

Calculation of the Wavelet Decomposition using Quadrature Formulae

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In many applications concerning wavelets, inner products of arbitrary functions $f(x)$ with wavelets and scaling functions have to be calculated. This paper involves the calculation of these inner products from function evaluations of $f(x)$.

The relationship between the scaling function $\phi(x)$, its values at the integers and the scaling parameters h_k is investigated. This results in the construction of a one point quadrature formula for the calculation of inner products of scaling functions and arbitrary functions.

Secondly, efficient multiple point quadrature formulae are constructed. An elegant method to solve the nonlinear system coming from this construction is presented. Since the construction of multiple point formulae using ordinary moments is ill-conditioned, a modified, well-conditioned construction using Chebyshev moments is presented.

1. NOTATION

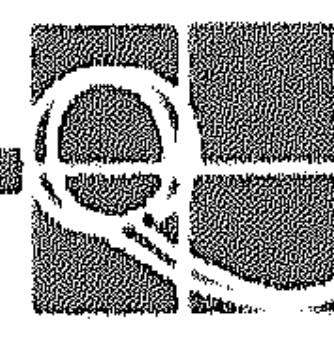
In this text we will use the notations \mathbb{N} , \mathbb{Z} , \mathbb{R} , $L^2(\mathbb{R})$, $l^2(\mathbb{Z})$ for the set of naturals, integers, reals, square integrable functions and square summable sequences respectively. We will only use real valued functions and the usual definition of inner product and norm in $L^2(\mathbb{R})$. If no bounds for a parameter k are indicated under a sum, $k \in \mathbb{Z}$ is assumed.

2. WAVELET MULTIREOLUTION APPROXIMATION

2.1. The scaling function

In order to construct a wavelet, one first needs a scaling function $\phi(x)$. In this

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volume this function is also referred to as the “father function”. It has the following properties:

- It has a non-vanishing integral:

$$\int_{-\infty}^{+\infty} \phi(x) dx \neq 0.$$

- A sequence $(h_k) \in l^2(\mathbb{Z})$ exists such that the scaling function satisfies the *refinement equation*:

$$\phi(x) = \sqrt{2} \sum_k h_k \phi(2x - k). \quad (1)$$

This equation is also called the *dilation equation* [19]. We refer to the coefficients h_k as the *scaling parameters*. In this paper we only consider the case where a finite number of scaling parameters is non-zero and the scaling function consequently has finite support.

The properties of the function $\phi(x)$ and the scaling parameters h_k are closely related. The refinement equation has a solution $\phi(x)$ if:

$$\sum_k h_{2k} = \sum_k h_{2k+1} = 1/\sqrt{2}. \quad (2)$$

In this case the scaling function satisfies:

$$\forall x \in \mathbb{R} : \sum_l \phi(x - l) = \sum_l \phi(l).$$

The scaling function is called *orthonormal* if its translates $\{\phi(x - l)\}_{l \in \mathbb{Z}}$ form an orthonormal set of functions:

$$\langle \phi(x), \phi(x - l) \rangle = \delta_{l0} \quad l \in \mathbb{Z}. \quad (3)$$

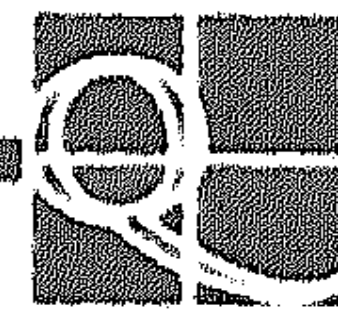
This orthogonality property can be written as a property of the scaling parameters h_k :

$$\sum_k h_k h_{k-2l} = \delta_{l0} \quad l \in \mathbb{Z}. \quad (4)$$

Equation (4) is necessary but not sufficient for the orthogonality property (3) [13]. The functions $\{\phi(x - l)\}_{l \in \mathbb{Z}}$ generate a vector space $V_0 \subset L^2(\mathbb{R})$.

2.2. Multiresolution analysis

We summarise the main elements of the concept of multiresolution analysis.



We refer to [12] and [15] for a more complete description of the subject. Our notation is slightly different from the one in these references. The basis of V_0 is formed by translations of the scaling function over multiples of a constant distance. The reciprocal of this distance is called the *resolution* of the basis. To obtain approximations of $f(x) \in L^2(\mathbb{R})$ at different resolutions, we can use the translated dilations of the scaling function, defined as:

$$\phi_{j,l}(x) = \sqrt{2^j} \phi(2^j x - l) \quad j, l \in \mathbb{Z}.$$

The set of orthogonal functions $\{\phi_{j,l}(x)\}_{l \in \mathbb{Z}}$ for a particular j , generates a space $V_j \subset L^2(\mathbb{R})$. Let P_j denote the orthogonal projection $L^2(\mathbb{R}) \rightarrow V_j$. The vector spaces V_j ($j \in \mathbb{Z}$) have the following properties defining a multiresolution analysis [15]:

1. $V_j \subset L^2(\mathbb{R})$.
2. $V_j \subset V_{j+1}$: this is a direct consequence of the refinement equation (1).
3. $\|f(x) - P_j f(x)\| = \min_{g(x) \in V_j} \|f(x) - g(x)\|$: by definition of P_j .
4. $v(x) \in V_j \Leftrightarrow v(2x) \in V_{j+1}$.
5. The projections $P_j f(x)$ converge to $f(x)$ as j tends to infinity:

$$\lim_{j \rightarrow \infty} P_j f(x) = f(x) \quad \text{or} \quad \bigcup_{j=0}^{\infty} V_j \text{ is dense in } L^2(\mathbb{R}).$$

The orthogonal complement of V_j in V_{j+1} is denoted by W_j and satisfies $W_j \perp V_j$ and $W_j \oplus V_j = V_{j+1}$. Consequently:

$$\bigoplus_{j=-\infty}^{+\infty} W_j \text{ is dense in } L^2(\mathbb{R}).$$

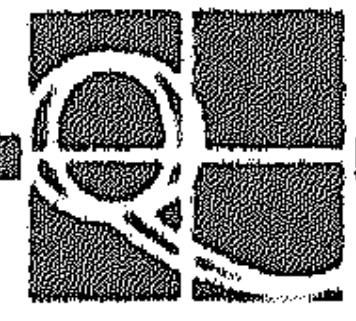
The orthonormal wavelet function, also referred to as the “mother function” is defined as:

$$\psi(x) = \sqrt{2} \sum_k g_k \phi(2x - k) \quad \text{with} \quad g_k = (-1)^k h_{1-k}. \quad (5)$$

It can be proven that the functions $\{\psi_{j,l}(x)\}_{l \in \mathbb{Z}}$ with $\psi_{j,l}(x) = \sqrt{2^j} \psi(2^j x - l)$ form an orthonormal basis of W_j [12] [19]. As a consequence of equations (2) and (5) the wavelet function $\psi(x)$ satisfies:

$$\int_{-\infty}^{+\infty} \psi(x) dx = 0.$$

Also:



$$\sqrt{2} \phi(2x) = \sum_l h_{2l} \phi(x+l) + \sum_l g_{2l} \psi(x+l). \quad (6)$$

This is the inverse of equations (5) and (1).

3. APPROXIMATION OF POLYNOMIALS BY SCALING FUNCTIONS

It is possible to represent certain polynomials with the functions $\{\phi(x-l)\}_{l \in \mathbf{Z}}$. In this case the sequence of coefficients does not belong to $l^2(\mathbf{Z})$. Consider an indicator function $\chi_{[\alpha, \beta]}$. This function is clearly an element of $L^2(\mathbf{R})$ and can thus be represented in $V_0 \oplus \bigoplus_{j=1}^{\infty} W_j$. Since the integral of $\psi(x)$ vanishes, the coefficients of the $\psi_{j,l}(x)$ are zero except if their support contains α or β . Now, if α tends to $-\infty$ and β to $+\infty$, the constant function 1 is expressed in terms of the $\phi(x-l)$. Its coefficients are independent of l and are denoted by a :

$$1 = a \sum_l \phi(x-l) \quad \text{with} \quad a = \int_{-\infty}^{+\infty} \phi(x) dx.$$

Integrating the first equation over $[0, 1]$ yields:

$$1 = a \int_{-\infty}^{+\infty} \phi(x) dx.$$

Consequently:

$$\int_{-\infty}^{+\infty} \phi(x) dx = \pm 1.$$

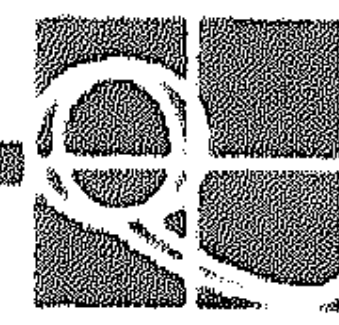
Notice that there are two possibilities of which we will retain the positive. As a result:

$$\forall x \in \mathbf{R} : \sum_l \phi(x-l) = 1. \quad (7)$$

Following a similar reasoning we can state that it is possible to represent a polynomial x^p with the set of functions $\{\phi(x-l)\}_{l \in \mathbf{Z}}$ if the first $p+1$ moments of the wavelet vanish. We will denote the number of vanishing moments of a wavelet function $\psi(x)$ with the symbol q . The number of vanishing moments is an important feature of the wavelet. In this context the following property is useful:

$$\int_{-\infty}^{+\infty} x^i \psi(x) dx = 0 \quad (i = 0, \dots, q-1) \Leftrightarrow \sum_k g_k k^i = 0 \quad (i = 0, \dots, q-1)$$

The number of vanishing moments defines the convergence properties of a wavelet series since if $f \in C^{(q)}$ [19]:



$$\|f(x) - P_j f(x)\| = O(2^{-jq}).$$

In order to find the coefficients of the representation of a polynomial in V_0 , one derives using (1) and (2) that for $p \geq 1$ [2]:

$$\int_{-\infty}^{+\infty} x^p \phi(x) dx = M_p = \frac{1}{2^p - 1} \sum_{i=1}^p \binom{p}{i} m_i M_{p-i}. \quad (8)$$

This is a p -terms recursion relation. The m_i denote the moments of the scaling parameters h_k :

$$m_i = \frac{1}{\sqrt{2}} \sum_k h_k k^i.$$

If necessary, we can write this dependency explicitly:

$$\begin{aligned} M_1 &= m_1 \\ M_2 &= \frac{2m_1^2 + m_2}{3} \\ M_3 &= \frac{2m_1^3 + 4m_1 m_2 + m_3}{7} \\ &\dots \end{aligned} \quad (9)$$

4. DECOMPOSITION AND RECONSTRUCTION ALGORITHM

Since V_j is equal to $V_{j-1} \oplus W_{j-1}$, a function $v_j(x) \in V_j$ can be written uniquely as the sum of a function $v_{j-1}(x) \in V_{j-1}$ and a function $w_{j-1}(x) \in W_{j-1}$:

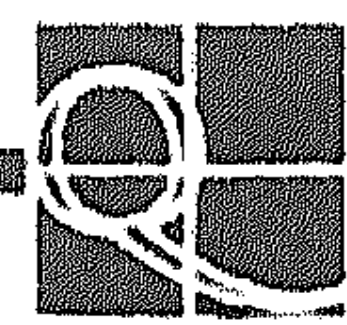
$$\begin{aligned} v_j(x) &= \sum_k \nu_{j,k} \phi_{j,k}(x) = v_{j-1}(x) + w_{j-1}(x) \\ &= \sum_l \nu_{j-1,l} \phi_{j-1,l}(x) + \sum_l \mu_{j-1,l} \psi_{j-1,l}(x). \end{aligned}$$

There is a one-to-one relationship between the coefficients of these functions that can be derived easily using (1), (5), and (6). The decomposition step is:

$$\nu_{j-1,l} = \sum_k h_{k-2l} \nu_{j,k} \quad \text{and} \quad \mu_{j-1,l} = \sum_k g_{k-2l} \nu_{j,k}. \quad (10)$$

The reconstruction step consists of calculating the $\nu_{j,k}$ out of the $\nu_{j-1,l}$ and the $\mu_{j-1,l}$:

$$\nu_{j,k} = \sum_l h_{k-2l} \nu_{j-1,l} + \sum_l g_{k-2l} \mu_{j-1,l}. \quad (11)$$



These operations are one step in the multiresolution algorithm described in [14] [15].

To use this in a practical algorithm we assume that the h_k are non-zero for $0 \leq k \leq L$, and that the g_k are chosen as $(-1)^k h_{L-k}$, so the g_k too are non-zero for $0 \leq k \leq L$. Suppose we want an approximation in V_n of a function $f(x)$ in the interval $[0, 1]$. This means that we need 2^n coefficients $\nu_{n,l}$. The next sections of this paper involve the calculation of these coefficients using function evaluations of $f(x)$. From these coefficients, the coefficients of the multiresolution basis can be calculated. We start with 2^n coefficients $\nu_{n,l}$ with $0 \leq l < 2^n$ at level n . At level j , we will calculate 2^j coefficients $\mu_{j,l}$ and $\nu_{j,l}$ with $0 \leq l < 2^j$. We can write the decomposition formulae (10) as:

$$\nu_{j-1,l} = \sum_{k=0}^L h_k \nu_{j,k+2l} \quad \text{and} \quad \mu_{j-1,l} = \sum_{k=0}^L g_k \nu_{j,k+2l}.$$

These formulae show that in order to calculate the 2^{j-1} coefficients $\nu_{j-1,l}$ and $\mu_{j-1,l}$, $L-1$ coefficients, namely $\nu_{j,2^j} \dots \nu_{j,2^j+L-2}$ are not calculated at the right boundary. For the reconstruction, coefficients will be missing at the left boundary. In order to solve this problem several possibilities can be considered (assume $0 \leq j \leq n$):

- Setting these coefficients equal to zero: $\nu_{j,i} = 0$ for $i < 0$ or $i > 2^j - 1$.
- Using symmetry at the boundaries:

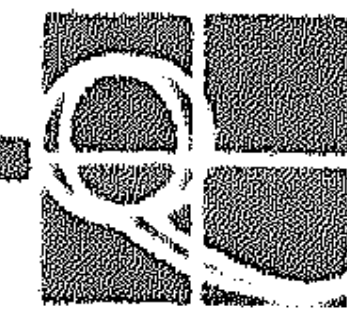
$$\begin{cases} \nu_{j,i} = \nu_{j,-i} & i < 0 \\ \nu_{j,i} = \nu_{j,2(2^j-1)-i} & i > 2^j - 1. \end{cases}$$

- Considering the sequences as periodic: $\nu_{j,i} = \nu_{j,i \bmod 2^j}$.

Similar possibilities hold for the $\mu_{j,l}$. The first two possibilities are an ad hoc solution and do not allow an exact reconstruction. This fact is called the *boundary effect*. The more levels used in the decomposition, the more the reconstruction is contaminated by the boundary effect. Only the last possibility allows an exact reconstruction. In order to understand this, we show that the algorithms that use the periodic sequences correspond to an orthonormal multiresolution basis of $L^2([0, 1])$. This construction is also described in [16]. All formulae derived hereafter only hold for positive level numbers ($j \geq 0$).

Define a 2^j -dimensional vector space V_j^* generated by the functions $\phi_{j,l}^*(\cdot)$ with $0 \leq l < 2^j$ which are defined as:

$$\phi_{j,l}^*(x) = \chi_{[0,1]}(x) \sum_{m=0}^{L-1} \phi_{j,l}(x+m) \quad 0 \leq l < 2^j. \quad (12)$$



If the support of $\phi_{j,l}(x)$, which is $[l/2^j, (l+L)/2^j]$, is a subset of $[0, 1]$, then $\phi_{j,l}^*(x) = \phi_{j,l}(x)$. Otherwise $\phi_{j,l}(x)$ is cut at the integers into pieces of length 1, which are shifted onto $[0, 1]$ and added up, yielding $\phi_{j,l}^*(x)$. The maximum number of these pieces is equal to L , since $j \geq 0$. Also:

$$\chi_{[0,1]}(x) \sum_{m=0}^{L-1} \phi_{j,l}(x+m) = \phi_{j,l \bmod 2^j}^*(x) \quad 0 \leq l \leq L(2^j - 1). \quad (13)$$

The space W_j^* is defined as the orthogonal complement of V_j^* in V_{j+1}^* . This vector space is generated by orthogonal functions $\psi_{j,l}^*$ which are defined similarly to (12).

The multiresolution coefficients are defined as:

$$\nu_{j,l} = \langle f(x), \phi_{j,l}^*(x) \rangle \quad \text{and} \quad \mu_{j,l} = \langle f(x), \psi_{j,l}^*(x) \rangle.$$

In fact a notation with asterisk for $\nu_{j,l}$ and $\mu_{j,l}$ would be more consistent, but the asterisk is omitted in order not to complicate the notation. Using (13) the refinement equation (1) can be written as:

$$\phi_{j-1,l}^*(x) = \sum_{k=0}^L h_k \phi_{j,(k+2l) \bmod 2^j}^*(x).$$

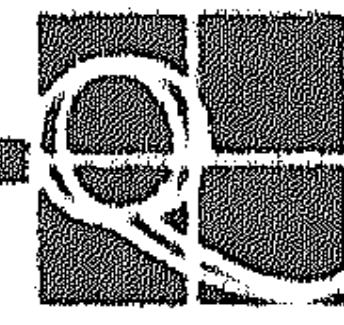
Taking the inner product of this equation with $f(x)$, yields:

$$\nu_{j-1,l} = \sum_{k=0}^L h_k \nu_{j,(k+2l) \bmod 2^j}.$$

This is the formula used in the periodic decomposition algorithm. The second decomposition formula and the reconstruction formula can be derived similarly. The complete periodic algorithm is described in [20]. Now:

$$\begin{aligned} \nu_{j,l} &= \langle f(x), \phi_{j,l}^*(x) \rangle = \sum_{m=0}^{L-1} \langle f(x), \phi_{j,l}(x+m) \chi_{[0,1]}(x) \rangle \\ &= \left\langle \sum_{m=0}^{L-1} \chi_{[m,m+1]}(x) f(x-m), \phi_{j,l}(x) \right\rangle = \langle f^*(x), \phi_{j,l}(x) \rangle. \end{aligned}$$

This means that we actually calculated the coefficients, as considered in the infinite case, of a function $f^*(x)$, a limited periodic expansion of the restriction of $f(x)$ to $[0, 1]$. Since $f^*(x)$ is square integrable, its projections in V_j will converge to $f^*(x)$ if j tends to infinity. Therefore the projections of $f(x)$ in V_j^* will converge to $f(x) \chi_{[0,1]}(x)$ if j tends to infinity, or:



$$\bigoplus_{j=-\infty}^{+\infty} W_j^* \text{ is dense in } L^2([0, 1]).$$

If $f^*(x)$ is a sufficiently differentiable function, the wavelet coefficients $\mu_{j,l}$ decay as $O(\eta_j^q)$ with $\eta_j = 2^{-j}$. However, if $f(0) \neq f(1)$, $f^*(x)$ is not continuous at the integers and the convergence will be slow since the wavelet coefficients at the boundary decay as $O(\sqrt{\eta_j})$.

The functions $\phi_{j,l}^*(x)$ satisfy:

$$\sum_{l=0}^{2^j-1} \phi_{j,l}^*(x) = \sqrt{2^j} \chi_{[0,1]}. \quad (14)$$

This is a consequence of equation (7). Taking the inner product of equation (14) with $f(x)$ yields:

$$\sqrt{2^{-j}} \sum_{l=0}^{2^j-1} \nu_{j,l} = \nu_{0,0} = \int_0^1 f(x) dx.$$

In this last coefficient the influence of the scaling function has disappeared !

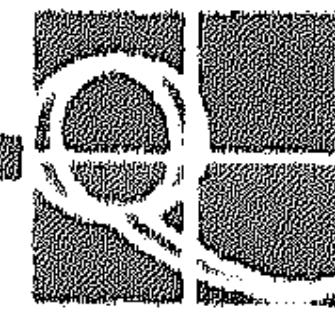
5. EXAMPLES AND GENERALISATIONS

A well-known example of orthogonal wavelets with compact support are the ones constructed by Ingrid Daubechies [7]. This is a family of wavelets with N vanishing moments and $2N$ scaling parameters h_k ($N \in \mathbb{N}$). The h_k are non-zero for $0 \leq k \leq 2N - 1$, and the support of $\phi(x)$ is $[0, 2N - 1]$. The support of the corresponding $\psi(x)$ is $[-(N - 1), N]$.

A generalisation of orthogonal wavelets are *biorthogonal* wavelets. This concept is introduced in [5] [6]. In this case the wavelets are not orthogonal, but with each wavelet $\psi_{j,l}(x)$ a dual wavelet $\tilde{\psi}_{j,l}(x)$ is associated such that the coefficient of $\psi_{j,l}(x)$ in the wavelet series of a function $f(x)$ is given by $\langle f(x), \tilde{\psi}_{j,l}(x) \rangle$. The dual wavelets are the translated dyadic dilations of one dual mother wavelet $\tilde{\psi}(x)$ which is derived from a dual scaling function $\tilde{\phi}(x)$ that satisfies a refinement equation. As a result different coefficients will appear in the decomposition and reconstruction step.

The construction in [6] has the advantage that all filters used in the wavelet transformation have linear phase and finite length. The filter coefficients are also rational numbers which can facilitate implementation. The scaling functions here are cardinal B-splines which are smoother than the Daubechies' scaling functions with the same number of scaling parameters.

A special case of the biorthogonal wavelets are the *semi-orthogonal* wavelets or *pre-wavelets* [5] [18]. In this case the wavelets that belong to one subspace



W_j are not orthogonal but the subspaces W_j are still orthogonal. The wavelets constructed in [5] have this property. Also here the scaling functions are cardinal B-splines, but the dual filters do not have finite length any more.

6. ONE POINT FORMULAE

In order to obtain the approximation of a function $f(x) \in L^2(\mathbb{R})$ in V_j one has to calculate the coefficients:

$$\nu_{j,l} = \int_{-\infty}^{+\infty} f(x) \phi_{j,l}(x) dx.$$

In this section we construct a one point quadrature formula for the calculation of these integrals, but first we will prove a theorem essential in this construction.

THEOREM 1. *If $\phi(x)$ is an orthogonal scaling function with compact support and with $q \geq 3$, then:*

$$\int_{-\infty}^{+\infty} x^2 \phi(x) dx = \left(\int_{-\infty}^{+\infty} x \phi(x) dx \right)^2. \quad (15)$$

PROOF Since $q \geq 3$ we have:

$$x = \sum_k a_k \phi(x - k) \quad \text{with} \quad a_k = \langle x, \phi(x - k) \rangle = M_1 + k M_0,$$

and

$$x^2 = \sum_k b_k \phi(x - k) \quad \text{with} \quad b_k = M_2 + 2k M_1 + k^2 M_0.$$

Thus

$$\sum_k k \phi(x - k) = x - M_1 \quad \text{and} \quad \sum_k k \phi(k) = M_1,$$

and

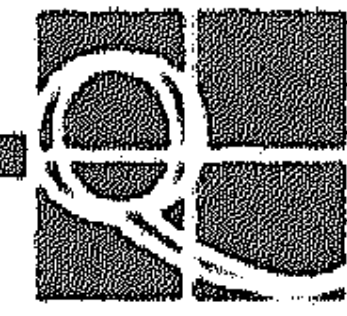
$$\begin{aligned} \sum_k k^2 \phi(x - k) &= x^2 - \sum_k (M_2 + 2k M_1) \phi(x - k) \\ &= x^2 - 2M_1 x - M_2 + 2M_1^2, \end{aligned}$$

and

$$\sum_k (x - k)^2 \phi(x - k) = 2M_1^2 - M_2.$$

Integrating this last equation over $[0, 1]$ yields:

$$M_2 = 2M_1^2 - M_2 \quad \text{or} \quad M_2 = M_1^2$$



Another way to express this theorem according to (9) is $m_2 = m_1^2$ or $M_2 = m_2 = \sum_k k^2 \phi(k)$. If f is sufficiently differentiable we have:

$$\begin{aligned} \nu_{j,l} &= \int_{-\infty}^{+\infty} f(x) \phi_{j,l}(x) dx = \sqrt{\eta_j} \int_{-\infty}^{+\infty} f(\eta_j(x+l)) \phi(x) dx \\ &= \sqrt{\eta_j} [f(\alpha_{j,l}) + \eta_j f'(\alpha_{j,l}) M_{1,\alpha} + \eta_j^2 f''(\alpha_{j,l}) M_{2,\alpha} / 2 + O(\eta_j^3)]. \end{aligned}$$

with $\eta_j = 2^{-j}$, $\alpha_{j,l} = \eta_j(\alpha + l)$ and $M_{p,\alpha} = \int_{-\infty}^{+\infty} (x - \alpha)^p \phi(x) dx$.

The $M_{p,\alpha}$ are called the *shifted moments*. A consequence of theorem is:

$$\forall \alpha \in \mathbb{R} : M_{2,\alpha} = M_{1,\alpha}^2$$

We can take $\sqrt{\eta_j} f(\alpha_{j,l})$ as an approximation of the integral. We still have a free parameter α which can be chosen such that $M_{1,\alpha}$ and consequently the first error term vanish. This happens if α is chosen equal to M_1 . If $\phi(x)$ is interpreted as a mass distribution along the x -axis, α is the mass center. But if the scaling function satisfies the conditions of theorem, also the second error term vanishes and the one point quadrature formula becomes:

$$\int_{-\infty}^{+\infty} f(x) \phi_{j,l}(x) dx = \sqrt{\eta_j} [f(\alpha_{j,l}) + K_{j,l} \eta_j^3] \quad (16)$$

with

$$K_{j,l} = \frac{1}{6} \int_{-\infty}^{+\infty} f'''(\eta_j(\xi(x) + l)) (x - \alpha)^3 \phi(x) dx$$

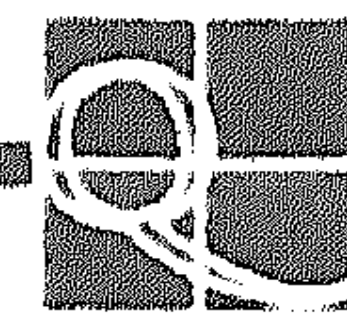
and $\xi(x)$ between x and α .

The *degree of accuracy* of a quadrature formula is p , if all polynomials of degree less than or equal to p are integrated exactly. The degree of accuracy of this one point quadrature formula is 2. This approximation improves for smoother f and increasing j .

In practical cases usually a number of discrete samples a_i , with $0 \leq i < 2^n$, is given. In that case a function $a(x) \in V_n$ is constructed to start the multiresolution scheme [15]:

$$a(x) = \sqrt{\eta_n} \sum_{l=0}^{2^n-1} a_l \phi_{n,l}(x) \quad \text{with} \quad \eta_n = 2^{-n}.$$

The quadrature formula can help us to find the relationship between the function $a(x)$ and the discrete samples a_i . Indeed: $\sqrt{\eta_n} a_l = \langle a(x), \phi_{n,l}(x) \rangle$ and



$\langle a(x), \phi_{n,l}(x) \rangle = \sqrt{\eta_n} [a(\eta_n (\alpha + l)) + O(\eta_n^3)]$, so $a_l = a(\eta_n (\alpha + l)) + O(\eta_n^3)$. This means that we have a quasi-interpolating property with an error of $O(\eta_n^3)$ [5].

In [8] scaling functions are constructed that have $q-1$ vanishing shifted moments:

$$M_{j,\alpha} = 0 \quad j = 1, \dots, q-1, \quad (17)$$

where q is the number of vanishing wavelet moments. In this case the one point quadrature formula has an error of $O(\eta_n^q)$. However, these functions have the disadvantage that the number of non-zero scaling parameters is 50% higher than in the original Daubechies scaling functions with the same number of vanishing wavelet moments. This results in an increase of 50% in calculation time. These formulae are also described in [2].

7. MULTIPLE POINT FORMULA

7.1. Principle

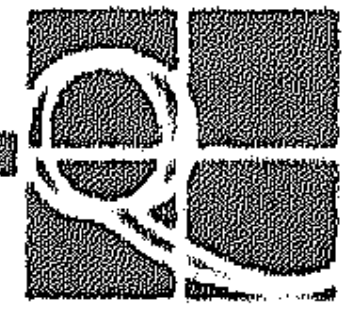
If the accuracy of the one point quadrature formula of the previous section is not sufficient, we can use multiple point integration rules with higher degree of accuracy. In this section we assume that $\phi(x)$ is an integrable function with compact support $[0, L]$, which satisfies a refinement equation (1) with $L+1$ non-zero coefficients. Notice that the orthogonality demand is not required here. This makes the results applicable to a wider range of scaling functions, including biorthogonal functions, B-spline functions [4] [5] and their autocorrelation functions.

The quadrature formula will be used at the finest level of the multiresolution tree to calculate the $\nu_{n,l}$. In order to keep the notations simple though, we will elaborate on the calculation of the $\nu_{0,l}$ and in particular of $\nu_{0,0}$. The quadrature formulae for other coefficients can be found using a linear transformation. The abscissae should be chosen equidistant for two reasons:

- In some applications, especially in signal and image processing, the function $f(x)$ is only known at equidistant abscissae.
- One usually wants to calculate a number of integrals $\nu_{0,l}$. Each pair of adjacent functions $\phi(x-l)$ and $\phi(x-(l+1))$ overlap in an interval of length $L-1$. If the abscissae are equidistant, it is possible that quadrature formulae for different integrals have common points.

We will choose the abscissae equidistant with spacing 2^s . Additionally, inspired by the idea in the previous section, we try to shift the abscissae over a distance τ in order to increase the degree of accuracy without increasing the work load. We construct an r point quadrature formula for the calculation of:

$$\nu_{0,0} = \int_0^L \phi(x) f(x) dx \approx \sum_{k=1}^r w_k f(x_k) = Q_r[f]$$



with $x_k = d_k - \tau$, $d_k = (k - 1)2^s$ and $(r - 1)2^s - L < \tau < 0$. The range of the shift τ is determined by the requirement that no abscissae should fall outside the integration interval or on its boundary. In order to have a non-zero range for the shift τ , the parameters r and s should be chosen such that $(r - 1)2^s < L$. The degree of accuracy of the quadrature formula is p , if:

$$Q_r[x^i] = M_i \quad 0 \leq i \leq p.$$

This technique to construct quadrature formulae is also used in [2] but here the shift τ is given a fixed value.

7.2. Construction using ordinary moments

In order to find the $r + 1$ unknowns $\{\tau, w_1, \dots, w_r\}$, one can impose the degree of accuracy to be r . This results in the following system which is nonlinear in the unknown τ .

$$\sum_{k=1}^r w_k [d_k - \tau]^i = M_i \quad 0 \leq i \leq r. \quad (18)$$

An elegant way to determine the value of the shift τ uses the product polynomial $\Pi(x)$. This polynomial is defined as:

$$\Pi(x) = \prod_{k=1}^r (x - x_k) = \prod_{k=1}^r (x + \tau - d_k) = \sum_{i=0}^r p_i(\tau) x^i,$$

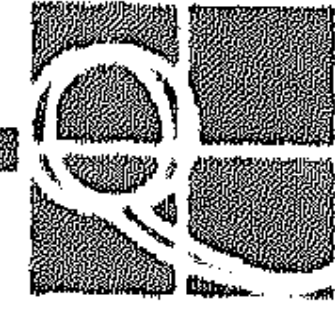
with $p_i(\tau)$ a polynomial of degree $r - i$ in τ and $p_r(\tau) \equiv 1$. Since the degree of accuracy has to be r , the product polynomial $\Pi(x)$ has to be integrated exactly. The quadrature formula yields zero for this polynomial. Thus:

$$\begin{aligned} 0 &= Q_r[\Pi] = \int_0^L \phi(x) \Pi(x) dx = \int_0^L \phi(x) \sum_{i=0}^r p_i(\tau) x^i dx \\ &= \sum_{i=0}^r p_i(\tau) M_i = \Gamma(\tau). \end{aligned}$$

This latter expression is a polynomial of degree r in τ . For the quadrature formula to exist, $\Gamma(\tau)$ must have a real root in the interval $((r - 1)2^s - L, 0)$. However, there is no theoretical certainty that $\Gamma(\tau)$ will have a real root in this interval. If there is no root in this interval, an arbitrary value for τ has to be chosen and one degree of accuracy is lost. We will use the notation $Q_r^*[f]$ for this case. Once τ is determined, the weights are the solution of the system of linear equations formed by the first r equations of (18).

7.3. Calculation of $\Gamma(\tau)$.

Let



$$p_i(\tau) = \sum_{j=0}^{r-i} p_{i,j} \tau^j \quad \text{and} \quad \Gamma(\tau) = \sum_{i=0}^r \sum_{j=0}^{r-i} p_{i,j} \tau^j M_i = \sum_{j=0}^r \left(\sum_{i=0}^{r-j} M_i p_{i,j} \right) \tau^j.$$

The coefficients $p_{i,j}$ are symmetric ($p_{i,j} = p_{j,i}$) since the product polynomial is symmetric in τ and x . The coefficients $p_{i,j}$ can be found as $p_{i,j}^{(r)}$ in:

$$\Pi^{(m)}(x) = \prod_{k=1}^m (x + \tau - d_k) = \sum_{i=0}^m \sum_{j=0}^{m-i} p_{i,j}^{(m)} \tau^j x^i.$$

An algorithm can be derived by writing:

$$\Pi^{(m)}(x) = (x + \tau - d_m) \Pi^{(m-1)}(x),$$

and identifying the coefficients of the powers of x and τ . The complete description of the algorithm can be found in [20].

A disadvantage of this construction is that the system of equations (18) is ill-conditioned if r is large. In the construction of Q_{14} for the Daubechies scaling function with $N = 4$, the condition number of the linear system to find the weights is 9.10^{15} . This problem can be overcome if we construct a system by imposing that the Chebyshev polynomials of degree less than or equal to r are integrated exactly [10] [17]. This is done in the next section.

8. MODIFIED CONSTRUCTION OF THE QUADRATURE FORMULA

The Chebyshev polynomial $T_n(x)$ of degree n is defined by the recursive relations $T_0(x) = 1$, $T_1(x) = x$ and $T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x)$ for $n > 1$ [1]. Since the interesting properties of the Chebyshev polynomials only hold in the interval $[-1, 1]$, one first needs to transform the scaling function $\phi(x)$ to this interval yielding a function $\bar{\phi}(y)$. We will use the notation y to indicate an independent variable that varies between -1 and 1 :

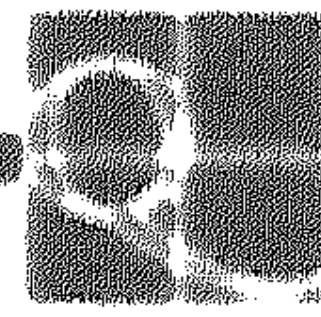
$$\bar{\phi}(y) = \frac{L}{2} \phi(x) \quad \text{with} \quad x = \frac{L(y+1)}{2} \quad \text{or} \quad y = \frac{2x-L}{L}.$$

The transformed refinement equation (1) becomes:

$$\bar{\phi}(y) = \sqrt{2} \sum_k h_k \bar{\phi}(2y - 2k/L + 1).$$

We will try to construct an r point quadrature formula for the calculation of:

$$\begin{aligned} \nu_{0,0} &= \int_0^L \phi(x) f(x) dx = \int_{-1}^{+1} \bar{\phi}(y) f\left(\frac{L(y+1)}{2}\right) dy \\ &= \int_{-1}^{+1} \bar{\phi}(y) \bar{f}(y) dy \approx \sum_{k=1}^r \bar{w}_k \bar{f}(y_k) = \sum_{k=1}^r w_k f(x_k) = Q_r[f], \end{aligned}$$



with $y_k = d_k - \bar{\tau}$, $\bar{d}_k = 2d_k/L - 1$, $\bar{\tau} = 2\tau/L$ and $\bar{w}_k = w_k$.

Let \tilde{M}_p denote the transformed ordinary moments and M_p^* the modified moments:

$$\tilde{M}_p = \langle y^p, \bar{\phi}(y) \rangle \quad \text{and} \quad M_p^* = \langle T_p(y), \bar{\phi}(y) \rangle.$$

The system to be solved can be written as:

$$\sum_{k=1}^r w_k T_i(\bar{d}_k - \bar{\tau}) = M_i^* \quad 0 \leq i \leq r. \quad (19)$$

The solution procedure is similar to the previous case. We can construct a polynomial $\bar{\Gamma}(\bar{\tau})$, written as a linear combination of Chebyshev polynomials, and try to find one of its roots in the appropriate interval.

8.1. Calculation of the modified moments.

It is possible to calculate the transformed ordinary moments as in (8) and calculate the modified moments from them using the coefficients $t_i^{(p)}$ of the Chebyshev polynomials:

$$T_p(y) = \sum_{i=0}^p t_i^{(p)} y^i \quad \text{and} \quad M_p^* = \sum_{i=0}^p t_i^{(p)} \tilde{M}_i. \quad (20)$$

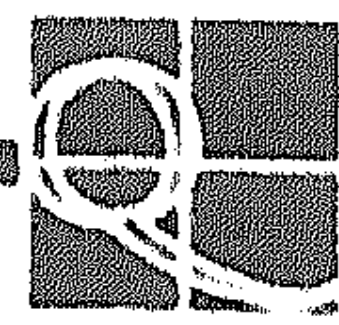
However, a considerable loss of significant digits will occur in the evaluation of M_p^* using (20). We need a formula to calculate the modified moments directly:

$$\begin{aligned} M_p^* &= \int_{-1}^{+1} T_p(y) \bar{\phi}(y) dy \\ &= \sqrt{2} \sum_k h_k \int_{k/L-1}^{k/L} T_p(y) \bar{\phi}(2y + 1 - 2k/L) dy \\ &= \sum_k \frac{h_k}{\sqrt{2}} \int_{-1}^{+1} T_p\left(\frac{u-1+2k/L}{2}\right) \bar{\phi}(u) du. \end{aligned} \quad (21)$$

In order to find a recursion formula, we write this last, shifted and dilated Chebyshev polynomial as a sum of Chebyshev polynomials of degree less than or equal to p :

$$T_p\left(\frac{y-1+2k/L}{2}\right) = 2^{-p} \sum_{i=0}^p w_i^{(p)}(k) T_i(y). \quad (22)$$

Substituting (22) in (21) yields:



$$M_p^* = \frac{1}{2^p - 1} \sum_{i=0}^{p-1} \left(\sum_{k=0}^L \frac{h_k}{\sqrt{2}} w_i^{(p)}(k) \right) M_i^*.$$

This appears to be a stable recursion formula. By using standard properties of the Chebyshev polynomials, the coefficients $w_i^{(p)}(k)$ can be calculated recursively. Details can be found in [20]. The same reference gives the construction of the coefficients of $\bar{\Gamma}(\bar{\tau})$ as a linear combination of Chebyshev polynomials.

The condition number of the system for the construction of the same Q_{14} formula as in the previous section is now $2 \cdot 10^3$! The roots of the polynomial $\bar{\Gamma}(\bar{\tau})$ can be found as the eigenvalues of its Chebyshev companion matrix. The effects of an orthogonal basis on the condition of the roots of a polynomial is discussed in [11]. It is stated there that the interval of orthogonality should contain the roots of interest. This condition is satisfied in most cases here.

9. ERROR ANALYSIS OF THE QUADRATURE RULE

Let x_0 be an arbitrary point of the interval $(0, L)$ not equal to one of the abscissae and let $P_r(x)$ be the polynomial of degree r which interpolates the function $f(x)$ in x_0, \dots, x_r . If $f(x) \in C^{r+1}[0, L]$ then [3]:

$$\forall x \in [0, L] : \exists \xi(x) \in]0, L[: f(x) = P_r(x) + e_r(x)$$

with

$$e_r(x) = \frac{\Pi(x)(x-x_0)}{(r+1)!} f^{(r+1)}(\xi(x)).$$

Let $E_r[f]$ denote the error of the integration rule:

$$\langle f(x), \phi(x) \rangle = Q_r[f] + E_r[f].$$

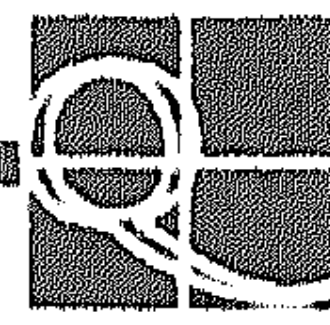
Then:

$$\begin{aligned} E_r[f] &= E_r[P_r + e_r] = E_r[P_r] + E_r[e_r] = E_r[e_r] \\ &= \langle \phi(x), e_r(x) \rangle - Q_r[e_r] \\ &= \frac{1}{(r+1)!} \int_0^L \phi(x) \Pi(x) (x-x_0) f^{(r+1)}(\xi(x)) dx \end{aligned}$$

with $\xi(x) \in (0, L)$.

We do not use this formula to estimate the error. The most important factor in this error formula will appear to be the $(r+1)$ th derivative of $f(x)$. Another way to see that this factor appears in the error is Peano's theorem [9]. In general ($\eta_j = 2^{-j}$):

$$\nu_{j,l} = \sqrt{\eta_j} \left[\sum_{k=1}^r w_k f(\eta_j(x_k + l)) + K_{j,l} \eta_j^{r+1} \right] \quad (23)$$



with

$$K_{j,l} = \frac{1}{(r+1)!} \int_0^L f^{(r+1)}(\eta_j(\xi(x)+l)) \Pi(x)(x-x_0) \phi(x) dx.$$

Using a generalisation of Bernoulli polynomials it is possible to derive an asymptotic error expansion for the quadrature formula [21]. As a result it is possible to use extrapolation techniques similar to Romberg integration.

10. FITTING THE FORMULAE IN THE MULTIREOLUTIONSCHEME

10.1. Using a quadrature formula at the finest level

Assume we have to calculate T coefficients $\nu_{n,l}$ yielding a function $v_n(x)$:

$$v_n(x) = \sum_{l=0}^{T-1} \nu_{n,l} \phi_{n,l}(x) \quad \text{with} \quad \nu_{n,l} = \langle \phi_{n,l}, f(x) \rangle.$$

The quadrature formula and error estimation yield:

$$\nu_{n,l} = \sqrt{\eta_n} \left[\sum_{k=1}^r w_k f(\eta_n((k-1)2^s - \tau + l)) + K_{n,l} \eta_n^{r+1} \right]. \quad (24)$$

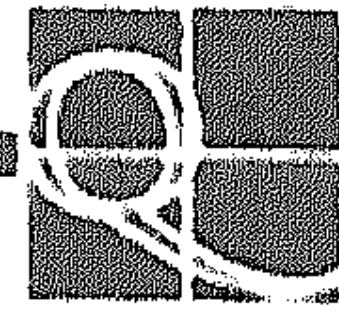
In order to calculate the coefficients, the function $f(x)$ is evaluated (or “sampled”) at a resolution 2^n . This means that $s \geq 0$. The total number of evaluations for the calculation of the T integrals is equal to $T + (r-1)2^s = N^{eval}$. Notice that the total number of evaluations is dominated by the first term and is only slightly dependent on r . As a result the one point quadrature formula, which needs T evaluations can always be replaced with a quadrature formula with higher degree of accuracy which requires in total almost the same number of evaluations. The work load is equal to Tr multiplications and $T(r-1)$ additions.

The highest order formula is the one with maximal r for $s \geq 0$. The maximal value of r for a specific choice of s is found by the requirement that the admittance interval of τ is not empty: $(r-1)2^s < L$ or $r = \lceil L/2^s \rceil$. Thus r is maximal if s is minimal. This yields $s = 0$, $r = L$ and $N^{eval} = T + (L-1)$. The error in this case is of order $O(\eta_n^{L+1})$.

Now one can calculate the coefficients of the multiresolution wavelet basis using the tree algorithm described in section . The errors on all the coefficients calculated in the tree are of order $O(\eta_n^{r+1})$.

10.2. Using a quadrature formula at the one but finest level

This section describes a method to obtain a higher accuracy in one part of the tree at the cost of a lower accuracy in the other part. The part of the tree in which the accuracy increases is the subtree formed by the $\nu_{n-1,l} \dots \nu_{0,l}$ and the



$\mu_{n-2,l} \cdots \mu_{0,l}$. The part of the tree in which the accuracy decreases is formed by the $\nu_{n,l}$ and the $\mu_{n-1,l}$.

The idea is to use function evaluations at resolution 2^n in a quadrature formula for the calculation of the coefficients at the one but finest level, namely the $\nu_{n-1,l}$. To get this resolution of evaluations, s has to be chosen equal to -1 . The highest order formula is the one with $r = 2L$.

$$\nu_{n-1,l} = \sqrt{2\eta_n} \left[\sum_{k=1}^{2L} w_k f(\eta_n((k-1) - 2\tau + 2l)) + K_{n-1,l} \eta_n^{2L+1} \right]. \quad (25)$$

The shift τ and the weights W_k are determined as described in section . The number of evaluations for the calculation of T inner products is $2T + 2(L - 1)$. From these coefficients the multiresolution coefficients of the first part of the tree can be calculated using the decomposition scheme. The error on the coefficients of this part is $O(\eta_n^{2L+1})$.

For the calculation of the $\nu_{n,l}$, we use a quadrature rule with $s = 0$ and $r = L$. The degree of accuracy is $r - 1$ since the value of the parameter τ is already determined by the first quadrature formula. The $\mu_{n-1,l}$ can be calculated with one step of the decomposition scheme. The error of the coefficients of the second part of the tree is $O(\eta_n^L)$.

10.3. Remark

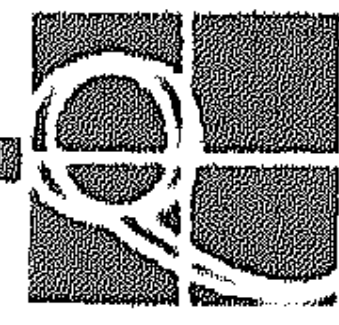
The quadrature formulae (16) and (23) show that the coefficients $\nu_{j,l}$ are $O(\sqrt{\eta_j})$. This is due to the fact that the $\phi_{j,l}(x)$ are scaled to have a unit L_2 -norm and their integral is thus equal to $\sqrt{\eta_j}$. This factor always appears outside the brackets of the quadrature rule to get a fair comparison between different formulae. In order to have coefficients of the same range in the whole tree, a factor $1/\sqrt{2}$ is introduced in the decomposition scheme and a factor $\sqrt{2}$ in the reconstruction scheme.

11. EXISTENCE OF THE QUADRATURE FORMULAE

As mentioned above, there is no guarantee that a quadrature formula with degree of accuracy r exists. In case of the Daubechies scaling functions we verified that the quadrature formula Q_{2N-1} exists for $2 \leq N \leq 10$. The quadrature formula Q_{4N-2} exists for $2 \leq N \leq 5$. For $N > 5$ this rule is not useful since it involves too many points. The weights of these formulae can vary in sign, but their absolute value doesn't grow too large when the number of points increases. For B-spline scaling functions, the quadrature formula Q_L exists for $2 \leq L \leq 10$ and Q_{2L} exists for $2 \leq L \leq 4$. The weights of the latter formulae are all positive.

12. EXAMPLE

We now compare the different quadrature rules in a practical example. We construct several multiresolution trees, each with coarsest level 0 and finest level n , and this for several n . We compare each time $\nu_{0,0}$ for the different quadrature rules.



n	5.2^n	trapezoidal rule	one point formula	Q_5^*	Q_5	Q_{10}
0	5	7.08e-04	1.17e-02	6.13e-04	2.15e-03	-
1	10	4.17e-03	1.43e-03	9.78e-05	4.40e-05	1.03e-08
2	20	7.96e-04	1.76e-04	4.30e-06	6.51e-07	1.11e-12
3	40	1.15e-04	2.19e-05	1.52e-07	9.38e-09	4.21e-15
4	80	1.53e-05	2.74e-06	5.03e-09	1.38e-10	9.99e-16
5	160	1.98e-06	3.43e-07	1.61e-10	2.09e-12	-
6	320	2.50e-07	4.28e-08	5.10e-12	3.19e-14	-
7	640	3.15e-08	5.35e-09	1.60e-13	1.11e-16	-
8	1280	3.96e-09	6.69e-10	4.66e-15	-	-
9	2560	4.96e-10	8.37e-11	2.22e-16	-	-
10	5120	6.20e-11	1.04e-11	-	-	-

TABLE 1. Errors of the integration rules.

Sufficient coefficients $\nu_{j,l}$ are provided at level j to calculate the coefficients of the next level $j-1$. As one can see from the decomposition formula, the number of coefficients $\nu_{j,l}$ has to be twice the number of coefficients $\nu_{j-1,l}$ plus $L-1$. Thus, $L2^j - (L-1)$ coefficients are needed on level j to retain one coefficient at level 0.

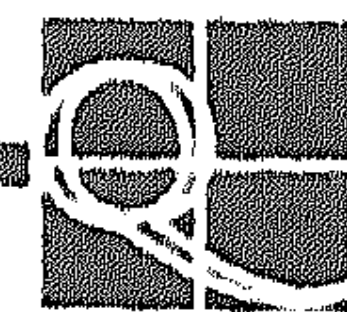
The one point formula based integration thus needs $L2^n - L + 1$ evaluations. In case we use an L -point quadrature formula at the finest level the number of evaluations is equal to the number of coefficients of the finest level augmented with $L-1$, yielding $L2^n$. In case we use a $2L$ -point quadrature formula at the one but finest level the number of evaluations is equal to twice the number of coefficients of the one but finest level augmented with $2(L-1)$, yielding also $L2^n$. Note that, if we would use the trapezoidal rule with evaluations at resolution 2^n directly for the calculation of $\nu_{0,0}$, the total number of evaluation points would be equal to $L2^n - 1$.

As an example we take for $\phi(x)$ the Daubechies scaling function with $N=3$, $f(x) = \sin(x)$ and:

$$\nu_{0,0} = \int_0^5 \phi(x) \sin(x) dx = 0.741104421925905 \dots \quad (26)$$

We compare the one point formula, the quadrature formulae Q_5^* (with $\tau = -1/2$), Q_5 , Q_{10} , and the trapezoidal rule directly applied for the calculation of this integral. The total number of evaluations is then respectively $5.2^n - 4$, 5.2^n , 5.2^n , 5.2^n and $5.2^n - 1$. The results are given in table 1. They show that for sufficiently differentiable functions $f(x)$, it is useful to search for the optimal value for the shift τ .

The trapezoidal rule has an error of $O(\eta_n^3)$ here, instead of $O(\eta_n^2)$ as we would expect. This is also a consequence of theorem .



13. CONCLUSION

In this paper we described multiresolution wavelet approximation and we developed several quadrature formulae for the calculation of a wavelet decomposition. We showed that a one point quadrature formula can be deduced from properties of orthogonal scaling functions. Multiple point quadrature formula can be constructed by solving a nonlinear system. This construction can become ill-conditioned when using ordinary moments. We tackled this problem by using Chebyshev modified moments. Which formula is suited in a practical situation depends on the kind of application and the required accuracy. Some questions still remain unanswered, such as the accuracy of the formulae and the influence of the shift τ if $f(x)$ is not sufficiently differentiable or if $f(x)$ contains singularities.

ACKNOWLEDGMENTS

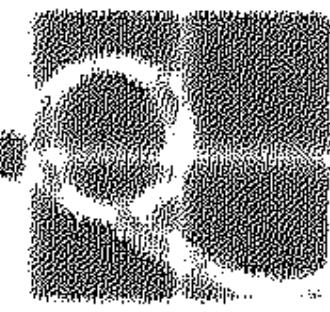
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REMARK

While preparing the final draft of this text, we were informed that the first theorem was also proven independently by R. Gopinath. Unfortunately at this time we cannot indicate the right reference.

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