Universal Representation of the Behavior of Linear Systems in Both Discrete and Continuous Time: a Unified Overview

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Starting in the 1960's and continuing through the 1970's, a fundamental research direction was the use of universal constructions to represent the algebraic aspects of automata and dynamical systems which operate in discrete time. In this paper, we first summarize some of the principal results of this research for discrete-time linear systems (based largely upon the work of Kalman and of Arbib and Manes), and we then provide an overview of our own more recent work on translating these results to linear systems which operate in continuous time.

0 Introduction

There are many important models of information processors in computer science [19]. Perhaps the most fundamental is that of a *sequential machine*, which is a six-tuple $M = (Q, \delta, I, Y, h, q_o)$, with Q the *state set*, I the *input set*, and $\delta : Q \times I \to Q$ the *state-transition* function. Y is the *output set* and $h : Q \to Y$, also a function, is the *output map*. $q_o \in Q$ is the *initial state* of M. We may thus think of the internal transitions of M as being described by equations of the form

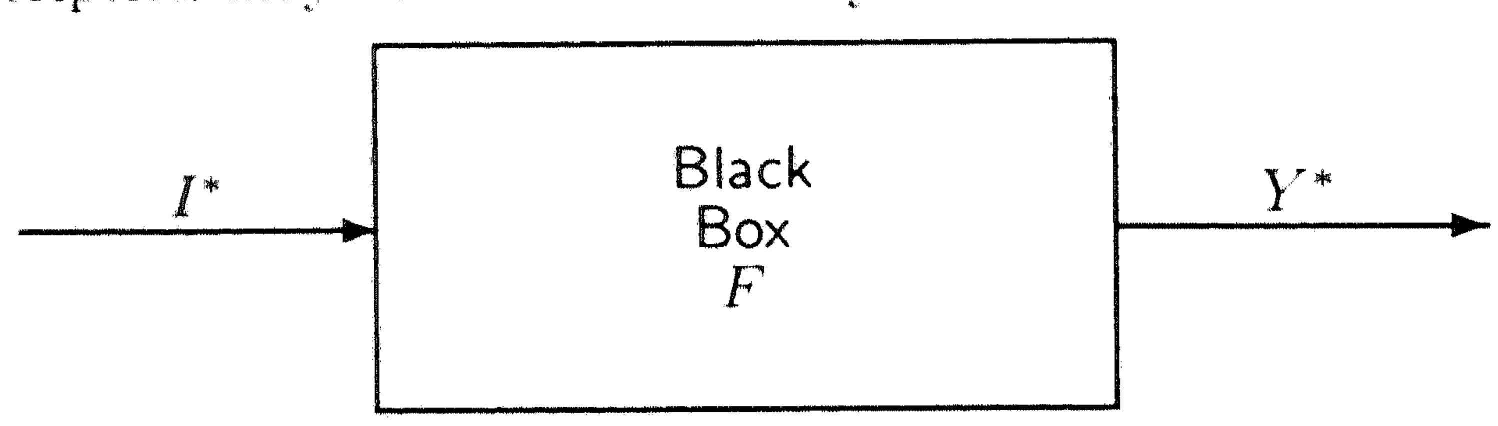
$$\begin{aligned} q(t+1) &= \delta(q(t), i(t)) \\ y(t) &= h(q(t)). \end{aligned} \tag{1}$$

In the theory of control and dynamical systems [30, Ch. 2], the same set of equations is studied, although Q, I, and Y are usually taken to be linear spaces in this case. Unquestionably, the most important special case in this latter context is that of *linear systems* [47], in which δ is an affine mapping, leading to equations of the following form.

$$\begin{aligned} q(t+1) &= f(q(t)) + g(i(t)) \\ y(t) &= h(q(t)) \end{aligned} \tag{2}$$

Both (1) and (2) describe systems which operate in *discrete time*, meaning that transitions of the system occur in discrete steps which we assume to be of unit time duration; therefore time is modelled by the nonnegative integers $\mathbf{N}_+$, the integer k representing time $t = k$. Here $q(t)$, $q(t+1) \in Q$, $i(t) \in I$, and $y(t) \in Y$ represent the values of the state, input, and output, at the times specified by the variable $t \in \mathbf{N}_+$.

Although the detailed use of these models in computer science may differ substantially from that in the theory of dynamical systems, it is nonetheless an inescapable observation that there is common ground. In each case, we may speak of the *behavior* of the system. Specifically, let I^* denote the set of all finite sequences of elements of I, and define Y^* similarly for Y. In an *input/output* representation or *behavior*, we think of the machine as being described as a sort of "black box" function $F : I^* \to Y^*$, in which an input sequence $i_0 i_1 .. i_k \in I^*$ represents a stream of inputs to the system, with i_t occurring at time t. The resulting output is also a sequence $y_0 y_1 .. y_k \in Y^*$, with y_t occurring at time t. Such a description may be visualized thusly.



For a sequential machine M, starting in the state q_o and described by the equations (1), the associated behavior $F_M : i_o i_1 .. i_k \mapsto y_0 y_1 .. y_k$ is given by $y_0 = h(q_o)$, $y_1 = h(\delta(q_o, i_0))$, $y_2 = h(\delta(\delta(q_o, i_0), i_1))$, etc. The fundamental question which we ask in a universal theory of behavior is how the construction $M \mapsto F_M$ arises using a particular type of algebraic construction, known as a universal construction.

In the other direction, we ask the question of *realization* of a behavior. That is, given a black box F as depicted above, we seek to identify a *canonical* sequential machine M with $F_M = F$. We must eschew a formal definition of canonical until later, but, informally, the canonical realization of F (for a sequential machine) is the "best" realization in the sense that it has the minimal number of states possible for a machine with behavior F.

The thesis that this bidirectional association between realizations and behaviors may be addressed in a unified fashion for systems arising from both computer science and from control and dynamical systems was first put forth in the literature by Arbib in 1965 [1]. Subsequently, the decade of the 1970's saw the development of a generalized algebraic theory of behavior, realization, and duality[1] for such systems based upon the foundational mathematical discipline of category theory [37, 25], which is the natural setting for universal

[1] The theory of *duality* in system theory develops the thesis that important concepts about systems come in pairs; we will not address duality to any significant degree in this paper.

constructions. Some of the more prominent papers on this topic include the work of Arbib and Manes [3, 4, 6, 7, 8], Bainbridge [10], Ehrig and his co-workers [16, 17] and Goguen [18]. The scope of the types of systems covered in this work includes not only sequential machines of the form (1) and linear machines of the form (2), but also fuzzy machines, tree automata, algebra automata, and group machines, just to name a few. Indeed, the strength of this theory lies in its ability to characterize, in a unified fashion, realization and behavior for a very wide class of machines. It also provided needed insight into the correct interpretation of these notions for more complex notions of systems. For example, the correct interpretation of realization and behavior for some classes of machines, such as group machines [2] and fuzzy automata [6], were only understood after placing them in this unifying categorical framework. An excellent summary of the key work in this field, with an extensive bibliography, may be found in [9].

A very significant feature of all of the work cited above is that it deals exclusively with discrete-time systems. While computer science is concerned almost exclusively with discrete-time systems, other dynamical models are often continuous in time. If we change the equations (1) above to their continuous-time equivalents, in which we replace $\mathbf{N}_+$ by the nonnegative reals $\mathbf{R}_+$ and we replace iteration by differentiation, we get the following.[2]

$$\begin{aligned} dq(t)/dt &= \delta(q(t), i(t)) \\ y(t) &= h(q(t)) \end{aligned} \tag{3}$$

In this context, it becomes highly nontrivial to construct a behavior, and global characterizations are not known, except in quite special cases. The equations (3) above can only be solved under very special circumstances, and then so often only locally. Furthermore, although our intuition suggests that I^* and Y^* must be replaced with spaces of continuous functions on the nonnegative real numbers, the precise nature of these functions is not immediate.[3] The universal approach to realization, by its very nature, requires a uniform characterization, in which every internal dynamics (such as those represented by equations (3)) has a natural[4] behavior associated with it, and conversely. However, equations of the form (3) have solutions only under special conditions, and even then such solutions are often only local in nature. Therefore, a general representation of the realization/behavior correspondence (as outlined above for discrete-time systems) of such systems is simply not feasible at this time. To make any headway in the continuous-time case, we must restrict our attention to special cases. In this work, we focus upon the following continuous-time equivalents of the linear equations (2).

[2]Of course, we must now assume that all spaces are topological vector spaces over the real numbers $\mathbf{R}$ or the complex numbers $\mathbf{C}$, at least locally.

[3]Indeed, this intuition is not quite correct; we shall see that I^* must be replaced by a space of *generalized* continuous functions, or distributions.

[4]Our use of the term *natural* here and throughout the paper has no special technical significance here; we simply intend to convey that the correspondence has some meaningful structure.

$$\begin{aligned} dq(t)/dt &= f(q(t)) + g(i(t)) \\ y(t) &= h(q(t)) \end{aligned} \tag{4}$$

Even then, the underlying mathematics is far more involved than in the discrete-time case, as it requires results from topological algebras and topological vector spaces, as well as the supporting algebraic framework of algebra and category theory. As a consequence, most of the research in continuous time has addressed the many important details of the systems modelled by equations of the form (4), rather than placing it within the global algebraic context cited above for discrete-time systems. It is the primary goal of a research program, which we have been following for some time, to develop a categorical theory of behavior, realization, and duality for continuous-time systems which completely parallels the already existing theory for discrete-time. Our principal results in this direction have been reported in [22] and [23].

The overall goal of this paper is to provide, for the nonspecialist, an overview of the algebraic theory of both discrete-time and continuous-time linear systems, with particular emphasis on how the comparatively simple constructions of the discrete-time case may be translated to continuous time. Insofar as possible, we have tried to avoid becoming involved in the rather technical details which are an essential part of a full presentation, particularly in the continuous-time case, where the details are extremely technical. As a consequence, proofs are generally omitted, and the finer points of definitions and theorems are often not spelled out completely. The interested reader can refer to the cited references to fill in the details.

No knowledge of category theory is necessary for the reading of this paper, although an acquaintance with the basic definitions, as may be found in [5] or [25], would be beneficial. We must assume an elementary knowledge of the theory of rings and modules, but the presentation found in [27] should prove sufficient. We also assume some knowledge of the basic definitions of locally convex topological vector spaces [35, 36], although we have made every effort to avoid difficult details.

To understand the continuous-time case, it is necessary to have some understanding of its discrete-time counterpart. Therefore, we start, in Section 1, with a brief but fairly rigorous exposition of the principal results of the discrete-time theory. Then, in Section 2, we show how these ideas may be translated to the continuous-time context. Because of the much more technical nature of the continuous-time case, the presentation in Section 2 is less rigorous than that of Section 1. However, we hope that it is substantial enough to give the interested reader a feel for the nature and complexity of the results.

We do not assume any specific knowledge of mathematical system theory. However, we expect that the reader will have had some exposure to the ideas of modelling dynamical systems by differential equations, and hopefully by difference equations as well.

1 Discrete-time linear systems

In this section, we provide a brief overview of the algebraic theory of discrete-time linear systems. The fundamentals of these results are based upon the pioneering work of Kalman [28, 29], while the representation and generalization within category theory is due to Arbib and Manes [3]. We have borrowed freely from both sources in preparing this section. Our discussion is limited to those aspects of this theory required as a foundation for our presentation of the continuous-time theory. The reader who is interested in more detail is encouraged to consult the above references. For more elementary information on linear systems from an algebraic point of view, the reader is referred to [38].

Basic definitions for linear systems

We fix a commutative ring $\mathbf{K}$ with unit.[5] Formally, a *discrete-time linear system over* $\mathbf{K}$ is a 6-tuple $M = (Q, f, I, g, Y, h)$ where Q (the *state space*), I (the *input space*), and Y (the *output space*) are all $\mathbf{K}$-modules, and $f : Q \to Q$ (the *state-transition map*), $g : I \to Q$ (the *input map*), and $h : Q \to Y$ (the *output map*) are all $\mathbf{K}$-linear mappings. Throughout this section, we fix for reference notation such a discrete-time linear system M. The dynamics are described by the equations (2), and the initial state is implicitly taken to be 0.

Since all of our systems in this section will be over $\mathbf{K}$, we shall refer to them as simply *discrete-time linear systems.* Such systems have been very widely studied in the literature on decision and control, particularly when $\mathbf{K}$ is the field of real or complex numbers, and I, Y, and Q are finite-dimensional vector spaces. See, *e.g.*, [38].

Following [3], let us call any pair (P, γ) with P a $\mathbf{K}$-module and $\gamma : P \to P$ a $\mathbf{K}$-*dynamics.* The *internal dynamics* of the system M as defined above is just (Q, f).

Given dynamics (P, γ) and (P', γ'), a *dynamorphism* $\lambda : (P, \gamma) \to (P', \gamma')$ is a $\mathbf{K}$-linear function $\lambda : P \to P'$ such that the following diagram commutes.

$$\begin{tikzcd} P \arrow[r, "\gamma"] \arrow[d, "\lambda"'] & P \arrow[d, "\lambda"] \\ P' \arrow[r, "\gamma'"'] & P' \end{tikzcd} \tag{5}$$

Let $\mathbf{K}[z]$ denote the space of all (formal) polynomials in the single variable z with coefficients in $\mathbf{K}$. A typical element of $\mathbf{K}[z]$ is written as $\sum_{k=0}^{\infty} a_k z^k$ with $a_k \in \mathbf{K}$; by the definition of polynomial only finitely many of the a_k's may be nonzero. We often drop the bounds and just write $\sum a_k z^k$. It is well known that $\mathbf{K}[z]$ admits a natural ring structure, corresponding exactly to the well-known addition and multiplication of ordinary polynomials over the real numbers [27,

[5]Typically, in an application, we would expect $\mathbf{K}$ to be a field, often the real numbers $\mathbf{R}$ or the complex numbers $\mathbf{C}$. Thus, when we speak of $\mathbf{K}$-modules, the reader may intuitively think of them as generalizing vector spaces over familiar fields.

Sec. 2.10]. The multiplication operation is called *convolution*, which we denote by "$*$"; thus

$$\begin{aligned} * : \mathbf{K}[z] \times \mathbf{K}[z] &\rightarrow \mathbf{K}[z] \\ (\sum_{k\geq 0} a_k z^k, \sum_{k\geq 0} b_k z^k) &\mapsto \sum_{k\geq 0} \sum_{m+n=k} a_m b_n z^k \end{aligned} \tag{6}$$

Clearly, $\mathbf{K}[z]$ also admits the structure of a $\mathbf{K}$-module. The following critical observation is due to Kalman.

LEMMA 1.1 (STRUCTURAL LEMMA FOR DISCRETE DYNAMICS) *There is a natural bijective correspondence between $\mathbf{K}$-dynamics and $\mathbf{K}[z]$-modules. With the $\mathbf{K}$-dynamics (Q, f) we associate the $\mathbf{K}[z]$-module whose action on Q is given by $(\sum a_k z^k)q = \sum a_k \cdot f^k(q)$, with the latter a "real" (as opposed to formal) sum, which is well defined since only finitely many of the a_k's are nonzero.* □

The preceding lemma is essentially a restatement of the well-known fact that z freely generates the ring $\mathbf{K}[z]$, or, speaking categorically, that $\mathbf{K}[z]$ is the free $\mathbf{K}[z]$ module over $\mathbf{K}$ [27, Thm. 2.11]. In the dynamics (P, γ), the mapping γ just corresponds to multiplication by z. From that, we can identify uniquely multiplication by z^k for any $k \in \mathbf{N}_+$, and hence recover the $\mathbf{K}[z]$ action.

The system-theoretic significance of 1.1 is that it provides two distinct representations for the dynamics of a discrete-time linear system. We may think of the $\mathbf{K}$-dynamics representation (Q, f) as a "local" representation, which tells us how to do a transition of a single time step, and the $\mathbf{K}[z]$-module representation as a "global" description of how to process sequences of transitions. Interestingly, the entire development of discrete-time behavior and realization is possible while working solely with $\mathbf{K}$-dynamics, as witnessed by the elegant presentation of Arbib and Manes [3]. However, as we shall see, the ability to have two distinct representations, one for local dynamics and another for global behavior, becomes crucial in the continuous-time case. In anticipation of paralleling our discrete-time methods in continuous-time, we therefore develop both representations in this section.

Universal reconstruction of the reachability map

We now turn to the issue of reachability for the discrete-time linear system $M = (Q, f, I, g, Y, h)$. Informally, the reachability map of M sends input sequences to the state resulting from that input, assuming that the machine begins in the zero state. To formalize this notion, it is mathematically most convenient to think of sequences of inputs as ending at time 0, rather than starting at 0. Thus, we represent a sequence of inputs to M as $i_k, i_{k-1} \ldots i_1, i_0$, with i_j occurring at time $-j$. Now, with the system initially in state $0 \in Q$, after applying input i_k, we see directly from equations (2) that at time $-k$ the system will be in state $g(i_k)$. At time $-k+1$, after applying i_{k-1}, M will be in state $f(g(i_k)) + g(i_{k-1})$. Proceeding along in this fashion, we arrive at the conclusion that, at time 0, M will be in the state $f^k(g(i_k)) + f^{k-1}(g(i_{k-1})) + \ldots + f(g(i_1)) + g(i_0) = \sum_{i=0}^{k} f^i(g(i_j))$.

We would like to express reachability as a linear map. To do so, define $I[z]$ to be the space of all polynomials over I in the single variable z. $I[z]$ is very much like $\mathbf{K}[z]$, except of course that it is not in general a ring, since we have no way to multiply the elements of I. Of course, it is a $\mathbf{K}$-module under the obvious action. We regard $I[z]$ as the linear space of all possible input sequences to M, with z^k identifying time $t = -k$ in the above convention. Thus, $\sum_{k=0}^{\infty} i_k z^k$ represents an input to M with i_k occurring at time $t = -k$. Since only finitely many of the i_k's are nonzero, the input has a finite starting time. Specifically, we regard the input as starting at $t = -k$, where $i_k \neq 0$ and $i_m = 0$ for all $m > k$. We define the *reachability map* ρ_M of M as the following $\mathbf{K}$-linear map.

$$\begin{aligned} \rho_M : I[z] &\rightarrow Q \\ \sum i_k z^k &\mapsto \sum f^k(g(i_k)), \end{aligned} \tag{7}$$

Clearly this recaptures the system-theoretic notion described above.

Observe that $(I[z], z)$ forms a $\mathbf{K}$-dynamics, with z denoting the following shift operator.

$$\begin{aligned} z : I[z] &\rightarrow I[z] \\ \sum i_k z^k &\mapsto \sum i_k z^{k+1} \end{aligned} \tag{8}$$

Given a $\mathbf{K}$-module I, a *free dynamics* over I is a pair $((I^{\S}, z), \eta_I)$, with $(I^{\S}, z)$ a dynamics and $\eta_I : I \rightarrow I^{\S}$ a $\mathbf{K}$-linear mapping, such that for any other dynamics (Q, f) and $\mathbf{K}$-linear mapping $g : I \rightarrow Q$, there is a unique dynamorphism $\rho : (I^{\S}, z) \rightarrow (Q, f)$ rendering the following diagram commutative. (Note that the rectangle is commutative just because ρ is a dynamorphism. The central condition here is that the triangle commutes.)

$$\begin{tikzcd} I \arrow[r, "\eta_I"] \arrow[dr, "g"'] & I^{\S} \arrow[r, "z"] \arrow[d, "\rho"] & I^{\S} \arrow[d, "\rho"] \\ & Q \arrow[r, "f"] & Q \end{tikzcd} \tag{9}$$

THEOREM 1.2 (EXISTENCE OF FREE DYNAMICS) *Let I be a $\mathbf{K}$-module. The free dynamics $((I^{\S}, z), \eta_I)$ over I exists, and is given by $I^{\S} = I[z]$. The mapping $z : I[z] \rightarrow I[z]$ is just multiplication by z. If $M = (Q, f, I, g, Y, h)$ is a discrete time-linear system, then the resulting dynamorphism $\rho : (I^{\S}, z) \rightarrow (Q, F)$ is exactly the reachability map ρ_M of M.* □

The construction of a free object (such as a free dynamics) is called a *universal construction* in category theory. Thus, the reachability map arises as the part of a universal construction, as the unique dynamorphism from the free dynamics over I to the machine dynamics (Q, f), relative to the input map g. This is a critical observation, because it gives us an abstract construction. In the continuous-time case, where we will have much less of an intuitive idea what the reachability map should be, we will imitate this construction within the appropriate context.

The representation of the free dynamics in the $\mathbf{K}[z]$-module interpretation is exactly the free $\mathbf{K}[z]$ module over I. It is well known that this module is the algebraic tensor product $\mathbf{K}[z] \otimes I$, with the module action just the extension of convolution [14, Ch. 2]

$$\begin{array}{rcl} \mathbf{K}[z] \times (\mathbf{K}[z] \otimes I) & \to & \mathbf{K}[z] \otimes I \\ (\alpha,\ \beta \otimes i) & \mapsto & (\alpha * \beta) \otimes i\,, \end{array} \tag{10}$$

extended by linearity. The natural isomorphism between $I[z]$ and $\mathbf{K}[z] \otimes I$ is simply

$$\begin{array}{rcl} I[z] & \to & \mathbf{K}[z] \otimes I \\ \sum i_k z^k & \mapsto & \sum (z^k \otimes i_k). \end{array} \tag{11}$$

Universal reconstruction of the observability map

Observability is dual to reachability. Intuitively, the observability map tells us, for each state q, the output sequence that we will see when M is started in state q with only zero inputs applied. More precisely, from state q, if we apply no further inputs, equations (2) tell us that we will observe the output sequence $h(q), h(f(q)), h(f^2(q)), \ldots$. This sequence is often called the *natural response* of M from state q.

To formalize this notion algebraically, let $Y[[z^{-1}]]$ denote the $\mathbf{K}$-linear space of all formal power series in the variable z^{-1}. We regard an element of this space as a formal sum $\sum_{k=0}^{\infty} y_k z^{-k}$, with no finiteness restrictions. (Note that z^{-1} is just a symbol like z; we use the variable z^{-1} rather than z for compatibility reasons to become apparent shortly.) We may identify $Y[[z^{-1}]]$ with the set of possible output sequences of M by regarding $\sum_{k=0}^{\infty} y_k z^{-k}$ as defining the output sequence whose value at time k is y_k.[6] Formally, the *observability map* σ_M of M is defined as follows.

$$\begin{array}{rcl} \sigma_M : Q & \to & Y[[z^{-1}]] \\ q & \mapsto & \displaystyle\sum_{k=0}^{\infty} h(f^k(q)) z^{-k}, \end{array} \tag{12}$$

We now show how this arises as a universal construction. Let $(Y[[z^{-1}]], z)$ denote the $\mathbf{K}$-dynamics with z the following shift operator.

$$\begin{array}{rcl} z : Y[[z^{-1}]] & \to & Y[[z^{-1}]] \\ \sum y_k z^{-k} & \mapsto & \sum y_{k+1} z^{-k} \end{array} \tag{13}$$

Note that the symbol z is serving double duty; it also denotes the shift operator in the dynamics $(I[z], z)$. Since it denotes "multiplication by z" in each case,

[6] In light of this convention and the previous one for $I[z]$, it would perhaps be more logical to reverse the roles of z and z^{-1}, and work with $I[z^{-1}]$ and $Y[[z]]$, for then z^k would correspond to time k and not $-k$. However, the convention which we use here is that introduced by Kalman, and has become standard in the literature.

this overloading is a natural one. Note also that, unlike the shift for the $(I[z], z)$ case, z here is not injective, as y_0 is lost in the above mapping.

Given a $\mathbf{K}$-module Y, a *cofree dynamics* over Y is a pair $((Y_\S, z), \varepsilon_Y)$, with $(Y_\S, z)$ a dynamics and $\varepsilon_Y : Y_\S \rightarrow Y$ a $\mathbf{K}$-linear mapping such that for any other dynamics (Q, f) and $\mathbf{K}$-linear mapping $h : Q \rightarrow Y$, there is a unique dynamorphism $\sigma : (Q, f) \rightarrow (Y_\S, \varepsilon_Y)$ such that the following diagram commutes. (Again, it is the triangle which is central here; the rectangle commutes by virtue of σ being a dynamorphism.)

$$\begin{tikzcd}
Q \arrow[r, "f"] \arrow[d, "\sigma"'] & Q \arrow[d, "\sigma"] \arrow[dr, "h"] & \\
Y_\S \arrow[r, "z"] & Y_\S \arrow[r, "\varepsilon_Y"] & Y
\end{tikzcd} \tag{14}$$

THEOREM 1.3 (EXISTENCE OF COFREE DYNAMICS) *Let Y be a $\mathbf{K}$-module. The cofree dynamics $((Y_\S, z), \varepsilon_Y)$ over Y exists, and is given by $Y_\S = Y[[z^{-1}]]$. The mapping $z : Y[[z^{-1}]] \rightarrow Y[[z^{-1}]]$ is just multiplication by z. If $M = (Q, f, I, g, Y, h)$ is a discrete time-linear system, then the resulting dynamorphism $\sigma : (Q, F) \rightarrow (Y_\S, z)$ is exactly the observability map σ_M of M.* □

The shift z here is forward in time, as in the similar operation for the free dynamics, but it has the opposite effect. Here it discards the z^0 term y_0 of a series $\sum_{k=0}^{\infty} y_k z^{-k}$ and shifts the rest down, yielding $\sum_{k=0}^{\infty} y_{k+1} z^{-k}$.

To recapture the cofree dynamics within the $\mathbf{K}[z]$-module context, let $L(\mathbf{K}[z], Y)$ denote the space of all $\mathbf{K}$-linear maps from $\mathbf{K}[z]$ to Y. It is easy to see that we have a natural isomorphism

$$\begin{array}{rcl} L(\mathbf{K}[z], Y) & \cong & Y[[z^{-1}]] \\ \varphi & \mapsto & \sum_{k \geq 0} \varphi(z^k) z^{-k}. \end{array} \tag{15}$$

The $\mathbf{K}[z]$-module action in this context translates to

$$\begin{array}{rcl} \mathbf{K}[z] \times L(\mathbf{K}[z], Y) & \rightarrow & L(\mathbf{K}[z], Y) \\ (\alpha, \varphi) & \mapsto & (\beta \mapsto \varphi(\alpha * \beta)) \end{array} \tag{16}$$

A construction which yields a cofree object (such as the cofree dynamics) is called a co-universal construction. Although the notion of duality has not been formally introduced, it is not difficult to see that diagram (14) may be obtained from diagram (9) by replacing input with output and turning around all of the arrows. Thus, universal and co-universal constructions are closely related, and we sometimes refer to them both as universal constructions.

Universal representation of behavior
We have thus shown how both the reachability map ρ_M and the observability map σ_M of a discrete-time linear system arise as universal constructions. Now we define the *total behavior* B_M of M as the composition $\sigma_M \circ \rho_M : I^{\S} \to Y_{\S}$. This provides us with a completely algebraic construction of the behavior of M, which we may represent pictorially by gluing diagrams (9) and (14) together (and replacing σ and ρ with σ_M and ρ_M, respectively), as follows.

$$
\begin{tikzcd}
I \arrow[r, "\eta_I"] \arrow[dr, "g"'] & I^{\S} \arrow[r, "z"] \arrow[d, "\rho_M"'] & I^{\S} \arrow[d, "\rho_M"] & \\
 & Q \arrow[r, "f"] \arrow[d, "\sigma_M"'] & Q \arrow[d, "\sigma_M"] \arrow[dr, "h"] & \\
 & Y_{\S} \arrow[r, "z"] & Y_{\S} \arrow[r, "\varepsilon_Y"] & Y
\end{tikzcd}
\tag{17}
$$

In the next section, we shall imitate these constructions in the context of continuous-time linear systems to discover what the reachability map, observability map, and total behavior should look like for such systems.

Algebraic representation of realization
We know that every discrete-time linear system M naturally defines its behavior B_M. Now let us examine the converse. Suppose that we are given $\mathbf{K}$-modules I and Y, and a dynamorphism

$$B : (I[z], z) \to (Y[[z^{-1}]], z). \tag{18}$$

It is natural to ask under what conditions (18) defines the behavior of a discrete-time linear system. The immediate answer is always, for we may define $M_{B_{\S}} = (I[z], z, I, \eta_I, Y, \varepsilon_Y \circ B)$ and $M_{B^{\S}} = (Y[[z^{-1}]], z, Y, B \circ \eta_I, Y, \varepsilon_Y)$; it is trivially verified that each has behavior B. These are extreme cases; the former is called the *free realization* of B, and the latter the *cofree realization*. Thus, each behavior yields at least two realizations. Generally, it will have many more. However, there is one realization which is most natural.

Given a discrete-time linear system M, we say that it is *reachable* if its reachability map ρ_M is surjective, and we say that it is *observable* if its observability map σ_M is injective. Intuitively, if M is reachable, there are no "useless" states which cannot be reached from 0 by the application of any input. Similarly, if M is observable, no two states are equivalent in the sense that they lead to exactly the same output sequences. We say that M is *canonical* if it is both reachable and observable. Intuitively, a canonical system is minimal in that it contains the fewest possible states. We have the following algebraic representation of canonicity.

THEOREM 1.4 (EXISTENCE AND UNIQUENESS OF CANONICAL REALIZATIONS) *Let I and Y be $\mathbf{K}$-modules, and let $B : (I[z], z) \longrightarrow (Y[[z^{-1}]], z)$ be a dynamorphism. Then, up to isomorphism (a renaming of states), there is a unique canonical discrete-time linear system whose behavior is B.*

PROOF OUTLINE: The idea is quite simple. We know that we may factor B as a function as $I[z] \xrightarrow{\rho_B} Q \xrightarrow{\sigma_B}$, with ρ_B surjective and σ_B injective. We simply take Q to be $I[z]/B$, the set of blocks of the equivalence relation defined by the function underlying B. Furthermore, Q is unique, up to renaming, as a set. The question is whether or not Q may be uniquely endowed with the structure of a dynamics. In other words, we ask whether or not we may fill in the diagram below with the dashed arrow labelled f, so that (Q, f) becomes a dynamics.

$$\begin{tikzcd} I[z] \arrow[r, "z"] \arrow[d, "\rho_B"'] & I[z] \arrow[d, "\rho_B"] \\ Q \arrow[r, dashed, "f"] \arrow[d, "\sigma_B"'] & Q \arrow[d, "\sigma_B"] \\ Y[[z^{-1}]] \arrow[r, "z"] & Y[[z^{-1}]] \end{tikzcd} \tag{19}$$

This is guaranteed by a result known as the *dynamorphic image lemma* [3, 4.4], which is a generalization of what was known in earlier contexts as the *Zeiger fill-in lemma* [29, 6.2]. This may also be recovered on purely algebraic grounds [37, Prop. 1, p. 195]. □

Summary

The following is the key identification for discrete-time linear systems, which we wish to extend to continuous time.

THEOREM 1.5 (SYSTEM REPRESENTATION THEOREM) *Given any $\mathbf{K}$-modules I, and Y, there is a natural bijective correspondence between behaviors $B : (I[z], z) \longrightarrow (Y[[z^{-1}]], z)$ and isomorphic equivalence classes of canonical systems with input space I and output space Y.* □

In words, behaviors and canonical realizations are coextensive, up to mathematical equivalence. Knowing the input/output action of a canonical system is sufficient to allow us to recover its internal structure, and conversely.

2 DYNAMICS AND BEHAVIOR OF CONTINUOUS-TIME LINEAR SYSTEMS

In this section, we attempt to illustrate the main ideas behind our efforts to transport the theory of the previous section to continuous-time linear systems. Due to the extremely technical nature of the results, we have adopted a fairly informal style of presentation in which we illustrate the principal ideas without becoming unnecessarily involved in details. The reader who finds the presentation insufficiently rigorous or incomplete, or who simply wants to learn more of the details, is invited to consult the references [22, 23, 24]. Virtually all of the presentation given here is based upon [23], and so we have not made explicit citations to that reference.

The reader may legitimately ask at this point why the abstract representation of the continuous-time case is not just a simple recasting of that of discrete time. In a *very* abstract sense it is, save that we must show that the continuous-time analogs of $I^{\S}$ and $Y_{\S}$ exist. However, demonstrating this existence requires that we pay close attention to topological as well as algebraic aspects, and this complicates the picture substantially. But in a more concrete sense, we seek not just an existence proof, but rather concrete representations of the analogs of $I^{\S}$, $Y_{\S}$, ρ_M, σ_M, and so forth. Obtaining these concrete representations (which require a translation from the discrete-time concept of iteration to the continuous-time notion of integration), requires that we employ a rather complex area of mathematics known as *distribution theory.*

The internal model

To place the equations (4) on more precise ground, we must take $\mathbf{K}$ to be either $\mathbf{R}$ (the field of real numbers) or else $\mathbf{C}$ (the field of complex numbers). Beyond that, several further choices must be made. First of all, we must identify a suitable setting for the spaces Q, I, and Y. There is a number of possibilities. One would be to work within the setting of Hilbert spaces or Banach spaces, as this is a natural setting for much of infinite-dimensional linear system theory [11], due to its enormously rich associated theory of one-parameter semigroups of operators [26]. Unfortunately, it does not appear to be possible to develop a completely satisfactory translation of the discrete-time results to this setting, although [21] contains some suggestions for pursuing a theory of behavior within the Banach space framework. Our approach, rather, is to allow the underlying spaces to be quite arbitrary locally convex topological vector spaces, in the sense of [35, 36]. A locally convex topological vector space, or *locally convex space* for short, is a topological vector space (= topological module) over $\mathbf{K}$ for which the operations of addition and scalar multiplication are continuous, and which in addition satisfies a technical condition known as local convexity. Almost all classes of spaces which arise in practice are locally convex, including all Banach and Hilbert spaces.

We must also state what the mappings f, g, and h are to be. The most obvious choice is to require each of them to be linear and continuous. Indeed, this is exactly what we did in our earlier work [22]. Linearity is essential — after all, we are dealing with linear systems — but this assumption of continuity turns out to be needlessly restrictive. Rather, we need to ask which conditions are necessary in order that the equations (4) admit a unique solution. It is well-known from the classical theory of semigroups of operators on Banach spaces [39, 13.35] that f need not be continuous to provide existence and uniqueness of solutions; rather, certain closed but not necessarily continuous operators suffice. The key is that f must be the infinitesimal generator of a semigroup of operators parameterized by $\mathbf{R}_+$, the nonnegative reals. Kōmura [34] has demonstrated that a similar approach applies in the more general context of locally convex spaces. Specifically, a *locally equicontinuous semigroup of operators of class* (C_o) is a pair (T, E) in which E is a locally convex space and $T : \mathbf{R}_+ \rightarrow \mathbf{L}(E)$ is a function from the nonnegative reals $\mathbf{R}_+$ into the space $\mathbf{L}(E)$ of all continuous linear mappings on E, which is a monoid homomorphism (*i.e.*, $T(0) = \mathbf{1}_E$ and $T(t_1) \circ T(t_2) = T(t_1 + t_2)$ for all $t_1, t_2 \in \mathbf{R}_+$), which is pointwise continuous at zero ($\lim_{t \rightarrow 0} T(t)e = e$ for any $e \in E$), and which is *locally equicontinuous* ($\{T(t) \mid 0 \leq t \leq \varepsilon\}$ is equicontinuous for some $\varepsilon > 0$.) The *infinitesimal generator* of such a semigroup is the function $\mathbf{g}_T : E \rightarrow E$ given by

$$\mathbf{g}_T(e) = \begin{cases} \lim\limits_{t \rightarrow 0} \frac{T(t)e - e}{t} & \text{if the limit exists;} \\ \text{undefined} & \text{otherwise.} \end{cases} \tag{20}$$

It is important to note that $\mathbf{g}_T$ need not be total, although it is closed, and if E is sequentially complete (*i.e.*, Cauchy sequences converge), the domain of definition is dense in E [34, Sec. 1]. Hereafter, we shall abbreviate "locally equicontinuous semigroup of class (C_o)" to just *semigroup*.

THEOREM 2.1 (EXISTENCE AND UNIQUENESS OF SOLUTIONS) *In the equations* (4), *assume that* Q, I, *and* Y *are locally convex spaces, that* f *is the infinitesimal generator of a semigroup* T, *that* g *and* h *are continuous and linear, and that* Q *is sequentially complete. Then* f *and* T *determine each other uniquely, and for every continuous function* $i : \mathbf{R}_+ \rightarrow I$, *the function defined by*

$$q(t) = T(t)q_0 + \int_0^t T(t-s)g(i(s))ds \tag{21}$$

is the unique continuously differentiable solution to (4) *with initial condition* q_0 *at* $t = 0$. □

The solution equation (21) is well-known in the more classical case in which all spaces are finite dimensional [38, 6-1, Thm. 1]. Actually, as we shall see later, 2.1 holds even when Q is not sequentially complete, provided that other conditions are met. Therefore, we shall not explicitly require state spaces to be sequentially complete.

Formally, a *continuous-time linear system* is a 6-tuple $M = (Q, f, I, g, Y, h)$ where Q (the *state space*), I (the *input space*), and Y (the *output space*) are all locally convex spaces, $f : Q \rightarrow Q$ (the *state-transition map*) is the infinitesimal generator of a semigroup, and $g : I \rightarrow Q$ (the *input map*) and $h : Q \rightarrow Y$ (the *output map*) are both continuous linear mappings. We call (Q, f) a *smooth dynamics* to emphasize that it has special properties to allow us to reconstruct the continuous-time behavior. A *dynamorphism* is defined as per diagram (5), with λ a continuous linear mapping, but we must be careful since γ and γ' need not be total functions. Precisely, commutativity in this case means that whenever one path is defined, then so is the other, and they are equal.

A simple example
Before moving on to the abstract theory, we present a simple example which will help to illustrate both the generality and the limitations of this framework. For this example, we assume some familiarity with the use of distributions (in the sense of [42]) to represent the modelling of wave phenomena. For those without the requisite background, this entire example may safely be skipped without loss of continuity. This example is also discussed in [23].

The setting is a unit length of ideal transmission line with series inductance L, series resistance R, shunt capacitance C, and shunt conductance A, all per unit length. The line is driven at its right endpoint $x = 1$ and terminated at its left endpoint $x = 0$ with a short circuit. It is assumed that the behavior of this line is governed by the standard wave equations, and that the loss is small enough that the characteristic impedance is given by $Z_o = \sqrt{L/C}$. See, *e.g.*, [13]. We let $V(x,t)$ represent the voltage across the line at position x $(0 \leq x \leq 1)$ at time t, and let $J(x,t)$ similarly represent the current on the line. The state space Q is $\{(V(x), J(x)) \in \mathcal{C}'_c(0,1] \times \mathcal{C}'_c[0,1] \mid V(x) - Z_o \cdot J(x) = 0$ in some neighborhood of $x = 1\}$. Because the right end of the line is short-circuited, the voltage is 0 at $x = 0$; this is recaptured by the fact that the voltage distribution be in $\mathcal{C}'_c(0,1]$, the space of all scalar-valued distributions with compact support contained in the half-open interval $(0,1]$. $\mathcal{C}'_c[0,1]$ is defined similarly, and $V(x) - Z_o \cdot J(x) = 0$ in some neighborhood of 1 recaptures the stipulation that the impedance at $x = 1$ be matched.

The line is driven with an impedance matched generator at $x = 1$, so that the input space I takes values in the field $\mathbf{K}$, and the input over time (the continuous-time equivalent of $I^{\S}$) has the form $\begin{pmatrix} Z_o \\ -1 \end{pmatrix} \cdot i(t)$, with $i(t)$ a real-valued signal (represented as a distribution). (The current carries a negative sign since it is travelling to the left.) The internal dynamics of this system are represented by the following differential equation.

$$\frac{d}{dt}\begin{pmatrix} V(x,t) \\ I(x,t) \end{pmatrix} = \begin{pmatrix} -\frac{R}{L} & -\frac{1}{L}\cdot\frac{\partial}{\partial x} \\ -\frac{1}{C}\cdot\frac{\partial}{\partial x} & -\frac{A}{C} \end{pmatrix}\begin{pmatrix} V(x,t) \\ J(x,t) \end{pmatrix} + \begin{pmatrix} Z_o \\ -1 \end{pmatrix} i(t) \quad (22)$$

As for modelling the output of this system, we would like to observe the voltage-current pair at $x = 1$. However, the state is a distribution which may not be representable as an ordinary function of time. Therefore, it is not possible to employ a representation which samples the values at $x = 1$ directly. There are two ways around this difficulty. The first is to regard the output space itself as a space of distributions about some small half-open interval $(1 - \iota, 1]$, where ι is some small number. This output is easily obtained by restricting the state distribution by using the natural surjections $\varrho_1 : \mathcal{C}'_c(0,1] \to \mathcal{C}'_c(1-\iota, 1]$ and $\varrho_2 : \mathcal{C}'_c[0,1] \to \mathcal{C}'_c(1-\iota,1]$ (see [43]). In this case, the output space Y is $\mathcal{C}'_c(1-\iota,1]$, and we get an output equation of the form

$$y(t) = \begin{pmatrix} \varrho_1 & 0 \\ 0 & \varrho_2 \end{pmatrix} q(t) \quad (23)$$

The second approach is to regard the output space Y as consisting of just pairs of real numbers $(V, I) \in \mathbf{K} \times \mathbf{K}$, representing values averaged about the point right endpoint $x = 1$. In this case, the output mapping h is an evaluation of the state distribution on a certain test function pair $(\varphi_1, \varphi_2) \in \mathcal{C}'_c(1 - \iota, 1]^2$. Typically, this test function would have unit area. The output equation would then take the following form, with "$*$" representing distributional convolution.

$$y(t) = \begin{pmatrix} \varphi_1 * (-) & 0 \\ 0 & \varphi_2 * (-) \end{pmatrix} q(t) \tag{24}$$

Perhaps the key observation to be made from this example is that the equations (3) are quite general; we may even recapture systems modelled by partial differential equations with delays (as is our example).

$\mathbf{R}_+$ *rings and modules*

In translating the discrete-time theory to continuous time, the key step is to replace $\mathbf{K}[z]$ with any member of a class of commutative topological rings, which we term $\mathbf{R}_+$-rings. In these rings, the multiplication is not necessarily continuous, but rather it is hypocontinuous with respect to the precompact sets, or *c-hypocontinuous*. That is, if R is an $\mathbf{R}_+$ ring, then the multiplication is continuous when restricted to sets of the form $K \times R$ and $R \times K$, with K a precompact subset of R.

To identify an $\mathbf{R}_+$-ring, we start by specifying an $\mathbf{R}_+$*-ring generator* $\mathcal{F}$, which is an operator which gives, for each locally convex space E, a locally convex space $\mathcal{F}(\mathbf{R}_+, E)$ of continuous functions from $\mathbf{R}_+$ into E. This space characterizes the desired system responses; for a continuous-time linear system $M = (Q, f, I, g, Y, h)$ modelled under $\mathcal{F}$, the outputs over time (*i.e.*, the continuous-time analog of $Y_{\S}$) will be precisely $\mathcal{F}(\mathbf{R}_+, Y)$. The range of choices for $\mathcal{F}$ is extensive, and includes most of the spaces of differentiable vector-valued functions identified in the classic paper of L. Schwartz [40]. Perhaps the two most important examples for $\mathcal{F}$ are $\mathcal{C}$ and $\mathcal{E}$. $\mathcal{C}(\mathbf{R}_+, Y)$ denotes the space of all continuous functions from $\mathbf{R}_+$ into Y, endowed with the topology of uniform convergence on compact subsets of $\mathbf{R}_+$ [43, Ch. 40]. $\mathcal{E}(\mathbf{R}_+, Y)$ denotes the subspace of $\mathcal{C}(\mathbf{R}_+, Y)$ consisting of the infinitely differentiable functions, and endowed with the topology of uniform convergence on compact sets of all derivatives. There is a multitude of other choices which we will mention later.

The $\mathbf{R}_+$-ring corresponding to $\mathcal{F}$ is obtained as the dual space $\mathcal{F}'(\mathbf{R}_+)$ (space of all continuous linear functionals) of $\mathcal{F}(\mathbf{R}_+, \mathbf{K}) = \mathcal{F}(\mathbf{R}_+)$. $\mathcal{C}'(\mathbf{R}_+)$ consists of all measures on $\mathbf{R}_+$, in the sense of Bourbaki [12], which have compact support. $\mathcal{E}'(\mathbf{R}_+)$ consists of all infinitely differentiable measures or *distributions* in the sense of L. Schwartz [42]. In each case, the ring multiplication is convolution of measures. See [43, Chap. 27] for an extensive discussion of convolution. The topology in all cases is that of uniform convergence on precompact subsets of $\mathcal{F}(\mathbf{R}_+)$.

Each $\mathbf{R}_+$-ring contains, as a (topologically) dense subring, a special ring which we denote by $\mathbf{K}\langle z\rangle$. This ring is exactly like $\mathbf{K}[z]$, save that instead of allowing only integer exponents of z, we allow any nonnegative real number.

More precisely, define the ring $\mathbf{K}\langle z\rangle$ to be the linear space of all formal polynomials of the form $\sum_{t\in\mathbf{R}_+} a_t z^t$, with each $a_t \in \mathbf{K}$. It is important to realize that, as with $\mathbf{K}[z]$, the sum must be finite in the sense that all but finitely many of the a_t's must be zero. The ring multiplication is defined analogously to that of equation (6).

$$\begin{array}{rcl} * : \mathbf{K}\langle z\rangle \times \mathbf{K}\langle z\rangle & \to & \mathbf{K}\langle z\rangle \\ (\sum_{t\in\mathbf{R}_+} a_t z^t, \sum_{t\in\mathbf{R}_+} b_t z^t) & \mapsto & \sum_{t\in\mathbf{R}_+} \sum_{r+s=t} a_r b_s z^t \end{array} \tag{25}$$

In all cases, the element z^t identifies the point (or Dirac) measure δ_t, defined by $\delta_t(f) = f(t)$. δ_0 is the multiplicative identity element of the ring $\mathcal{F}'(\mathbf{R}_+)$.

On each $\mathbf{R}_+$-ring $R = \mathcal{F}'(\mathbf{R}_+)$, we define the *differentiation operator* $D : R \to R$ as the transpose of differentiation of functions. That is, if $\mu \in \mathcal{F}'(\mathbf{R}_+)$, then for $\varphi \in \mathcal{F}(\mathbf{R}_+)$,

$$D(\mu)(\varphi) = \begin{cases} \mu(d\varphi/dt) & \text{if } \varphi \text{ is differentiable} \\ \text{undefined} & \text{otherwise} \end{cases} . \tag{26}$$

In $\mathcal{E}(\mathbf{R}_+)$, D is a continuous and everywhere defined function. However, in $\mathcal{C}(\mathbf{R}_+)$, it is not everywhere defined, since not every continuous function is differentiable. Nonetheless, it is closed and densely defined on any $\mathbf{R}_+$-ring.

Given an $\mathbf{R}_+$-ring R, an *(R)-module* is a pair (E, b) in which E is a locally convex space and $b : R \times E \to E$ is a bilinear mapping which is c-hypocontinuous and which satisfies the usual axioms for a module. Given such an (R)-module, the *associated semigroup* $\mathbf{T}_b : \mathbf{R}_+ \to \mathbf{L}(E)$ is defined by $t \mapsto (e \mapsto b(z^t, e))$. A semigroup realized in this fashion is termed an *(R)-semigroup.*

THEOREM 2.2 *Let $R = \mathcal{F}'(\mathbf{R}_+)$ be an $\mathbf{R}_+$-ring, and let (E, b) be an (R)-module. Then $\mathbf{T}_b$ is a semigroup, and for any $e \in E$, the function $t \mapsto \mathbf{T}_b(t)e$ is in $\mathcal{F}(\mathbf{R}_+, E)$.* □

Thus, for *any* $\mathbf{R}_+$-ring R, each (R)-module defines a semigroup in a natural way. This is effectively the continuous-time analog of one direction of 1.1. To get the other direction, for a semigroup $T : \mathbf{R}_+ \to \mathbf{L}(E)$, define the $\mathbf{K}\langle z\rangle$-module $\mathbf{b}_T$ as follows.

$$\begin{array}{rcl} \mathbf{b}_T : \mathbf{K}\langle z\rangle \times E & \to & E \\ (\sum a_t z^t, e) & \mapsto & \sum a_t \cdot T(t)e \end{array} \tag{27}$$

THEOREM 2.3 *Let $R = \mathcal{F}'(\mathbf{R}_+)$ be an $\mathbf{R}_+$-ring, and let T be a semigroup on E. Then $\mathbf{b}_T$ extends uniquely to an (R)-module action $\mathbf{b}_{T,R} : R \times E \to E$ if and only if the function $t \mapsto T(t)e$ is in $\mathcal{F}(\mathbf{R}_+, E)$ for each $e \in E$.* □

We are now in a position to assert the full continuous-time counterpart to 1.1.

LEMMA 2.4 (STRUCTURAL LEMMA FOR CONTINUOUS DYNAMICS) *There is a natural bijective correspondence between (R)-semigroups and (R)-modules. With the (R)-semigroup (Q, f) we associate the (R)-module whose action on Q is given by $(\sum a_t z^t)q = \sum a_t \mathbf{T}_f(t)q$, extended by continuity using the denseness of $\mathbf{K}\langle z \rangle$ in R.* □

A smooth dynamics (Q, f) in which f is the infinitesimal generator of an (R)-semigroup is called an *(R)-dynamics*. The fact that we are working relative to an $\mathbf{R}_+$-ring automatically guarantees that f will be densely defined and will determine a unique semigroup. There is no need to explicitly require Q to be sequentially complete.

The reachability map in continuous time

We fix an $\mathbf{R}_+$-ring R. A free (R)-dynamics is defined in exact analogy to the discrete-time case; namely, given a locally convex space I, a *free (R)-dynamics* over I is a pair $((I^\$, d), \eta_I)$, with $(I^\$, d)$ an (R)-dynamics and $\eta_I : I \to I^\$$ a continuous linear mapping, such that for any other (R)-dynamics (Q, f) and continuous-linear mapping $g : I \to Q$, there is a unique dynamorphism ρ : $(I^\$, d) \to (Q, f)$ rendering the following diagram commutative.

$$\begin{tikzcd}
I \arrow[r, "\eta_I"] \arrow[dr, "g"'] & I^\$ \arrow[r, "d"] \arrow[d, "\rho"] & I^\$ \arrow[d, "\rho"] \\
& Q \arrow[r, "f"] & Q
\end{tikzcd} \tag{28}$$

If $M = (Q, f, I, g, Y, h)$ is a continuous-time linear system with (Q, f) an (R)-dynamics, the function ρ is called the *reachability map* of M, and is denoted ρ_M, in exact analogy to the discrete-time case. Note that ρ_M depends upon R, although our notation does not make this dependency explicit.

We construct the free (R)-dynamics using the module representation. Given two locally convex spaces E and F, the *c-hypocontinuous tensor product* $E \otimes_c F$ has as underlying space the algebraic tensor product $E \otimes F$, and carries the strongest locally convex topology which renders the canonical bilinear mapping $E \times F \to E \otimes F$ defined by $(e, f) \mapsto e \otimes f$ c-hypocontinuous.

THEOREM 2.5 (EXISTENCE OF FREE DYNAMICS) *Let I be a locally convex space. The free dynamics $((I^\$, d), \eta_I)$ over I exists, and is given by $I^\$ = R \otimes_c I$, with*

$$\begin{aligned} d : R \otimes_c I &\to R \otimes_c I \\ \mu \otimes i &\mapsto D(\mu) \otimes i. \end{aligned} \tag{29}$$

If $M = (Q, f, I, g, Y, h)$ is a continuous time-linear system with (Q, f) an (R)-dynamics, then the resulting dynamorphism $\rho_M : (I^\$, d) \to (Q, f)$ is defined on *the subspace* $\mathbf{K}\langle z\rangle \otimes I \subseteq R \otimes I$ *by*

$$\begin{array}{rcl} \rho_M : \mathbf{K}\langle z\rangle \otimes I & \to & Q \\ \sum (a_t z^t \otimes i_t) & \mapsto & \sum a_t \cdot T_f(t) i_t, \end{array} \tag{30}$$

and is extended to all of $R \otimes I$ *by the density of* $\mathbf{K}\langle z\rangle$ *in* R. □

The (R)-module action of the free (R)-dynamics is completely analogous to that of equations (10). We simply replace $\mathbf{K}[z]$ by R to get

$$\begin{array}{rcl} R \times (R \otimes I) & \to & R \otimes I \\ (\alpha,\ \beta \otimes i) & \mapsto & (\alpha * \beta) \otimes i\ , \end{array} \tag{31}$$

Note that the definition of ρ_M on $\mathbf{K}\langle z\rangle$ is independent of the choice of R, and closely parallels the discrete-time case. It is the extension to R, using the topological density of $\mathbf{K}\langle z\rangle$ in R, which admits the smooth inputs which makes the continuous-time case so much richer. To elaborate, let $M = (Q, f, I, g, Y, h)$ be a continuous-time linear system with (Q, f) an (R)-dynamics. Let us first consider inputs in $\mathbf{K}\langle z\rangle \otimes I$. Regard the element $z^t \otimes i$ as an impulse input applied to the system at time $-t$ with weight i. (Note the time reversal.) The response $\rho_M(z^t \otimes i)$ is the resulting state at time 0. Thus, $\mathbf{K}\langle z\rangle \otimes I$ is regarded as an input space of finite linear combinations of I-valued impulses occurring at times ≤ 0. The reachability map ρ_M gives the response at time 0 to such a train of impulses. That is, $\rho_M(\sum z^{t_k} \otimes i_k) = \sum \mathbf{b}_{\mathbf{T}_f, R}(z^{t_k}, g(i_k)) = \sum \mathbf{T}_f(t_k) g(i_k)$. Note the direct analogy to the discrete-time case, in which an input may be regarded to be of the form $\sum z^k \otimes i_k$ and the response to be $\sum \mathbf{T}_f(k) g(i_k) = \sum f^k g(i_k)$, where $f = \mathbf{T}_f(1)$. The only difference, other than the topological considerations required for differentiation, is that the set of times at which an input is allowed to occur is the nonpositive reals $\mathbf{R}_-$ in the continuous-time case (recall the time reversal) and the nonpositive integers $\mathbf{N}_-$ in the discrete-time case, and that f assumes the rôle of a continuous rather than discrete generator. Thus, if we first examine the input signals and reachability map of a continuous-time linear system in terms of its skeleton input set $\mathbf{K}\langle z\rangle \otimes I$, we see that it is not all that different from its discrete-time counterpart. What does make continuous-time linear systems richer than their discrete-time counterparts is the ability to complete $\mathbf{K}\langle z\rangle$ and $\mathbf{K}\langle z\rangle \otimes I$ to obtain a much more diversified set of inputs, tailored to the specific situation.

Let $r \otimes i \in R \otimes_c I$. If $r = z^t$ for some t, we already know that $\rho_M(r \otimes i) = \mathbf{T}_f(t) g(i)$. If $r \in R$ more generally, we can approximate r as closely as desired by a sum of the form $\sum_{j=1}^n a_j \cdot z^{t_j}$ with $a_j \in \mathbf{K}$, since $\mathbf{K}\langle z\rangle$ is dense in R. $(\sum_{j=1}^n a_j \cdot z^{t_j}) \otimes i$ then approximates $r \otimes i$. Since ρ_M is continuous, $\rho_M(r \otimes i)$ is approximated by $\sum_{j=1}^n a_j \cdot \mathbf{T}_f(t_j) g(i)$.

Now suppose that r is represented by a function $\varphi_r : \mathbf{R}_+ \to \mathbf{K}$. Then, using the density of $\mathbf{K}\langle z\rangle$ in R, there must be a net of sums $\sum_{j=1}^{n_\alpha} a_j \cdot \mathbf{T}_f(t_j) g(i)$ which converges to the Q-valued integral $\int_0^\infty \varphi_r(t) \mathbf{T}_f(t) g(i) dt$. Regarding $\varphi_r(t)$ as an input signal to the system M, the time scale is reversed, so if we view $\varphi : \mathbf{R}_- \to \mathbf{K}$, the state at time $t = 0$ is $q(0) = \int_{-\infty}^0 \mathbf{T}_f(-t)(\varphi_r(t) \cdot g(i)) dt$, which is in agreement with (21). The extension to a sum of the form $\sum_{k=1}^{n} r_k \otimes i_k$ of such inputs is by simple linearity. The inputs in the most general case may be regarded as vector-valued distributions in the sense of Schwartz [41].

The observability map in continuous time

The construction of the observability map in continuous time is no more difficult than in discrete time. With respect to an $\mathbf{R}_+$-ring R, given a locally convex space Y, a *cofree* (R)*-dynamics* over Y is a pair $((Y_\$, d), \varepsilon_Y)$, with $(Y_\$, d)$ a dynamics and $\varepsilon_Y : Y_\$ \to Y$ a continuous linear mapping such that for any other (R)-dynamics (Q, f) and continuous linear mapping $h : I \to Q$, there is a unique dynamorphism $\sigma : (Q, f) \to (Y_\$, \varepsilon_Y)$ such that the following diagram commutes.

$$\begin{tikzcd} Q \arrow[r, "f"] \arrow[d, "\sigma"'] & Q \arrow[d, "\sigma"] \arrow[dr, "h"] & \\ Y_\$ \arrow[r, "d"] & Y_\$ \arrow[r, "\varepsilon_Y"] & Y \end{tikzcd} \tag{32}$$

If $M = (Q, f, I, g, Y, h)$ is a discrete time-linear system, then the resulting dynamorphism $\sigma : (Q, F) \to (Y_\$, d)$ is exactly what we define to be the *observability map* σ_M of M.

THEOREM 2.6 (EXISTENCE OF COFREE DYNAMICS) *Let* $R = \mathcal{F}'(\mathbf{R}_+)$ *be an* $\mathbf{R}_+$*-ring, and let* Y *be a locally convex space. The cofree dynamics* $((Y_\$, d), \varepsilon_Y)$ *over* Y *exists, and is given by* $Y_\$ = \mathcal{F}(\mathbf{R}_+, Y)$. *The mapping* $D : \mathcal{F}(\mathbf{R}_+, Y) \to \mathcal{F}(\mathbf{R}_+, Y)$ *is the differentiation operator* D, *which may be a partial function. If* $M = (Q, f, I, g, Y, h)$ *is a continuous-time linear system with* (Q, f) *an* (R)*-dynamics, then the resulting dynamorphism* $\sigma_M : (Q, f) \to (Y_\$, d)$ *is defined by*

$$\begin{array}{rcl} \sigma_M : Q & \to & \mathcal{F}(\mathbf{R}_+, Y) \\ q & \mapsto & (t \mapsto h(\mathbf{T}_f(t)q)) \end{array} \tag{33}$$

□

Thus, the observability maps just reads out the "natural response" of the semigroup of the dynamics (Q, f), after the output function h has been applied. σ_M depends upon R only to the extent of defining its range.

In analogy to (15), there is an natural isomorphism of locally convex spaces

$$\begin{array}{rcl} \mathbf{L}_c(\mathcal{F}'(\mathbf{R}_+), Y) & \cong & \mathcal{F}(\mathbf{R}_+, Y) \\ \varphi & \mapsto & (t \mapsto \varphi(z^t)). \end{array} \tag{34}$$

Here $\mathbf{L}_c(\mathcal{F}'(\mathbf{R}_+), Y)$ denotes the space of all continuous linear maps from $\mathcal{F}'(\mathbf{R}_+)$ into Y, with the topology of uniform convergence on precompact subsets of $\mathcal{F}'(\mathbf{R}_+)$. The (R)-module action in this context translates to

$$\begin{array}{rcl} R \times \mathbf{L}_c(R, Y) & \to & \mathbf{L}_c(R, Y) \\ (\alpha, \varphi) & \mapsto & (\beta \mapsto \varphi(\alpha * \beta)) \end{array} \tag{35}$$

Other contexts

We remarked earlier that there are a great many possible choices for the $\mathbf{R}_+$-ring R. It is impossible to give a full account of this here, but the following table gives some idea of the possibilities.

Type of Dynamics	Inputs in $I^\$$	Outputs in $Y_\$$
locally equicontinuous semigroups	measures with compact support	continuous functions
infinitely differentiable semigroups	distributions with compact support	C^∞ functions
equicontinuous semigroups	uniform bounded additive measures	bounded and uniformly continuous functions
bounded semigroups	L^1 measures	bounded continuous functions
stable semigroups	L^1 measures	continuous functions which vanish at ∞
finite-response semigroups	Radon measures	continuous functions with scalar-compact support

Note that there is a fundamental measure/test-function duality between $I^\$$ and $Y_\$$, so that, in general, the more measures in $I^\$$, the fewer functions in $Y_\$$. There is no single choice of topological ring which provides a biggest version of each. Rather, the choice is a modelling problem; one of selecting the best framework for the systems being considered. Note also that bounded semigroups and stable semigroups take the same space of inputs. The distinction comes in that this space is topologized differently in each case, so that the underlying *topological* ring is not the same. In all cases, though, the inputs are allowed to be not only continuous functions, but also continuous measures. Therefore, the restriction of "sufficiently smooth" identified in 2.1 is really not much of a restriction at all. Of course, to interpret equation (21) pointwise rather than operationally, the input must be an ordinary function with certain constraints.

It is also possible to work within the context of locally convex spaces which possess a certain degree of completion (such as complete or quasi-complete spaces), rather than with all locally convex spaces. Basically, we just apply the appropriate completion operator to all constructions. We refer the reader to the complete papers for details.

Algebraic connection of behavior and realization

In the discrete-time case, we said that a system M was reachable if its reachability map ρ_M is surjective, and observable it its observability map σ_M is injective. This definition is not adequate in the continuous-time case, since (surjection,injection) factorizations of continuous linear mappings are not unique up to isomorphism. To get uniqueness, we must take the topological aspects into account. Categorically speaking, we must work with an *image-factorization system* (or $(\mathcal{E},\mathcal{M})$-*system* [25, §33]) for the category of locally convex spaces.[7]

[7] Actually, such factorizations are necessary in discrete time as well if one works with topologized systems. See [20] for details.

The three most fundamental of such systems are:

(i) (topologically dense mappings, closed topological embeddings)

(ii) (surjections, topological embeddings)

(iii) (closed surjections, injections)

Since the dynamorphic image lemma [3, 4.4] is formulated abstractly in terms of image-factorization systems, any continuous linear mapping between locally convex spaces has a unique (up to isomorphism) factorization in any of these systems. Thus, within the context of a particular image-factorization system, 1.4 extends directly to the continuous-time context, provided we speak of $(\mathcal{E}, \mathcal{M})$-canonical systems. We then have the following continuous-time analog of 1.5.

THEOREM 2.7 (SYSTEM REPRESENTATION FOR CONTINUOUS TIME) *Given an $\mathbf{R}_+$-ring $R = \mathcal{F}'(\mathbf{R}_+)$, an image factorization system $(\mathcal{E}, \mathcal{M})$, and locally convex spaces I, and Y, there is a natural bijective correspondence between behaviors $B : (\mathcal{F}'(\mathbf{R}_+) \otimes_c I, d) \to (\mathcal{F}(\mathbf{R}_+, Y), d)$ and isomorphic equivalence classes of $(\mathcal{E}, \mathcal{M})$-canonical systems with input space I and output space Y.* □

Remarks on the literature

In addition to our own work, there was substantial earlier work on the algebraic theory of continuous-time linear systems using a module-based approach. In an early paper, Kalman and Hautus [31], a theory with $R = \mathcal{E}(\mathbf{R}_+)$ and I and Y finite dimensional is presented. However, the constructions were purely algebraic, with no attention paid to the topological aspects, so that a reconstruction of the infinitesimal dynamics was not possible. Also, Kamen developed an early theory based upon the module $\mathcal{E}(\mathbf{R}_+)$ [32], and later upon the space $\mathcal{D}(\mathbf{R}_+)$ of distributions which do not necessarily have compact support [33], but he too did not address the topological issues, and so was unable to recover the infinitesimal dynamics.

In later work, Yamamoto [45, 46, 44] developed a theory of continuous-time linear systems which did construct infinitesimal dynamics from behaviors. However, because his work emphasized other aspects of continuous-time linear systems, his definition of behavior did not involve universal constructions. Thus, he was not able to formulate a natural bijective correspondence between behavior and internal dynamics.

There has also been much other recent work on the algebraic theory of continuous-time linear systems, particularly over Hilbert and Banach spaces. In [15], Curtain provides a survey of the major results along these lines. However, this work does does not provide a *universal* approach to the representation of behavior as we have described in our paper. Indeed, if we require that all of the spaces with which we work be Hilbert spaces (or even Banach spaces), then it is possible to show that free and cofree dynamics do not always exist. It is for this reason that the direction surveyed in [15] must take a fundamentally different perspective on the problems of system representation. Nonetheless, there are important connections between the two approaches, which are unfortunately beyond the scope of this survey.

The theory of continuous-time linear systems is an incredibly rich one, and we have only mentioned a few of the most relevant papers. The reader interested in further information should use the bibliographies of these references, as well as our own papers, as a guide. In addition, we should point out that there are many aspects of the categorical theory of discrete-time linear systems which we have not touched upon. In particular, the papers [7] and [8] contain very elegant generalizations of key ideas.

REFERENCES

1. M. A. ARBIB, A common framework for automata theory and control theory, *SIAM J. Control Optim.*, **3**(1965), 206–222.
2. M. A. ARBIB, Coproducts and decomposable machines, *J. Comput. System Sci.*, **7**(1973), 278–287.
3. M. A. ARBIB AND E. G. MANES, Foundations of system theory: decomposable systems, *Automatica – J. IFAC*, **10**(1974), 285–302.
4. M. A. ARBIB AND E. G. MANES, Adjoint machines, state-behavior machines, and duality, *J. Pure Appl. Algebra*, **6**(1975), 313–344.
5. M. A. ARBIB AND E. G. MANES, *Arrows, Structures, and Functors*, Academic Press, 1975.
6. M. A. ARBIB AND E. G. MANES, Fuzzy machines in a category, *Bull. Austral. Math. Soc.*, **13**(1975), 169–210.
7. M. A. ARBIB AND E. G. MANES, Intertwined recursion, tree transformations, and linear systems, *Inform. and Control*, **40**(1979), 144–180.
8. M. A. ARBIB AND E. G. MANES, Foundations of system theory: the Hankel matrix, *J. Comput. System Sci.*, **20**(1980), 330–378.
9. M. A. ARBIB AND E. G. MANES, Machines in a category, *J. Pure Appl. Algebra*, **19**(1980), 9–20.
10. E. S. BAINBRIDGE, A unified minimal realization theory with duality, Technical Report 140, Logic of Computers Group, Department of Computer and Communication Sciences. University of Michigan, 1972. Also Ph.D. dissertation.
11. A. V. BALAKRISHNAN, *Applied Functional Analysis*, Springer-Verlag, 1976.
12. N. BOURBAKI, *Eléments de Mathématique, Intégration*, Hermann, 1965.
13. R. G. BROWN, R. A. SHARPE, AND W. L. HUGHES, *Lines, Waves, and Antennas*, Ronald Press, 1961.
14. H. CARTAN AND S. EILENBERG, *Homological Algebra*, Princeton University Press, 1956.
15. R. F. CURTAIN, Representations of infinite-dimensional linear systems, in *Three Decades of Mathematical System Theory* (J. M. Schumacher and H. Nijmeijer, eds.), pp. 101–128, Springer-Verlag, 1989.
16. H. EHRIG, K. D. KIERMEIER, H. KREOWSKI, AND W. KÜHNEL, *Universal Theory of Automata*, Teubner, 1974.
17. H. EHRIG AND H. KREOWSKI, Systematic approach to reduction and minimization in automata and system theory, *J. Comput. System Sci.*,

12(1976), 269–303.

18. J. A. GOGUEN, Discrete-time machines in closed monoidal categories. I, *J. Comput. System Sci.*, **10**(1975), 1–43.
19. M. A. HARRISON, *Introduction to Formal Language Theory*, Addison-Wesley, 1978.
20. S. J. HEGNER, Duality theory for discrete-time linear systems, *J. Comput. System Sci.*, **17**(1978), 116–143.
21. S. J. HEGNER, An algebraic approach to continuous-time linear systems defined over Banach spaces, in *Proceedings of the Fourteenth Annual Conference on Information Sciences and Systems*, pp. 347–352, 1980.
22. S. J. HEGNER, Linear decomposable systems in continuous time, *SIAM J. Math. Anal.*, **12**(1981), 243–273.
23. S. J. HEGNER, Algebraic representation of dynamics and behavior for continuous-time linear systems, *Math. Systems Theory*, **22**(1989), 227–254.
24. S. J. HEGNER, Algebraic representation of classes of (C_o) semigroups and the behavior of inhomogeneous linear differential equations, 1991, manuscript in preparation.
25. H. HERRLICH AND G. E. STRECKER, *Category Theory*, Allyn and Bacon, 1973.
26. E. HILLE AND R. S. PHILLIPS, *Functional Analysis and Semigroups*, American Mathematical Society, 1957.
27. N. JACOBSON, *Basic Algebra I*, W. H. Freeman, second edition, 1985.
28. R. E. KALMAN, Algebraic structure of linear dynamical systems. I. the module of Σ, *Proc. Nat. Acad. Sci. U.S.A.*, **54**(1965), 1503–1508.
29. R. E. KALMAN, Advanced theory of linear systems, in *Topics in Mathematical System Theory* (R. E. Kalman, P. L. Falb, and M. A. Arbib, eds.), pp. 235–339, McGraw-Hill, New York, 1969.
30. R. E. KALMAN, P. L. FALB, AND M. A. ARBIB, *Topics in Mathematical System Theory*, McGraw-Hill, 1969.
31. R. E. KALMAN AND M. L. J. HAUTUS, Realization of continuous-time linear dynamical systems: rigorous theory in the style of Schwartz, in *Ordinary Differential Equations, 1971 NRL-MRC Conference* (L. Weiss, ed.), pp. 151–164, Academic Press, 1972.
32. E. W. KAMEN, A distributional-module theoretic representation of linear dynamical continuous-time systems, Technical Report 6650-24, Information Systems Laboratory, Stanford Electronic Laboratories, 1971. Also Ph.D. dissertation.
33. E. W. KAMEN, Module structure of infinite-dimensional systems with applications to controllablity, *SIAM J. Control Optim.*, **14**(1976), 389–408.
34. T. KŌMURA, Semigroups of operators in locally convex spaces, *J. Funct. Anal.*, **2**(1968), 248–296.
35. G. KÖTHE, *Topological Vector Spaces I*, Springer-Verlag, 1969.
36. G. KÖTHE, *Topological Vector Spaces II*, Springer-Verlag, 1979.
37. S. MAC LANE, *Categories for the Working Mathematician*, Springer-Verlag, 1971.

38. L. Padulo and M. A. Arbib, *System Theory: A Unified Approach to Continuous and Discrete Systems*, Hemisphere, 1974.
39. W. Rudin, *Functional Analysis*, McGraw-Hill, 1973.
40. L. Schwartz, Espaces de fonctions différentiables à valeurs vectorielles, *J. Analyse. Math.*, **4**(1954–55), 88–148.
41. L. Schwartz, Théorie des distributions à valeurs vectorielles, Chap. 1, *Ann. Inst. Fourier (Grenoble)*, **7**(1957), 1–141.
42. L. Schwartz, *Théorie des Distributions*, troisième édition, Hermann, 1966.
43. F. Treves, *Topological Vector Spaces, Distributions, and Kernels*, Academic Press, 1967.
44. Y. Yamamoto, Module structure of constant linear systems and its application to controllablity, *J. Math. Anal. Appl.*, **83**(1981), 411–437.
45. Y. Yamamoto, Realization theory of infinite-dimensional linear systems, part I, *Math. Systems Theory*, **15**(1981), 55–77.
46. Y. Yamamoto, Realization theory of infinite-dimensional linear systems, part II, *Math. Systems Theory*, **15**(1981), 169–190.
47. L. A. Zadeh and C. A. Desoer, *Linear System Theory*, McGraw-Hill, 1963.