Some recent results on adjoint semigroups

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In this expository paper, we discuss some recent results in the theory of adjoint semigroups.

Introduction

A *C_0-semigroup of linear operators* on a Banach space X is a family $\mathbf{T} = \{T(t)\}_{t\geq 0}$ of bounded linear operators on X which satisfies

[(S1)] $T(0) = I$;

[(S2)] $T(s)T(t) = T(s+t)$ for all $s, t \geq 0$.

[(S3)] $\lim_{t\downarrow 0} \|T(t)x - x\| = 0$ for all $x \in X$.

The *generator* of $\mathbf{T}$ is the linear operator A with domain $D(A)$ defined by

$$\begin{aligned} D(A) &:= \{x \in X : \lim_{t\downarrow 0} \tfrac{1}{t}\big(T(t)x - x\big) \text{ exists}\}; \\ Ax &:= \lim_{t\downarrow 0} \tfrac{1}{t}\big(T(t)x - x\big), \quad x \in D(A). \end{aligned}$$

One of the motivations to study C_0-semigroups stems from the theory of (partial) differential equations. The reason is not hard to see: for $x \in D(A)$, the map $u(t) = T(t)x$ is the unique C^1-solution of the initial value problem

$$\begin{aligned} \frac{du}{dt}(t) &= Au(t), \quad t > 0, \\ u(0) &= x. \end{aligned} \tag{0.1}$$

By a C^1-solution we mean a continuously differentiable map $u : [0,\infty) \to X$ satisfying equation (0.1); differentiation is with respect to the norm of X. In fact, generators of C_0-semigroups are characterized by this property as follows. Let A be a densely defined linear operator on a Banach space X and assume that the resolvent set of A is not empty. Then the problem (0.1) has a unique C^1-solution for every $x \in D(A)$ if and only if A is the generator of a C_0-semigroup. In that case, the solution is given by $u(t) = T(t)x$.

For example, if $A \in M_n(\mathbb{C}^n)$ is an $n \times n$ matrix, then the solution to the initial value problem

$$\begin{aligned} \frac{du}{dt}(t) &= Au(t), \quad t > 0, \\ u(0) &= x, \end{aligned}$$

where $x \in \mathbb{C}^n$, is given by $u(t) = e^{tA}x$. Clearly, $T(t) = e^{tA}$ is a C_0-semigroup with generator A on the Banach space $X = \mathbb{C}^n$.

As a second example, let $X = C_0(\mathbb{R})$, the Banach space of complex-valued continuous functions on $\mathbb{R}$ with the sup-norm, and consider the family $\mathbf{T}$ defined by

$$T(t)f(s) = f(s+t), \quad f \in C_0(\mathbb{R}), s \in \mathbb{R}, t \geq 0.$$

One easily verifies that $\mathbf{T}$ is a C_0-semigroup on $C_0(\mathbb{R})$, the so-called translation semigroup. Its generator A is given by

$$\begin{aligned} D(A) &= \{f \in C_0(\mathbb{R}) \cap C^1(\mathbb{R}) : f' \in C_0(\mathbb{R})\}; \\ Af &= f', \quad f \in D(A). \end{aligned}$$

In this example, for initial values $f \in D(A)$, the semigroup $\mathbf{T}$ is related to the solutions u of the partial differential equation

$$\begin{aligned} \frac{\partial u}{\partial t}(t,s) &= \frac{\partial u}{\partial s}(t,s), \quad s \in \mathbb{R}, t > 0, \\ u(0,\cdot) &= f, \end{aligned} \tag{0.2}$$

by the relation $u(t,\cdot) = T(t)f$. By writing it in the form (0.1), equation (0.2) can be regarded as an equation on the Banach space $X = C_0(\mathbb{R})$.

As a third example, we mention the fact that the Laplacian Δ generates a C_0-semigroup on $X = C_0(\mathbb{R})$. In a similar way it corresponds to the solutions of the heat-equation

$$\begin{aligned} \frac{\partial u}{\partial t}(t,s) &= \Delta u(t,s), \quad s \in \mathbb{R}, t > 0, \\ u(0,\cdot) &= f \end{aligned}$$

and again, this equation can be regarded as a special case of (0.1).

Let $\mathbf{T}$ be a C_0-semigroup on X. The *adjoint semigroup* is the family $\mathbf{T}^* = \{T^*(t)\}_{t \geq 0}$ of operators on the dual space X^* defined by $T^*(t) := (T(t))^*$, $t \geq 0$. The abstract properties of the adjoint semigroup were first studied by PHILLIPS[14] and DE LEEUW[7]. Only in recent years, the interest for adjoint semigroups became more widespread, after useful applications were found in several fields. For example, Amann used adjoint semigroup techniques to prove that certain second order elliptic operators generate analytic semigroups in $L^1(\Omega)$. Also, adjoint semigroups proved to be a useful tool in studying certain delay equations and approximation problems. Through the work of CLÉMENT, DIEKMANN, HEIJMANS, GYLLENBERG and THIEME [3] it was realized that certain perturbation problems arising in population dynamics have a natural functional analytic setting in terms of adjoint semigroups.

As the latter work is the main motivation for the recent interest in adjoint semigroups, let us describe its heuristics in more detail. Consider a population whose individuals are parametrized by their age. More precisely, for each time t, we have a function $n(t,\cdot) \in L^1[0, a_{\max})$ describing the age-distribution of the population; here $a_{\max}$ is the maximal age the individuals can attain. Thus, at time t, the number of individuals whose age is between a_0 and $a_0 + \epsilon$ is given by

$$\int_{a_0}^{a_0+\epsilon} n(t,a)\, da.$$

The fact that the state space is $L^1[0, a_{\max})$ reflects the assumption that the total population size is finite at each time. If no new births occur and no individuals die before the age $a_{\max}$, the function n satisfies the relation

$$n(t+\epsilon, a) = \begin{cases} n(t, a-\epsilon), & a-\epsilon \geq 0, \\ 0, & a-\epsilon < 0. \end{cases}$$

Let $n_0 = n(0,\cdot) \in L^1[0, a_{\max})$ be the age-distribution at time $t = 0$. Defining $\mathbf{T}$ by

$$(T(t)n_0)(a) = n(t,a), \tag{0.3}$$

one easily verifies that $\mathbf{T}$ is a C_0-semigroup in $L^1[0, a_{\max})$ with generator A given by

$$D(A) = \{f \in AC[0, a_{\max}) : f(0) = 0\}$$

and $Af = -f'$. Note that the derivative exists a.e. and defines an L^1-function, f being absolutely continuous. This semigroup corresponds to the partial differential equation

$$\begin{aligned} \frac{\partial n}{\partial t}(t,a) &= -\frac{\partial n}{\partial a}(t,a), \quad a \in [0, a_{\max}),\ t > 0, \\ n(t,0) &= 0, \quad t \geq 0, \\ n(0,\cdot) &= n_0. \end{aligned}$$

If we now assume that the individuals reproduce at an age-dependent rate $\beta(\cdot) \in L^\infty[0, a_{\max})$, the equation governing the population becomes

$$\begin{aligned} \frac{\partial n}{\partial t}(t,a) &= -\frac{\partial n}{\partial a}(t,a), \\ n(t,0) &= \int_0^{a_{\max}} \beta(a) n(t,a)\, da, \\ n(0,\cdot) &= n_0. \end{aligned} \tag{0.4}$$

Thus, one can think of the births as a perturbation of the boundary condition at $a = 0$.

If we try to rewrite equation (0.4) as an ordinary differential equation in the Banach space $X = L^1[0, a_{\max})$, we run into the difficulty of how to deal with the boundary condition, as ordinary differential equations do not have boundary conditions. But thinking of L^1-functions as (absolutely continuous) measures, we can identify $n(t, \cdot) \in L^1[0, a_{\max})$ with the measure $N(t) \in M[0, a_{\max})$ whose density is $n(t, \cdot)$. Then, at least formally, we can rewrite (0.4) as

$$\begin{aligned} \frac{dN}{dt}(t) &= A_0\big(N(t)\big) + \Big(\int_0^{a_{\max}} \beta(a)\ d\big(N(t)\big)(a)\Big)\ \delta_0, \\ N(0) &= N_0, \end{aligned} \tag{0.5}$$

where the derivative is, e.g., the weak*-derivative of the measure-valued function $N(\cdot)$, A_0 is the Radon-Nikodym derivative, and δ_0 is the Dirac measure concentrated at $a = 0$. In this way we can interpret equation (0.5) as an equation in the Banach space $M[0, a_{\max})$ of bounded Borel measures on $[0, a_{\max})$. The perturbation caused by birth becomes an additive perturbation by a bounded linear operator $B_0 : L^1[0, a_{\max}) \to M[0, a_{\max})$, given by

$$B_0 f = \Big(\int_0^{a_{\max}} \beta(a) f(a)\ da\Big)\ \delta_0.$$

In order to deal with equation (0.5) in a rigorous setting of semigroups $\mathbf{T}$ on some Banach space X, one needs a perturbation theory in which perturbations are allowed to be of the form $B : X \to Y$, where Y is some 'larger' Banach space containing X as a subspace. Precisely this can be done by means of adjoint semigroup theory. It turns out that a perturbation theory can be constructed for the case $Y = X^{\odot\times}$. This is a space that can be canonically constructed by means of duality from the pair $(X, \mathbf{T})$; the precise definition is given in Section 4. For the above semigroup on $X = L^1[0, a_{\max})$ we have $X^{\odot\times} = M[0, a_{\max})$, so this example indeed fits into that theory.

The relation to adjoint semigroup theory becomes even more apparent if one considers the equation dual to (0.4), which is

$$\begin{aligned} \frac{\partial m}{\partial t}(t, a) &= \frac{\partial m}{\partial a}(t, a) + \beta(a) m(t, 0), \\ m(t, a_{\max}) &= 0, \\ m(0, \cdot) &= m_0. \end{aligned} \tag{0.6}$$

The perturbation by births now appears as a 'genuine' additive perturbation. The proper state space for this problem is $C_0[0, a_{\max})$, the space of continuous functions on $[0, a_{\max})$ vanishing at $a = a_{\max}$ (the deeper reason for this is that this space is the $\odot$-dual $X^\odot$ of $X = L^1[0, a_{\max})$. The C_0-semigroup associated to equation (0.6) is $\mathbf{T}^\odot$, the $\odot$-adjoint of $\mathbf{T}$; cf. Section 1). The boundary condition $m(t, a_{\max}) = 0$ is then built into the state space. Analogously to what we did in (0.5), we now think of $C_0[0, a_{\max})$ as embedded in the larger space $L^\infty[0, a_{\max})$, and regard the birth perturbation as an additive perturbation by the bounded linear operator

$$B_1 : C_0[0, a_{\max}) \to L^\infty[0, a_{\max}), \qquad B_1 f = f(0)\beta$$

(recall our assumption $\beta \in L^\infty[0, a_{\max})$).

The dual equation fits into the $\odot\times$-perturbation theory equally well, because for the semigroup $\mathbf{T}^\odot$ on $X^\odot = C_0[0, a_{\max})$ we have $(X^\odot)^{\odot\times} = L^\infty[0, a_{\max})$ ($= X^*$, cf. Theorem 4.3).

Thus, it was recognized that problems of the type discussed above can be succesfully dealt with in an abstract framework of adjoint semigroup theory. These applications also caused a renewed interest for the abstract functional analysis of adjoint semigroups. Many new results were proved by, e.g., De Pagter, Grabosch and Nagel, Greiner, Schep and the author [5, 6, 9, 10, 11, 13]. It was found that adjoint semigroups are interesting objects in their own right and that much can be said about them by using results and methods from Banach space theory.

In this note, we will mainly deal with the abstract theory and highlight some of its most interesting results. We return to the above example only in Section 4. For further details about the application of adjoint semigroups to age-dependent populations we refer to [3].

Most proofs of the results presented here and further results can be found in [9], which is based on my Ph.D. thesis prepared at the CWI in Amsterdam. I would like to thank my colleagues, especially Hans Heesterbeek and my supervisor Odo Diekmann, for making it such a wonderful place to work. Also, I would like to thank Ben de Pagter for his constant interest and encouragement.

1. Strong continuity of the adjoint semigroup

It is immediate to verify that the adjoint semigroup $\mathbf{T}^*$ has properties (S1) and (S2). Property (S3), strong continuity, need not hold, however.

Example 1.1.

(i) Let $\mathbf{T}$ be the translation semigroup on $X = C_0(\mathbb{R})$ as defined in the introduction. Its adjoint on $X^* = M(\mathbb{R})$, the space of bounded Borel measures on $\mathbb{R}$, is given by $(T^*(t)\mu)(F) = \mu(F - t)$.

(ii) Let $\mathbf{T}$ be the semigroup on $X = L^1[0, a_{\max})$ defined in equation (0.3). Then for $f \in X^* = L^\infty[0, a_{\max})$ one has

$$(T^*(t)f)(a) = \begin{cases} f(a+t), & a + t \le a_{\max}; \\ 0, & a + t > a_{\max}. \end{cases}$$

∎

Consider the semigroup in Example 1.1 (i) and let μ be a Dirac measure δ. Then it is clear that $\lim_{t\downarrow 0} \|T^*(t)\delta - \delta\| = 2$. Thus, $\mathbf{T}^*$ fails to be strongly continuous. In fact, $\lim_{t\downarrow 0}\|T^*(t)\mu-\mu\|=0$ if and only if μ is absolutely continuous with respect to the Lebesgue measure.

We are led to the following definition:

$$X^\odot := \{x^*\in X^* : \lim_{t\downarrow 0}\|T^*(t)x^*-x^*\|=0\}.$$

Thus, $X^\odot$ is precisely the subspace of X^* on which the action of $\mathbf{T}^*$ is strongly continuous. It is easy to see that $X^\odot$ is a closed, $\mathbf{T}^*$-invariant subspace. The restricted semigroup $\mathbf{T}^\odot$ defined by $T^\odot(t) := T^*(t)|_{X^\odot}$ is a C_0-semigroup on $X^\odot$. Starting from this semigroup, we can repeat this procedure and define $X^{\odot *}, X^{\odot\odot}$, and $\mathbf{T}^{\odot *}$ and $\mathbf{T}^{\odot\odot}$. The generators of $\mathbf{T}^\odot$ and $\mathbf{T}^{\odot\odot}$ will be denoted by $A^\odot$ and $A^{\odot\odot}$, repectively.

The natural map $j: X\to X^{\odot *}$, defined by

$$\langle jx, x^\odot\rangle := \langle x^\odot, x\rangle, \quad x^\odot\in X^\odot,$$

can be shown to be an embedding. Thus one can identify X isomorphically with a closed subspace of $X^{\odot *}$. The map j need not be isometric; cf. Example 1.6 below. If $X = X^{\odot\odot}$, then we say that X is *$\odot$-reflexive with respect to* $\mathbf{T}$. Trivially, if X is reflexive, then it is $\odot$-reflexive with respect to every C_0-semigroup. The following characterization of $\odot$-reflexivity is due to de PAGTER [12] and improves an earlier result of HILLE and PHILLIPS.

THEOREM 1.2. *X is $\odot$-reflexive with respect to $\mathbf{T}$ if and only if the resolvent $R(\lambda, A) = (\lambda - A)^{-1}$ is weakly compact for some $\lambda\in\varrho(A)$.*

EXAMPLE 1.3. Here are some easy examples.

(i) Let $X = C_0(\mathbb{R})$ and $\mathbf{T}$ translation. Then $X^\odot$ consists of all finite Borel measures on $\mathbb{R}$ which are absolutely continuous with the Lebesgue measure. Hence, by the Radon-Nikodym theorem, we can identify $X^\odot$ with $L^1(\mathbb{R})$. Furthermore, one has $X^{\odot\odot} = BUC(\mathbb{R})$, the space of all bounded, uniformly continuous functions on $\mathbb{R}$. Thus, X is not $\odot$-reflexive with respect to $\mathbf{T}$.

Similarly, one can consider rotation on $X = C(\Gamma)$, Γ being the unit circle. One has $X^\odot = L^1(\Gamma)$ and, due to the compactness of Γ, $X^{\odot\odot} = C(\Gamma)$. So in this case, X is $\odot$-reflexive with respect to $\mathbf{T}$. This can also be seen directly from the fact that the resolvent $(\lambda - d/d\theta)^{-1}$ is compact for each $\lambda > 0$.

(ii) Let $\mathbf{T}$ be defined on $X = L^1[0, a_{\max})$ by equation (0.3). Then $X^\odot = C_0[0, a_{\max})$ and $X^{\odot\odot} = X = L^1[0, a_{\max})$. Thus, X is $\odot$-reflexive with respect to $\mathbf{T}$; this depends on the fact that we assume $a_{\max} < \infty$. In the case $a_{\max} = \infty$, the space $L^1[0, a_{\max})$ is not $\odot$-reflexive with respect to $\mathbf{T}$.

(iii) Let X be a Banach space with a Schauder basis $\{x_n\}_{n=1}^\infty$ and define $\mathbf{T}$ by $T(t)x_n = e^{-nt}x_n$. Then $\mathbf{T}$ is a C_0-semigroup on X. Let $\{x_n^*\}_{n=1}^\infty$ be the coordinate functionals corresponding to this basis, i.e., x_n^* is the (bounded) functional defined by $\langle x_n^*, \sum_{k=1}^\infty \alpha_k x_k\rangle = \alpha_n$. The space $X^\odot$ is precisely the closed linear span of $\{x_n^*\}_{n=1}^\infty$ in X^*. In particular, $\{x_n^*\}_{n=1}^\infty$ is a Schauder basis for $X^\odot$ and we have $T^\odot(t)x_n^* = e^{-nt}x_n^*$. Therefore, $X^{\odot\odot}$ is precisely the closure of the coordinate functionals of this basis, which are given by $\{x_n\}_{n=1}^\infty$. It follows that X is $\odot$-reflexive with respect to $\mathbf{T}$.

(iv) If X is $\odot$-reflexive with respect to $\mathbf{T}$, then $X^\odot$ is $\odot$-reflexive with respect to $\mathbf{T}^\odot$. ■

Although $\mathbf{T}^*$ need not be strongly continuous, the inequality

$$|\langle T^*(t)x^* - x^*, x\rangle| \le \|x^*\|\,\|T(t)x - x\|$$

shows that $\mathbf{T}^*$ is weak*-continuous. Hence, if X is a reflexive Banach space, then $\mathbf{T}$ is weakly continuous. By a standard theorem of semigroup theory, weakly continuous semigroups are strongly continuous, and we obtain the following classical result due to PHILLIPS [14]:

THEOREM 1.4. *If $\mathbf{T}$ is a C_0-semigroup on a reflexive Banach space, then $X^\odot = X^*$.*

Another, more elementary proof of Theorem 1.4 is as follows: first one proves that $X^\odot = \overline{D(A^*)}$, where A^* is the adjoint of the generator A. Since A is always densely defined and closed, $D(A^*)$ and hence also $X^\odot$ is weak*-dense in X^*. Thus, by reflexivity, $X^\odot$ is weakly dense. But $X^\odot$ is also norm-closed, hence weakly closed, and therefore $X^\odot = X^*$.

The converse of Theorem 1.4 is false: there are non-reflexive Banach spaces on which the adjoint of every C_0-semigroup is strongly continuous. In fact, there is a well-known theorem of Lotz [Lo] that every C_0-semigroup on $L^\infty[0,1]$ is uniformly continuous, i.e. $\lim_{t\downarrow 0}\|T(t)-I\| = 0$. Of course, the adjoint of such a semigroup is uniformly continuous as well, and hence strongly continuous. However, if X is a non-reflexive Banach space with a Schauder basis, then there exists a C_0-semigroup on X whose adjoint fails to be strongly continuous. In fact, X has a (probably different) Schauder basis whose coordinate functionals span a *proper* closed subspace of X^*. Thus, for a large class of spaces, reflexivity is the only sufficient criterion that guarantees $X^\odot = X^*$. For special classes of Banach spaces or semigroups sometimes more can be said, however. An example is the following theorem about c_0, the Banach space of all scalar sequences which converge to 0 with the sup-norm.

THEOREM 1.5. *Let $\mathbf{T}$ be a C_0-semigroup on c_0. If there exist $M < 2$ and $\omega \in \mathbb{R}$ such that $\|T(t)\| \le Me^{\omega t}$ for all $t \ge 0$, then $c_0^\odot = c_0^*$.*

It is an easy consequence of the uniform boundedness theorem and the semigroup property (S2) that for every C_0-semigroup $\mathbf{T}$ there are constants $M \ge 1$ and $\omega \in \mathbb{R}$ such that $\|T(t)\| \le Me^{\omega t}$ for all $t \ge 0$. The point of the theorem is that M should be less than 2. The constant 2 is optimal, as is shown by the following example, which is also useful for the discussion in the next section.

EXAMPLE 1.6. Let x_n be the nth unit vector of c_0 and put $y_n = \sum_{k=1}^n x_k$. The sequence $\{y_n\}_{n=1}^\infty$ can be shown to be a Schauder basis for c_0. The formula $T(t)y_n = e^{-nt}y_n$ then defines a C_0-semigroup on c_0 satisfying $\|T(t)\| \le 2$ for all t. Moreover, $c_0^\odot$ is the closed linear span of the coordinate functionals of $\{y_n\}_{n=1}^\infty$, which is a co-dimension one subspace of c_0^*. ∎

This semigroup has the further pathological property that the natural map $j : c_0 \to c_0^{\odot *}$ fails to be isometric. In general, it is an easy consequence of the bipolar theorem that $j : X \to X^{\odot *}$ is isometric if and only if the closed unit ball B_X of X is closed in the weak topology induced by $X^\odot$. In the above example, one can show directly that $(2, 0, 0, 0, ...) \in \overline{B_{c_0}}^{\sigma(c_0, c_0^\odot)}$.

2. THE CO-DIMENSION OF $X^\odot$ IN X^*

Knowing that $X^\odot$ can be a proper subspace of X^*, the question arises what can be said about its 'relative size' in X^*. We noted already in the introduction that $X^\odot$ is weak*-dense in X^*, but with respect to the norm-topology the situation is far more subtle. In that case, the natural object of study is the size of the quotient space $X^*/X^\odot$. We start with noting that there is a nice description of the quotient norm of $X^*/X^\odot$. Let $q : X^* \to X^*/X^\odot$ be the quotient map.

THEOREM 2.1. *Let* $\mathbf{T}$ *be a* C_0*-semigroup on a Banach space* X. *Then*

$$|||qx^*||| = \limsup_{t \downarrow 0} \|T^*(t)x^* - x^*\|$$

defines an equivalent norm on $X^*/X^\odot$.

Example 1.6 seems to indicate that not very much can be said about the size of $X^*/X^\odot$. Indeed, for the semigroup there one has $\dim c_0^*/c_0^\odot = 1$, and by taking direct sums it is possible to construct semigroups for which $X^*/X^\odot$ can have any finite dimension. Let us analyse this example more closely. The adjoint semigroup is easily seen to be strongly continuous for $t > 0$. This is equivalent to saying that $T^*(t)x^* \in c_0^\odot$ for every $t > 0$ and $x^* \in l^1$. Letting $q : c_0^* \to c_0^*/c_0^\odot$ be the quotient map, this is in turn equivalent to saying that $q(T^*(t)x^*) = 0$ for all $t > 0$ and $x^* \in l^1$.

On $X^*/X^\odot$, there is a natural quotient semigroup $T_q(t)$, defined by $T_q(t)qx^* = q(T^*(t)x^*)$. Thus, in the above example, all orbits of $T_q(t)$ are zero for $t > 0$. This is a special case of the following result. We say that a Banach space valued function is *separably valued* if its range is contained in some separable subspace.

THEOREM 2.2. *Let* $\mathbf{T}$ *be a* C_0*-semigroup on a Banach space* X *and let* $x^* \in X^*$. *If the orbit* $t \mapsto T_q(t)qx^*$ *is separably-valued, then* $T_q(t)qx^* = 0$ *for all* $t > 0$.

This theorem implies that non-zero orbits of the quotient semigroup cannot be strongly continuous. An elementary proof of this is given in [10].

COROLLARY 2.3. *If* $\mathbf{T}$ *extends to a* C_0*-group, then* $X^*/X^\odot$ *is either zero or non-separable.*

Indeed, if $X^*/X^\odot$ is separable, then $\mathbf{T}^*$ is strongly continuous for $t > 0$ by Theorem 2.2, and since $\mathbf{T}$ extends to a group, this implies that $\mathbf{T}^*$ is strongly continuous for all t, i.e. $X^\odot = X^*$.

Let us make a few comments on the proof of Corollary 2.3 (Theorem 2.3 can be proved in a similar way by refining the argument a little bit). So let us assume that $X^*/X^\odot$ is separable. It is a well-known fact that a strongly measurable semigroup is strongly continuous for $t > 0$. Therefore the idea is to try to use the Pettis measurability theorem: if (Ω, Σ, μ) is a σ-finite measure space and $f : \Omega \to X$ is a weakly measurable, separably valued map into the Banach space X, then f is strongly measurable. Now let us look at a quotient orbit $t \mapsto T_q(t)qx^*$. Since we assume that it is separably valued, the strategy is to try to prove some kind of weak measurability property. By using the tool of so-called Baire-1 functionals, this can indeed be done after changing to a weaker norm (in which the completion of $X^*/X^\odot$ is again separable). Recall that a functional $x^{**} \in X^{**}$ is called *Baire-1* if it is the weak*-limit of a *sequence* of elements in X. The idea is to construct sufficiently many Baire-1 functionals in the annihilator $X^{\odot\perp}$ of $X^\odot$. Identifying $X^{\odot\perp}$ with $(X^*/X^\odot)^*$, the quotient semigroup on $X^*/X^\odot$ is measurable with respect to any such functional. One obtains that in the norm induced by these functionals, the quotient orbit is weakly measurable, hence strongly measurable by Pettis's theorem, and hence strongly continuous for $t > 0$.

The second difficulty is to show that $T_q(t)qx^* = 0$ for $t > 0$. The idea is to turn to the dual space of $X^*/X^\odot$ and to show that $\langle T_q^*(t)x^{\odot\perp}, qx^*\rangle = 0$ for each of the Baire-1 functionals $x^{\odot\perp}$ discussed above. The proof of this is based on the fact that there is a natural isomorphism

$$X_{\odot\odot} \simeq X^{\odot\odot} \oplus (X^*/X^\odot)^\odot,$$

where $X_{\odot\odot}$ is the space of strong continuity of the bi-adjoint semigroup $\{T^{**}(t)\}_{t\geq 0}$ on X^{**} and $(X^*/X^\odot)^\odot$ is the space of strong continuity of the adjoint of the quotient semigroup.

In this way one is led to the question in how far $X^{\odot\odot}$ and $X_{\odot\odot}$ can differ. The point is that, a priori, the space $(X^*/X^\odot)^\odot$ could be zero, since the quotient semigroup on $X^*/X^\odot$ is not a C_0-semigroup. There are indeed examples where this happens; for instance if $\mathbf{T}^*$ is strongly continuous for $t > 0$. More generally, one can prove that this happens if each of the orbits $t \mapsto T^*(t)x^*$ is locally Pettis integrable. It is a result of this type that finally leads to Theorem 2.2.

By Theorem 3.3 below, an example of a semigroup for which $X^{\odot\odot}$ is a proper subspace of $X_{\odot\odot}$ is the translation group on $X = C_0(\mathbb{R})$.

3. THE ADJOINT OF A POSITIVE SEMIGROUP

Many semigroups encountered in applications are positive, i.e. they map positive elements to positive elements. Throughout this section, we assume that $\mathbf{T}$ is a positive C_0-semigroup on a Banach lattice E.

It is for this class of semigroups for which the most detailed abstract results have been obtained. We refer the reader to [5], [9], [10], [11] and [13] for more details. One of the interesting discoveries was that several results concerning the behaviour of Borel measures on $\mathbb{R}$ can be generalized to results about the adjoints of positive C_0-semigroups on Banach lattices. We will deal with these results below.

The first question we address is whether $E^\odot$ has some nice lattice properties if $\mathbf{T}$ does. For example, one might hope that $E^\odot$ is a sublattice if $\mathbf{T}$ is positive. This was an open problem for some time and was finally solved to the negative by GRABOSCH and NAGEL [5], who constructed the following counterexample.

EXAMPLE 3.1. Let $E := L^1[0,1] \times L^1[0,1]$ with norm $\|(f,g)\| := \|f\| + \|g\|$. Consider the operator

$$A = \begin{pmatrix} d/dx & 0 \\ 0 & d/dx \end{pmatrix}$$

with domain

$$D(A) = \left\{ \begin{pmatrix} f \\ g \end{pmatrix} \in E : \ f, g \in AC[0,1], \ \begin{pmatrix} f(1) \\ g(1) \end{pmatrix} = B \begin{pmatrix} f(0) \\ g(0) \end{pmatrix} \right\}.$$

Here $AC[0,1]$ denotes the linear space of all absolutely continuous functions on $[0,1]$, and B is a real 2×2 matrix. The operator A generates a positive C_0-semigroup on E. One can show that

$$E^\odot = \left\{ \begin{pmatrix} \phi \\ \psi \end{pmatrix} \in C[0,1] \times C[0,1] : \ \begin{pmatrix} \phi(0) \\ \psi(0) \end{pmatrix} = B^t \begin{pmatrix} \phi(1) \\ \psi(1) \end{pmatrix} \right\}.$$

It follows that $E^\odot$ is a sublattice of $C[0,1] \times C[0,1]$, and hence of E^*, if and only if B is a lattice homomorphism on $\mathbb{R}^2$. This is the case if and only if B is a positive diagonal- or off-diagonal matrix. In fact, in this example, $E^\odot$ is a Banach lattice with respect to its own ordering if and only it is a sublattice of E^*, so in general $E^\odot$ need not even be a Banach lattice in its own right. ∎

On Banach lattices which are 'sufficiently different' from L^1-spaces, one has the following positive result.

THEOREM 3.2. *Let $\mathbf{T}$ be a positive C_0-semigroup on a Banach lattice E. If E^* has order continuous norm, then $E^\odot$ is a projection band in E^*.*

Examples of spaces whose duals have order continuous norm are the space of continuous functions $C(K)$ and $C_0(\Omega)$. One can say more about adjoints of positive semigroups on $C(K)$:

THEOREM 3.3. *Let $\mathbf{T}$ be a positive C_0-semigroup on $E = C(K)$, K compact Hausdorff. Then the following are equivalent:*

(i) $t \mapsto T^*(t)x^*$ *is weakly Borel measurable for each* $x^* \in E^*$*;*
(ii) $\mathbf{T}^*$ *is strongly continuous for* $t > 0$*;*
(iii) $E^{\odot\odot} = E_{\odot\odot}$.

In particular, if $\mathbf{T}$ extends to a C_0-group, then $\mathbf{T}^*$ is weakly Borel measurable if and only if $\mathbf{T}^*$ is strongly continuous. Theorem 3.3 is non-trivial; it depends on a deep result result of Riddle, Saab and Uhl that a weakly Borel measurable map taking values in the dual of a separable Banach space is Pettis integrable. One might wonder whether weak (i.e. weak *Lebesgue*) measurability already implies strong continuity for $t > 0$. Under certain set-theoretical assumptions, this is true, but it is an open question whether this can be proved directly.

Also non-trivial is the following beautiful result of Talagrand [15], which is an orbit-wise analogue of Theorem 3.3 for the case $E = L^1(\Gamma)$.

THEOREM 3.4. *Let $\mathbf{T}$ be the rotation group on $L^1(\Gamma)$, Γ the unit circle. If for some $f \in L^\infty(\Gamma)$ the orbit $t \mapsto T^*(t)f$ is weakly measurable, then f is equal a.e. to a Riemann measurable function.*

Recall that an (everywhere defined) function is Riemann measurable if it is continuous a.e. Assuming Martin's Axiom (MA), the following orbitwise generalization of Theorem 3.3 can be proved [11]:

THEOREM 3.5 (MA). *Let $\mathbf{T}$ be a positive C_0-semigroup on a Banach lattice E. If, for some $x^* \in X^*$, the map $t \mapsto T^*(t)x^*$ is weakly measurable, then$T^*(t)x^*$ belongs to the band generated by $E^\odot$ for all $t > 0$. If $\mathbf{T}^*$ is a lattice semigroup, in particular if $\mathbf{T}$ extends to a positive group, then x^* itself belongs to this band as well.*

We recall the fact that Martin's Axiom is implied by (but does not imply) the Continuum Hypothesis. Applied to the group of translations on $C_0(\mathbb{R})$, Theorem 3.5 implies that translation of a bounded Borel measure μ on $\mathbb{R}$ is weakly measurable if and only if μ is absolutely continuous with respect to the Lebesgue measure (in which case translation of μ is strongly continuous).

After these 'weak implies strong' results, we turn to the lattice properties of individual orbits of $\mathbf{T}^*$. The most interesting results are concerned with the behaviour of elements in E^* which are disjoint from $E^\odot$.

THEOREM 3.6. *Let $\mathbf{T}$ be a positive C_0-semigroup on a Banach lattice E. Suppose that either E has a quasi-interior point of E^* has order continuous norm. If $x^* \perp E^\odot$, then $T^*(t)x^* \perp x^*$ for almost all $t \geq 0$.*

Recall that $u \in E$ is a *quasi-interior point* if the ideal generated by u is norm dense in E. Every separable Banach lattice and every L^∞-space have quasi-interior points. In the special case where $\mathbf{T}$ is the translation group on $E = C_0(\mathbb{R})$, we have $\mu \perp E^\odot = L^1(\mathbb{R})$ if and only if μ is singular with respect to the Lebesgue measure (Example 1.1), and the theorem reduces to the classical theorem of Wiener and Young [WY] that a singular measure on $\mathbb{R}$ is disjoint to almost all of its translates.

THEOREM 3.7. *Let $\mathbf{T}$ be a positive C_0-semigroup on a Banach lattice E. If $x^* \perp E^\odot$, then*

$$\limsup_{t \downarrow 0} \|T^*(t)x^* - x^*\| \geq 2\|x^*\|.$$

In the case $E = C(K)$, this follows easily from Theorem 3.6: $E^{\odot}$ is a projection band in $M(K)$, hence lattice isometric to some L^1-space. If in an L^1-space we have $T^*(t)x^* \perp x^*$, then $\|T^*(t)x^* - x^*\| = \|T^*(t)x^*\| + \|T^*(t)x^*\|$, and by weak*-continuity it follows that $\limsup_{t\downarrow 0} \|T^*(t)x^*\| \geq \|x^*\|$. However, arbitrary dual Banach lattices do not have additive norm and, what is more, Theorem 3.6 fails for arbitrary Banach lattices.

Our final result is concerned with multiplication semigroups. A C_0-semigroup on a Banach lattice E is called a *multiplication semigroup* if each operator $\mathbf{T}$ is a band preserving operator. The reason for this terminology is that in most classical function spaces, an operator is band preserving if and only if it can be represented as multiplication with some (continuous, measurable) function. If $\mathbf{T}$ is a multiplication semigroup, then $\mathbf{T}$ is positive, $E^{\odot}$ is an ideal in E^* and $\mathbf{T}^*$ is strongly continuous for $t > 0$.

There are two trivial examples of $\odot$-reflexive multiplication semigroups: those on reflexive Banach lattices E, and multiplication semigroups of the form $T(t)x_n = e^{-k_n t}x_n$, where $\{x_n\}_{n=1}^{\infty}$ is an unconditional Schauder basis for E and (k_n) is a sequence of real numbers which is bounded from below. Note that in both cases, E has order continuous norm. The following theorem states that these are essentially the only examples:

THEOREM 3.8. *If E is $\odot$-reflexive with respect to a multiplication semigroup $\mathbf{T}$, then E has order continuous norm. Furthermore, if E does not contain a reflexive projection band, then E has an unconditional Schauder basis $\{x_n\}_{n=1}^{\infty}$ and $\mathbf{T}$ is of the form $T(t)x_n = e^{-k_n t}x_n$, where (k_n) is a sequence of real numbers which is bounded from below.*

In general, if E is $\odot$-reflexive with respect to a positive C_0-semigroup $\mathbf{T}$, then E need not have order continuous norm, even if $\mathbf{T}$ is disjointness preserving, as is shown be the rotation group on $C(\Gamma)$. However, if a Banach space X is $\odot$-reflexive with respect to a C_0-semigroup $\mathbf{T}$, then X does not contain a closed subspace isomorphic to l^{∞}. Therefore, by the general theory of Banach lattices, if a $\odot$-reflexive E is σ-Dedekind complete, it must have order continuous norm.

4. THE SPACE $X^{\odot\times}$

In the introduction we were led to the study of initial value problems of the type

$$\begin{aligned} \frac{du}{dt}(t) &= Au + Bu, \\ u(0) &= x \end{aligned} \tag{4.1}$$

where A is the generator of a C_0-semigroup on a Banach space X and $B : X \to Y$ is a perturbation taking its values in a Banach space Y containing X as a closed subspace. In the particular case studied there we had $X = L^1[0, a_{\max})$ and $Y = M[0, a_{\max})$. For the dual equation, we had $X = C_0[0, a_{\max})$ and $Y = L^{\infty}[0, a_{\max})$. We already observed that, in both these cases, X is $\odot$-reflexive with respect to $\mathbf{T}$ and that the relation $Y = X^{\odot\times} = X^{\odot *}$ holds. In this section, we show how equation (4.1) can be given a precise meaning and how it can be solved by means of abstract methods.

We start with the introduction of the canonical space $X^{\odot\times}$ associated with a C_0-semigroup. This space is a closed subspace between X and $X^{\odot *}$: we have natural inclusions of closed subspaces $X \subset X^{\odot\times} \subset X^{\odot *}$. We define $X^{\odot\times}$ to be the subspace of $X^{\odot *}$ which is mapped into X by the adjoint of the resolvent of $\mathbf{T}^{\odot}$:

$$X^{\odot\times} = \{x^{\odot *} \in X^{\odot *} : R(\lambda, A^{\odot})^{*} x^{\odot *} \in X\}.$$

The space $X^{\odot\times}$ is a closed subspace of $X^{\odot *}$. It is an easy consequence of the resolvent identity that $X^{\odot\times}$ is independent of the choice of $\lambda \in \varrho(A)$. Obviously, if X is $\odot$-reflexive with respect to $\mathbf{T}$, then $X^{\odot\times} = X^{\odot *}$.

THEOREM 4.1. *Let $\mathbf{T}$ be a C_0-semigroup on X.*

(i) If $B : X \to X^{\odot\times}$ is a bounded operator, then the part of $A^{\odot *} + B$ in X generates a C_0-semigroup $\mathbf{U}$ on X. Moreover, we have

$$\|U(t) - T(t)\| = O(t), \quad t \downarrow 0. \tag{4.2}$$

(ii) Conversely, if $\mathbf{U}$ and $\mathbf{T}$ are two C_0-semigroups on X such that (4.2) holds, then these semigroups have the same space $X^{\odot\times}$, and there exists a bounded operator $B : X \to X^{\odot\times}$ such that the generator $A_{\mathbf{U}}$ is precisely the part of $A_{\mathbf{T}}^{\odot *} + B$ in X.

Moreover, if A is a generator and $B : X \to X^{\odot *}$ is bounded, then the part of $A^{\odot *} + B$ generates a C_0-semigroup on X if and only if B takes its values in $X^{\odot\times}$. These results are essentially contained in [3, Part I] and [4].

Assertion (i) shows how equation (4.1) can be given a meaning: the 'correct' initial value problem is

$$\begin{aligned} \frac{du}{dt}(t) &= A^{\odot\times} u + Bu, \\ u(0) &= x. \end{aligned} \tag{4.3}$$

A priori, this equation makes sense as an equation in the space $X^{\odot\times}$. But thanks to Theorem 4.1, the part of $A^{\odot\times} + B$ in X is the generator of a C_0-semigroup $\mathbf{U}$ on X. Thus, for x in the domain of this part, we can regard equation (4.3) as an initial value problem on X, the solution of which is given by $u(t) = U(t)x$. In the concrete example on $X = L^1[0, a_{\max})$ of the introduction, we have $A_0 = A^{\odot\times}$ and $B_0 : X \to X^{\odot\times}$. Hence, the part of the operator $A_0 + B_0$ in $L^1[0, a_{\max})$ generates a C_0-semigroup $\mathbf{U}$ on $L^1[0, a_{\max})$. For initial values $n(0, \cdot)$ in its domain, equation (0.4) admits the solution $u(t) = U(t)n(0, \cdot)$. The dual equation can be dealt with analogously.

We will now continue with some further properties of the space $X^{\odot\times}$, all of which show that it truly is an important intrinsic object associated to X and $\mathbf{T}$, rather than just some *ad hoc* object.

For the next result, we need the fact that there is a natural embedding $k : X^{\odot\odot} \to X^{**}$, given by

$$\langle kx^{\odot\odot}, x^* \rangle := \lim_{\lambda\to\infty} \langle x^{\odot\odot}, \lambda R(\lambda, A^*)x^* \rangle.$$

The limit always exists, so this definition makes sense. By means of the map k, each $x^{\odot\odot} \in X^{\odot\odot}$ acts as a bounded linear functional on X^*. The following example illustrates this in the case of the translation group.

EXAMPLE 4.2. Let $\mathbf{T}$ be the translation group on $X = C_0(\mathbb{R})$. We know that $X^{\odot\odot} = BUC(\mathbb{R})$. One can verify by direct computation that for $f \in X^{\odot\odot}$ and a finite Borel measure $\mu \in X^*$ we have

$$\langle kf, \mu \rangle = \int_{\mathbb{R}} f(t)\ d\mu(t),$$

where the integral is the abstract Lebesgue integral. Thus, we recover the natural pairing of $BUC(\mathbb{R})$ with the space of finite Borel measures $M(\mathbb{R})$. ■

The other way around, one can regard each $x^* \in X^*$ as a bounded functional on $X^{\odot\odot}$. In doing so X^* can be identified with a closed subspace of $X^{\odot\odot *} = (X^{\odot})^{\odot *}$. The following result tells us which subspace:

THEOREM 4.3. *Under the above identifications,* $X^* = (X^{\odot})^{\odot\times}$.

Thus, the ${}^{\odot\times}$-spaces occur 'naturally'. Our final result gives one more striking example of a class of spaces which turn out to be ${}^{\odot\times}$-spaces. To this end, we briefly recall some concepts from interpolation theory. For the details, we refer to [2]. Let $\mathbf{T}$ be a C_0-semigroup on X. For $0 < \alpha \leq 1$ we define

$$\begin{aligned} X_\alpha &:= \{x \in X : \limsup_{t\downarrow 0} \tfrac{1}{t^\alpha}\|T(t)x - x\| = 0\}; \\ X_{\alpha,\infty} &:= \{x \in X : \limsup_{t\downarrow 0} \tfrac{1}{t^\alpha}\|T(t)x - x\| < \infty\}. \end{aligned}$$

With respect to appropriately chosen norms, these spaces are Banach spaces, which can be thought of as abstract little- and big Hölder spaces of exponent α. Clearly, X_α is $\mathbf{T}$-invariant; the restriction of $\mathbf{T}$ defines a C_0-semigroup $\mathbf{T}_\alpha$ on X_α. For this semigroup we can prove:

THEOREM 4.4. *There exists a natural isomorphism* $X_{\alpha,\infty} \simeq (X_\alpha)^{\odot\times}$.

In fact, if we let A_α denote the generator of $\mathbf{T}_\alpha$, the isomorphism is given by $(\lambda - A)R(\lambda, A_\alpha^{\odot})^*|_{(X_\alpha)^{\odot\times}}$; this map does not depend on the choice of $\lambda \in \varrho(A)$. For the specialists, this is why Theorem 4.4 works: one can show that $X_{\alpha,\infty}$ can be identified with the so-called Favard class of the extrapolation space $(X_\alpha)_{-1}$ of X_α; on the other hand, for every C_0-semigroup on a Banach space X one has $Fav(X) = D(A^{\odot *}) \cap X$, so $X^{\odot\times}$ is the inverse image under $A^{\odot *}$ of the Favard class of X, and this implies that $X^{\odot\times}$ can be identified with the Favard class of X_{-1}.

If X is $\odot$-reflexive with respect to $\mathbf{T}$, then one can show that X_α is $\odot$-reflexive with respect to $\mathbf{T}_\alpha$. In that case, Theorem 4.4 gives a natural isomorphism $X_{\alpha,\infty} \simeq (X_\alpha)^{\odot *}$; in other words, $X_{\alpha,\infty}$ is a dual space in a natural way.

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