Quantum Grassmannians and $q$-hypergeometric series

dedicated to Prof. T.H. Koornwinder

Masatoshi Noumi
Department of Mathematical Sciences, University of Tokyo
Komaba 3-8-1, Meguro-Ku, Tokyo 153, Japan
e-mail address: noumi@tansei.cc.u-tokyo.ac.jp

INTRODUCTION
One of the important aspects of quantum groups is the link with the theory of $q$-special functions. It is already known that various $q$-special functions arise naturally in the framework of quantum groups. Roughly speaking, quantum groups, as well as other group-theoretic frameworks, have at least two ways to provide special functions: one as spherical functions on coset spaces, and the other as connection coefficients describing the tensor products of representations.

As to the study of connections of quantum groups with $q$-special functions, one of the most important contributors is Prof. T.H. Koornwinder, who was among the first to recognize that the unitary representations of the quantum $SU(2)$ group give a natural interpretation of the little $q$-Jacobi polynomials as matrix coefficients ([9], see also Vaksman-Soibelman [24] and Masuda et al [13]). He also showed, in a joint work with Koelink [8], that the $q$-Hahn polynomials arise naturally as the Clebsch-Gordan coefficients for the quantum $SU(2)$ group (see also [7]). More impressive to the author is his realization of the continuous $q$-Legendre polynomials and a two-parameter family of Askey-Wilson polynomials as zonal spherical functions on the quantum $SU(2)$ group ([10], [11]). This discovery of Koornwinder has brought out a new stage of links between the nature of $q$-special functions and geometric problems in quantum groups. For these subjects, we refer the reader to the survey paper [12] by Koornwinder himself (see also Noumi [15] and Noumi-Mimachi [16]–[19]).

In this article, we would like to discuss a slightly different point relating quantum groups to $q$-special functions. This subject was motivated by a recent work of Horikawa [5], who showed that the contiguity relations for the general $q$-hypergeometric series $2\varphi_1$ can be described by the quantized universal enveloping algebra $U_q(gl(4))$. He found this fact by investigating $q$-analogues of Gelfand’s interpretation of Gauss’ hypergeometric functions by the Grassmann manifold $Grass(2,4)$. This work is remarkable because it implies that quantum groups are involved in $q$-analysis from its very starting points. He has also obtained analogous results [6].

293
for the case of Grass\((k,n)\), where some \(q\)-hypergeometric series, including a \(q\)-analogue of Lauricella’s \(F_D\), and their contiguity relations are discussed. It is not clear, however, why quantum groups arise as contiguity relations for \(q\)-hypergeometric series. Horikawa’s presentation is in fact given by factorizing directly the \(q\)-difference equations for them.

This article is an attempt to give an intrinsic explanation of the phenomenon that the quantized universal enveloping algebras arise as the contiguity relations for \(q\)-hypergeometric series. Our approach is carried out by considering a \textit{quantum} analogue of the Grassmann manifolds and the Gelfand hypergeometric functions associated with them. In this article, we discuss exclusively the case of Grass\((2,n)\) and show that a \(q\)-analogue of Lauricella’s hypergeometric series \(F_D\) of \(n-3\) variables falls into this framework. From this interpretation, we see that the quantized universal enveloping algebra \(U_q(\mathfrak{gl}(n))\) arises naturally as the algebra describing the contiguity relations of these \(q\)-hypergeometric series. The exposition in this article is still experimental and incomplete in many respects. The author hopes that the framework of Gelfand’s generalized hypergeometric functions will activate mathematical interactions even between quantum groups and \(q\)-special functions. The author is grateful to Prof. Horikawa for valuable suggestions and for communicating his preprints before publication.

1. Gelfand’s hypergeometric equations
In this section, we give a brief review on Gelfand’s generalized hypergeometric equations ([(1), (2)], associated with the Grassmannian Grass\((k,n)\) \((k \leq n)\). In what follows, we denote by Grass\((k,n)\), or simply by \(G_{k,n}\), the Grassmannian consisting of all \(k\)-dimensional subspaces of the \(n\)-dimensional complex vector space \(\mathbb{C}^n\). Let us consider the general \(k \times n\) complex matrices

\[
T = \begin{pmatrix} t_{11} & t_{12} & \cdots & t_{1n} \\
   t_{21} & t_{22} & \cdots & t_{2n} \\
   \cdots & \cdots & \cdots & \cdots \\
   t_{k1} & t_{k2} & \cdots & t_{kn} 
\end{pmatrix}. \tag{1}
\]

The \textit{Gelfand hypergeometric equation} of type \(G_{k,n}\) is the following system of differential equations defined on the matrix space Mat\((k,n)\):

\[
\sum_{j=1}^{n} t_{rj} \frac{\partial}{\partial t_{sj}} \Phi(T) = -\delta_{rs} \Phi(T) \quad (1 \leq r, s \leq k), \tag{2}
\]

\[
\sum_{r=1}^{k} t_{rj} \frac{\partial}{\partial t_{rj}} \Phi(T) = \lambda_j \Phi(T) \quad (1 \leq j \leq n), \tag{3}
\]

\[
\frac{\partial^2}{\partial t_{ri} \partial t_{sj}} \Phi(T) = \frac{\partial^2}{\partial t_{si} \partial t_{rj}} \Phi(T) \quad (1 \leq r < s \leq k, 1 \leq i < j \leq n), \tag{4}
\]

where \(\lambda_j\) \((1 \leq j \leq n)\) are complex parameters such that \(\sum_{j=1}^{n} \lambda_j = -k\). Multi-valued holomorphic solutions \(\Phi(T)\) of the system (2)–(4), defined on an appropriate open subset of the matrix space Mat\((k,n)\), are called Gelfand’s hyperge-
ometric functions of type $G_{k,n}$. Note that equation (2) is equivalent to saying that $\Phi(T)$ has the relative invariance

$$\Phi(gT) = \det(g)^{-1}\Phi(T) \quad (g \in \text{GL}(k)), \quad (5)$$

under the left action of $\text{GL}(k)$. On the other hand, equation (3) requires the homogeneity

$$\Phi(T\text{diag}(c_1, \ldots, c_n)) = \Phi(T)c_1^{\lambda_1} \cdots c_n^{\lambda_n} \quad (6)$$

under the action of the diagonal subgroup of $\text{GL}(n)$. Equation (4) is the essential part of the Gelfand hypergeometric equation. In fact, from (4), many examples of classical hypergeometric functions in some appropriate variables can be obtained by eliminating the relative invariance (2) and (3).

For simplicity, we consider the case of the Grassmannian Grass($2,n$). In this case, the Gelfand hypergeometric equation above is related to Lauricella’s hypergeometric functions $F_D$ in $n-3$ variables ([2]). Note first that the generic $2 \times n$ matrix $T$ has the following decomposition:

$$T = \begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 & \cdots & 1 \\ 0 & 1 & x_3 & \cdots & x_n \end{pmatrix} \times \text{diag}(1,1,-\xi_{23}/\xi_{12}, \ldots, -\xi_{2n}/\xi_{12}), \quad (7)$$

where $\xi_{ij} = t_{1i}t_{2j} - t_{2i}t_{1j}$ for $1 \leq i, j \leq n$ and and $x_j = -\xi_{1j}/\xi_{12}$ for $j = 3, \ldots, n$. The relative invariance (5), (6) implies that $\Phi(T)$ can be written in the form

$$\Phi(T) = G(x_3, \ldots, x_n)\xi_{12}^{\lambda_1+\lambda_2+1}(-\xi_{23})^{\lambda_3} \cdots (-\xi_{2n})^{\lambda_n}, \quad (8)$$

for some function $G(x_3, \ldots, x_n)$ in the variables $(x_3, \ldots, x_n)$. Note here that $\sum_{j=1}^n \lambda_j = -2$. Furthermore, the function $G(x_3, \ldots, x_n)$ should be homogeneous of degree $-\lambda_2 - 1$ from the condition (6). Hence, $G(x_3, \ldots, x_n)$ takes the form

$$G(x_3, \ldots, x_n) = F(z_4, \ldots, z_n)x_3^{-\lambda_2-1}, \quad (9)$$

where $z_j = x_j/x_3$ for $j = 4, \ldots, n$. In these coordinates $z = (z_4, \ldots, z_n)$, it is known that equation (4) gives rise to the following differential equation to be satisfied by the function $F(z_4, \ldots, z_n)$:

$$\left\{\theta_k \left(\sum_{j=4}^n \theta_j + \lambda_2 + \lambda_3 + 1\right) - z_k(\theta_k - \lambda_k)\left(\sum_{j=4}^n \theta_j + \lambda_2 + 1\right)\right\}F(z_4, \ldots, z_n) = 0, \quad (10)$$

for $k = 4, \ldots, n$, where $\theta_j = z_j\partial/\partial z_j$ ($j = 4, \ldots, n$). A solution to this equation, holomorphic near the origin $z = 0$, is given by Lauricella’s hypergeometric series

$$F_D\left(\begin{array}{c} \lambda_2 + 1; -\lambda_4, \ldots, -\lambda_n \\ \lambda_2 + \lambda_3 + 2 \end{array}; z_4, \ldots, z_n\right). \quad (11)$$
Lauricella’s hypergeometric series $F_D$ are defined as follows:

$$F_D\left(\begin{array}{c} \alpha; \beta_1, \ldots, \beta_m \\ \gamma \end{array} ; x_1, \ldots, x_m \right) = \sum_{\nu_1, \ldots, \nu_m \geq 0} \frac{(\alpha)_{\nu_1 + \ldots + \nu_m} (\beta_1)_{\nu_1} \cdots (\beta_m)_{\nu_m}}{(\gamma)_{\nu_1 + \ldots + \nu_m} \nu_1! \cdots \nu_m!} x_1^{\nu_1} \cdots x_m^{\nu_m},$$

(12)

where $(\alpha)_\nu = \alpha(\alpha+1) \cdots (\alpha+\nu-1)$.

It should be noted that the system of equations (4) is covariant under the adjoint action of $GL(n)$. From this property, it is known that the contiguity relations for Lauricella’s $F_D$ are described by the Lie algebra $gl(n)$. For the details, see [4], [22], [14] for example.

2. **Quantum Grassmannians**

We now start to consider the quantum analogue of Gelfand’s framework of generalized hypergeometric functions. For this purpose, we investigate a $q$-deformation of the graded algebra of homogenous functions on the Grassmannian.

Recall that the coordinate ring $A_q(Mat(k,n))$ of the quantum matrix space $Mat_q(k,n)$ is the algebra generated by the “canonical coordinates” $t_{rj}$ ($1 \leq r \leq k, 1 \leq j \leq n$) with commutation relations:

$$
t_{rj}t_{sj} = q^{s_j}t_{sj}t_{rj} \quad (1 \leq r < s \leq k, 1 \leq j \leq n),
$$

$$
t_{ri}t_{rj} = q^{s_i}t_{rj}t_{ri} \quad (1 \leq r \leq k, 1 \leq i < j \leq n),
$$

$$
t_{rj}t_{si} = t_{si}t_{rj},
$$

$$
t_{rj}t_{sj} - t_{sj}t_{rj} = (q - q^{-1})t_{rj}t_{si} \quad (1 \leq r < s \leq k, 1 \leq i < j \leq n).$$

(13)

These generators correspond to the matrix coordinates $T = (t_{rj})_{1 \leq r \leq k, 1 \leq j \leq n}$ as in (1). It will be helpful to note that, for each $1 \leq r < s \leq k$ and $1 \leq i < j \leq n$, the $2 \times 2$ matrix of generators

$$
\begin{pmatrix}
a & b \\
c & d
\end{pmatrix} = 
\begin{pmatrix}
t_{ri} & t_{rj} \\
t_{si} & t_{sj}
\end{pmatrix}
$$

satisfies the $Mat_q(2)$-relations:

$$
ab = qba, ac = qca, bd = qdb, cd = qdc,
$$

$$
bc = cb, ad - da = (q - q^{-1})bc.
$$

(14)

In what follows, we fix a complex number $q$ with $0 < |q| < 1$.

We recall that the algebra $A_q(Mat(k,n))$ has a natural structure of a two-sided comodule over the pair of Hopf algebras $(A_q(GL(k)), A_q(GL(n)))$. Accordingly, it becomes a bimodule over the pair $(U_q(gl(n)), U_q(gl(k)))$. By using this structure, we can consider the quantum analogue of the coset space $SL(k) \backslash Mat(k,n)$. Note that the quotient space $SL(k) \backslash Mat'(k,n)$ of the open subset $Mat'(k,n)$ of all matrices of maximal rank defines a $C^*$-bundle over the Grassmannian.
Grass$(k, n) = GL(k) \backslash \text{Mat}(k, n)$ with the natural projection. Let $\mathcal{A}$ be the following subalgebra of left $SL_q(k)$-invariants in $A_q(\text{Mat}(k, n))$: 

$$\mathcal{A} = A_q(SL(k) \backslash \text{Mat}(k, n)) = \{ \varphi \in A_q(\text{Mat}(k, n)); \varphi a = \varepsilon(a) \varphi \text{ for all } a \in U_q(sl(k)) \}. \quad (16)$$

By the standard monomial theory for this case, one can show that the algebra $\mathcal{A}$ is generated by the quantum minor determinants

$$\xi_{j_1, j_2, \ldots, j_k} = \sum_{w \in S_k} (-q)^{\ell(w)} t_{w(1)j_1} t_{w(2)j_2} \cdots t_{w(k)j_k} \quad (1 \leq j_1, j_2, \ldots, j_k \leq n) \quad (17)$$

where $S_k$ stands for the permutation group of $k$ letters and, for each $w \in S_k$, $\ell(w)$ denotes the number of inversions in $w$. These quantum minors are analogues of the Plücker coordinates of the Grassmannian Grass$(k, n)$. For the classical Plücker coordinates of the Grassmannian, see [3], for example. The general Plücker relations for our quantum minors can be found in [21] and [23].

In the case of the Grassmannian Grass$(2, n)$, the structure of the algebra $\mathcal{A} = A_q(SL(2) \backslash \text{Mat}(2, n))$ is simply described. We remark first that the quantum minors

$$\xi_{ij} = t_{11} t_{2j} - q t_{2i} t_{1j} \quad (1 \leq i, j \leq n) \quad (18)$$

satisfy the following two types of Plücker relations:

$$\xi_{12} \xi_{34} - q \xi_{13} \xi_{24} + q^2 \xi_{14} \xi_{23} = 0,$$

$$\xi_{34} \xi_{12} - q^{-1} \xi_{24} \xi_{13} + q^{-2} \xi_{23} \xi_{14} = 0. \quad (19)$$

The other Plücker relations are obtained by replacing the sequence $1 < 2 < 3 < 4$ by an arbitrary increasing sequence $i < j < r < s$ of indices. The typical commutation relations among $\xi_{ij}$ are given as follows:

$$\xi_{11} = 0, \quad q \xi_{12} + \xi_{21} = 0,$$

$$\xi_{12} \xi_{13} = q \xi_{13} \xi_{12}, \quad \xi_{12} \xi_{23} = q \xi_{23} \xi_{12}, \quad \xi_{13} \xi_{23} = q \xi_{23} \xi_{13},$$

$$\xi_{12} \xi_{34} = q^2 \xi_{34} \xi_{12}, \quad \xi_{13} \xi_{24} - q \xi_{24} \xi_{13} = (q - q^{-1}) \xi_{14} \xi_{23}, \quad \xi_{14} \xi_{23} = \xi_{23} \xi_{14}. \quad (20)$$

To get the other commutation relations, replace $1 < 2 < \cdots$ by any increasing sequence $i < j < \cdots$. It is interesting to see that the $2 \times 2$ matrix

$$\left( \begin{array}{cc} a & b \\ c & d \end{array} \right) = \left( \begin{array}{cc} \xi_{13} & \xi_{14} \\ \xi_{23} & \xi_{24} \end{array} \right) \quad (21)$$

again satisfies the Mat$_q(2)$-relations (15). In this case the quantum determinant is given by

$$ad - qbc = \xi_{13} \xi_{24} - q \xi_{14} \xi_{23} = q^{-1} \xi_{12} \xi_{34} \quad \text{or}$$

$$da - q^{-1}cb = \xi_{24} \xi_{13} - q \xi_{23} \xi_{14} = q \xi_{34} \xi_{12}. \quad (22)$$

The last equality is the Plücker relation.
3. **Localization**

In order to consider the analogue of the decomposition (7) of the matrix \( T \), we need to introduce some localizations of the algebra \( \mathcal{A} = A_q(\text{SL}(2) \backslash \text{Mat}(2, n)) \).

The first step is to invert the principal minor \( \xi_{12} \). This procedure is easily understood since \( \xi_{12} \xi_{ij} \) is a constant multiple of \( \xi_{ij} \xi_{12} \) for each \( i < j \). It should be noted here that we have

\[
\xi_{ij} = (q^{-1} \xi_{ij} \xi_{11} - q^{-2} \xi_{ij} \xi_{12}) \xi_{12}^{-1} \quad \text{for} \quad 3 \leq i < j, \quad (23)
\]

by the Plücker relations as in (19). By this fact, one can show that the localization \( \mathcal{A} [\xi_{12}^{-1}] \) has the monomial basis

\[
\xi_{2n}^{a_n} \cdots \xi_{23}^{a_3} \xi_{11}^{a_2} \cdots \xi_{13}^{a_3} \xi_{12}^\mu
\]

where \( a_j, b_j \in \mathbb{N} \) and \( \mu \in \mathbb{Z} \). In other words, the algebra \( \mathcal{A} [\xi_{12}^{-1}] \) is isomorphic to a noncommutative tensor product of \( A_q(\text{Mat}(2, n - 2)) \) with the algebra of Laurent polynomials. This type of localization corresponds to restricting the C*-bundle \( \text{SL}(2) \backslash \text{Mat}'(2, n) \) to the biggest cell of the Grassmannian \( \text{Grass}(2, n) \).

The next step is to adjoin the inverses of the quantum minors \( \xi_{23}, \ldots, \xi_{2n} \). This time one should be careful in treating the inverses since the commutation relations involving \( \xi_{2j} \)'s are not trivial. For each \( i < j \), the commutation relation between \( \xi_{ij} \) and \( \xi_{ij}^{-1} \) should be

\[
\xi_{ij} \xi_{ij}^{-1} = \xi_{ij}^{-1} \xi_{ii} + q^{-2} (q - q^{-1}) \xi_{2j}^{-2} \xi_{ij} \xi_{ij}^{-1}. \quad (25)
\]

To understand the structure of the localization \( \mathcal{A} [\xi_{12}^{-1}, \xi_{23}^{-1}, \ldots, \xi_{2n}^{-1}] \), it is worthwhile to recall the commutation relations

\[
a^r b^s = \sum_{k \geq 0} (-1)^k q^{k(2r + 2s - 2k - 1)} \frac{(q^{-2}; q^2)_k (q^{-2}; q^2)_k (q^{-s}; q^2)_k (q^{-s}; q^2)_k}{(q^2; q^2)_k} \quad (26)
\]

for \( r, s \in \mathbb{N} \) in \( A_q(\text{Mat}(2)) \), where \( (x; q)_k = (1 - x) \cdots (1 - q^{k-1} x) \). Note that the summation on the right-hand side terminates as long as either \( r \) or \( s \) is a nonnegative integer even if the other is an arbitrary parameter. This means that one can consistently adjoin one of the inverses \( d^{-1} \) or \( a^{-1} \) to the algebra \( A_q(\text{Mat}(2)) \), although one cannot handle these two inverses simultaneously in the algebraic sense.

In our algebra \( \mathcal{A} [\xi_{12}^{-1}, \xi_{23}^{-1}, \ldots, \xi_{2n}^{-1}] \), the commutation relations between \( \xi_{ii} \) and \( \xi_{ij}^{-1} \) with \( i < j \) are described by (26) for \( a = \xi_{11}, d = \xi_{2j} \) and \( r \in \mathbb{N}, s \in \mathbb{Z} \).

By this fact, one can show that the algebra \( \mathcal{A} [\xi_{12}^{-1}, \xi_{23}^{-1}, \ldots, \xi_{2n}^{-1}] \) also has the monomial basis of type (24) with \( a_j \in \mathbb{N} \) and \( b_j, \mu \in \mathbb{Z} \).

After these localizations, the matrix \( T = (t_{ij})_{1 \leq i \leq 2, 1 \leq j \leq n} \) of generators for \( A_q(\text{Mat}(2, n)) \) can be decomposed in a similar way to (7):

\[
T = \begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix} \left( \begin{array}{cccc} 1 & 0 & 1 & \cdots & 1 \\ 0 & 1 & u_3 & \cdots & u_n \end{array} \right) \times \text{diag}(1, 1, -q^{-1} \xi_{12}^{-1} \xi_{23}, \ldots, -q^{-1} \xi_{12}^{-1} \xi_{2n}), \quad (27)
\]

298
where \( u_j = -\xi_{j+1}^{-1}\xi_{j+1}^{-1} \) for \( j = 3, \ldots, n \). Let us denote by \( \mathcal{R} \) the subalgebra of \( \mathcal{A}[\xi_{12}^{-1}, \xi_{23}^{-1}, \ldots, \xi_{2n}^{-1}] \) generated by the elements \( u_3, \ldots, u_n \), regarding them as a quantum analogue of the homogeneous coordinates of the projective \( n-3 \) space. By the argument above, we see that the algebra \( \mathcal{A}[\xi_{12}^{-1}, \xi_{23}^{-1}, \ldots, \xi_{2n}^{-1}] \) has the decomposition

\[
\mathcal{A}[\xi_{12}^{-1}, \xi_{23}^{-1}, \ldots, \xi_{2n}^{-1}] = \bigoplus_{\mu, \lambda_3, \ldots, \lambda_n \in \mathbb{Z}} \mathcal{A}[\xi_{12}^{-1}, \xi_{23}^{-1}, \ldots, \xi_{2n}^{-1}] \cdot \xi_{12}^\mu \xi_{23}^{\lambda_3} \ldots \xi_{2n}^{\lambda_n} \mathcal{R}.
\]

(28)

Although \( \mathcal{R} \) is still a noncommutative subalgebra, its structure is not too complicated. In fact, the generators \( u_3, \ldots, u_n \) satisfy the commutation relations

\[
(u_i - u_j)u_k = q^2 u_k (u_i - u_j) \quad (3 \leq i, j \leq k \leq n).
\]

(29)

If one takes another system of generators

\[
v_3 = u_3 - u_4, \ldots, v_{n-1} = u_{n-1} - u_n \quad \text{and} \quad v_n = u_n,
\]

(30)

then their commutation relations are simply given by \( v_iv_j = q^2 v_jv_i \) for \( 3 \leq i < j \leq n \).

4. A REPRESENTATION OF \( U_q(\mathfrak{gl}(n)) \)

Recall that \( U_q(\mathfrak{gl}(n)) \) is an algebra generated by the symbols \( q^{\pm e_j} \) \((1 \leq j \leq n)\), \( e_j \), \( f_j \) \((1 \leq j \leq n - 1)\). Hereafter, we use the notation \( q^h = q^{a_1 e_1 + \cdots + a_n e_n} \) for any linear combination \( h = a_1 e_1 + \cdots + a_n e_n \) with integral coefficients. Then the fundamental relations for the generators of \( U_q(\mathfrak{gl}(n)) \) are given by

\[
q^0 = 1, \quad q^hq^{h'} = q^{h+h'},
\]

\[
q^he_jq^{-h} = q^{\delta_{ij} > 1} e_j, \quad q^hf_jq^{-h} = q^{\delta_{ij} > 1} f_j \quad (1 \leq j \leq n-1),
\]

\[
e_{i}f_{j} - f_{j}e_{i} = \delta_{ij}(q^{i+1} - q^{-i+1})/(q - q^{-1}) \quad (1 \leq i, j \leq n-1),
\]

\[
e_{i}e_{j} = e_{j}e_{i}, \quad f_{i}f_{j} = f_{j}f_{i} \quad (|i - j| \geq 2),
\]

\[
e_{i}^2e_{j} - (q + q^{-1})e_{i}e_{j}e_{i} + e_{i}e_{j}^2e_{i} = 0 \quad (|i - j| = 1),
\]

\[
f_{i}^2f_{j} - (q + q^{-1})f_{i}f_{j}f_{i} + f_{i}f_{j}^2f_{i} = 0 \quad (|i - j| = 1),
\]

(31)

where \( a_j = e_j - e_{j+1} \) for \( 1 \leq j \leq n-1 \) and \( < , > \) stands for the canonical symmetric bilinear form such that \( < e_i, e_j > = \delta_{ij} \). The last two cubic equations of (31) are sometimes referred to as the Serre relations. The algebra \( U_q(\mathfrak{gl}(n)) \) has a Hopf algebra structure. We take here the following convention of the coproduct \( \Delta \):

\[
\Delta(q^h) = q^h \otimes q^h,
\]

\[
\Delta(e_j) = e_j \otimes 1 + q^{\delta_{ij} > 1} e_j \otimes e_j \quad (1 \leq j \leq n-1),
\]

\[
\Delta(f_j) = f_j \otimes q^{-\delta_{ij} + \delta_{ij+1}} + 1 \otimes f_j \quad (1 \leq j \leq n-1).
\]

(32)

We denote its counit and antipode by \( \varepsilon \) and \( S \), respectively:

\[
\varepsilon(q^h) = 1, \quad \varepsilon(e_j) = 0, \quad \varepsilon(f_j) = 0,
\]

\[
S(q^h) = q^{-h}, \quad S(e_j) = -q^{-\delta_{ij} + \delta_{ij+1}} e_j, \quad S(f_j) = -f_j q^{\delta_{ij} + \delta_{ij+1}}
\]

(33)
for $1 \leq j \leq n - 1$.

The coordinate ring $A_q(Mat(2, n))$ of the quantum matrix space $Mat_q(2, n)$ has the structure of a left $U_q(gl(n))$-module. Furthermore, it is an algebra with $U_q(gl(n))$-symmetry in the sense that

(a) $a.1 = \xi(a)1$ for all $a \in U_q(gl(n))$;

(b) if $a \in U_q(gl(n))$ and $\Delta(a) = \sum_i a_i^\prime \otimes a_i^{\prime\prime}$, then

$$a.(\varphi \psi) = \sum_i a_i^\prime \varphi a_i^{\prime\prime} \psi$$

for all $\varphi, \psi \in A_q(Mat(2, n))$.

The actions of $q^h$ and $e_k, f_k$ on the generators $t_{rj}(1 \leq r \leq 2, 1 \leq j \leq n)$ of $A_q(Mat(2, n))$ are given by

$$q^h t_{rj} = q^{<h,r,j>} t_{rj}, \quad e_k t_{rj} = \delta_{k+1,j} t_{rk}, \quad f_k t_{rj} = \delta_{k,j} t_{r,k+1},$$

(34)

respectively. Their actions on an arbitrary element $\varphi \in A_q(Mat(2, n))$ are then determined by a successive use of the rule (b) above. It is clear that the invariant ring $A = A_q(SL(2) \setminus Mat(2, n))$ is a left $U_q(gl(n))$-submodule of $A_q(Mat(2, n))$.

The action of $U_q^{-1}(gl(n))$ on the quantum minors is determined by

$$q^h \xi_{ij} = q^{<h,i,j>} \xi_{ij},$$

$$e_k \xi_{ij} = \delta_{k+1,j} \xi_{kj} + b_{k+1,j} q^{<\alpha_{k,j}>} \xi_{ik},$$

$$f_k \xi_{ij} = \delta_{k,j} q^{<\alpha_{k,j}>} \xi_{k+1,j} + b_{k,j} \xi_{ik+1}.$$  

(35)

We now consider the localization $A[\xi_{12}^{-1}, \xi_{23}^{-1}, \ldots, \xi_{2n}^{-1}]$ of $A$ discussed in the previous section. This algebra has a unique structure of an algebra with $U_q^{-1}(gl(n))$-symmetry containing $A$ as a $U_q^{-1}(gl(n))$-submodule. Suppose that $\varphi$ is a weight vector of weight $\mu$ in $A$: $q^h \varphi = q^{<h,\mu>} \varphi$. If $\varphi$ is invertible in a localization of $A$, then we must have

$$q^h \varphi^{-1} = q^{<h,\mu>} \varphi^{-1},$$

$$e_k \varphi^{-1} = q^{<\alpha_{k,j}>} e_k \varphi^{-1},$$

$$f_k \varphi^{-1} = q^{<\alpha_{k,j}>} f_k \varphi^{-1},$$

(36)

provided that the properties (a),(b) above are preserved. We can extend the action of $U_q^{-1}(gl(n))$ on $A$ to $A[\xi_{12}^{-1}, \xi_{23}^{-1}, \ldots, \xi_{2n}^{-1}]$ by the rules (36) for $\varphi = \xi_{12}, \xi_{23}, \ldots, \xi_{2n}$, so that the conditions (a), (b) are satisfied in the localization $A[\xi_{12}^{-1}, \xi_{23}^{-1}, \ldots, \xi_{2n}^{-1}]$ as well.

Recall that the algebra $A[\xi_{12}^{-1}, \xi_{23}^{-1}, \ldots, \xi_{2n}^{-1}]$ has the decomposition

$$A[\xi_{12}^{-1}, \xi_{23}^{-1}, \ldots, \xi_{2n}^{-1}] = \bigoplus_{\mu_1, \ldots, \mu_n} \mathbb{Z}^{\xi_{23}^{-1}, \ldots, \xi_{12}^{-1}}.$$  

(37)

Note that the subalgebra $\mathcal{R}$, generated by the elements $u_j = -\xi_{2j}^{-1} \xi_{1j}$ ($3 \leq j \leq n$), has a monomial basis

$$u_{n-1} \nu_{n-1} \nu_{n-2} \ldots u_3 \nu_3$$

(38)

We consider the commutative algebra $\mathbb{C}[x_3, \ldots, x_n]$ of polynomials in $n-3$ variables and define an isomorphism of vector spaces $\phi : \mathbb{C}[x_3, \ldots, x_n] \rightarrow \mathcal{R}$ by the normal ordering
\[ \phi(x_3^{\nu_3} \cdots x_n^{\nu_n}) = u_1^{\nu_1} \cdots u_3^{\nu_3}. \] (39)

In order to translate the action of \( U_q(gl(n)) \) into the terms of usual \( q \)-difference operators, we consider a copy of the polynomial ring \( \mathbb{C}[x_3, \cdots, x_n] \):

\[ \mathcal{G} = \bigoplus_{\mu, \lambda_3, \cdots, \lambda_n} \mathcal{G}_{\mu, \lambda_3, \cdots, \lambda_n}, \] (40)

where \( \mathcal{G}_{\mu, \lambda_3, \cdots, \lambda_n} = \mathbb{C}[x_3, \cdots, x_n] \) for all \( \mu, \lambda_3, \cdots, \lambda_n \in \mathbb{Z} \). Then we define an isomorphism of vector spaces

\[ w: \mathcal{G} \rightarrow \mathcal{A}[\xi_{12}^{-1}, \xi_{23}^{-1}, \cdots, \xi_{2n}^{-1}] \] (41)

as the direct sum of linear mappings

\[ w_{\mu, \lambda_3, \cdots, \lambda_n}: \mathbb{C}[x_3, \cdots, x_n] \rightarrow \xi_{2n}^{\lambda_n} \cdots \xi_{23}^{\lambda_3} \xi_{12}^{-1} \mathcal{R} \] (42)

such that

\[ w_{\mu, \lambda_3, \cdots, \lambda_n}(G) = \xi_{2n}^{\lambda_n} \cdots \xi_{23}^{\lambda_3} \xi_{12}^{-1} \phi(G) \] (43)

for all \( G \in \mathbb{C}[x_3, \cdots, x_n] \). Through the isomorphism \( w \) thus defined, we obtain a left \( U_q(gl(n)) \)-module structure on the vector space \( \mathcal{G} \).

After some explicit computations, we see that the action of \( U_q(gl(n)) \) on \( \mathcal{G} \) is actually described in terms of the \( q \)-difference operators with respect to the variables \( x_3, \cdots, x_n \). Suppose that \( a \) is an element of \( U_q(gl(n)) \) such that

\[ q^h a q^{-h} = q^{\kappa_h} a \] for some \( \kappa = \kappa_1 e_1 + \cdots + \kappa_n e_n (\kappa_j \in \mathbb{Z}) \). If \( a \) is an element of weight \( \kappa \) in this sense, then its action on \( \mathcal{G} \) is represented by a family of operators

\[ \tilde{\rho}_{\mu, \lambda_3, \cdots, \lambda_n}(a): \mathcal{G}_{\mu, \lambda_3, \cdots, \lambda_n} \rightarrow \mathcal{G}_{\mu', \lambda_1', \cdots, \lambda_n'}, \] (44)

where \( \mu' = \mu + \kappa_1 + \kappa_2 \), \( \lambda_j' = \lambda_j + \kappa_j \) (\( j \geq 3 \)). These operators \( \tilde{\rho}_{\mu, \lambda_3, \cdots, \lambda_n}(a) \) are in fact expressed by a \( q \)-difference operator depending polynomially on \( q^{\pm \mu}, q^{\pm \lambda_3}, \cdots, q^{\pm \lambda_n} \). We remark that a commutation relation among the operators \( \tilde{\rho}_{\mu, \lambda_3, \cdots, \lambda_n}(a) \) (\( a \in U_q(gl(n)) \)) makes sense even for non-integral values of \( \mu, \lambda_3, \cdots, \lambda_n \), if it depends polynomially on \( q^{\pm \mu}, q^{\pm \lambda_3}, \cdots, q^{\pm \lambda_n} \) and is valid for all \( \mu, \lambda_3, \cdots, \lambda_n \in \mathbb{Z} \). The explicit formulas of the operators \( \tilde{\rho}_{\mu, \lambda_3, \cdots, \lambda_n}(a) \) for \( a = q^{e_i}, e_j, f_j \) will be given in the next section after we restrict them to some spaces of homogeneous functions.

5. A \( q \)-ANALOGUE OF LAURICELLA’S \( F_D \)

In view of the formulas (8) and (9) in the classical case, we take the homogeneity of \( G = G(x_3, \cdots, x_n) \) into our consideration. Let \( U \) be an open set of the algebraic torus \( (\mathbb{C}^*)^{n-2} \) with coordinates \( x = (x_3, \cdots, x_n) \) and suppose that \( U \) is stable under the action of the multiplicative group \( (q\mathbb{Z})^{n-2} \). For each \( n \)-tuple \( \lambda = (\lambda_1, \lambda_2, \cdots, \lambda_n) \) of complex numbers with \( \sum_{j=1}^n \lambda_j = -2 \), we denote by \( \mathcal{F}_\lambda \) the vector space of all holomorphic functions \( G = G(x_3, \cdots, x_n) \) on \( U \) such that

\[ T_{q, x_3} \cdots T_{q, x_n} G = q^{-\lambda_3 - 1} G. \] (45)

301
We take here the general convention to denote by $T_{q,x_j}$ the $q$-shift operator in the variable $x_j$:

$$T_{q,x_j} G(x_3, \ldots, x_n) = G(x_3, \ldots, qx_j, \ldots, x_n). \quad (46)$$

For each element $a \in U_q(\mathfrak{gl}(n))$ of weight $\kappa$, we define an operator

$$\rho_\lambda(a) : \mathcal{F}_\lambda \to \mathcal{F}_{\lambda + \kappa} \quad (47)$$

by setting

$$\rho_\lambda(a) = \tilde{\rho}_{\lambda_1 + \lambda_2 + 1, \ldots, \lambda_n}(a). \quad (48)$$

Here the right-hand side is understood as a $q$-difference operator depending polynomially on the parameters $q^{\pm \lambda_1}, \ldots, q^{\pm \lambda_n}$. With these operators $\rho_\lambda(a)$, we obtain a representation $\mathcal{F} = \bigoplus_\lambda \mathcal{F}_\lambda$ of the quantized universal enveloping algebra $U_q(\mathfrak{gl}(n))$.

We give below the explicit formulas for the $q$-difference operators

$$\rho_\lambda(q^{e_j}) : \mathcal{F}_\lambda \to \mathcal{F}_\lambda \quad (1 \leq j \leq n),$$

$$\rho_\lambda(e_j) : \mathcal{F}_\lambda \to \mathcal{F}_{\lambda + \alpha_j} \quad (1 \leq j \leq n - 1),$$

$$\rho_\lambda(f_j) : \mathcal{F}_\lambda \to \mathcal{F}_{\lambda - \alpha_j} \quad (1 \leq j \leq n - 1), \quad (49)$$

where $\alpha_j = e_j - e_{j+1}$. For simplicity, we set $T_j = T_{q^j,x_j}, T_{x_j} = T_{j+1}T_{j+2} \cdots T_n$ for $j = 3, \cdots, n$. With these notations, we have

$$\rho_\lambda(q^{e_j}) = q^{\lambda_j} \quad (1 \leq j \leq n),$$

$$(q - q^{-1}) \rho_\lambda(e_1) = q^{\lambda_1 - \lambda_2} x_3 - q^{-\lambda_1 - \lambda_3 - 2} \sum_{k=3}^{n} (x_k - x_{k+1}) q^{-2\lambda_{k+1}} T_{k},$$

$$(q - q^{-1}) \rho_\lambda(e_2) = q^{-\lambda_1 + \lambda_2 - \lambda_3 + 1} \frac{1}{x_3}(1 - T_3),$$

$$(q - q^{-1}) \rho_\lambda(e_j) = q^{\lambda_j + 1} (1 - q^{-2\lambda_{j+1}} T_j) - q^{-\lambda_{j+1}} \frac{x_j}{x_{j+1}} (1 - T_{j+1}), \quad (j \geq 3)$$

$$(q - q^{-1}) \rho_\lambda(f_1) = q^{2\lambda_2 + 1} \sum_{k=3}^{n} \frac{1}{x_k} (1 - T_k) T_{k+1},$$

$$(q - q^{-1}) \rho_\lambda(f_2) = -q^{\lambda_1} x_3 + q^{-\lambda_1 - 2\lambda_2 - 2} \sum_{k=3}^{n} (x_k - x_{k+1}) q^{-2\lambda_{k+1}} T_{k+1},$$

$$(q - q^{-1}) \rho_\lambda(f_j) = -q^{-\lambda_j} (1 - q^{2\lambda_j} T_j^{-1}) + q^{\lambda_j} \frac{T_{j+1}}{x_j} (1 - T_j^{-1}), \quad (j \geq 3). \quad (50)$$

Here we set $x_{n+1} = 0$ and $\lambda_{n+1} = \lambda_{k+1} + \cdots + \lambda_n$. Note that we have $T_3 \cdots T_n = q^{-2\lambda_2 - 2}$ on $\mathcal{F}_\lambda$.

With these operators, we consider the $q$-difference equation corresponding to the equation (10) for Lauricella’s $F_D$. We recall that, in the classical setting of Section 1, we have a remarkable identity
for each \( i < j \), where \((E_{ij})_{1 \leq i,j \leq n}\) is the basis for the Lie algebra \( \mathfrak{gl}(n) \), corresponding to the matrix units. In view of this equality of Capelli type, we look at the elements

\[
(q - q^{-1})^2 C_j = (q^{t_{j+1}} - q^{-t_{j+1}})(q^{t_{j+1}} - q^{-t_{j+1}}) - (q - q^{-1})^2 f_j e_j,
\]

for \( 1 \leq j \leq n - 1 \). For each \( j \), \( C_j \) is a central element of the subalgebra \( C[q^{\pm t_{j}}, q^{\pm (t_{j+1})}, e_j, f_j] \) of \( U_q(\mathfrak{gl}(n)) \). By the explicit formulas above, we compute

\[
(q - q^{-1})^2 \rho \lambda(C_j) = (x_j - x_{j+1})
\]

\[
\times \{ x_j^{-1}(1 - T_j^{-1})(1 - q^{-2 \lambda_{j+1}} T_{j+1}) - x_{j+1}^{-1}(1 - q^{2 \lambda_j} T_j^{-1})(1 - T_{j+1}) \}
\]

for \( j \geq 3 \). Hence, it is reasonable to investigate the following \( q \)-difference equation for the unknown function \( G = G(x) \):

\[
T_3 \cdots T_n G(x) = q^{-2 \lambda_2 - 2} G(x) \quad \text{and}
\]

\[
x_j^{-1}(1 - T_j^{-1})(1 - q^{-2 \lambda_{j+1}} T_{j+1}) G(x)
\]

\[
= x_{j+1}^{-1}(1 - q^{2 \lambda_j} T_j^{-1})(1 - T_{j+1}) G(x) \quad (3 \leq j \leq n).\]

The \( q \)-difference equation (54) above can be regarded as a homogeneous form of a \( q \)-analogue of the equation (10) for Lauricella’s hypergeometric series \( F_D \). Let us consider the following \( q \)-analogue of \( \varphi_D \) of \( F_D \):

\[
\varphi_D \left( \begin{array}{c}
a_1; b_1, \cdots, b_m; c; t_1, \cdots, t_m \end{array} \right)
\]

\[
= \sum_{v_1, \cdots, v_m \geq 0} \frac{(a_1; q)_{v_1} \cdots (b_m; q)_{v_m} (c; q)_{v_1 + \cdots + v_m} (q; q)_{v_1} \cdots (q; q)_{v_m} \cdot t_1^{v_1} \cdots t_m^{v_m}}{(q; q)_{v_1} \cdots (q; q)_{v_m}}.
\]

This \( q \)-hypergeometric series defines a holomorphic function around the origin \( t_1 = \cdots = t_m = 0 \). Under the condition that \( \lambda_2 + \lambda_3 \) is not an integer, one can directly check that the \( q \)-difference equation (54) has a solution

\[
G(\lambda; x_3, \cdots, x_n) = x_3^{-\lambda_2 - 1} \varphi_D \left( \begin{array}{c}
q^{2(\lambda_2 + 1)}; q^{-2 \lambda_4}; \cdots; q^{-2 \lambda_n}; q^2; z_4, \cdots, z_n \end{array} \right),
\]

where the variables \( z_4, \cdots, z_n \) are given by

\[
z_j = q^{2(1 + \lambda_3 + \lambda_4 + \cdots + \lambda_j)} x_j/x_3 \quad (j = 4, \cdots, n).
\]

By a direct verification, we can also show that, if \( a \) is an element of \( U_q(\mathfrak{gl}(n)) \) of weight \( \kappa \), the \( q \)-difference operator \( \rho \lambda(a) \) transforms the solution \( G(\lambda; x) \) to a constant multiple of \( G(\lambda + \kappa; x) \). In this sense, the family of \( q \)-difference operators \( \rho \lambda(a) \) describes the contiguity relations for the \( q \)-hypergeometric series \( \varphi_D \). It is also checked that \( \rho \lambda(C_j) G(\lambda; x) = 0 \) for all \( 1 \leq j \leq n - 1 \). For convenience, we will give in the last section the explicit formulas of the contiguity relations for the \( q \)-hypergeometric series \( \varphi_D \) appearing in (56).
Our derivation of the \( q \)-hypergeometric equation (54) is still incomplete since it is based only on the elements \( C_j \) of (52). It is expected that a more systematic approach to the \( q \)-analogue of Gelfand’s hypergeometric equations can be carried out by using the quantum analogue of the differential operators as in [20].

6. CONTINUITY RELATIONS FOR \( \varphi_D \)

We consider the following \( q \)-analogue of Lauricella’s hypergeometric series \( F_D \):

\[
\varphi_D\binom{a;b_1,\ldots,b_m}{c}\left(q; t_1,\ldots,t_m\right) = \sum_{\nu_1,\ldots,\nu_m \geq 0} \binom{c}{\nu_1+\ldots+\nu_m} \binom{a}{b_1; \ldots, b_m; t_1, \ldots, t_m} \cdot (q; q)^{\nu_1+\ldots+\nu_m}. \tag{58}
\]

Note that the \( q \)-hypergeometric series \( \varphi = \varphi_D \) satisfies the \( q \)-difference equation

\[
\begin{align*}
\{(1 - cq^{-1}T_{q,t})(1 - T_{q,t_j}) - t_j(1 - aT_{q,t})(1 - b_jT_{q,t_j})\}\varphi &= 0 \quad (1 \leq j \leq m), \\
\{t_i(1 - b_iT_{q,t_i})(1 - T_{q,t_j}) - t_j(1 - T_{q,t_i})(1 - b_jT_{q,t_i})\}\varphi &= 0 \quad (1 \leq i < j \leq m),
\end{align*} \tag{59}
\]

where \( T_{q,t} = T_{q,t_1} \cdots T_{q,t_m} \). For each \( \lambda = (\lambda_1, \ldots, \lambda_n) \) with \( \sum_{j=1}^n \lambda_j = -2 \), we define a series \( F(\lambda; z) \) in the \( n - 3 \) variables \( z = (z_{4}, \ldots, z_{n}) \) by

\[
F(\lambda; z) = \varphi_D \left( \frac{q^{2(\lambda_3+1)}; q^{2\lambda_4}; \ldots, q^{2\lambda_n}}{q^{2(\lambda_2+\lambda_3+2)}; q^2; z_4, \ldots, z_n} \right), \tag{60}
\]

assuming that \( \lambda_2 + \lambda_3 \) is not an integer.

We give below explicit formulas for the family of \( q \)-difference operators that describes the contiguity relations for \( F(\lambda; z) \). For each \( \lambda = (\lambda_1, \ldots, \lambda_n) \) with \( \sum_{j=1}^n \lambda_j = -2 \), we define the operators \( \pi_{\lambda}(q^{e_j}) \) \((1 \leq j \leq n)\), \( \pi_{\lambda}(c_{j}) \), \( \pi_{\lambda}(f_{j}) \) \((1 \leq j \leq n - 1)\) as follows:

\[
\pi_{\lambda}(q^{e_j}) = q^{\lambda_j} \quad (1 \leq j \leq n),
\]

\[
(q - q^{-1})\pi_{\lambda}(c_{1}) = q^{-\lambda_1-\lambda_2}(1 - q^{2(\lambda_2+\lambda_3+1)}T) - q^{\lambda_1+\lambda_2} \sum_{k=4}^{n} z_k (1 - q^{-2\lambda_{k+1}}T_k)T_{>k},
\]

\[
(q - q^{-1})\pi_{\lambda}(c_{2}) = -q^{\lambda_1+\lambda_2-\lambda_3-1}(1 - q^{2(\lambda_2+1)}T),
\]

\[
(q - q^{-1})\pi_{\lambda}(c_{3}) = q^{\lambda_3} \left\{ (1 - q^{-2\lambda_4}T_4) - q^{2\lambda_3+1} \frac{1}{z_4} (1 - T_4) \right\},
\]

\[
(q - q^{-1})\pi_{\lambda}(c_{j}) = q^{\lambda_{j+1}} \left\{ (1 - q^{-2\lambda_{j+1}}T_{j+1}) - \frac{q^{-2\lambda_{j+1}}}{z_{j+1}} (1 - T_{j+1}) \right\} T_{>j+1}^{-1} \quad (j \geq 4),
\]

\[
(q - q^{-1})\pi_{\lambda}(f_{1}) = -q(1 - q^{2\lambda_3+1}T) + q^{1-2\lambda_1} \sum_{k=4}^{n} z_k (1 - q^{-2\lambda_k}T_k)T_{>k},
\]

\[
(q - q^{-1})\pi_{\lambda}(f_{2}) = \left\{ -q^{\lambda_1}(1 - q^{2(\lambda_3+1)}T) + q^{-\lambda_1} \sum_{k=4}^{n} z_k (1 - q^{-2\lambda_k}T_k)T_{>k} \right\} T^{-1},
\]

304
\[(q - q^{-1}) \pi_\lambda(f_0) = q^{-\lambda_3} \left\{ -\left(1 - q^{2(\lambda_2 + \lambda_3 + 1)}T \right) + q^{-2(\lambda_4 + 1)}z_4 \left(1 - q^{2(\lambda_2 + 1)}T \right) \right\}, \]
\[(q - q^{-1}) \pi_\lambda(f_j) = q^{\lambda_j} \left\{ \left(1 - q^{-2\lambda_j}T_j \right) - q^{-2(\lambda_{j+1} + 1)}\frac{\tilde{z}_{j+1}}{z_j} \left(1 - T_j \right) \right\} \quad (j \geq 4), \quad (61)\]

where \(T_k = T_{q^2}z_k, \ T_{>k} = T_{k+1}T_{k+2} \cdots T_n, \ T = T_1T_2 \cdots T_n \) and \(\lambda_{>k} = \lambda_{k+1} + \lambda_{k+2} + \cdots + \lambda_n. \) These \(q\)-difference operators, obtained from those in (50), give a representation of the quantized universal enveloping algebra \(U_q(gl(n)). \)

For each \(\lambda \) with \(\sum_{j=1}^n \lambda_j = -2, \) let \(\mathcal{H}_\lambda \) be the same vector space of all germs of holomorphic functions near \(z = 0 \) and consider the \(q\)-difference operators \(\pi_\lambda(q^{f_j}) : \mathcal{H}_\lambda \to \mathcal{H}_\lambda, \ \pi_\lambda(e_j) : \mathcal{H}_\lambda \to \mathcal{H}_{\lambda + \alpha_j}, \ \pi_\lambda(f_j) : \mathcal{H}_\lambda \to \mathcal{H}_{\lambda - \alpha_j} \) for \(\alpha_j = \epsilon_j - \epsilon_{j+1}. \) These operators define a left \(U_q(gl(n))\)-module structure on the direct sum \(\bigoplus_\lambda \mathcal{H}_\lambda. \)

The contiguity relations for these \(q\)-hypergeometric series \(F(\lambda; z)\) defined above are given explicitly as follows:

\[
\begin{align*}
\pi_\lambda(q^{f_j})F(\lambda; z) &= q^{\lambda_j}F(\lambda; z) \quad (1 \leq j \leq n), \\
\pi_\lambda(e_1)F(\lambda; z) &= -q^{\lambda_1 + \lambda_3 + 1}[\lambda_2 + \lambda_3 + 1]F(\lambda + \alpha_1; z), \\
\pi_\lambda(e_2)F(\lambda; z) &= q^{-\lambda_1 - \lambda_3}[\lambda_2 + 1]F(\lambda + \alpha_2; z), \\
\pi_\lambda(e_3)F(\lambda; z) &= q^{-\lambda_2 - 1}[\lambda_3 + 1][\lambda_4]_{\lambda_2 + \lambda_3 + 2}F(\lambda + \alpha_3; z), \\
\pi_\lambda(e_j)F(\lambda; z) &= [\lambda_{j+1}][\lambda_j + 1]F(\lambda + \alpha_j; z) \quad (j \geq 4), \\
\pi_\lambda(f_1)F(\lambda; z) &= -q^{-\lambda_1 - \lambda_3}[\lambda_1][\lambda_2 + 1]_{\lambda_2 + \lambda_3 + 2}F(\lambda - \alpha_1; z), \\
\pi_\lambda(f_2)F(\lambda; z) &= q^{\lambda_1 + \lambda_3 + 1}[\lambda_3 + 1]F(\lambda - \alpha_2; z), \\
\pi_\lambda(f_3)F(\lambda; z) &= q^{\lambda_2 + 1}[\lambda_2 + \lambda_3 + 1]F(\lambda - \alpha_3; z), \\
\pi_\lambda(f_j)F(\lambda; z) &= [\lambda_j][\lambda_j + 1]F(\lambda - \alpha_j; z) \quad (j \geq 4),
\end{align*}
\]

where \([a] = (q^a - q^{-a})/(q - q^{-1}).\)

References


21. M. Noumi, H. Yamada and K. Mimachi. Finite dimensional representations of the quantum group $GL_q(n; \mathbb{C})$ and the zonal spherical functions on $U_q(n-1)\setminus U_q(n)$, to appear in *Japanese J. Math.*


23. E. Taft and J. Towber (1991). Quantum deformation of flag schemes and
Grassmann schemes I. A $q$-deformation of the shape algebra for $GL(n)$, *J. Algebra* 142, 1–36.