Quantum Grassmannians and q-hypergeometric series

dedicated to Prof. T.H. Koornwinder

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INTRODUCTION

One of the important aspects of quantum groups is the link with the theory of q-special functions. It is already known that various q-special functions arise naturally in the framework of quantum groups. Roughly speaking, quantum groups, as well as other group-theoretic frameworks, have at least two ways to provide special functions: one as spherical functions on coset spaces, and the other as connection coefficients describing the tensor products of representations.

As to the study of connections of quantum groups with q-special functions, one of the most important contributors is Prof. T.H. Koornwinder, who was among the first to recognize that the unitary representations of the quantum $SU(2)$ group give a natural interpretation of the little q-Jacobi polynomials as matrix coefficients ([9], see also VAKSMAN-SOIBELMAN [24] and MASUDA et al [13]). He also showed, in a joint work with KOELINK [8], that the q-Hahn polynomials arise naturally as the Clebsch-Gordan coefficients for the quantum $SU(2)$ group (see also [7]). More impressive to the author is his realization of the continuous q-Legendre polynomials and a two-parameter family of Askey-Wilson polynomials as zonal spherical functions on the quantum $SU(2)$ group ([10], [11]). This discovery of Koornwinder has brought out a new stage of links between the nature of q-special functions and geometric problems in quantum groups. For these subjects, we refer the reader to the survey paper [12] by Koornwinder himself (see also NOUMI [15] and NOUMI-MIMACHI [16]–[19]).

In this article, we would like to discuss a slightly different point relating quantum groups to q-special functions. This subject was motivated by a recent work of HORIKAWA [5], who showed that the contiguity relations for the general q-hypergeometric series ${}_2\varphi_1$ can be described by the quantized universal enveloping algebra $U_q(gl(4))$. He found this fact by investigating q-analogues of Gelfand's interpretation of Gauss' hypergeometric functions by the Grassmann manifold $Grass(2,4)$. This work is remarkable because it implies that quantum groups are involved in q-analysis from its very starting points. He has also obtained analogous results [6] for the case of $Grass(k,n)$, where some q-hypergeometric series, including a q-analogue of Lauricella's F_D, and their contiguity relations are discussed. It is not clear, however, why quantum groups arise as contiguity relations for q-hypergeometric series. Horikawa's presentation is in fact given by factorizing directly the q-difference equations for them.

This article is an attempt to give an intrinsic explanation of the phenomenon that the quantized universal enveloping algebras arise as the contiguity relations for q-hypergeometric series. Our approach is carried out by considering a *quantum* analogue of the Grassmann manifolds and the Gelfand hypergeometric functions associated with them. In this article, we discuss exclusively the case of $Grass(2,n)$ and show that a q-analogue of Lauricella's hypergeometric series F_D of $n-3$ variables falls into this framework. From this interpretation, we see that the quantized universal enveloping algebra $U_q(gl(n))$ arises naturally as the algebra describing the contiguity relations of these q-hypergeometric series. The exposition in this article is still experimental and incomplete in many respects. The author hopes that the framework of Gelfand's generalized hypergeometric functions will activate mathematical interactions even between quantum groups and q-special functions. The author is grateful to Prof. Horikawa for valuable suggestions and for communicating his preprints before publication.

1. Gelfand's hypergeometric equations

In this section, we give a brief review on Gelfand's generalized hypergeometric equations ([1], [2]), associated with the Grassmannian $Grass(k,n)$ $(k \le n)$. In what follows, we denote by $Grass(k,n)$, or simply by $G_{k,n}$, the Grassmannian consisting of all k-dimensional subspaces of the n-dimensional complex vector space $\mathbb{C}^n$. Let us consider the general $k \times n$ complex matrices

$$T = \begin{pmatrix} t_{11} & t_{12} & \cdots & t_{1n} \\ t_{21} & t_{22} & \cdots & t_{2n} \\ & \cdots & \cdots & \\ t_{k1} & t_{k2} & \cdots & t_{kn} \end{pmatrix}. \tag{1}$$

The *Gelfand hypergeometric equation* of type $G_{k,n}$ is the following system of differential equations defined on the matrix space $Mat(k,n)$:

$$\sum_{j=1}^{n} t_{rj} \frac{\partial}{\partial t_{sj}} \Phi(T) = -\delta_{rs}\Phi(T) \qquad (1 \le r,s \le k), \tag{2}$$

$$\sum_{r=1}^{k} t_{rj} \frac{\partial}{\partial t_{rj}} \Phi(T) = \lambda_j \Phi(T) \qquad (1 \le j \le n), \tag{3}$$

$$\frac{\partial^2}{\partial t_{ri}\partial t_{sj}} \Phi(T) = \frac{\partial^2}{\partial t_{si}\partial t_{rj}} \Phi(T) \qquad (1 \le r < s \le k, 1 \le i < j \le n), \tag{4}$$

where λ_j $(1 \le j \le n)$ are complex parameters such that $\sum_{j=1}^{n} \lambda_j = -k$. Multi-valued holomorphic solutions $\Phi(T)$ of the system (2)–(4), defined on an appropriate open subset of the matrix space $Mat(k,n)$, are called Gelfand's hypergeometric functions of type $G_{k,n}$. Note that equation (2) is equivalent to saying that $\Phi(T)$ has the relative invariance

$$\Phi(gT) = \det(g)^{-1}\Phi(T) \quad (g \in GL(k)), \tag{5}$$

under the left action of $GL(k)$. On the other hand, equation (3) requires the homogeneity

$$\Phi(T\mathrm{diag}(c_1, ..., c_n)) = \Phi(T){c_1}^{\lambda_1} \cdots {c_n}^{\lambda_n} \tag{6}$$

under the action of the diagonal subgroup of $GL(n)$. Equation (4) is the essential part of the Gelfand hypergeometric equation. In fact, from (4), many examples of classical hypergeometric functions in some appropriate variables can be obtained by eliminating the relative invariance (2) and (3).

For simplicity, we consider the case of the Grassmannian $Grass(2, n)$. In this case, the Gelfand hypergeometric equation above is related to Lauricella's hypergeometric functions F_D in $n-3$ variables ([2]). Note first that the generic $2 \times n$ matrix T has the following decomposition:

$$\begin{aligned} T &= \begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 & \cdots & 1 \\ 0 & 1 & x_3 & \cdots & x_n \end{pmatrix} \\ &\quad \times \mathrm{diag}(1, 1, -\xi_{23}/\xi_{12}, \cdots, -\xi_{2n}/\xi_{12}), \end{aligned} \tag{7}$$

where $\xi_{ij} = t_{1i}t_{2j} - t_{2i}t_{1j}$ for $1 \leq i, j \leq n$ and and $x_j = -\xi_{1j}/\xi_{2j}$ for $j = 3, \cdots, n$. The relative invariance (5), (6) implies that $\Phi(T)$ can be written in the form

$$\Phi(T) = G(x_3, \cdots, x_n)\xi_{12}^{\lambda_1+\lambda_2+1}(-\xi_{23})^{\lambda_3} \cdots (-\xi_{2n})^{\lambda_n}, \tag{8}$$

for some function $G(x_3, ..., x_n)$ in the variables $(x_3, \cdots, x_n)$. Note here that $\sum_{j=1}^{n} \lambda_j = -2$. Furthermore, the function $G(x_3, \cdots, x_n)$ should be homogeneous of degree $-\lambda_2 - 1$ from the condition (6). Hence, $G(x_3, \cdots, x_n)$ takes the form

$$G(x_3, \cdots, x_n) = F(z_4, \cdots, z_n){x_3}^{-\lambda_2-1}, \tag{9}$$

where $z_j = x_j/x_3$ for $j = 4, \cdots, n$. In these coordinates $z = (z_4, \cdots, z_n)$, it is known that equation (4) gives rise to the following differential equation to be satisfied by the function $F(z_4, \cdots, z_n)$:

$$\{\vartheta_k(\sum_{j=4}^{n} \vartheta_j + \lambda_2 + \lambda_3 + 1) - z_k(\vartheta_k - \lambda_k)(\sum_{j=4}^{n} \vartheta_j + \lambda_2 + 1)\}F(z_4, \cdots, z_n) = 0, \tag{10}$$

for $k = 4, \cdots, n$, where $\vartheta_j = z_j\partial/\partial z_j$ $(j = 4, \cdots, n)$. A solution to this equation, holomorphic near the origin $z = 0$, is given by *Lauricella's hypergeometric series*

$$F_D\left(\begin{matrix} \lambda_2 + 1; -\lambda_4, \cdots, -\lambda_n \\ \lambda_2 + \lambda_3 + 2 \end{matrix} ; z_4, \cdots, z_n \right). \tag{11}$$

Lauricella's hypergeometric series F_D are defined as follows:

$$F_D\left(\begin{matrix}\alpha;\beta_1,\cdots,\beta_m\\ \gamma\end{matrix};x_1,\cdots,x_m\right)$$
$$=\sum_{\nu_1,\cdots,\nu_m\geq 0}\frac{(\alpha)_{\nu_1+\cdots+\nu_m}(\beta_1)_{\nu_1}\cdots(\beta_m)_{\nu_m}}{(\gamma)_{\nu_1+\cdots+\nu_m}\nu_1!\cdots\nu_m!}x_1{}^{\nu_1}\cdots x_m{}^{\nu_m}, \tag{12}$$

where $(\alpha)_\nu=\alpha(\alpha+1)\cdots(\alpha+\nu-1)$.

It should be noted that the system of equations (4) is covariant under the adjoint action of $GL(n)$. From this property, it is known that the contiguity relations for Lauricella's F_D are described by the Lie algebra $gl(n)$. For the details, see [4], [22], [14] for example.

2. Quantum Grassmannians

We now start to consider the quantum analogue of Gelfand's framework of generalized hypergeometric functions. For this purpose, we investigate a q-deformation of the graded algebra of homogenous functions on the Grassmannian.

Recall that the coordinate ring $A_q(Mat(k,n))$ of the quantum matrix space $Mat_q(k,n)$ is the algebra generated by the "canonical coordinates" t_{rj} $(1\leq r\leq k, 1\leq j\leq n)$ with commutation relations:

$$\begin{aligned}
t_{rj}t_{sj}&=qt_{sj}t_{rj} && (1\leq r<s\leq k,1\leq j\leq n),\\
t_{ri}t_{rj}&=qt_{rj}t_{ri} && (1\leq r\leq k,1\leq i<j\leq n),\\
t_{rj}r_{si}&=t_{si}t_{rj},\\
t_{ri}t_{sj}-t_{sj}t_{ri}&=(q-q^{-1})t_{rj}t_{si} && (1\leq r<s\leq k,1\leq i<j\leq n).
\end{aligned}\tag{13}$$

These generators correspond to the matrix coordinates $T=(t_{rj})_{1\leq r\leq k,1\leq j\leq n}$ as in (1). It will be helpful to note that, for each $1\leq r<s\leq k$ and $1\leq i<j\leq n$, the 2×2 matrix of generators

$$\begin{pmatrix}a&b\\c&d\end{pmatrix}=\begin{pmatrix}t_{ri}&t_{rj}\\t_{si}&t_{sj}\end{pmatrix}\tag{14}$$

satisfies the $Mat_q(2)$-relations:

$$\begin{aligned}
&ab=qba, ac=qca, bd=qdb, cd=qdc,\\
&bc=cb, ad-da=(q-q^{-1})bc.
\end{aligned}\tag{15}$$

In what follows, we fix a complex number q with $0<|q|<1$.

We recall that the algebra $A_q(Mat(k,n))$ has a natural structure of a two-sided comodule over the pair of Hopf algebras $(A_q(GL(k)),A_q(GL(n)))$. Accordingly, it becomes a bimodule over the pair $(U_q(gl(n)),U_q(gl(k)))$. By using this structure, we can consider the quantum analogue of the coset space $SL(k)\backslash Mat(k,n)$. Note that the quotient space $SL(k)\backslash Mat'(k,n)$ of the open subset $Mat'(k,n)$ of all matrices of maximal rank defines a $\mathbf{C}^*$-bundle over the Grassmannian

$Grass(k,n) = GL(k)\backslash Mat'(k,n)$ with the natural projection. Let $\mathcal{A}$ be the following subalgebra of left $SL_q(k)$-invariants in $A_q(Mat(k,n))$:

$$\begin{aligned}\mathcal{A} &= A_q(SL(k)\backslash Mat(k,n)) \\ &= \{\varphi \in A_q(Mat(k,n)); \varphi.a = \varepsilon(a)\varphi \quad \text{for all} \quad a \in U_q(sl(k))\}. \end{aligned} \tag{16}$$

By the standard monomial theory for this case, one can show that the algebra $\mathcal{A}$ is generated by the *quantum minor determinants*

$$\xi_{j_1 j_2 \cdots j_k} = \sum_{w \in \mathbf{S}_k} (-q)^{\ell(w)} t_{w(1)j_1} t_{w(2)j_2} \cdots t_{w(k)j_k} (1 \leq j_1, j_2, \cdots, j_k \leq n) \tag{17}$$

where $\mathbf{S}_k$ stands for the permutation group of k letters and, for each $w \in \mathbf{S}_k$, $\ell(w)$ denotes the number of inversions in w. These quantum minors are analogues of the Plücker coordinates of the Grassmannian $Grass(k,n)$. For the classical Plücker coordinates of the Grassmannian, see [3], for example. The general Plücker relations for our quantum minors can be found in [21] and [23].

In the case of the Grassmannian $Grass(2,n)$, the structure of the algebra $\mathcal{A} = A_q(SL(2)\backslash Mat(2,n))$ is simply described. We remark first that the quantum minors

$$\xi_{ij} = t_{1i}t_{2j} - qt_{2i}t_{1j} \quad (1 \leq i,j \leq n) \tag{18}$$

satisfy the following two types of Plücker relations:

$$\begin{aligned}&\xi_{12}\xi_{34} - q\xi_{13}\xi_{24} + q^2\xi_{14}\xi_{23} = 0, \\ &\xi_{34}\xi_{12} - q^{-1}\xi_{24}\xi_{13} + q^{-2}\xi_{23}\xi_{14} = 0.\end{aligned} \tag{19}$$

The other Plücker relations are obtained by replacing the sequence $1 < 2 < 3 < 4$ by an arbitrary increasing sequence $i < j < r < s$ of indices. The typical commutation relations among ξ_{ij} are given as follows:

$$\begin{aligned}&\xi_{11} = 0, \; q\xi_{12} + \xi_{21} = 0, \\ &\xi_{12}\xi_{13} = q\xi_{13}\xi_{12}, \; \xi_{12}\xi_{23} = q\xi_{23}\xi_{12}, \xi_{13}\xi_{23} = q\xi_{23}\xi_{13}, \\ &\xi_{12}\xi_{34} = q^2\xi_{34}\xi_{12}, \; \xi_{13}\xi_{24} - \xi_{24}\xi_{13} = (q - q^{-1})\xi_{14}\xi_{23}, \; \xi_{14}\xi_{23} = \xi_{23}\xi_{14}.\end{aligned} \tag{20}$$

To get the other commutation relations, replace $1 < 2 < \cdots$ by any increasing sequence $i < j < \cdots$. It is interesting to see that the 2×2 matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \xi_{13} & \xi_{14} \\ \xi_{23} & \xi_{24} \end{pmatrix} \tag{21}$$

again satisfies the $Mat_q(2)$-relations (15). In this case the quantum determinant is given by

$$\begin{aligned} ad - qbc &= \xi_{13}\xi_{24} - q\xi_{14}\xi_{23} = q^{-1}\xi_{12}\xi_{34} \quad \text{or} \\ da - q^{-1}cb &= \xi_{24}\xi_{13} - q^{-1}\xi_{23}\xi_{14} = q\xi_{34}\xi_{12}.\end{aligned} \tag{22}$$

The last equality is the Plücker relation.

3. Localization

In order to consider the analogue of the decomposition (7) of the matrix T, we need to introduce some localizations of the algebra $\mathcal{A} = A_q(SL(2)\backslash Mat(2,n))$.

The first step is to invert the principal minor ξ_{12}. This procedure is easily understood since $\xi_{12}\xi_{ij}$ is a constant multiple of $\xi_{ij}\xi_{12}$ for each $i < j$. It should be noted here that we have

$$\xi_{ij} = (q^{-1}\xi_{2j}\xi_{1i} - q^{-2}\xi_{2i}\xi_{1j})\xi_{12}^{-1} \quad \text{for} \quad 3 \leq i < j, \tag{23}$$

by the Plücker relations as in (19). By this fact, one can show that the localization $\mathcal{A}[\xi_{12}{}^{-1}]$ has the monomial basis

$$\xi_{2n}{}^{b_n} \cdots \xi_{23}{}^{b_3}\xi_{1n}{}^{a_n} \cdots \xi_{13}{}^{a_3}\xi_{12}{}^{\mu} \tag{24}$$

where $a_j, b_j \in \mathbb{N}$ and $\mu \in \mathbb{Z}$. In other words, the algebra $\mathcal{A}[\xi_{12}{}^{-1}]$ is isomorphic to a noncommutative tensor product of $A_q(Mat(2,n-2))$ with the algebra of Laurent polynomials. This type of localization corresponds to restricting the $\mathbf{C}^*$-bundle $SL(2)\backslash Mat'(2,n)$ to the biggest cell of the Grassmannian $Grass(2,n)$.

The next step is to adjoin the inverses of the quantum minors $\xi_{23}, \cdots, \xi_{2n}$. This time one should be careful in treating the inverses since the commutation relations involving ξ_{2j}'s are not trivial. For each $i < j$, the commutation relation between ξ_{1i} and $\xi_{2j}{}^{-1}$ should be

$$\xi_{1i}\xi_{2j}{}^{-1} = \xi_{2j}{}^{-1}\xi_{1i} + q^{-2}(q - q^{-1})\xi_{2j}{}^{-2}\xi_{2i}\xi_{1j}. \tag{25}$$

To understand the structure of the localization $\mathcal{A}[\xi_{12}^{-1}, \xi_{23}^{-1}, \cdots, \xi_{2n}^{-1}]$, it is worthwhile to recall the commutation relations

$$a^r d^s = \sum_{k \geq 0} (-1)^k q^{k(2r+2s-2k+1)} \frac{(q^{-2r}; q^2)_k (q^{-2s}; q^2)_k}{(q^2; q^2)_k} d^{s-k} c^k b^k a^{r-k} \tag{26}$$

for $r, s \in \mathbb{N}$ in $A_q(Mat(2))$, where $(x;q)_k = (1-x)(1-qx)\cdots(1-q^{k-1}x)$. Note that the summation on the right-hand side terminates as long as either r or s is a nonnegative integer even if the other is an arbitrary parameter. This means that one can consistently adjoin one of the inverses d^{-1} or a^{-1} to the algebra $A_q(Mat(2))$, although one cannot handle these two inverses simultaneously in the algebraic sense.

In our algebra $\mathcal{A}[\xi_{12}^{-1}, \xi_{23}^{-1}, \cdots, \xi_{2n}^{-1}]$, the commutation relations between ξ_{1i} and $\xi_{2j}{}^{-1}$ with $i < j$ are described by (26) for $a = \xi_{1i}$, $d = \xi_{2j}$ and $r \in \mathbb{N}$, $s \in \mathbb{Z}$. By this fact, one can show that the algebra $\mathcal{A}[\xi_{12}^{-1}, \xi_{23}^{-1}, \cdots, \xi_{2n}^{-1}]$ also has the monomial basis of type (24) with $a_j \in \mathbb{N}$ and $b_j, \mu \in \mathbb{Z}$.

After these localizations, the matrix $T = (t_{rj})_{1\leq r\leq 2, 1\leq j\leq n}$ of generators for $A_q(Mat(2,n))$ can be decomposed in a similar way to (7):

$$\begin{aligned} T \;=\; & \begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 & \cdots & 1 \\ 0 & 1 & u_3 & \cdots & u_n \end{pmatrix} \\ & \times \mathrm{diag}(1, 1, -q^{-1}\xi_{12}{}^{-1}\xi_{23}, \cdots, -q^{-1}\xi_{12}{}^{-1}\xi_{2n}), \end{aligned} \tag{27}$$

where $u_j = -\xi_{2j}{}^{-1}\xi_{1j}$ for $j = 3, \cdots, n$. Let us denote by $\mathcal{R}$ the subalgebra of $\mathcal{A}[\xi_{12}^{-1}, \xi_{23}^{-1}, \cdots, \xi_{2n}^{-1}]$ generated by the elements $u_3, \cdots, u_n$, regarding them as a quantum analogue of the homogeneous coordinates of the projective $n-3$ space. By the argument above, we see that the algebra $\mathcal{A}[\xi_{12}^{-1}, \xi_{23}^{-1}, \cdots, \xi_{2n}^{-1}]$ has the decomposition

$$\mathcal{A}[\xi_{12}{}^{-1}, \xi_{23}{}^{-1}, \cdots, \xi_{2n}{}^{-1}] = \bigoplus_{\mu, \lambda_3, \cdots, \lambda_n \in \mathbf{Z}} \xi_{2n}{}^{\lambda_n} \cdots \xi_{23}{}^{\lambda_3} \xi_{12}{}^{\mu} \mathcal{R}. \tag{28}$$

Although $\mathcal{R}$ is still a *noncommutative* subalgebra, its structure is not too complicated. In fact, the generators $u_3, \cdots, u_n$ satisfy the commutation relations

$$(u_i - u_j)u_k = q^2 u_k (u_i - u_j) \quad \text{for} \quad (3 \le i, j \le k \le n). \tag{29}$$

If one takes another system of generators

$$v_3 = u_3 - u_4, \cdots, v_{n-1} = u_{n-1} - u_n \quad \text{and} \quad v_n = u_n, \tag{30}$$

then their commutation relations are simply given by $v_i v_j = q^2 v_j v_i$ for $3 \le i < j \le n$.

4. A REPRESENTATION OF $U_q(gl(n))$

Recall that $U_q(gl(n))$ is an algebra generated by the symbols $q^{\pm \epsilon_j}$ $(1 \le j \le n), e_j, f_j$ $(1 \le j \le n-1)$. Hereafter, we use the notation $q^h = q^{a_1 \epsilon_1} \cdots q^{a_n \epsilon_n}$ for any linear combination $h = a_1 \epsilon_1 + \cdots + a_n \epsilon_n$ with integral coefficients. Then the fundamental relations for the generators of $U_q(gl(n))$ are given by

$$\begin{aligned}
&q^0 = 1, \quad q^h q^{h'} = q^{h+h'},\\
&q^h e_j q^{-h} = q^{<h,\alpha_j>} e_j, \quad q^h f_j q^{-h} = q^{-<h,\alpha_j>} f_j \quad (1 \le j \le n-1),\\
&e_i f_j - f_j e_i = \delta_{ij}(q^{\epsilon_i - \epsilon_{i+1}} - q^{-\epsilon_i + \epsilon_{i+1}})/(q - q^{-1}) \quad (1 \le i, j \le n-1),\\
&e_i e_j = e_j e_i, \quad f_i f_j = f_j f_i \quad (|i-j| \ge 2),\\
&e_i{}^2 e_j - (q + q^{-1}) e_i e_j e_i + e_j e_i{}^2 = 0 \quad (|i-j| = 1),\\
&f_i{}^2 f_j - (q + q^{-1}) f_i f_j f_i + f_j f_i{}^2 = 0 \quad (|i-j| = 1),
\end{aligned} \tag{31}$$

where $\alpha_j = \epsilon_j - \epsilon_{j+1}$ for $1 \le j \le n-1$ and $<\ ,\ >$ stands for the canonical symmetric bilinear form such that $< \epsilon_i, \epsilon_j >= \delta_{ij}$. The last two cubic equations of (31) are sometimes referred to as the *Serre relations*. The algebra $U_q(gl(n))$ has a Hopf algebra structure. We take here the following convention of the coproduct Δ:

$$\begin{aligned}
\Delta(q^h) &= q^h \otimes q^h,\\
\Delta(e_j) &= e_j \otimes 1 + q^{\epsilon_j - \epsilon_{j+1}} \otimes e_j \quad (1 \le j \le n-1),\\
\Delta(f_j) &= f_j \otimes q^{-\epsilon_j + \epsilon_{j+1}} + 1 \otimes f_j \quad (1 \le j \le n-1).
\end{aligned} \tag{32}$$

We denote its counit and antipode by ε and S, respectively:

$$\begin{aligned}
&\varepsilon(q^h) = 1, \quad \varepsilon(e_j) = 0, \varepsilon(f_j) = 0,\\
&S(q^h) = q^{-h}, \quad S(e_j) = -q^{-\epsilon_j + \epsilon_{j+1}} e_j, S(f_j) = -f_j q^{\epsilon_j - \epsilon_{j+1}}
\end{aligned} \tag{33}$$

for $1 \leq j \leq n-1$.

The coordinate ring $A_q(Mat(2,n))$ of the quantum matrix space $Mat_q(2,n)$ has the structure of a left $U_q(gl(n))$-module. Furthermore, it is an *algebra with* $U_q(gl(n))$*-symmetry* in the sense that

(a) $a.1 = \varepsilon(a)1$ for all $a \in U_q(gl(n))$;
(b) if $a \in U_q(gl(n))$ and $\Delta(a) = \sum_i a'_i \otimes a''_i$, then
$a.(\varphi\psi) = \sum_i a'_i.\varphi\ a''_i.\psi$ for all $\varphi, \psi \in A_q(Mat(2,n))$.

The actions of q^h and e_k, f_k on the generators $t_{rj} (1 \leq r \leq 2, 1 \leq j \leq n)$ of $A_q(Mat(2,n))$ are given by

$$q^h.t_{rj} = q^{<h,\epsilon_j>}t_{rj}, \quad e_k.t_{rj} = \delta_{k+1,j}t_{rk}, \quad f_k.t_{rj} = \delta_{k,j}t_{r,k+1}, \tag{34}$$

respectively. Their actions on an arbitrary element $\varphi \in A_q(Mat(2,n))$ are then determined by a successive use of the rule (b) above. It is clear that the invariant ring $\mathcal{A} = A_q(SL(2)\backslash Mat(2,n))$ is a left $U_q(gl(n))$-submodule of $A_q(Mat(2,n))$. The action of $U_q(gl(n))$ on the quantum minors is determined by

$$\begin{aligned} q^h.\xi_{ij} &= q^{<h,\epsilon_i+\epsilon_j>}\xi_{ij}, \\ e_k.\xi_{ij} &= \delta_{k+1,i}\xi_{kj} + \delta_{k+1,j}q^{<\alpha_k,\epsilon_i>}\xi_{ik}, \\ f_k.\xi_{ij} &= \delta_{k,i}q^{-<\alpha_k,\epsilon_j>}\xi_{k+1,j} + \delta_{k,j}\xi_{ik+1}. \end{aligned} \tag{35}$$

We now consider the localization $\mathcal{A}[\xi_{12}^{-1}, \xi_{23}^{-1}, \cdots, \xi_{2n}^{-1}]$ of $\mathcal{A}$ discussed in the previous section. This algebra has a unique structure of an algebra with $U_q(gl(n))$-symmetry containing $\mathcal{A}$ as a $U_q(gl(n))$-submodule. Suppose that φ is a weight vector of weight μ in $\mathcal{A} : q^h.\varphi = q^{<h,\mu>}\varphi$. If φ is invertible in a localization of $\mathcal{A}$, then we must have

$$\begin{aligned} q^h.\varphi^{-1} &= q^{-<h,\mu>}\varphi^{-1}, \\ e_k.\varphi^{-1} &= -q^{-<\alpha_k,\mu>}\varphi^{-1}\ e_k.\varphi\ \varphi^{-1}, \\ f_k.\varphi^{-1} &= -q^{<\alpha_k,\mu>}\varphi^{-1}.\ f_k.\varphi\ \varphi^{-1}, \end{aligned} \tag{36}$$

provided that the properties (a),(b) above are preserved. We can extend the action of $U_q(gl(n))$ on $\mathcal{A}$ to $\mathcal{A}[\xi_{12}^{-1}, \xi_{23}^{-1}, \cdots, \xi_{2n}^{-1}]$ by the rules (36) for $\varphi = \xi_{12}, \xi_{23}, \cdots, \xi_{2n}$, so that the conditions (a), (b) are satisfied in the localization $\mathcal{A}[\xi_{12}^{-1}, \xi_{23}^{-1}, \cdots, \xi_{2n}^{-1}]$ as well.

Recall that the algebra $\mathcal{A}[\xi_{12}^{-1}, \xi_{23}^{-1}, \cdots, \xi_{2n}^{-1}]$ has the decomposition

$$\mathcal{A}[\xi_{12}^{-1}, \xi_{23}^{-1}, \cdots, \xi_{2n}^{-1}] = \bigoplus_{\mu,\lambda_3,\cdots,\lambda_n \in \mathbb{Z}} {\xi_{2n}}^{\lambda_n} \cdots {\xi_{23}}^{\lambda_3} {\xi_{12}}^{\mu}\mathcal{R}. \tag{37}$$

Note that the subalgebra $\mathcal{R}$, generated by the elements $u_j = -{\xi_{2j}}^{-1}\xi_{1j}$ $(3 \leq j \leq n)$, has a monomial basis

$${u_n}^{\nu_n} {u_{n-1}}^{\nu_{n-1}} \cdots {u_3}^{\nu_3} \quad (\nu_3, \cdots, \nu_n \in \mathbf{N}). \tag{38}$$

We consider the *commutative* algebra $\mathbf{C}[x_3, \cdots, x_n]$ of polynomials in $n-3$ variables and define an isomorphism of vector spaces $\phi : \mathbf{C}[x_3, \cdots, x_n] \tilde{\rightarrow} \mathcal{R}$ by the normal ordering

$$\phi({x_3}^{\nu_3}\cdots{x_n}^{\nu_n}) = {u_n}^{\nu_n}\cdots{u_3}^{\nu_3}. \tag{39}$$

In order to translate the action of $U_q(gl(n))$ into the terms of usual q-difference operators, we consider a copy of the polynomial ring $\mathbf{C}[x_3,\cdots,x_n]$:

$$\mathcal{G} = \bigoplus_{\mu,\lambda_3,\cdots,\lambda_n\in Z}\mathcal{G}_{\mu,\lambda_3,\cdots,\lambda_n}, \tag{40}$$

where $\mathcal{G}_{\mu,\lambda_3,\cdots,\lambda_n} = \mathbf{C}[x_3,\cdots,x_n]$ for all $\mu,\lambda_3,\cdots,\lambda_n \in \mathbb{Z}$. Then we define an isomorphism of vector spaces

$$w : \mathcal{G}\tilde{\rightarrow}\mathcal{A}[\xi_{12}^{-1},\xi_{23}^{-1},\cdots,\xi_{2n}^{-1}] \tag{41}$$

as the direct sum of linear mappings

$$w_{\mu,\lambda_3,\cdots,\lambda_n} : \mathbf{C}[x_3,\cdots,x_n]\tilde{\rightarrow}\xi_{2n}^{\lambda_n}\cdots\xi_{23}^{\lambda_3}\xi_{12}^{\mu}\mathcal{R} \tag{42}$$

such that

$$w_{\mu,\lambda_3,\cdots,\lambda_n}(G) = \xi_{2n}^{\lambda_n}\cdots\xi_{23}^{\lambda_3}\xi_{12}^{\mu}\phi(G) \tag{43}$$

for all $G \in \mathbf{C}[x_3,\cdots,x_n]$. Through the isomorphism w thus defined, we obtain a left $U_q(gl(n))$-module structure on the vector space $\mathcal{G}$.

After some explicit computations, we see that the action of $U_q(gl(n))$ on $\mathcal{G}$ is actually described in terms of the q-difference operators with respect to the variables $x_3,\cdots,x_n$. Suppose that a is an element of $U_q(gl(n))$ such that $q^h a q^{-h} = q^{<h,\kappa>}a$ for some $\kappa = \kappa_1\epsilon_1 + \cdots + \kappa_n\epsilon_n$ ($\kappa_j \in \mathbb{Z}$). If a is an element of weight κ in this sense, then its action on $\mathcal{G}$ is represented by a family of operators

$$\tilde{\rho}_{\mu,\lambda_3,\cdots,\lambda_n}(a) : \mathcal{G}_{\mu,\lambda_3,\cdots,\lambda_n} \longrightarrow \mathcal{G}_{\mu',\lambda_3',\cdots,\lambda_n'}, \tag{44}$$

where $\mu' = \mu + \kappa_1 + \kappa_2, \lambda_j' = \lambda_j + \kappa_j$ ($j \geq 3$). These operators $\tilde{\rho}_{\mu,\lambda_3,\cdots,\lambda_n}(a)$ are in fact expressed by a q-difference operator depending polynomially on $q^{\pm\mu}, q^{\pm\lambda_3},\cdots,q^{\pm\lambda_n}$. We remark that a commutation relation among the operators $\tilde{\rho}_{\mu,\lambda_3,\cdots,\lambda_n}(a)$ ($a \in U_q(gl(n))$) makes sense even for non-integral values of $\mu,\lambda_3,\cdots,\lambda_n$, if it depends polynomially on $q^{\pm\mu}, q^{\pm\lambda_3},\cdots,q^{\pm\lambda_n}$ and is valid for all $\mu,\lambda_3,\cdots,\lambda_n \in \mathbb{Z}$. The explicit formulas of the operators $\tilde{\rho}_{\mu,\lambda_3,\cdots,\lambda_n}(a)$ for $a = q^{\epsilon_j}, e_j, f_j$ will be given in the next section after we restrict them to some spaces of homogeneous functions.

5. A q-ANALOGUE OF LAURICELLA'S F_D

In view of the formulas (8) and (9) in the classical case, we take the homogeneity of $G = G(x_3,\cdots,x_n)$ into our consideration. Let U be an open set of the algebraic torus $(\mathbf{C}^*)^{n-2}$ with coordinates $x = (x_3,\cdots,x_n)$ and suppose that U is stable under the action of the multiplicative group $(q^{\mathbb{Z}})^{n-2}$. For each n-tuple $\lambda = (\lambda_1,\lambda_2,\cdots,\lambda_n)$ of complex numbers with $\sum_{j=1}^n \lambda_j = -2$, we denote by $\mathcal{F}_\lambda$ the vector space of all holomorphic functions $G = G(x_3,\cdots,x_n)$ on U such that

$$T_{q,x_3}\cdots T_{q,x_n}G = q^{-\lambda_2-1}G. \tag{45}$$

We take here the general convention to denote by T_{q,x_j} the q-shift operator in the variable x_j:

$$T_{q,x_j}G(x_3,\cdots,x_n)=G(x_3,\cdots,qx_j,\cdots,x_n). \tag{46}$$

For each element $a\in U_q(gl(n))$ of weight κ, we define an operator

$$\rho_\lambda(a):\mathcal{F}_\lambda\to\mathcal{F}_{\lambda+\kappa} \tag{47}$$

by setting

$$\rho_\lambda(a)=\tilde{\rho}_{\lambda_1+\lambda_2+1,\lambda_3,\cdots,\lambda_n}(a). \tag{48}$$

Here the right-hand side is understood as a q-difference operator depending polynomially on the parameters $q^{\pm\lambda_1},\cdots,q^{\pm\lambda_n}$. With these operators $\rho_\lambda(a)$, we obtain a representation $\mathcal{F}=\bigoplus_\lambda \mathcal{F}_\lambda$ of the quantized universal enveloping algebra $U_q(gl(n))$.

We give below the explicit formulas for the q-difference operators

$$\begin{aligned}\rho_\lambda(q^{\epsilon_j}) &: \mathcal{F}_\lambda\to\mathcal{F}_\lambda \quad (1\le j\le n),\\ \rho_\lambda(e_j) &: \mathcal{F}_\lambda\to\mathcal{F}_{\lambda+\alpha_j} \quad (1\le j\le n-1),\\ \rho_\lambda(f_j) &: \mathcal{F}_\lambda\to\mathcal{F}_{\lambda-\alpha_j} \quad (1\le j\le n-1),\end{aligned} \tag{49}$$

where $\alpha_j=\epsilon_j-\epsilon_{j+1}$. For simplicity, we set $T_j=T_{q^2,x_j}, T_{>j}=T_{j+1}T_{j+2}\cdots T_n$ for $j=3,\cdots,n$. With these notations, we have

$$\begin{aligned}\rho_\lambda(q^{\epsilon_j})&=q^{\lambda_j} \quad (1\le j\le n),\\ (q-q^{-1})\rho_\lambda(e_1)&=q^{\lambda_1-\lambda_2}x_3-q^{-\lambda_1-\lambda_2-2}\sum_{k=3}^{n}(x_k-x_{k+1})q^{-2\lambda_{>k}}T_{>k},\\ (q-q^{-1})\rho_\lambda(e_2)&=q^{-\lambda_1+\lambda_2-\lambda_3+1}\frac{1}{x_3}(1-T_3),\\ (q-q^{-1})\rho_\lambda(e_j)&=q^{\lambda_{j+1}}(1-q^{-2\lambda_{j+1}}T_j)-q^{-\lambda_{j+1}}\frac{x_j}{x_{j+1}}(1-T_{j+1}),(j\ge 3)\\ (q-q^{-1})\rho_\lambda(f_1)&=q^{2\lambda_2+3}\sum_{k=3}^{n}\frac{1}{x_k}(1-T_k)T_{>k},\\ (q-q^{-1})\rho_\lambda(f_2)&=-q^{\lambda_1}x_3+q^{-\lambda_1-2\lambda_2-2}\sum_{k=3}^{n}(x_k-x_{k+1})q^{-2\lambda_{>k}}T_{>k},\\ (q-q^{-1})\rho_\lambda(f_j)&=-q^{-\lambda_j}(1-q^{2\lambda_j}T_j^{-1})+q^{\lambda_j}\frac{x_{j+1}}{x_j}(1-T_j^{-1})(j\ge 3).\end{aligned} \tag{50}$$

Here we set $x_{n+1}=0$ and $\lambda_{>k}=\lambda_{k+1}+\cdots+\lambda_n$. Note that we have $T_3\cdots T_n=q^{-2\lambda_2-2}$ on $\mathcal{F}_\lambda$.

With these operators, we consider the q-difference equation corresponding to the equation (10) for Lauricella's F_D. We recall that, in the classical setting of Section 1, we have a remarkable identity

$$(E_{ii}+1)E_{jj}-E_{ji}E_{ij}=(t_{1i}t_{2j}-t_{2i}t_{1j})(\frac{\partial^2}{\partial t_{1i}\partial t_{2j}}-\frac{\partial^2}{\partial t_{2i}\partial t_{1j}}), \tag{51}$$

for each $i<j$, where $(E_{ij})_{1\le i,j\le n}$ is the basis for the Lie algebra $gl(n)$, corresponding to the matrix units. In view of this equality of Capelli type, we look at the elements

$$(q-q^{-1})^2C_j=(q^{\epsilon_j+1}-q^{-\epsilon_j-1})(q^{\epsilon_{j+1}}-q^{-\epsilon_{j+1}})-(q-q^{-1})^2f_je_j, \tag{52}$$

for $1\le j\le n-1$. For each j, C_j is a central element of the subalgebra $\mathbf{C}[q^{\pm\epsilon_j},q^{\pm\epsilon_{j+1}},e_j,f_j]$ of $U_q(gl(n))$. By the explicit formulas above, we compute

$$\begin{aligned}(q-q^{-1})^2\rho_\lambda(C_j) &= (x_j-x_{j+1}) \\ &\times\{x_j^{-1}(1-T_j^{-1})(1-q^{-2\lambda_{j+1}}T_{j+1})-x_{j+1}^{-1}(1-q^{2\lambda_j}T_j^{-1})(1-T_{j+1})\}\end{aligned} \tag{53}$$

for $j\ge 3$. Hence, it is reasonable to investigate the following q-difference equation for the unknown function $G=G(x)$:

$$\begin{aligned}&T_3\cdots T_nG(x) = q^{-2\lambda_2-2}G(x) \quad\text{and}\\ &x_j^{-1}(1-T_j^{-1})(1-q^{-2\lambda_{j+1}}T_{j+1})G(x)\\ &=x_{j+1}^{-1}(1-q^{2\lambda_j}T_j^{-1})(1-T_{j+1})G(x)\quad(3\le j\le n).\end{aligned} \tag{54}$$

The q-difference equation (54) above can be regarded as a homogeneous form of a q-analogue of the equation (10) for Lauricella's hypergeometric series F_D. Let us consider the following q-analogue φ_D of F_D:

$$\begin{aligned}&\varphi_D\left(\begin{matrix}a;b_1,\cdots,b_m\\ c\end{matrix};q\;;t_1,\cdots,t_m\right)\\ &=\sum_{\nu_1,\cdots,\nu_m\ge 0}\frac{(a;q)_{\nu_1+\cdots+\nu_m}(b_1;q)_{\nu_1}\cdots(b_m;q)_{\nu_m}}{(c;q)_{\nu_1+\cdots+\nu_m}(q;q)_{\nu_1}\cdots(q;q)_{\nu_m}}{t_1}^{\nu_1}\cdots{t_m}^{\nu_m}.\end{aligned} \tag{55}$$

This q-hypergeometric series defines a holomorhic function around the origin $t_1=\cdots=t_m=0$. Under the condition that $\lambda_2+\lambda_3$ is not an integer, one can directly check that the q-difference equation (54) has a solution

$$G(\lambda;x_3,\cdots,x_n)=x_3^{-\lambda_2-1}\varphi_D\left(\begin{matrix}q^{2(\lambda_2+1)};q^{-2\lambda_4},\cdots,q^{-2\lambda_n}\\ q^{2(\lambda_2+\lambda_3+2)}\end{matrix};q^2;z_4,\cdots,z_n\right), \tag{56}$$

where the variables $z_4,\cdots,z_n$ are given by

$$z_j=q^{2(1+\lambda_3+\lambda_4+\cdots+\lambda_j)}x_j/x_3\quad(j=4,\cdots,n). \tag{57}$$

By a direct verification, we can also show that, if a is an element of $U_q(gl(n))$ of weight κ, the q-difference operator $\rho_\lambda(a)$ transform the solution $G(\lambda;x)$ to a constant multiple of $G(\lambda+\kappa;x)$. In this sense, the family of q-difference operators $\rho_\lambda(a)$ describes the contiguity relations for the q-hypergeometric series φ_D. It is also checked that $\rho_\lambda(C_j)G(\lambda;x)=0$ for all $1\le j\le n-1$. For convenience, we will give in the last section the explicit formulas of the contiguity relations for the q-hypergeometric series φ_D appearing in (56).

Our derivation of the q-hypergeometric equation (54) is still incomplete since it is based only on the elements C_j of (52). It is expected that a more systematic approach to the q-analogue of Gelfand's hypergeometric equations can be carried out by using the quantum analogue of the differential operators as in [20].

6. Contiguity relations for φ_D

We consider the following q-analogue of Lauricella's hypergeometric series F_D:

$$\varphi_D\left(\begin{matrix} a; b_1, \cdots, b_m \\ c \end{matrix} ; q ; t_1, \cdots, t_m\right) = \sum_{\nu_1,\cdots,\nu_m \geq 0} \frac{(a;q)_{\nu_1+\cdots+\nu_m}(b_1;q)_{\nu_1}\cdots(b_m;q)_{\nu_m}}{(c;q)_{\nu_1+\cdots+\nu_m}(q;q)_{\nu_1}\cdots(q;q)_{\nu_m}} t_1{}^{\nu_1}\cdots t_m{}^{\nu_m}. \tag{58}$$

Note that the q-hypergeometric series $\varphi = \varphi_D$ satisfies the q-difference equation

$$\begin{aligned}
&\{(1 - cq^{-1}T_{q,t})(1 - T_{q,t_j}) - t_j(1 - aT_{q,t})(1 - b_jT_{q,t_j})\}\varphi = 0 \ (1 \leq j \leq m), \\
&\{t_i(1 - b_iT_{q,t_i})(1 - T_{q,t_j}) - t_j(1 - T_{q,t_i})(1 - b_jT_{q,t_j})\}\varphi = 0 \ (1 \leq i < j \leq m),
\end{aligned} \tag{59}$$

where $T_{q,t} = T_{q,t_1}\cdots T_{q,t_m}$. For each $\lambda = (\lambda_1, \cdots, \lambda_n)$ with $\sum_{j=1}^n \lambda_j = -2$, we define a series $F(\lambda; z)$ in the $n-3$ variables $z = (z_4, \cdots, z_n)$ by

$$F(\lambda; z) = \varphi_D\left(\begin{matrix} q^{2(\lambda_2+1)}; q^{-2\lambda_4}, \cdots, q^{-2\lambda_n} \\ q^{2(\lambda_2+\lambda_3+2)} \end{matrix} ; q^2; z_4, \cdots, z_n\right), \tag{60}$$

assuming that $\lambda_2 + \lambda_3$ is not an integer.

We give below explicit formulas for the family of q-difference operators that describes the contiguity relations for $F(\lambda; z)$. For each $\lambda = (\lambda_1, \cdots, \lambda_n)$ with $\sum_{j=1}^n \lambda_j = -2$, we define the operators $\pi_\lambda(q^{\epsilon_j})$ $(1 \leq j \leq n)$, $\pi_\lambda(e_j)$, $\pi_\lambda(f_j)$ $(1 \leq j \leq n-1)$ as follows:

$$\pi_\lambda(q^{\epsilon_j}) = q^{\lambda_j} \quad (1 \leq j \leq n)$$

$$(q - q^{-1})\pi_\lambda(e_1) = q^{\lambda_1-\lambda_2}(1 - q^{2(\lambda_2+\lambda_3+1)}T) - q^{\lambda_1+\lambda_2}\sum_{k=4}^n z_k(1 - q^{-2\lambda_{>k}}T_k)T_{>k},$$

$$(q - q^{-1})\pi_\lambda(e_2) = -q^{-\lambda_1-\lambda_2-\lambda_3-1}(1 - q^{2(\lambda_2+1)}T),$$

$$(q - q^{-1})\pi_\lambda(e_3) = q^{\lambda_4}\{(1 - q^{-2\lambda_4}T_4) - q^{2(\lambda_3+1)}\frac{1}{z_4}(1 - T_4)\},$$

$$(q - q^{-1})\pi_\lambda(e_j) = q^{\lambda_{j+1}}\{(1 - q^{-2\lambda_{j+1}}T_{j+1}) - \frac{q^{-2}z_j}{z_{j+1}}(1 - T_{j+1})\}T_j{}^{-1} \ (j \geq 4),$$

$$(q - q^{-1})\pi_\lambda(f_1) = -q(1 - q^{2(\lambda_2+1)}T) + q^{1-2\lambda_1}\sum_{k=4}^n \frac{q^{-2\lambda_{>k}}}{z_k}(1 - T_k)T_{>k},$$

$$(q - q^{-1})\pi_\lambda(f_2) = \{-q^{\lambda_1}(1 - q^{2(\lambda_3+1)}T) + q^{-\lambda_1}\sum_{k=4}^n z_k(1 - q^{-2\lambda_k}T_k)T_{>k}\}T^{-1},$$

$$(q-q^{-1})\pi_\lambda(f_3) = q^{-\lambda_3}\{-(1-q^{2(\lambda_2+\lambda_3+1)}T) + q^{-2(\lambda_4+1)}z_4(1-q^{2(\lambda_2+1)}T)\},$$

$$(q-q^{-1})\pi_\lambda(f_j) = q^{\lambda_j}\{(1-q^{-2\lambda_j}T_j) - q^{-2(\lambda_{j+1}+1)}\frac{z_{j+1}}{z_j}(1-T_j)\}\ (j \geq 4), \quad (61)$$

where $T_k = T_{q^2,z_k}, T_{>k} = T_{k+1}T_{k+2}\cdots T_n, T = T_4T_5\cdots T_n$ and $\lambda_{>k} = \lambda_{k+1} + \lambda_{k+2} + \cdots + \lambda_n$. These q-difference operators, obtained from those in (50), give a representation of the quantized universal enveloping algebra $U_q(gl(n))$. For each λ with $\sum_{j=1}^n \lambda_j = -2$, let $\mathcal{H}_\lambda$ be the same vector space of all germs of holomorphic functions near $z=0$ and consider the q-difference operators $\pi_\lambda(q^{\epsilon_j}) : \mathcal{H}_\lambda \to \mathcal{H}_\lambda$, $\pi_\lambda(e_j) : \mathcal{H}_\lambda \to \mathcal{H}_{\lambda+\alpha_j}$, $\pi_\lambda(f_j) : \mathcal{H}_\lambda \to \mathcal{H}_{\lambda-\alpha_j}$ for $\alpha_j = \epsilon_j - \epsilon_{j+1}$. These operators define a left $U_q(gl(n))$-module structure on the direct sum $\bigoplus_\lambda \mathcal{H}_\lambda$.

The contiguity relations for these q-hypergeometric series $F(\lambda; z)$ defined above are given explicitly as follows:

$$\begin{aligned}
\pi_\lambda(q^{\epsilon_j})F(\lambda;z) &= q^{\lambda_j}F(\lambda;z) \quad (1 \leq j \leq n),\\
\pi_\lambda(e_1)F(\lambda;z) &= -q^{\lambda_1+\lambda_3+1}[\lambda_2+\lambda_3+1]F(\lambda+\alpha_1;z),\\
\pi_\lambda(e_2)F(\lambda;z) &= q^{-\lambda_1-\lambda_3}[\lambda_2+1]F(\lambda+\alpha_2;z),\\
\pi_\lambda(e_3)F(\lambda;z) &= q^{-\lambda_2-1}\frac{[\lambda_3+1][\lambda_4]}{[\lambda_2+\lambda_3+2]}F(\lambda+\alpha_3;z),\\
\pi_\lambda(e_j)F(\lambda;z) &= [\lambda_{j+1}]F(\lambda+\alpha_j;z) \quad (j \geq 4),\\
\pi_\lambda(f_1)F(\lambda;z) &= -q^{-\lambda_1-\lambda_3}\frac{[\lambda_1][\lambda_2+1]}{[\lambda_2+\lambda_3+2]}F(\lambda-\alpha_1;z),\\
\pi_\lambda(f_2)F(\lambda;z) &= q^{\lambda_1+\lambda_3+1}[\lambda_3+1]F(\lambda-\alpha_2;z),\\
\pi_\lambda(f_3)F(\lambda;z) &= q^{\lambda_2+1}[\lambda_2+\lambda_3+1]F(\lambda-\alpha_3;z),\\
\pi_\lambda(f_j)F(\lambda;z) &= [\lambda_j]F(\lambda-\alpha_j;z) \quad (j \geq 4),
\end{aligned} \quad (62)$$

where $[a] = (q^a - q^{-a})/(q-q^{-1})$.

References

1. I.M. Gelfand (1986). General theory of hypergeometric functions, *Dokl. Akad. Nauk SSSR* **288**, 14–18.
2. I.M. Gelfand, A.V. Zelevinsky and M.M. Kapranov (1989). Equations of hypergeometric type and toric varieties, *Func. Anal. Appl.* **23**, 3–38.
3. P. Griffiths and F. Adams (1974). Topics in Algebraic Geometry and Analytic Geometry, *Mathematical Notes* **13**, Princeton University Press.
4. E. Horikawa *Transformations and contiguity relations for Gelfand's hypergeometric functions*, preprint.
5. E. Horikawa (1992). Contiguity relations for q-hypergeometric function and related quantum group, *Proc. Japan Acad.* **68**, 157–160.
6. E. Horikawa (1992). *Lauricella's F_D, its q-analogue and quantum group symmetry*, preprint.

7. A.N. KIRILLOV and N.YU. RESHETIKHIN (1989). Representations of the algebra $U_q(sl(2))$, q-orthogonal polynomials and invariants of links, *Infinite-dimensional Lie algebras and groups*, ed. V.G. KAC, World Scientific, 285–339.
8. H.T. KOELINK and T.H. KOORNWINDER (1989). The Clebsch-Gordan coefficients for the quantum $S_\mu U(2)$ group and q-Hahn polynomials, *Proc. K. Ned. Akad. Wet.*, Ser.A **92**, 443–456.
9. T.H. KOORNWINDER (1989). Representations of the twisted quantum $SU(2)$ quantum group and some q-hypergeometric orthogonal polynomials, *Proc. K. Ned. Akad. Wet.*, Ser.A **92**, 97–117.
10. T.H. KOORNWINDER (1989). Continuous q-Legendre polynomials as spherical matrix elements of irreducible representations of the quantum $SU(2)$ group, *CWI Quaterly* **2**, 171–173.
11. T.H. KOORNWINDER (1990). Askey-Wilson polynomials as zonal spherical functions on the $SU(2)$ quantum group, *CWI report AM-R9013.*
12. T.H. KOORNWINDER (1990). Orthogonal polynomials in connection with quantum groups, *Orthogonal polynomials, Theory and Practice*, ed. P. NEVAI, NATO ASI Series, 257-292, Kluwer Academic Publishers.
13. T. MASUDA, K. MIMACHI, Y. NAKAGAMI, M. NOUMI and K. UENO (1991). Representations of the quantum group $SU_q(2)$ and the little q-Jacobi polynomials, *J. Functional Anal.* **99**, 357–386.
14. W. MILLER JR. (1972). Lie theory and the Lauricella functions F_D, *J. Math. Phys.*, **13**, 1393–1399.
15. M. NOUMI (1991). Quantum groups and q-orthogonal polynomials — Towards a realization of Askey-Wilson polynomials on $SU_q(2)$, ICM90 Satellite Conference Proceedings: *Special Functions*, eds. M. KASHIWARA and T. MIWA, Springer-Verlag, 260–288.
16. M. NOUMI and K. MIMACHI (1990). Quantum 2-spheres and big q-Jacobi polynomials, *Commun. Math. Phys.* **128**, 521–531.
17. M. NOUMI and K. MIMACHI (1992). Spherical functions on a family of quantum 3-spheres, *Compositio Mathematica* **83**, 19–42.
18. M. NOUMI and K. MIMACHI (1991). Rogers's q-ultraspherical polynomials on a quantum 2-sphere, *Duke Mathematical Journal* **63**, 65–80.
19. M. NOUMI and K. MIMACHI (1990). Askey-Wilson polynomials and the quantum group $SU_q(2)$, *Proc. Japan Acad. Ser.A* **66**, 146–149.
20. M. NOUMI, T. UMEDA and M. WAKAYAMA. (1991). A quantum analogue of the Capelli identity and an elementary differential calculus on $GL_q(n)$, preprint *J-TOKYO-MATH*, 91-16.
21. M. NOUMI, H. YAMADA and K. MIMACHI. Finite dimensional representations of the quantum group $GL_q(n; \mathbf{C})$ and the zonal spherical functions on $U_q(n-1)\backslash U_q(n)$, to appear in *Japanese J. Math.*
22. T. SASAKI (1991). Contiguity relations of Aomoto-Gel'fand hypergeometric function and applications to Appell's system F_3 and Goursat's system ${}_3F_2$, *SIAM J. Math. Anal.* **22**821–846.
23. E. TAFT and J. TOWBER (1991). Quantum deformation of flag schemes and

Grassmann schemes I. A q-deformation of the shape algebra for $GL(n)$, *J. Algebra* **142**, 1–36.
24. L.L. Vaksman and Ya.S. Soibelman (1988). Algebra of functions on the quantum group $SU(2)$, *Functional Anal. Appl.* **22**, 170–181.