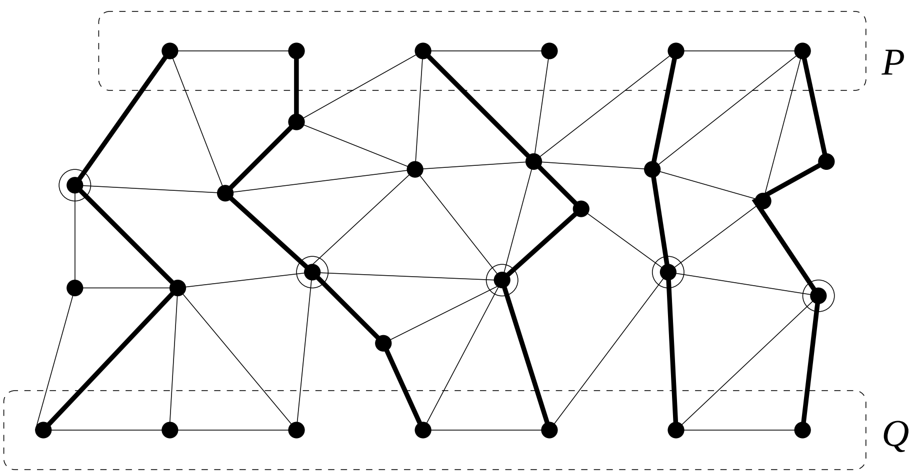
Paths and Flows—a Historical Survey

Alexander Schrijver
CWI
Kruislaan 413, 1098 SJ Amsterdam
The Netherlands
email: lex@cwi.nl
and
Department of Mathematics, University of Amsterdam
Plantage Muidergracht 24, 1018 TV Amsterdam
The Netherlands

1. Menger's theorem

In 1927, the topologist Karl Menger published an article called *Zur allgemeinen Kurventheorie* (On the general theory of curves), in which he stated a remarkable result, now one of the most fundamental results in graph theory:

> Satz β. *Ist K ein kompakter regulär eindimensionaler Raum, welcher zwischen den beiden endlichen Mengen P und Q n-punktig zusammenhängend ist, dann enthält K n paarweise fremde Bögen, von denen jeder einen Punkt von P und einen Punkt von Q verbindet.*[1]



The result can be formulated as a maximum-minimum theorem in terms of graphs, as follows: Menger's theorem. *Let $G = (V, E)$ be an undirected*

[1] Theorem β. *If K is a compact regular one-dimensional space which is n-point connected between the two finite sets P and Q, then K contains n pairwise disjoint curves, each of which connects a point in P and a point in Q.*

graph and let $P, Q \subseteq V$. *Then the maximum number of pairwise disjoint* $P-Q$ *paths is equal to the minimum cardinality* n *of any set of vertices that intersects each* $P-Q$ *path.*

Here V denotes the vertex set of G and E the edge set. A $P-Q$ *path* is a path starting in P and ending in Q. Two paths are *disjoint* if they do not have any vertex or edge in common. The result became also known as the *n-chain theorem* or the *n-arc theorem.* KNASTER [20] observed that (by an easy construction) Menger's theorem is equivalent to:

MENGER'S THEOREM (variant). *Let* $G=(V,E)$ *be an undirected graph and let* $s,t \in V$ *with* $st \notin E$. *Then the maximum number of pairwise internally disjoint* $s-t$ *paths is equal to the minimum cardinality of any subset of* $V \setminus \{s,t\}$ *that intersects each* $s-t$ *path.*

(This theorem was proved by RUTT [33] for *plane* curves, on a suggestion of J.R. Kline.)

Here an $s-t$ *path* is a path starting in s and ending in t. Two paths are *internally disjoint* if they do not have a vertex or edge in common, except for the end vertices.

Why was Menger interested in this question? In his article he investigates a certain class of topological spaces called 'curves': a *curve* is a connected compact topological space X with the property that for each $x \in X$ and each neighbourhood N of x there exists a neighbourhood $N' \subseteq N$ of x such that $|\mathrm{bd}(N')|$ is totally disconnected. Here bd stands for 'boundary'; a space is *totally disconnected* if each point forms an open set. Notice that each graph, considered as a topological space, is a curve.

In particular, Menger was motivated by characterizing a certain furcation number of curves. To this end, a curve X is called *regular* if for each $x \in X$ and each neighbourhood N of x there exists a neighbourhood $N' \subseteq N$ of x such that $|\mathrm{bd}(N')|$ is finite. The *order* of a point $x \in X$ is equal to the minimum natural number n such that for each neighbourhood N of x there exists a neighbourhood $N' \subseteq N$ of x satisfying $|\mathrm{bd}(N')| \leq n$.

According to Menger:

> Eines der wichtigsten Probleme der Kurventheorie ist die Frage nach die Beziehungen zwischen der Ordnungszahl eines Punktes der regulären Kurve K und der Anzahl der im betreffenden Punkt zusammenstossenden und sonst fremden Teilbögen von K.[2]

In fact, Menger used 'Satz β' to show that if a point in a regular curve K has order n, then there exists a topological n*-leg* with p as top; that is, K contains n arcs $P_1, \ldots, P_n$ such that $P_i \cap P_j = \{p\}$ for all i,j with $i \neq j$.

[2] One of the most important problems of the theory of curves is the question of the relations between the order of a point of a regular curve K and the number of subarcs of K meeting in that point and else disjoint.

The proof idea is as follows. There exists a series $N_1 \supset N_2 \supset \cdots$ of open neighbourhoods of p such that $N_1 \cap N_2 \cap \cdots = \{p\}$ and $|\mathrm{bd}(N_i)| = n$ for all $i = 1, 2, \ldots$ and such that

(1) $|\mathrm{bd}(N)| \geq n$ for each neighbourhood $N \subseteq N_1$.

This follows quite directly from the definition of order.

Now Menger showed that we may assume that the space $G_i := \overline{N_i} \setminus N_{i+1}$ is a (topological) graph. For each i let $Q_i := \mathrm{bd}(N_i)$. Then (1) gives with Menger's theorem that there exist n pairwise disjoint paths $P_{i,1}, \ldots, P_{i,n}$ in G such that each $P_{i,j}$ runs from Q_i to Q_{i+1}. Properly connecting these paths for $i = 1, 2, \ldots$ we obtain n arcs forming the required n-leg.

It was however noticed by KŐNIG [22] that Menger gave a lacunary proof of 'Satz β'. Menger applies induction on $|E|$, where E is the edge set of the graph G. Menger first claims that one easily shows that $|E| \geq n$, and that if $|E| = n$ then G consists of n disjoint arcs connecting P and Q. He states that if $|E| > n$ then there is a vertex $s \notin P \cup Q$, or in his words (where the 'Grad' denotes $|E|$):

> Wir nehmen also an, der irreduzibel n-punktig zusammenhängende Raum K' besitze den Grad $g(> n)$. Offenbar enthält dann K' ein punktförmiges Stück s, welches in der Menge $P + Q$ nicht enthalten ist.[3]

Indeed if such a vertex s exists one is done: If s is not contained in any set W intersecting each $P - Q$ path such that $|W| = n$, then we can delete s and the edges incident with s without decreasing the minimum in the theorem. If s is contained in a set W intersecting each $P - Q$ path such that $|W| = n$, then we can split G into two subgraphs G_1 and G_2 that intersect in W in such a way that $P \subseteq G_1$ and $Q \subseteq G_2$. By the induction hypothesis, there exist n pairwise disjoint $P - W$ paths in G_1 and n pairwise disjoint $W - Q$ paths in G_2. By pairwise sticking these paths together we obtain paths as required.

However, such a vertex s need not exist. It might be that V is the disjoint union of P and Q in such a way that each edge connects P and Q. In that case, G is a bipartite graph, and what should be shown is that G contains a matching of size n. This is a nontrivial basis of the proof.

It is unclear when Menger became aware of the hole. In his reminiscences on the origin of the n-arc theorem MENGER [27] writes:

> In the spring of 1930, I came through Budapest and met there a galaxy of Hungarian mathematicians. In particular, I enjoyed making the acquaintance of Dénes Kőnig, for I greatly admired the work on set theory of his father, the late Julius Kőnig—to this day one of the most significant contributions to the continuum problem—and I had read with interest some of Dénes papers. Kőnig told me that he was about to finish a book that would include all that was known about graphs. I assured him that

[3]Thus we assume that the irreducibly n-point-connected space K' has degree $g(> n)$. Obviously, in that case K' contains a point-shaped piece s, that is not contained in the set $P + Q$.

> such a book would fill a great need; and I brought up my n-Arc Theorem which, having been published as a lemma in a curve-theoretical paper, had not yet come to his attention. Kőnig was greatly interested, but did not believe that the theorem was correct. "This evening," he said to me in parting, "I won't go to sleep before having constructed a counterexample." When we met the next day he greeted me with the words, "A sleepless night!" and asked me to sketch my proof for him. He then said that he would add to his book a final section devoted to my theorem. This he did; and it is largely thanks to Kőnig's valuable book that the n-Arc Theorem has become widely known among graph theorists.

Dénes Kőnig was a pioneer in graph theory and in applying graphs to other areas like set theory, matrix theory, and topology. He had published in the 1910s theorems on perfect matchings and factorizations of regular bipartite graphs in relation to the study of determinants by Frobenius.

At the meeting of 26 March 1931 of the Eötvös Loránd Matematikai és Fizikai Társulat (Loránd Eötvös Mathematical and Physical Society) in Budapest KŐNIG [21] presented a new result that formed in fact the induction basis for Menger's theorem:

> Páros körüljárású graphban az éleket kimerítő szögpontok minimális száma megegyezik a páronként közös végpontot nem tartalmazó élek maximális számával.[4]

In other words:

KŐNIG'S THEOREM. *In a bipartite graph $G = (V, E)$, the maximum size of a matching is equal to the minimum number of vertices needed to cover all edges.*

[The result can also be derived by some direct construction from the theorem of FROBENIUS [12] that a bipartite graph with colour classes each of size n has a perfect matching if and only if one cannot select a set of $n - 1$ vertices that intersects each edge.]

Kőnig did not mention in his paper that this result provided the missing link in Menger's proof, although he finishes with:

> Megemlítjük végül, hogy eredményeink szorosan összefüggnek FROBENIUS-nak determinánsokra és MENGERnek graphokre vonatkozó némely vizsgálatával. E kapcsolatokra másutt fogunk kiterjeszkedni.[5]

'Másutt' (elsewhere) came in 1933 [22]. In this paper, Kőnig gives again a proof of Kőnig's theorem, and he also gives a full proof of Menger's theorem. At this point, he adds the following footnote:

[4] In an even circuit graph, the minimal number of vertices that exhaust the edges agrees with the maximal number of edges that pairwise do not a contain common end point.

[5] We finally mention that our results are closely connected to some investigations of FROBENIUS on determinants and of MENGER on graphs. We will enlarge on these connections elsewehere.

> Der Beweis von MENGER enthält eine Lücke, da es vorausgesetzt wird (S. 102, Zeile 3–4) daß "K' ein punktförmiges Stück s enthält, welches in der Menge $P+Q$ nicht enthalten ist", während es recht wohl möglich ist, daß —mit der hier gewählten Bezeichnungsweise ausgedrückt—jeder Knotenpunkt von G zu H_1+H_2 gehört. Dieser—keineswegs einfacher—Fall wurde in unserer Darstellung durch den Beweis des Satzes 13 erledigt. Die weiteren—hier folgenden—Überlegungen, die uns zum Mengerschen Satz führen werden, stimmen in Wesentlichen mit dem—sehr kurz gefaßten—Beweis von MENGER überein. In Anbetracht der Allgemeinheit und Wichtigkeit des Mengerschen Satzes wird im Folgenden auch dieser Teil ganz ausführlich und den Forderungen der *rein-kombinatorischen* Graphentheorie entsprechend dargestellt.
>
> [Zusatz bei der Korrektur, 10.V.1933] Herr MENGER hat die Freundlichkeit gehabt—nachdem ich ihm die Korrektur meiner vorliegenden Arbeit zugeschickt habe—mir mitzuteilen, daß ihm die oben beanstandete Lücke seines Beweises schon bekannt war, daß jedoch sein vor Kurzem erschienenes Buch *Kurventheorie* (Leipzig, 1932) einen vollkommen lückenlosen und rein kombinatorischen Beweis des Mengerschen Satzes (des "n-Kettensatzes") enthält. Mir blieb dieser Beweis bis jetzt unbekannt.[6]

This book of MENGER [26] was published in 1932, and contains a complete proof of Menger's theorem. Menger did not refer to any hole in his proof, but remarked:

> Über den n-Kettensatz für Graphen und die im vorangehenden zum Beweise verwendete Methode vgl. Menger (Fund. Math. 10, 1927, S. 101 f.). Die obige detaillierte Ausarbeitung und Darstellung stammt von Nöbeling.[7]

In his book *Theorie der endlichen und unendlichen Graphen* (Theory of finite and infinite graphs), published in 1936, KŐNIG [23] calls his theorem *ein wichtiger Satz* (an important theorem), and he emphasizes the chronological order of the proofs of Menger's theorem and of Kőnig's theorem that follows from Menger's theorem:

[6]The proof of MENGER contains a hole, as it is assumed (page 102, line 3–4) that "K' contains a point-shaped piece s that is not contained in the set $P+Q$", while it is quite well possible that—expressed in the notation chosen here—every node of G belongs to H_1+H_2. This—by no means simple—case is settled in our presentation by the proof of Theorem 13. The further arguments following here that will lead us to Menger's theorem, agree essentially with the—very briefly couched—proof of MENGER. In view of the generality and the importance of Menger's theorem, also this part is exhibited in the following very extensively and answering to the progress of the *purely combinatorial* graph theory.

[Added in proof, May 10, 1933] Mr. MENGER has had the kindness—after I have sent him the galley proofs of my present work—to inform me that the hole in his proof objected above, was known to him already, but that his recently appeared book *Curve Theory* (Leipzig, 1932) contains a completely holeless and purely combinatorial proof of the Menger theorem (the "n-chain theorem"). As yet this proof remained unknown to me.

[7]On the n-chain theorem for graphs and the method used in the foregoing for the proof, cf. Menger (Fund. Math. 10, 1927, p. 101 ff.). The detailed elaboration and explanation above originates from Nöbeling.

> Ich habe diesen Satz 1931 ausgesprochen und bewiesen, s. Kőnig [9 und 11]. 1932 erschien dann der erste lückenlose Beweis des Mengerschen Graphensatzes, von dem in §4 die Rede sein wird und welcher als eine Verallgemeinerung dieses Satzes 13 (falls dieser *nur für endliche* Graphen formuliert wird) angesehen werden kann.[8]

We finally mention that a result related to Menger's theorem was presented by Whitney on 28 February 1931 to the American Mathematical Society ([35]): a graph is n-connected if and only if any two vertices are connected by n internally disjoint paths. While referring to the papers of Menger and Rutt, Whitney also gave a direct proof.

Other proofs of Menger's theorem were given by NÖBELING [28] and HAJÓS [14].

2. FLOWS IN NETWORKS

In the beginning of the 1950s, T.E. Harris at the RAND Corporation (the think tank of the U.S. Air Force in Santa Monica, California) called attention for the following problem:

> Consider a rail network connecting two cities by way of a number of intermediate cities, where each link of the network has a number assigned to it representing its capacity. Assuming a steady state condition, find a maximal flow from one given city to the other.

This question raised a stream of research at RAND. The problem can be formalized as follows.

Let be given a directed graph $D = (V, A)$, with two special vertices, a 'source' s and a 'sink' or 'terminal' t. Then an $s-t$ *flow* is a function $f : A \longrightarrow \mathbb{R}_+$ such that for each vertex $v \neq s, t$ the *flow conservation law* holds; that is:

$$(2) \qquad \sum_{a \in \delta^+(v)} f(a) = \sum_{a \in \delta^-(v)} f(a).$$

Here $\delta^+(v)$ denotes the set of arcs entering v and $\delta^-(v)$ denotes the set of arcs leaving v. The *value* of f is equal to the net flow leaving s; that is:

$$(3) \qquad \text{value}(f) := \sum_{a \in \delta^-(s)} f(a) - \sum_{a \in \delta^+(s)} f(a).$$

It is not difficult to prove that this quantity should be equal to the net flow entering t.

If moreover a 'capacity' function $c : A \longrightarrow \mathbb{R}_+$ is given, one says that f is *subject to* c if $f(a) \leq c(a)$ for each arc a.

Now the MAXIMUM FLOW PROBLEM can be formulated:

[8]I have enunciated and proved this theorem in 1931, see Kőnig [9 und 11]. Next in 1932 the first holeless proof of the Menger theorem appeared, of which will be spoken in §4 and which can be considered as a generalization of this Theorem 13 (in case this is formulated *only for finite* graphs).

(4) given: a directed graph $D = (V, A)$, vertices $s, t \in V$, and a 'capacity' function $c : A \longrightarrow \mathbb{R}_+$;

find: a flow f subject to c maximizing value(f).

In their basic paper "Maximal flow through a network" (published as a RAND Report of 19 November 1954), Ford and Fulkerson [8] observed that this is just a linear programming problem, and hence can be solved with Dantzig's simplex method. Indeed, it is a problem that is very close to the transportation and the transshipment problems studied in the 1940 by Kantorovich, Hitchcock, and Koopmans, which by themselves formed an important motivation for studying linear programming.

Main result of Ford and Fulkerson's paper is the famous *max-flow min-cut theorem*. To this end, the concept of a *cut* is defined.

Let U is any set with $s \in U$ and $t \notin U$. Then $\delta(U)$ (the set of all edges with one end in U and the other in $V \setminus U$) is an *$s - t$ cut*. The *capacity* of the cut is the sum of all $c(e)$ for $e \in \delta(U)$.

It is clear that the capacity of any cut is an upper bound on the maximal value of $s - t$ cuts. What Ford and Fulkerson [8] showed is:

Max-flow min-cut theorem. *The maximal value of the $s - t$ flows is equal to the minimal capacity of the $s - t$ cuts.*

In this paper, Ford and Fulkerson also gave a simple algorithm for the maximal flow problem in case the graph, added with an extra edge connecting s and t, is planar.

In a report of 1 January 1955, Dantzig and Fulkerson [3] showed that the max-flow min-cut theorem can also be deduced from the duality theorem of linear programming and in a report of 1 April 1955 [13] they gave a simple computational method for the maximum flow problem based on the simplex method.

In the first report it was also observed that Menger's theorem follows as a consequence. It follows from the method of Dantzig [2] that if the capacity function is integer-valued, then there exists a maximum flow that is also integer-valued.

Indeed, by taking capacities ∞ on the edges and 0 in the vertices, and by observing that any integer flow can be decomposed into pairwise disjoint paths, we obtain Menger's theorem. It also yields a new variant of Menger's theorem:

Menger's theorem (variant). *Let $G = (V, E)$ be an undirected graph, and let $s, t \in V$. Then the maximum number of pairwise edge-disjoint $s - t$ paths is equal to the minimum cardinality of any $s - t$ cut.*

Here two paths are called *edge-disjoint* if they do not have any edge in common. This variant of Menger's theorem can be derived by decomposing any integer $s - t$ flow of value k as a sum of the incidence vectors of k $s - t$ paths.

There is also an easy reverse construction that gives the max-flow min-cut theorem as a consequence of Menger's theorem. So the two theorems are equivalent.

The max-flow min-cut theorem being also a combinatorial result, one was interested in obtaining combinatorial methods for finding maximum flows. A heuristic method for the maximum flow problem, the *flooding technique*, was presented by BOLDYREFF [1] on 3 June 1955 at the New York meeting of the Operations Research Society of America (RAND Report of 5 August 1955). The method was intuitive, and the author did not claim generality:

> It has been previously assumed that a highly complex railway transportation system, too complicated to be amenable to analysis, can be represented by a much simpler model. This was accomplished by representing each complete railway operating division by a point, and by joining pairs of such points by arcs (lines) with traffic carrying capacities equal to the maximum possible volume of traffic (expressed in some convenient unit, such as trains per day) between the corresponding operating divisions.
>
> In this fashion, a network is obtained consisting of three sets of points — points of origin, intermediate or junction points, and the terminal points (or points of destination) — and a set of arcs of specified traffic carrying capacities, joining these points to each other.

Boldyreff's arguments for designing a heuristic procedure are formulated as follows:

> In the process of searching for the methods of solving this problem the following objectives were used as a guide:
>
> **1.** That the solution could be obtained quickly, even for complex networks.
>
> **2.** That the method could be explained easily to personnel without specialized technical training and used by them effectively.
>
> **3.** That the validity of the solution be subject to easy, direct verification.
>
> **4.** That the method would not depend on the use of high-speed computing or other specialized equipment.

Boldyreff's 'flooding technique' pushes a maximum amount of flow greedily through the network. If at some vertex a 'bottleneck' arises (i.e., there are more trains arriving than can be pushed further through the network), it is eliminated by returning the excess trains to the origin.

It is empirical, not using backtracking, and not leading to a optimum solution in all cases:

> Whenever arbitrary decisions have to be made, ordinary common sense is used as a guide. At each step the guiding principle is to move forward the maximum possible number of trains, and to maintain the greatest flexibility for the remaining network.

Boldyreff speculates:

> In dealing with the usual railway networks a single flooding, followed by removal of bottlenecks, should lead to a maximal flow.

Boldyreff gives the example of a complex network the model of a real, comprehensive, railway transportation system 41 vertices and 85 arcs:

> The total time of solving the problem is less than thirty minutes.

His closing remarks are:

> Finally there is the question of a systematic formal foundation, the comprehensive mathematical basis for empiricism and intuition, and the relation of the present techniques to other processes, such as, for instance, the multistage decision process (a suggestion of Bellman's).
>
> All this is reserved for the future.

Soon after, Ford and Fulkerson presented in a RAND Report of 29 December 1955 [9] their 'very simple algorithm' for the maximum flow problem, based on finding 'augmenting paths'. The algorithm finds in a finite number of steps a maximum flow, if all capacities have rational values. After mentioning the maximum flow problem, they remark:

> This is of course a linear programming problem, and hence may be solved by Dantzig's simplex algorithm. In fact, the simplex computation for a problem of this kind is particularly efficient, since it can be shown that the sets of equations one solves in the process are always triangular [2]. However, for the flow problem, we shall describe what appears to be a considerably more efficient algorithm; it is, moreover, readily learned by a person with no special training, and may easily be mechanized for handling large networks. We believe that problems involving more than 500 nodes and 4,000 arcs are within reach of present computing machines.

Referring to Boldyreff's paper as by '[1]' Ford and Fulkerson opiniate in their report:

> (It is the opinion of the authors that if the problem is given in matrix form, no special attempts should be made to obtain a good starting solution. If, on the other hand, the problem can be pictured readily as a linear graph, a "flooding" idea described in [1] might be used to obtain a starting flow. By following the approach suggested in [1], which, however, calls for the exercise of judgement, an initial flow can be obtained that often is optimal for simple networks. If not, it might be used as a good starting point for initiating the procedure given in this paper.)

(This paragraph as well as the reference to Boldyreff's paper do not appear in the final paper.)

In the paper it is also observed that the max-flow min-cut theorem holds for directed graphs as well. We thus also obtain a directed version of Menger's theorem.

Alternative proofs of the max-flow min-cut theorem were given by Robacker [29] and by Elias, Feinstein and Shannon [6]. In this last paper it is claimed that the result was known by workers in communication theory:

> This theorem may appear almost obvious on physical grounds and appears to have been accepted without proof for some time by workers in communication theory. However, while the fact that this flow cannot be exceeded is indeed almost trivial, the fact that it can actually be achieved is by no means obvious. We understand that proofs of the theorem have been given by Ford and Fulkerson and Fulkerson and Dantzig. The following proof is relatively simple, and we believe different in principle.

An interesting equivalent form of the max-flow min-cut was shown by HOFFMAN [15], now called *Hoffman's circulation theorem.* Let $D = (V, A)$ be a directed graph. A function $f : A \longrightarrow \mathbb{R}$ is called a *circulation* if in each vertex v the flow conservation law (2) holds. Hoffman showed:

HOFFMAN'S CIRCULATION THEOREM. *Let $D = (V, A)$ be a directed graph and let $d, c : A \longrightarrow \mathbb{R}$, with $d(a) \leq c(a)$ for each $a \in A$. Then there exists a circulation $f : A \longrightarrow \mathbb{R}$ satisfying $d(a) \leq f(a) \leq c(a)$ for each arc a, if and only if*

$$(5) \qquad \sum_{a\in\delta^+(U)} d(a) \leq \sum_{a\in\delta^-(U)} c(a)$$

for each subset U of V.

Let us finally note that Ford and Fulkerson's augmenting path algorithm for the maximum-flow problem has implementations that give a polynomial-time algorithm. This was shown by DINITS [4] and EDMONDS and KARP [5].

3. MULTICOMMODITY FLOWS AND DISJOINT PATHS

Quite often in practice one is not purely interested in sending one type of flow through a network, but several types simultaneously. For instance, in a telephone network one wishes to transmit several phone calls simultaneously.

In mathematical terms, it means sending flows $f_1, \ldots, f_k$ simultaneously, where f_i runs from a given vertex s_i to another given t_i and should have a given value d_i (say), such that the *total* amount of flow through any arc does not exceed the capacity of that arc.

Thus we have the following MULTICOMMODITY FLOW PROBLEM:

(6) given: a directed graph $D = (V, A)$, pairs $s_1, t_1, \ldots, s_k, t_k$, 'demands' $d_1,\ldots,d_k$ and a 'capacity' function $c : A \longrightarrow \mathbb{R}_+$;

find: flows $f_1, \ldots, f_k$, where f_i is an $s_i - t_i$ flow of value d_i, such that

$$\sum_{i=1}^{k} f_i(a) \leq c(a)$$

for each arc a.

Again, this is is a special case of a linear programming problem. There is also an undirected variant where we require that the sum of the flows in both directions in any undirected arc does not exceed the capacity. (In fact, there is a straightforward reduction to the directed version.)

One of the first studies on multicommodity flows was presented by Robacker in a RAND Report of 26 September 1956 [30]. He in particular considered the MAXIMUM MULTICOMMODITY FLOW PROBLEM:

(7) given: a directed graph $D = (V, A)$, pairs $s_1, t_1, \ldots, s_k, t_k$ of vertices, and a capacity function $c : A \longrightarrow \mathbb{R}_+$;

find: flows $f_1, \ldots, f_k$, where f_i is an $s_i - t_i$ flow such that $\sum_{i=1}^{k} f_i(a) \leq c(a)$ for each $a \in A$ and such that $\sum_{i=1}^{k} \text{value}(f_i)$ is as large as possible.

Robacker observed that the following 'Decomposition theorem' applies: The maximum total value of flow in a multicommodity network is equal to

$$(8) \qquad \max_{c_1,\ldots,c_k} \sum_{i=1}^{k} \min_{C \in \mathcal{C}_i} c_i(C).$$

Here the maximum ranges over all k-tupels of vectors $c_1, \ldots, c_k$ in $\mathbb{R}_+^E$ such that $c_1 + \cdots + c_k = c$. Moreover, $\mathcal{C}_i$ denotes all $s_i - t_i$ cuts and $c_i(C)$ denotes the capacity of cut C with respect to the capacity function c_i. (In fact, Robacker restricted himself to undirected graphs.)

So the theorem decomposes the maximum multicommodity flow problem into k maximum single-commodity flow problems. The problem is reduced to finding the optimum decomposition of the capacity function c into k functions $c_1, \ldots, c_k$.

Robacker also notes:

> At present there are no computational techniques other than those of linear programming for determining maximal flow through multicommodity networks. It is hoped, however, that the decomposition theorem may lead to new methods as did the minimum-cut, maximum-flow theorem for single-commodity networks.

KALABA and JUNCASA [17] described in 1957 applications of the multicommodity flow problem to telecommunication networks. In particular they mention:

> In a system such as the Western Union System, which has some 15 regional switching centers all connected to each other, an optimal routing problem of this type would have about 450 conditions and involve around 3000 variables. If solved using the simplex method in its most general form, this would be at the threshold of the capacity of modern large-scale computers and would require several hours for solution.

It turned out, however, that the combinatorial techniques that made the single-commodity flow problem so tractable, did not work for multicommodity flows. FORD and FULKERSON [10] suggested a variant of the simplex method based on a column-generation technique, where each simplex step consists of determining a shortest path. Although they did not carry out computations, they expected that their method is more practicable than the direct simplex method, at least in space required.

Success was obtained in 1963 by HU [16] who extended Ford and Fulkerson's 1-commodity flow algorithm to *two* commodities. He moreover described a *max-biflow min-cut* theorem extending the max-flow min-cut theorem. With respect to extending his method to more than two commodities, Hu remarked:

> Although the algorithm for constructing maximum bi-flow is very simple, it is unlikely that similar techniques can be developed for constructing multicommodity flows. The linear programming approach used by Ford and Fulkerson to construct maximum multicommodity flows in a network is the only tool now available.

This last remarks still applies today.

Hu also showed that if all input data are integer and there exists a solution for the 2-commodity flow problem, then there exists a half-integer solution. Later, this was extended by ROTHSCHILD and WHINSTON [32] to the existence of an integer solution if the input data are integer and satisfy a certain parity condition.

It should be noted that generally even if all input data are integer, the existence of a fractional solution to the multicommodity flow problem does not imply the existence of an integer solution. Thus the question for integer-valued multicommodity flows is independent of the general, fractional multicommodity flow problem. Linear programming does not automatically yield an integer solution for the multicommodity flow problem with integer input data.

Neither does it give a solution for the following combinatorial version of the problem, the DISJOINT PATHS PROBLEM:

(9) given: a graph $G = (V, E)$ and k pairs of vertices $s_1, t_1, \ldots, s_k, t_k$;
find: pairwise disjoint paths $P_1, \ldots, P_k$ where P_i runs from s_i to t_i $(i = 1, \ldots, k)$.

This covers four variants of the problem: the graph can be directed or undirected, and 'disjoint' can mean: vertex-disjoint or edge-disjoint.

In 1974 D.E. KNUTH (see [18]) showed that the integer multicommodity flow problem is NP-complete. This destroys (for those believing NP$\neq$co-NP or NP$\neq$P) the hope for nice theorems (like the max-flow min-cut theorem) and fast algorithms for the integer multicommodity flow problem.

In fact, Knuth showed that the integer multicommodity flow problem is NP-complete even if we restrict ourselves to problems in which all capacities are equal to 1. That is, the edge-disjoint paths problem for undirected graphs is NP-complete (hence also for directed graphs). In addition, LYNCH [24] showed in 1975 that the vertex-disjoint paths problem is NP-complete even if we restrict ourselves to planar undirected graphs. So also for the disjoint paths problem no Menger-type theorem may be expected.

Moreover, EVEN, ITAI and SHAMIR [7] proved in 1976 that the integer 2-commodity flow problem is NP-complete. And in 1980 FORTUNE, HOPCROFT and WYLLIE [11] showed the NP-completeness of the vertex-disjoint paths problem for directed graphs, even when restricted to the case $k = 2$.

We finish with two positive results. In 1979, KHACHIYAN [19] showed that linear programming problems can be solved in polynomial time. This implies that the (fractional) multicommodity flow problem is solvable in polynomial time. (This result was sharpened by TARDOS [34] to: the multicommodity flow problem is solvable in *strongly* polynomial time, that is by a series arithmetic operations the number of which is bounded by a polynomial in the size of the graph.)

In 1986, ROBERTSON and SEYMOUR [31], as a result of their *Graph Minors* project, proved that for each *fixed* k, there exists a polynomial-time algorithm for the disjoint paths problem for undirected graphs. Their algorithm has running time bounded by $c_k|V|^3$, for some constant heavily depending on k.

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