(G)}$.*

- (a) (GERARDS, SEBŐ [16]) *If going along the boundary of \mathcal{M} we meet $s_1, \dots, s_k, t_1, \dots, t_k$ in this order, then the cut-condition plus the parity condition is sufficient for the existence of paths for $(G, D; c)$.*
- (b) (GERARDS [11, 12]) *If going along the boundary of \mathcal{M} we meet $s_1, \dots, s_k, t_k, \dots, t_1$ in this order, then the cut-condition plus the parity condition is sufficient for the existence of paths for $(G, D; c)$.*

The special location of the terminals on the boundary of the Möbius strip is illustrated in Figure 4. As we shall see below, (15a) relates to even face embed-

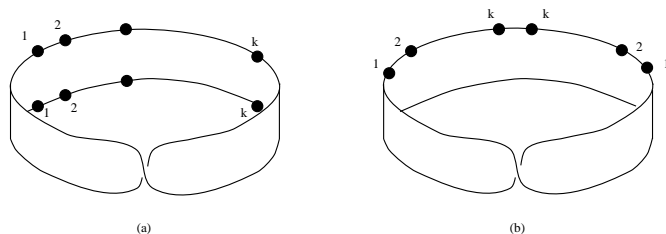


FIGURE 4. *terminals are indicated only by their indices.*

dings in the Klein bottle and (15b) to even face embeddings in the 2-pinned projective plane.

The Klein bottle

Consider the signed graph (G, D) obtained from (15a). This has an even face embedding in the Klein bottle. The reason is that if we identify “opposite” points on the Möbius strip, we obtain the Klein bottle. As the Möbius strip contains one-sided curves, any reasonably general instance of (G, D) will have even one-sided circuits in the Klein bottle, so will not be as in Case 5. But Case 5 and (15a) together imply:

(16) THEOREM (GERARDS, SEBŐ [16]). *Let (G, Σ) be a signed graph. If (G, Σ) has an even face embedding in the Klein bottle then it is weakly bipartite.*

We will sketch how this follows from the multicommodity flow theorems in Case 5 and in (15a). We only need to prove that for each $c \in \mathbb{Z}_+^{E(G)}$,

the minimum in (9) equals the maximum in (8). Choose an integral optimal solution x to (9) with a minimum number of positive coefficients. As mentioned before we may assume that x is the characteristic vector of $D := \delta(U) \triangle \Sigma$ for some $U \in V(G)$. Hence, (9) holds. So it remains to prove that (8) holds as well; or in other words, that the cut-condition is sufficient for the existence of flows in $(G, D; c)$. We do this by showing that G and D are as in the multicommodity flow problems in Case 5 and in (15a).

Select closed curves $\Gamma_1, \dots, \Gamma_k$ on the Klein bottle according to the following rules: The curves are pairwise disjoint and each of them is disjoint from $V(G)$ and meets G only in edges of D . The curves go from face to face by “crossing” the edges in D . Each edge in D is crossed by only one curve; this curve intersects the edge in a single point.

As (G, D) is isomorphic to (G, Σ) , it is embedded such that each face has an even number of edges in D on the boundary. From this it is quite easy to see that, indeed, the curves exist. We apply the following surgery on (G, D) and the Klein bottle. We cut the Klein bottle open along the curves. This yields a bordered surface S . Each time we cut through an edge $d \in D$ we create new terminals s_d and t_d located at the open ends of the two “half-edges” obtained by cutting through d (with s_d on the left-hand side of the curve cutting through d and t_d on the right-hand side). Between these two new terminals we add a new demand edge \hat{d} . Thus we get a new graph \hat{G} and a new collection of demand edges \hat{D} . Clearly, $\hat{G} \setminus \hat{D}$ is embedded in S with the new terminals on the boundary. It is also easy to see that each multicommodity flow problem on G with demands in D can be transformed to an equivalent one on \hat{G} with demands in \hat{D} . Moreover, we know that S is connected. (If not, D contains a cut; the symmetric difference of D with that cut would correspond to another optimal solution of (9), with fewer positive components.)

Suppose one of the curves, Γ_1 say, is two-sided. If we cut the Klein bottle *only* along that curve we get the annulus. Moreover, the terminals s_d created by cutting Γ_1 lie all on the outer boundary in clock-wise order (say) and the nodes t_d on the inner boundary in anti-clockwise order. So, if there are no other curves to cut along we end up as in Schrijver’s result in Case 5. But, there cannot be any other curve in our collection, as the annulus has no “non-separating” curves. So remains the case that all cutting curves are one-sided. If we cut along Γ_1 we obtain the Möbius strip. Moreover, all the terminals created by Γ_1 lie in the same order as in (15a). So, if there are no other cutting curves, we end up with a multicommodity flow as in (15a). If there is a second (again one-sided) curve and we cut also along that one, we get the annulus. As the annulus has no non-separating curves, we end up as in Okamura’s theorem in Case 5.

Recently, András Sebő observed that (15a) follows from the following “distance packing” result, which also implies the results in Case 5.

- (17) (SCHRIJVER [28]) *Let G be a bipartite graph embedded in the annulus. Then there exists a collection of pairwise edge-disjoint cuts $\delta(U_1), \dots, \delta(U_k)$, such that for each pair of nodes s and t of $V(G)$ which both lie on $I(\mathcal{A})$ or both lie on $O(\mathcal{A})$, the length of the shortest st -path in G is equal to the number of cuts among $\delta(U_1), \dots, \delta(U_k)$ with $|\{s, t\} \cap U_i| = 1$.*

Note that the essence of this result is that one collection of cuts can be selected which simultaneously satisfies the properties in (17) for all indicated pairs of nodes. The existence of such a collection for just a single pair, is an easy, old and well-known fact about distances in graphs. Recently, (16) has been extended by DE GRAAF and SCHRIJVER [4] (cf. DE GRAAF [3]).

The 2-pinched projective plane

Next consider an instance (G, D) of (15b). Now the signed graph, can be embedded in the projective plane with all but two faces even. To see this observe that identifying the boundary of the Möbius strip with the boundary of a disk yields the projective plane. Moreover, by the order of the terminals, we can embed the odd edges in such a glued disk. This suggests the following result.

- (18) THEOREM (GERARDS [11]). *Let (G, Σ) be embedded in the projective plane. If exactly two faces are bounded by an odd cycle in (G, Σ) , then (G, Σ) is weakly bipartite.*

By similar surgeries as used above one can derive this result from (15b) and Okamura's theorem in Case 5. The only difference is that now one of the cutting curves is not closed but starts in one odd face and ends in the other.

- (19) THEOREM (GERARDS, SCHRIJVER [15]). *Let (G, Σ) have an even face embedding in the projective plane. Then any signed graph obtained from (G, Σ) by identifying two of its vertices is weakly bipartite.*

Obviously, there are multicommodity flow theorems related to this: namely one on a pinched disk where the terminals of a demand edge lie oppositely on the boundary of the disk, and a multicommodity flow problem on the disk with all terminals on the boundary except for the terminals of one demand edge, which can be anywhere. But we did not use these to prove (19), instead we used a distance packing theorem.

- (20) THEOREM (GERARDS, SCHRIJVER [15]). *Let G be a bipartite graph embedded in the disk. Moreover, let $s_0, t_0 \in V(G)$. Then there exists a collection of pairwise edge-disjoint cuts $\delta(U_1), \dots, \delta(U_k)$, such that for each pair of nodes s and t of $V(G)$ which either both lie on the boundary of the disk or satisfy $s = s_0$ and $t = t_0$, the length of the shortest st -path in G is equal to the number of cuts among $\delta(U_1), \dots, \delta(U_k)$ with $|\{s, t\} \cap U_i| = 1$.*

Together (18) and (19) imply that all signed graphs with an even face embedding in the 2-pinched projective plane are weakly bipartite.

6. EPILOGUE

Motivated by Seymour's conjecture (3) and the relation between weakly bipartite signed graphs and classes of multicommodity flow problems for which the cutcondition is sufficient we searched for weakly bipartite graphs. This resulted in the following list:

Signed graphs (G, Σ) such that G has no K_5 -minor.

Signed graphs (G, Σ) with a vertex u such that $(G, \Sigma) \setminus \{u\}$ has no \widetilde{K}_4 -minor.

Signed graphs with even face embedding on the Klein bottle or on the 2-pinched projective plane.

As far as I know, no other weakly bipartite signed graphs are known at this moment: Seymour's conjecture (3) – and even more (4) – are still a challenge.

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