# Introductory Recommendations for the Study of Hopf Algebras in Mathematics and Physics

Michiel Hazewinkel *CWI*
*P.O. Box 4079, 1009 AB Amsterdam, The Netherlands*

In this article I discuss Hopf algebras and their occurrence in various parts of mathematics and theoretical physics. The main point of view is that Hopf algebras provide the setting for a more general idea of symmetry than that afforded by group actions. This is illustrated in a number of settings: $q$-special functions, quantum inverse scattering, representation theory.

## 1 Preamble

The following few pages are based on lectures I have given in 1988 and 1989 at various places, notably at the State University of Utrecht, at CWI in Amsterdam, at the University of Amsterdam, in Bukhta Peschanya on the shores of lake Baikal on the occasion of the 3rd Siberian School on Algebra and Analysis.

All of these lectures were introductory and addressed to rather mixed audiences desirous of finding out about Hopf algebras and quantum groups and what they are good for (if anything).

The present paper can be regarded as a very incomplete synopsis of these various lectures which concentrates on the motivational part. Hopefully it is a precursor of a much fuller treatment of the material.

A great deal of the modern motivation for studying Hopf algebras involves some idea of generalized symmetry. This takes many forms and includes such matters as the quantum inverse scattering method, the theory of $q$-special functions and the quantization of classical completely integrable systems. This is the subject matter of Section 3 below. Historically, generalized symmetry is also behind the appearance of Hopf algebras. This time in the form that the cohomology algebra (and the homology coalgebra) of a space with some sort of group structure is particularly nice (Section 4.1). Further motivation for the study of Hopf algebras comes from representation theory, particularly the representation theory of the symmetric groups (Section 4.2) and from combinatorics (Section 5).

## 2 COALGEBRAS, BIALGEBRAS AND HOPF ALGEBRAS

### 2.1 *Tensor products*

Let $V$ be a vector space (over a field $K$), with basis $e_1, e_2, \ldots$ Think of $K$ as $\mathbb{R}$ or $\mathbb{C}$. A good way to think of $V$ in connection with the remainder of this section is as follows. The $e_1, e_2, \ldots$ represent the pure states of some machine or (physical) system and the elements of $V$, i.e. finite sums $\Sigma x_i e_i$, are mixed states or distributions over the pure states. Such an interpretation occurs frequently in many domains of science; for instance quantum mechanics, game theory and computer science.

Now consider two machines (for instance) represented by the vector spaces $V, W$ with bases $e_1, e_2, \ldots$ and $f_1, f_2, \ldots$ respectively. The pure states of the combined machine are the ordered pairs $(e_i, f_j)$ and the corresponding vector space of mixed states consists of the finite sums $\Sigma c_{ij}(e_i, f_j)$. This vector space is the tensor product $V \otimes W$ of $V$ and $W$. The elements of the basis $\{(e_i, f_j)\}_{i,j}$ are conventionally written $e_i \otimes f_j$.

In all of the following $A$ will always be assumed commutative. Then the tensor product is associative in the sense that there is a natural isomorphism $(E \otimes_A F) \otimes_A G \simeq E \otimes_A (F \otimes_A G)$. It is also commutative: $\tau : e \otimes f \mapsto f \otimes e$ defines an isomorphism $E \otimes_A F \simeq F \otimes_A E$. The switch morphism $\tau$ and the associativity morphism satisfy all possible coherence conditions.

If $\alpha : E \to E'$ and $\beta : F \to F'$ are homomorphisms of $A$-modules then

$$(\alpha \otimes \beta)(e \otimes f) = \alpha(e) \otimes \beta(f)$$

defines a homomorphism of $A$-modules $E \otimes_A F \to E' \otimes_A F'$.

### 2.2 *Algebras*

An algebra $A$ over $K$ is a vector space over $K$ with a composition structure or multiplication, i.e., a $K$-bilinear map $A \times A \to A$. This is equivalent to giving a $K$-linear map $m : A \otimes A \to A$. A unit element is an element $1 \in A$ such that $1 \cdot a = a \cdot 1 = a$ for all $a \in A$. Specifying such an element $U$ in $A$ is equivalent to giving a $K$-linear map $K \to A$ (defined by $r \mapsto rU$, $r \in K$). The fact that $e : K \to A$ corresponds to a unit element can be expressed by the commutativity of the diagrams

$$(2.2.1) \qquad \begin{array}{ccc} A \otimes A & \xrightarrow{m} & A \\ \uparrow {\scriptstyle e \otimes id_A} & & \| \\ K \otimes A & \xrightarrow{\simeq} & A \end{array} \qquad\qquad \begin{array}{ccc} A \otimes A & \xrightarrow{m} & A \\ \uparrow {\scriptstyle id_A \otimes e} & & \| \\ A \otimes K & \xrightarrow{\simeq} & A \end{array}$$

where the lower arrows are the canonical isomorphisms $K \otimes A \to A$, $x \otimes a \mapsto xa$, and $A \otimes K \to A$. Associativity of the composition law $m : A \otimes A \to A$ is expressed by the commutativity of the diagram

$$(2.2.2) \qquad \begin{array}{ccc} A \otimes A \otimes A & \xrightarrow{m \otimes id_A} & A \otimes A \\ \downarrow {\scriptstyle id_A \otimes m} & & \downarrow {\scriptstyle m} \\ A \otimes A & \xrightarrow{m} & A \end{array}$$

If $\tau : A \otimes A \longrightarrow A \otimes A$ is the switch map determined by $a_1 \otimes a_2 \mapsto a_2 \otimes a_1$, then commutativity of $m : A \otimes A \longrightarrow A$ means that the following diagram is commutative.

(2.2.3)
$$\begin{array}{ccc} A \otimes A & \xrightarrow{\tau} & A \otimes A \\ & \searrow_m \quad \swarrow_m & \\ & A & \end{array}$$

It is useful to formulate these properties in terms of diagrams (categorical terms) (without elements) in order to see what the dual structures should be (coalgebras, the subject of the next subsection) and to see what algebras would be like in more general settings (categories) than vector spaces or modules.

Here are some examples of algebras.

*2.2.4 The matrix algebra*
Let $M_n(k)$ be the $n^2$ dimensional vector space with basis vectors $e_{ij} i, j = 1, \ldots, n$. An associative multiplication is given by

$$e_{ij} \otimes e_{kl} \mapsto \delta_{jk} e_{il}$$

where $\delta_{jk}$ is the Kronecker delta. There is a unit element, viz. $e_{11} + \cdots + e_{nn}$.

*2.2.5 The concatenation algebra*
Let $\Omega$ be some set (alphabet). Let $\Omega^*$ denote the set of all finite words over $\Omega$, i.e. $\Omega^*$ denotes the set of all finite strings (including the empty string $\phi$) $a_1 a_2 \cdots a_m$, $a_i \in \Omega$. Let $A(\Omega)$ be the vector space of all finite sums $\Sigma x_\omega \omega$, $\omega \in \Omega^*$, $x_\omega \in K$. An associative multiplication is given by

$$a_1 a_2 \cdots a_n \otimes b_1 b_2 \cdots b_m \mapsto a_1 a_2 \cdots a_n b_1 b_2 \cdots b_m.$$

The unit element is the basis vector corresponding to the empty word.

*2.2.6 The merge algebra*
There is a second, somewhat less familiar, algebra structure on $A(\Omega)$ known as 'merge' or 'shuffle'. Given two words $\omega_1 = a_1 \cdots a_n$, $\omega_2 = a_{n+1} \cdots a_{n+m}$. A merge or shuffle of $\omega_1$ and $\omega_2$ is a word of length $n+m$

$$\omega_3 = c_1 c_2 \cdots c_{n+m}$$

such that each of the $a_i, 1 \leq i \leq n$ and the $a_j$, $n+1 \leq j \leq n+m$ appears once in $\omega_3$ and such that within $\omega_3$ the $a_i, 1 \leq i \leq n$ occur in their original sequence and so do the $a_j, n+1 \leq j \leq n+m$. More formally let $Sh(n,m)$ be the set of all permutations $\sigma$ of $\{1, 2, \ldots, n+m\}$ such that $\sigma^{-1}(i) < \sigma^{-1}(j)$ if $1 \leq i < j \leq n$ and $\sigma^{-1}(n+k) < \sigma^{-1}(n+l)$ if $1 \leq k < l \leq m$. The shuffles of $\omega_1$ and $\omega_2$ are precisely the words

$$a_{\sigma(1)} a_{\sigma(2)} \cdots a_{\sigma(n+m)} = \sigma(a_1 \cdots a_{n+m})$$

for $\sigma \in Sh(n,m)$. The concept corresponds to the familiar rifle-shuffle in card playing. The merge algebra now has the multiplication given by

$$\omega_1 \otimes \omega_2 \mapsto \sum_{\sigma \in Sh(n,m)} \sigma(\omega_1 \omega_2)$$

if $\omega_1$ is a word of length $n$ and $\omega_2$ is a word of length $m$. The merges of the words $aba$ and $bc$ are

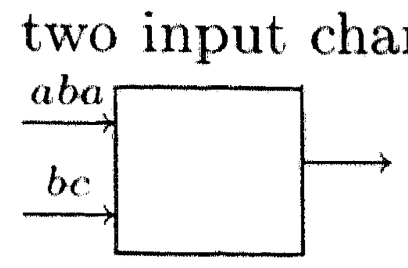
$$aba\underline{bc},\ ab\underline{b}a\underline{c},\ a\underline{b}ba\underline{c},\ \underline{b}aba\underline{c},\ ab\underline{bc}a,$$

$$a\underline{b}b\underline{c}a,\ \underline{b}ab\underline{c}a,\ a\underline{bc}ba,\ \underline{b}a\underline{c}ba,\ \underline{bc}aba$$

where I have underlined the letters coming from the second word. Note that there are repeats. Thus the merge product of $aba$ and $bc$ is equal to

$$ababc + 2abbac + babac + 2abbca + babca + abcba + bacba + bcaba.$$

To understand how this algebra structure might arise naturally, imagine a 'black box' with two input channels and one output channel which simply passes on whatever comes in on one of the two input channels. Depending on the timing of the various letters coming in, the result will be one of the merges of the words appearing at the two input channels and precisely all merges occur.




The merge multiplication is both commutative and associative. There is also a unit element given by the empty word.

### *2.3 Coalgebras*

A coalgebra over $K$ is a vector space $C$ with a decomposition structure, a way of cutting up the elements of $C$. More precisely a coalgebra structure is given by a $K$-linear map

$$\mu : C \longrightarrow C \otimes C.$$

A good way to think about $\mu(c) = \Sigma c_i' \otimes c_i''$ is to regard the right-hand side as giving all ways of cutting up $c$ into ordered pairs $(c_i', c_i'')$. A counit is a linear map $\epsilon : C \longrightarrow K$ such that the following two diagrams are commutative.

(2.3.1)
$$\begin{array}{ccc} C\otimes C & \overset{\mu}{\leftarrow} & C \\ \downarrow{\scriptstyle \epsilon\otimes id_C} & & \| \\ K\otimes C & \underset{\sim}{\quad} & C \end{array} \qquad\qquad \begin{array}{ccc} C\otimes C & \overset{\mu}{\leftarrow} & C \\ \downarrow{\scriptstyle id_C\otimes\epsilon} & & \| \\ C\otimes K & \underset{\sim}{\quad} & C \end{array}$$

The comultiplication is coassociative if the following diagram commutes

(2.3.2)
$$\begin{array}{ccc} C\otimes C\otimes C & \overset{\mu\otimes id_C}{\longleftarrow} & C\otimes C \\ \uparrow{\scriptstyle id_C\otimes\mu} & & \uparrow{\scriptstyle\mu} \\ C\otimes C & \overset{\mu}{\leftarrow} & C \end{array}$$

and the comultiplication $\mu$ is cocommutative if the diagram

(2.3.3)
$$\begin{array}{ccc} C\otimes C & \overset{T}{\rightarrow} & C\otimes C \\ \nwarrow{\scriptstyle\mu} & & \nearrow{\scriptstyle\mu} \\ & C & \end{array}$$

commutes. Associativity corresponds to the following idea. Consider breaking up an element into three pieces. There are two two-step ways of doing this.

First break up the object $c$ into two pieces $c'$ and $c''$ and then break up either the first part or the second part into two pieces. It is natural to require that the collection of three part decompositions obtained by going the first route is the same as the collection which arises by using the second one. This is coassociativity. Note the 'duality' of diagrams (2.3.1) - (2.3.3) with respect to diagrams (2.2.1)-(2.2.3).

Some examples of coalgebras follow.

*2.3.4 The matrix coalgebra*

Consider again the vector space $M_n(K)$ of $n \times n$ matrices with coefficients in $K$ with the basis $e_{i,j}, i,j = 1,\dots,n$. A comultiplication is defined on $M_n(K)$ by

$$\mu(e_{ij}) = \sum_k e_{ik} \otimes e_{kj}.$$

This is coassociative. There is a counit given by $\epsilon(e_{ii}) = 1$, $\epsilon(e_{ij}) = 0$ if $i \neq j$.

*2.3.5 The cut coalgebra*

Consider again the vector space $A(\Omega)$ of all distributions of words on the alphabeth $\Omega$. There is a comultiplication on $A(\Omega)$ which cuts up each word into all possible prefix-suffix pairs:

$$\mu(a_1 \cdots a_n) = 1 \otimes a_1 \cdots a_n + \sum_{i=1}^{n-1} a_1 \cdots a_i \otimes a_{i+1} \cdots a_n + a_1 \cdots a_n \otimes 1$$

where I have written 1 for (the basis vector corresponding to) the empty word. There is a counit $\epsilon : A(\Omega) \to K$, given by $\epsilon(\omega) = 0$ if $\omega$ is of length $\geq 1$ and $\epsilon(1) = 1$.

*2.3.6 The group coalgebra*

Let $G$ be a group. For the moment and for simplicity let $G$ be finite. Consider the vector space $A(G)$ of $K$-valued functions on $G$. The group multiplication $G \times G \to G$, induces a comultiplication $\mu : A(G) \to A(G \times G) = A(G) \otimes A(G)$. A basis of $A(G)$ is given by the delta functions $e_g, g \in G : e_g(g') = 1$ if $g = g'$ and $e_g(g') = 0$ if $g \neq g'$. In terms of this basis

$$\mu(e_g) = \sum_{g_1 g_2 = g} e_{g_1} \otimes e_{g_2}.$$

There is also a counit given by $\epsilon(e_g) = 0$ if $g$ is not the unit element of $G$ and $\epsilon(e_g) = 1$ if $g$ is the unit element of $G$.

If $G$ is an algebraic group (or group scheme) take for $A(G)$ the algebra of algebraic $K$-balued functions on $G$; then again the multiplication on $G$ induces a comultiplication on $A(G)$. If $G$ is a Lie group and one considers for instance the smooth functions Fun $(G)$ on $G$ there are (slight) technical difficulties because $\mathrm{Fun}(G \times G) \not\simeq \mathrm{Fun}(G) \otimes \mathrm{Fun}(G)$. Instead one has to take a suitable completed tensor product.

*2.4 Duality*

Let $C$ be a coalgebra over $K$. Consider $C^D = \mathrm{Hom}_K(C, K)$, the dual vector space of all linear functionals on $C$. There is a natural linear mapping

$$\varphi : C^D \otimes C^D \to (C \otimes C)^D$$

given by $\varphi(f \otimes g)(c_1 \otimes c_2) = f(c_1)g(c_2)$. If $C$ is finite dimensional this is an isomorphism of vector spaces. Now consider the composed map

$$m : C^D \otimes C^D \to (C \otimes C)^D \xrightarrow{\mu^D} C^D.$$

This defines a composition structure on $C^D$ turning $C^D$ into an algebra. If $\epsilon : C \to K$ is a counit then $e = \epsilon^D : K \to C^D$ is a unit. Obviously $C^D$ is associative, resp. commutative, if $C$ is coassociative, resp. cocommutative.

It is somewhat harder to obtain a coalgebra by dualizing an algebra. The reason is that there is no natural mapping $(A \otimes A)^D \to A^D \otimes A^D$, or, that $\varphi$ above is not an isomorphism if $C$ is an infinite dimensional vector space. Instead of $A^D$ consider $A^0 = \{f : A \to K : \mathrm{Ker}(f)$ contains an ideal of finite codimension in $A\}$. One (easily) proves that $m^D : A^D \to (A \otimes A)^D$ actually maps $A^0$ into $A^0 \otimes A^0 \subset (A \otimes A)^D$ and thus $m^D$ restricted to $A^0$ defines a coalgebra structure on $A^0$.

The pair of functors $C \mapsto C^D$ and $A \mapsto A^0$ establishes a duality between the categories of algebras over $K$ and coalgebras over $K$.

The matrix algebra and the matrix coalgebra are dual to each other. The dual of the group coalgebra of 2.3.6 is the group algebra $K[G] = \{\sum_g a_g g : a_g \in K\}$ with the multiplication $m(\sum_g a_g g, \sum_g b_g g) = \sum_g(\sum_h a_{gh^{-1}} b_h)g$ (given by $g \otimes h \mapsto gh$).

*2.5 Bialgebras*

A bialgebra $B$ over $K$ is a vector space with both a coassociative decomposition and an associative composition structure. Moreover these structures are compatible in the sense that $\epsilon : B \to K$ and $\mu : B \to B \otimes B$ are algebra homomorphisms or, equivalently, that $e : K \to B$ and $m : B \otimes B \to B$ are coalgebra homomorphisms. In terms of diagrams this means that the following diagrams commute

(2.5.1)
$$\begin{array}{ccc}
B \otimes B & \xrightarrow{\mu\otimes\mu} & B \otimes B \otimes B \otimes B \\
 & & \downarrow id_B \otimes T \otimes id_B \\
\downarrow m & & B \otimes B \otimes B \otimes B \\
 & & \downarrow m \otimes m \\
B & \xrightarrow{\mu} & B \otimes B
\end{array}$$

(2.5.2)
$$\begin{array}{ccc}
B \otimes B & \xrightarrow{\epsilon\otimes\epsilon} & K \otimes K \\
\downarrow m & & \wr| \\
B & \xrightarrow{\epsilon} & K
\end{array}$$

(2.5.3)
$$\begin{array}{ccc} K & \xrightarrow{e} & B \\ \downarrow \iota & & \downarrow \mu \\ K \otimes K & \xrightarrow{e \otimes e} & B \otimes B \end{array}$$

(2.5.4)
$$\begin{array}{ccc} K & \xrightarrow{e} & B \\ \downarrow \iota & \swarrow \epsilon & \\ K & & \end{array}$$

In words the first diagram says that the result of cutting up a product $a_1a_2$ in all possible ways is the same as that of cutting up the factors $a_1$ and $a_2$ and then combining each 'prefix-suffix pair' $(a_1', a_1'')$ of $a_1$ with each 'prefix-suffix pair' $(a_2', a_2'')$ to form a 'prefix-suffix pair' $(a_1'a_2', a_1''a_2'')$ of $a_1a_2$. The functor $B \mapsto B^0$ defines a duality of the category of bialgebras into itself.

The cut comultiplication and the merge multiplication of 2.3.5 and 2.2.6 together define a bialgebra structure on $A(\Omega)$. This bialgebra is called the merge-cut bialgebra (or shuffle-cut bialgebra); it is of importance in distributed and concurrent computing, cf [3,4]. In a certain sense, not the most obvious categorical one, they are free (there are no relations). It seems to me important to try to understand the sub and quotient bialgebras of this one (which relations are compatible with the given composition and decomposition structures).

As in 2.3.6 let $G$ be a group and $A(G)$ the vector space of (suitable) $K$-valued functions on $G$. Pointwise multiplication makes $A(G)$ an algebra which together with the comultiplication of 2.3.6 defines a bialgebra structure in $A(G)$.

The coalgebra structure on $M_n(K)$ of 2.3.4 and the algebra structure on $M_n(K)$ of 2.2.4 do *not* combine to define a bialgebra structure on $M_n(K)$. Indeed in this case diagram (2.5.1) is about maximally far from commuting. Nor do the cut coalgebra and concatenation algebra structure on $A(\Omega)$ combine to define a bialgebra structure (but there does exist the spray-concatenation bialgebra which is the dual of the merge-cut bialgebra).

### *2.6 Hopf algebras*

A Hopf algebra $H$ over $K$ is a bialgebra with an additional structure map $\iota : H \to H$ called an antipode which imitates the map induced by $e_g \mapsto e_{g^{-1}}$ in the case of the bialgebra $A(G)$ of a group (cf 2.5). More precisely the antipode $\iota : H \to H$ is a linear map such that the following diagrams commute.

$$\begin{array}{ccccc} H & \xrightarrow{\epsilon} & K & \xrightarrow{e} & H \\ \downarrow \mu & & & & \uparrow m \\ H \otimes H & & \xrightarrow{\iota \otimes id_H} & & H \otimes H \end{array} \qquad \begin{array}{ccccc} H & \xrightarrow{\epsilon} & K & \xrightarrow{e} & H \\ \downarrow \mu & & & & \uparrow m \\ H \otimes H & & \xrightarrow{id_H \otimes \iota} & & H \otimes H \end{array}$$

As already indicated $e_g \mapsto e_{g^{-1}}$ turns the bialgebra $A(G)$ into a Hopf algebra. The map $\iota : A(\Omega) \to A(\Omega)$ defined by $\iota(1) = 1$ and $\iota(a_1 \cdots a_n) = (-1)^n a_n a_{n-1} \cdots a_1$ turns the merge-cut bialgebra into a Hopf algebra.

Further examples of Hopf algebras and bialgebras will occur later in this paper.

The antipode $\iota$ of a Hopf algebra satisfies $\iota(1) = 1$ and $\iota(ab) = \iota(b)\iota(a)$. In particular if $H$ is not commutative it is not a homomorphism but an antihomomorphism of algebras. Correspondingly if $\mu(a) = \sum_i a_i' \otimes a_i''$, then

$\mu(\imath(a)) = \sum_i \imath(a_i'') \otimes \imath(a_i')$.

The functor $H \mapsto H^0$ defines a duality of the category of Hopf-algebras into itself.

Two monographs on Hopf algebras are [1,31].

## 3 Generalised symmetry

This section contains some brief 'motivational' sketches for the study of bialgebras plus one more extensive section on quantum inverse scattering. The last subsection can be seen as a worked out version of one of the more condensed parts of [7].

### *3.1 Harmonic analysis*

The standard setting for symmetry is embodied in the idea of a group acting on a space, a (suitable, for instance differentiable when $G$ is a Lie group and $S$ a differentiable manifold) mapping $\alpha : G \times S \to S$ where $G$ is a group and $S$ a space. The action $\alpha : G \times S \to S$ is required to satisfy $\alpha(1, s) = s$ and $\alpha(g, \alpha(h, s)) = \alpha(gh, s)$. As a rule one writes simply $gs$ instead of $\alpha(g, s)$, so that the last requirement is $g(hs) = (gh)s$.

As is frequently the case one attempts to understand such situations by decomposing the space $S$ into $G$-orbits. In the particular case of a group acting on itself (on the left) $m : G \times G \to G$, there is only one orbit, and nothing can be done at this level.

Things change drastically if instead of $S$ one considers the functions on $S$. The action of $G$ on $S$ induces a linear action on the algebra of functions on $S$ defined by $g(f)(s) = f(g^{-1}s)$, $f : S \to K$, $g \in G$, $s \in S$, and as a rule Func($S$) is not indecomposable. For instance in the case of the circle group $G = S^1$ acting on itself decomposing $f : S^1 \to \mathbb{R}$ (a periodic function) means decomposing $f$ into its Fourier components.

Instead of an action $\alpha : G \times S \to S$, we now have an action $G \times \mathrm{Func}(S) \to \mathrm{Func}(S)$, or, more algebraically, a linear mapping

$$(3.1.1) \quad K[G] \otimes \mathrm{Func}(S) \to \mathrm{Func}(S)$$

(with the extra property that for each $g$, the map $f \mapsto gf$ is a homomorphism of algebras). Thus Func($S$) becomes a module over the group algebra $K[G]$. This suggests two things. First that it might on occasion be a reasonable idea to switch for the study of a space to the study of functions on that space. Second instead of the Hopf algebra $K[G]$ of symmetry operators one could consider more general Hopf algebras. (The comultiplication on $K[G]$ is given by $g \mapsto g \otimes g$, the antipode by $g \mapsto g^{-1}$, and the counit by $\epsilon(g) = 1$.) The first suggestion is already a good reason to study Hopf algebras. For if $G$ is a group then Func($G$) (or $A(G)$) is a Hopf algebra.

As to the second suggestion, in the last 10 years or so it has become clear that more general ideas of symmetry than an action of a group on a space are indeed called for. Thus for instance in conformal field theory one nowadays encounters 'higher relatives' of the Virasoro Lie algebra as symmetry algebras and these are not group algebras (or their infinitesimal counterparts, universal enveloping algebras of Lie algebras, cf. e.g. [30] and references quoted there). Another 'symmetry without groups' occurs in the case of quasi crystals, which exhibit five-fold symmetry elements, an impossibility according to the crystallographic group symmetry theory, and the related Penrose tilings. Not that it is clear that these symmetries can be understood in terms of Hopf algebras. (But, without a great deal of evidence, I am still inclined to think so.) Other cases where symmetry in the form of Hopf algebras definitely plays a role will be indicated below (dressing method, $q$-special functions, quantum inverse scattering method). There are also still other parts of mathematics with objects of great regularity and symmetry without the presence of a symmetry group or Lie algebra.

### *3.2 Representations and measuring*

The second reason suggested above for considering more general symmetries than those coming from groups actually suggests a much more drastic generalization, viz. arbitrary algebras as symmetry algebras, i.e., the study of $A$-modules where $A$ is any algebra. There are, however, two good related reasons not to go that far. First for groups and Lie algebras there is a natural notion of the product of two representations, and that construction is a powerful tool in representation theory. For groups the construction is as follows. Let $\rho : G \to \mathrm{End}(V)$, $\sigma : G \to \mathrm{End}(W)$ be two representations. Then the product representation on $V \otimes W$ is given by $(\rho \otimes \sigma)(g)(v \otimes w) = \rho(g)(v) \otimes \sigma(g)(w)$. If $\sigma$ is a Lie algebra and $\rho$ and $\sigma$ are two representations of $\sigma$ on $V$ and $W$ respectively then the product representation of $\sigma$ on $V \otimes W$ is given by the formula $(\rho \otimes \sigma)(a)(v \otimes w) = \rho(a)v \otimes w + v \otimes \rho(a)w$.

In case $A$ is just an algebra there is no natural product of representations. Given two representations $A \otimes V \to V$ and $A \otimes W \to W$ there still is a natural representation of $A \otimes A$ in $V \otimes W$ but no way to derive a representation of $A$ itself from this.

But if $A$ is a bialgebra the comultiplication can be used to define a product representation $A \otimes V \otimes W \to V \otimes W$ as follows

$$A \otimes V \otimes W \xrightarrow{\mu_A \otimes id_V \otimes id_W} A \otimes A \otimes V \otimes W \xrightarrow{id_A \otimes \tau \otimes id_W} A \otimes V \otimes A \otimes W \xrightarrow{\rho \otimes \sigma} V \otimes W.$$

Of course for this to be a representation of the *algebra* $A$ it is essential that $\mu_A : A \to A \otimes A$ is a homomorphism of algebras.

The product structure on the module of all representations in the case of groups comes precisely from the bialgebra structure on $K[G]$ given by the coalgebra structure $\mu : g \mapsto g \otimes g, g \in G$. In the case of Lie algebras a $\mathfrak{g}$-module is the same as an $U\mathfrak{g}$ module where $U\mathfrak{g}$ is the universal enveloping algebra of $\mathfrak{g}$. And $U\mathfrak{g}$ also carries a bialgebra structure given by the coalgebra structure map $U\mathfrak{g} \to U\mathfrak{g} \otimes U\mathfrak{g}$ determined by $a \mapsto a \otimes 1 + 1 \otimes a$.

The second reason for not considering arbitrary algebras and modules as embodying the idea of symmetry is as follows. As already mentioned, the representation (3.1.1) of $K[G]$ on Func$(S)$ has an extra property, viz. that for each $g \in G$, the map $f \mapsto gf$ is an *algebra* endomorphism of Func$(S)$. This can be formulated in terms of the Hopf algebra structure. Let $C$ be a coalgebra and $A$ an algebra and let $\alpha : C \otimes A \to A$ be a $K$-linear map. Then one says that $(\alpha, C)$ *measures* $A$ to $A$ if the following diagrams commute.

$$
\begin{array}{ccc}
C \otimes A \otimes A & \xrightarrow{id_C \otimes m_A} & C \otimes A \\
\downarrow \mu_C \otimes id_A \otimes id_A & & \downarrow \alpha \\
C \otimes C \otimes A \otimes A & & A \\
\downarrow id_C \otimes \tau \otimes id_A & & \uparrow m_A \\
C \otimes A \otimes C \otimes A & \xrightarrow{\alpha \otimes \alpha} & A \otimes A
\end{array}
$$

$$
\begin{array}{ccc}
C \otimes K & \xrightarrow{\epsilon_C \otimes id_K} & K \otimes K \\
 & & \wr \\
\downarrow id_C \otimes e_A & & K \\
 & & \downarrow e_A \\
C \otimes A & \xrightarrow{\alpha} & A
\end{array}
$$

In case $C$ is a bialgebra, this can also be stated as follows. Let $A$ be an algebra which is also a $C$-module (via $\alpha$). Give $A \otimes A$ the product $C$-module structure. Then the commutativity of the diagrams above simply means that $m_A : A \otimes A \to A$ and $e_A : K \to A$ are $C$-module maps. The algebra $A$ is then also called a $C$-module algebra.

All this adds up to the suggestion that a natural setting for the study of generalized symmetry would be a bialgebra $B$ (or Hopf algebra $H$) and the 'representations' to study would be the $B$-module algebras.

### *3.3 The q-disease*

Special functions, such as various orthogonal polynomials like the Jacobi polynomials, turn up all over mathematics and physics and they satisfy all kinds of nice (differential or difference) relations. There are many aspects of special function theory, but a central one, and the one that concerns us here, is the relation with representation theory, a great discovery of the second half of this century (E. Wigner, 1955; N. Ya. Vilenkin, 1967; W. Miller Jr., 1968; cf [34,32,21,33]). Modulo quite a large amount of nontrivial mathematics the central theme is that (at least many) special functions occur as entries in the unitary representations of Lie groups and that their orthogonality relations come from the orthogonality of the matrix entries of unitary representations which comes from Schur's orthogonality relations. In the case of a compact group $G$ with Haar measure $d\mu(g)$ this takes the following form. If $\sigma = (a^{\sigma}_{ij}(g))$, $\rho = (b^{\rho}_{kl}(g))$ are two irreducible unitary representations then

$$\int_G a^{\sigma}_{ij}(g)\overline{b^{\rho}_{kl}(g)}d\mu(g) = 0$$

unless $\rho = \sigma$, $i = k, j = l$.

An exception to this philosophy is formed by the generalized hypergeometric functions themselves (from which most orthogonal polynomials are obtainable through specialization). This adds perhaps (this is very speculative) one more argument in favour of the idea of considering more general 'symmetry structures' than provided by group actions.

Quite early in the game (Heine, 1847) it was discovered that in many cases it was possible to insert an extra parameter $q$ in the formulas defining special functions and orthogonal polynomials and to do it in such a way that many characteristic relations and properties acquired a $q$-analogue, [2,8,10]. This became a challenge and a sport; according to some, a disease: for all orthogonal polynomials (special functions) find the appropriate $q$-analogue. This tremendous amount of work has recently acquired additional significance. Not only have $q$-series and polynomials popped up all over the place in lattice statistical physics, field theory, Lie algebra, transcendental number theory, elliptic functions, combinatorics,... but also a fundamental deeper understanding of their place in the general scheme of things has emerged (is beginning to emerge).

Basically $q$-special functions are to Hopf algebras as (ordinary) special functions are to Lie algebras. More precisely various classical groups like $SU(2)$ deform to quantum groups like $SU_q(2)$; this means that the commutative Hopf algebra of functions on $SU(2)$ can be deformed (in the technical sense of deformations of algebraic structures, [13]), cf. also below in section 3.4.9, to a noncommutative Hopf algebra denoted $SU_q(2)$. It is perhaps noteworthy (historically) that these quantum groups were discovered virtually simultaneously and certainly independently by three different authors motivated by quite different considerations [7,14,35,36]. The representations of $SU(2)$ deform right along (in a unique manner; a rather striking fact) and the entries 'are' the $q$-analogues. There is still orthogonality, harmonic analysis, differential operators, and so on; cf. [19] for an excellent recent survey.

It is a curious historical accident that $q$ was used as a parameter for $q$-special functions long before it was clear that they had to do anything with 'quantum'.

There is little doubt that the 'frequent popping up' of $q$ special functions in many areas and problems has to do with (hidden) symmetry of the Hopf algebra type (in the majority of cases). But in most instances the details, the precise way in which such a symmetry is present, are largely, even totally, unclear. A great deal of work remains to be done.

### *3.4 Dressing and quantization*

Besides other reasons, such as the relation between Hopf algebras and Yang Baxter solutions discussed in some detail below, a good reason for studying the deformations of symmetry groups which are Hopf algebras is that otherwise the so-called 'dressing method' of Zaharov-Shabat for classical integrable systems does not quantize. In the following I try to explain this.

#### *3.4.1 The dressing method* [37]

This technique is most easily understood at the most general 'generic' level. Consider an overdetermined system of equations

$$(3.4.2)\quad \varphi_x = u\varphi, \quad \varphi_t = v\varphi,$$

where $\varphi, u$, and $v$ are $N \times N$ matrix valued functions. The necessary and sufficient solvability condition is that the Zaharov-Shabat equation

(3.4.3) $u_t - v_x + [u, v] = 0$

is fulfilled. Think of $u, v$ as functions of $x, t$ and a spectral parameter $\lambda$ with fixed poles such as

(3.4.4) $u = u_0 + \sum_n \frac{u_n}{\lambda - a_n}, \quad v = v_o + \sum_n \frac{v_n}{\lambda - b_n}$

'Dressing-up' a (possibly trivial) solution $(\varphi_0, u_0, v_0)$ of (3.4.2)-(3.4.3) now goes as follows. Take a contour $\Gamma$ in the $\lambda$-plane which avoids the poles $a_n, b_n$. Take a suitable, say, piece-wise differentiable, function $g_0(\lambda)$ on $\Gamma$ with values in $GL(N, \mathbb{C})$. Now solve the family of Riemann Hilbert boundary value problems (one for each $x, t$) posed by the functions $\varphi_0 g_0 \varphi_0^{-1}$. That means, find $\Phi^-(x, t, \lambda)$ holomorphic in $\lambda$ outside $\Gamma$ (and invertible) and $\Phi^+(x, t, \lambda)$ holomorphic inside $\Gamma$ such that

$$\Phi^- = \varphi_0 g_0 \varphi_0^{-1} \Phi^+ \text{ on } \Gamma$$

(except at points of discontinuity in $\lambda$ of $\varphi_0 g_0 \varphi_0^{-1}$). Now take

(3.4.5) $\varphi_1 = (\Phi^-)^{-1}(\varphi_0)$

$$u = (\Phi^-)^{-1} u_0 \Phi^- x - (\Phi^-)^{-1} \Phi^-_x$$

$$v = (\Phi^-)^{-1} v_0 \Phi^- x - (\Phi^-)^{-1} \Phi^-_t$$

(gauge transformations). Then $u, v$ have the same pole structure in $\lambda$ as $u_0, v_0$ and are new solutions of (3.4.3). Moreover this defines an action of, say, $GL(N, \mathbb{C}[\lambda, \lambda^{-1}])$ (the Fourier transforms of two functions in the circle) on the solutions of (3.4.3) (and (3.4.2)).

Many classical integrable systems can be obtained by specializing the Zaharov-Shabat system (3.4.3); e.g., by imposing certain relations between the $u_0, \ldots, u_M$; $v_0, \ldots, v_M$. The problem remains when the action just described (or part of it) is compatible with such a specialization. This certainly happens in a number of cases. The general problem is known as the specialization problem and it is very far from being solved.

*3.4.6 Double Lie algebra and bi-Lie-algebras*

Now let us turn to one of the more general families of constructions known (so far) for classical completely integrable systems. The setting is a semi-simple real Lie algebra $\mathfrak{g}$ with a decomposition $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{p}$ where $\mathfrak{a}$ and $\mathfrak{p}$ are sub-Lie-algebras. Now, quite generally, given a Lie algebra $\mathfrak{h}$, there is a Poisson structure on $\mathfrak{h}^*$ the dual of $\mathfrak{h}$. I.e., there is a bracket $\{f, g\}$ on the differentiable functions $C^\infty(\mathfrak{h}^*)$ on $\mathfrak{h}^*$ which satisfies the Leibniz identity

(3.4.7) $\{fh, g\} = f\{h, g\} + \{f, g\}h$

and which makes $C^\infty(\mathfrak{h}^*)$ a Lie algebra. The definition goes back to Lie and it has been rediscovered several times (Kostant, Kirillov, Berezin, ... ). The definition is

$$(3.4.8)\quad \{f,g\}(X) = \langle [df(X),\ dg(X)], X\rangle.$$

Note that this makes sense $df(X)$, the differential of $f$ at $X \in \mathfrak{h}^*$ is an element of the cotangent space of $\mathfrak{g}^*$ at $X$, hence an element of $\mathfrak{h}^{**} = \mathfrak{h}$. Thus the bracket [,] makes sense and gives an element of $\mathfrak{h}$ which can be evaluated at $X \in \mathfrak{h}^*$. It turns out that on the orbits of the coadjoint action of $\mathfrak{h}$ on $\mathfrak{h}^*$ this defines a symplectic structure, so that the symplectic leaves of the Poisson structure (3.4.8) are the coadjoint orbits.

Now let $f$ be any coadjoint invariant function on $\mathfrak{g}^*$ where $\mathfrak{g}$ has a decomposition $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{p}$ as above. Restrict $f$ to an orbit $O$ of $\mathfrak{a}$ in $\mathfrak{a}^*$. As just explained there is a symplectic structure on $O$. This defines a Hamiltonian system on $O$ which is completely integrable. The reason is the so-called Adler-Kostant-Symes lemma which says that $ad^*$ invariant functions on $\mathfrak{g}^*$ when restricted to an $\mathfrak{a}$ orbit $O$ in $\mathfrak{a}^*$ are in involution with respect to the symplectic structure on $\mathfrak{a}^*$. (Here $\mathfrak{a}^*$ is identified with the orthogonal complement, $\mathfrak{p}^0$, of $\mathfrak{p} \subset \mathfrak{g}$.) One class of systems which arises this way is the class of Toda lattices. In the case $\mathfrak{g} = sl_n(\mathbb{R})$ we have $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{p}$, with $\mathfrak{p}$ lower triangular of trace zero, $\mathfrak{a}$ orthogonal.

Parenthetically let me remark here that there is as yet no clear way of recovering the Lie algebra $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{p}$ from the equations of the corresponding completely integrable system.

The details are not too important for what I want to describe next. What is important is that in fact on $\mathfrak{g}$ we have two Lie-algebra structures: the original one, and the one on the vector space $\mathfrak{g}$ obtained by taking the Lie algebra direct sum of $\mathfrak{a}$ and $\mathfrak{p}$. Moreover these two are compatible in a certain technical sense making $\mathfrak{g}$ of a so-called double Lie algebra, [28]. Take the invariant nondegenerate bilinear form on $\mathfrak{g}$ and identify $\mathfrak{g}$ and $\mathfrak{g}^*$ using it. Then the second Lie algebra structure on $\mathfrak{g}$ defines a Lie algebra structure on $\mathfrak{g}^*$ or, equivalently, a co-Lie-algebra structure $\mu : \mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g}$. The result is a bi-Lie-algebra which, by definition, is a vector space $V$ with a multiplication $m : V \otimes V \to V$ making $V$ a Lie-algebra, a comultiplication $\mu : V \to V \otimes V$ making $V$ a co-Lie-algebra (i.e. $V^*$ a Lie algebra) such that $m$ and $\mu$ are compatible in the sense that $\mu$ is a 1-cocycle (for the adjoint action of $V$ on $V \otimes V$). In the case just discussed where $V$ is semi-simple as a Lie algebra $\mathfrak{g}$ every cocycle is a coboundary and in fact $\mu$ is the coboundary of an element $r \in \mathfrak{g} \otimes \mathfrak{g}$ which satisfies a modified classical Yang Baxter equation.

The second important point is, cf. [28,29], that also in this setting there are dressing transformations. These are linked to those discussed above in 3.4.1 but, as far as I know, the specialization details still need to be worked out.

The situation then is that there is a symmetry group acting on the symplectic manifold $O$. If this action would preserve the symplectic structure one could hope to quantize the whole situation, keeping the symmetry. It is a profound observation of Michael Semenov-Tian-Shansky, loc.cit., that in fact the action does not preserve the symplectic structure. Instead we have a Poisson group acting on a Poisson manifold and to preserve symmetry both must be simultaneously deformed: the Poisson group to a Hopf algebra and the manifold (or rather the functions on it) to some associative algebra (of operators maybe). What all these phrases mean is the subject of the next subsubsection.

*3.4.9 Poisson groups and quantization*

A Poisson algebra $A$ over $K$ is a commutative-associative algebra with unit with a second multiplication $A \otimes A \to A$, $(f,g) \mapsto \{f,g\}$ which makes the vector space $A$ a Lie algebra and which moreover satisfies the Leibniz rule

$$\{fg,h\} = f\{g,h\} + \{f,h\}g.$$

A Poisson manifold is a smooth manifold $M$ with a Poisson algebra structure on the algebra $C^\infty(M)$ of smooth functions on $M$ (pointwise addition and multiplication). This is the natural setting for classical mechanics: one considers a Poisson algebra (Poisson manifold) $A$ over $\mathbb{R}$, an element $f \in A$, the Hamiltonian, differentiable functions $\mathbb{R} \to A$ and the equation to be studied is

$$(3.4.10) \quad \dot{\varphi} = \{f,\varphi\}.$$

Given Poisson algebras $A, B$, the tensor product $A \otimes B$ carries a Poisson structure defined by $\{a \otimes b,\ c \otimes d\} = \{a,c\} \otimes bd + ac \otimes \{b,d\}$. Modulo a bit of fiddling with completions this determines a Poisson manifold structure on $M \times N$ given Poisson structures on the manifolds $M$ and $N$. A Poisson algebra homomorphism $\varphi : A \to B$ is, of course, a homomorphism of algebras such that $\{\varphi(a), \varphi(b)\} = \varphi\{a,b\}$ for all $a, b \in A$.

A Poisson Lie group is now a Lie group $G$ with a Poisson structure (on its underlying manifold) such that the multiplication $m : G \times G \to G$, and the inverse $i : G \to G$ are mappings of Poisson manifolds.

Lie theory applies. The Lie algebra of a Poisson Lie group inherits a bi-Lie algebra structure and the Lie algebra Lie group correspondence makes bi-Lie algebras correspond with Poisson Lie groups.

Now let us turn to quantization. Given a Poisson algebra $A$ over $K$, a quantization of it is an associative algebra $S_h$ over $K[[h]]$ such that

$$(3.4.11) \quad A_h \otimes_{K[[h]]} K = A$$

i.e. if one sets $h = 0$ the original algebra is recovered, and such that

$$(3.4.12) \quad \{f,g\} = \text{class}(h^{-1}(\tilde{f}\tilde{g} - \tilde{g}\tilde{f})), \quad f,g \in A.$$

Here 'class' is the quotient homomorphism $A_h \to A$ and $\tilde{f}, \tilde{g}$ are any two elements of $A_h$ such that $\text{class}(\tilde{f}) = f$, $\text{class}(\tilde{g}) = g$. Note that (3.4.12) makes sense because $A$ is commutative and because of (3.4.11).

A quantization of (3.4.10) consists of a Hamiltonian $F \in A_h$ with $\text{class}(F) = f$ and one studies differentiable functions $\Phi : \mathbb{R} \to A_h$ such that

$$(3.4.13) \quad \dot{\Phi} = [F, \Phi] = F\Phi - \Phi F.$$

In the present setting it is immaterial whether $A_h$ is an algebra of operators or not (though of course it can be). And, indeed, in some of the approaches to quantization, cf e.g. a number of articles in [13], this is not the case.

Note that, apart from the fact that general associative algebras are a bit harder to understand than commutative ones, conceptually, the underlying situation of (3.4.13), an associative algebra, is a rather simpler object than that of (3.4.10), a Poisson algebra. From this point of view the Poisson structure (and hence the classical dynamics) are an infinitesimal (semiclassical) residues of the basic noncommutativeness of an algebra of observables.

Now, to conclude this section, let us turn back to the case of Adler, Kostant, Symes, Reiman, Semenov-Tian-Shansky integrable systems as described above. Here the situation is that of a Poisson Lie group $G$ (whose algebra of functions is a commutative Poisson Hopf algebra) acting on a Poisson manifold $M$: $G \times M \xrightarrow{\pi} M$ and instead of $G$ preserving the symplectic structure on the symplectic leaves of $M$ one has that $\pi$ is a morphism of Poisson manifolds. To quantize this both $M$ and $G$ must be quantized simultaneously which in the case of $G$ asks for a Hopf algebra which is a deformation (in the sense explained above) of the Poisson Hopf algebra $C^\infty(G)$.

Though it is quite well known what the quantum versions of, for instance the Toda lattices are, and though there is also no doubt whatever that the deformation of the group involved will be $SL_q(n;\mathbb{R})$ (or more precisely the loop algebra version) in that case, the program sketched above still remains to be carried out.

### *3.5 Quantum inverse scattering method and Hopf algebras*

In the quantum inverse scattering method constructed by L.D. Faddeev, E.K. Sklyanin, L.A. Tahtadžyan a.o., cf. e.g. [9], a very important role is played by relations of the following form (fundamental commutation relations)

$$(3.5.1)\quad R(\lambda,\mu)\,(T(\lambda)\times T(\mu)) = (T(\mu)\times T(\lambda))\,R(\lambda,\mu).$$

Here $T(\lambda)$ is a matrix of operators

$$(3.5.2)\quad T(\lambda) = \begin{pmatrix} T_1^1(\lambda) & \dots & T_n^1(\lambda) \\ \vdots & & \\ T_1^n(\lambda) & \dots & T_n^n(\lambda) \end{pmatrix}$$

$R(\lambda,\mu)$ is an $n^2\times n^2$ matrix of scalars, and $T(\lambda)\times T(\mu)$ stands for the Kronecker product. Thus

(3.5.3)

$$T(\lambda)\times T(\mu) = \begin{pmatrix} T_1^1(\lambda)T_1^1(\mu) & T_1^1(\lambda)T_2^1(\mu) & T_2^1(\lambda)T_1^1(\mu) & T_2^1(\lambda)T_2^1(\mu) \\ T_1^1(\lambda)T_1^2(\mu) & T_1^1(\lambda)T_2^2(\mu) & T_2^1(\lambda)T_1^2(\mu) & T_2^1(\lambda)T_2^2(\mu) \\ T_1^2(\lambda)T_1^1(\mu) & T_1^2(\lambda)T_2^1(\mu) & T_2^2(\lambda)T_1^1(\mu) & T_2^2(\lambda)T_2^1(\mu) \\ T_1^2(\lambda)T_1^2(\mu) & T_1^2(\lambda)T_2^2(\mu) & T_2^2(\lambda)T_1^2(\mu) & T_2^2(\lambda)T_2^2(\mu) \end{pmatrix}$$

in the case of $2\times 2$ matrices of operators. Finally in (3.5.1) the entries of $R(\lambda,\mu)$ are supposed to be scalars so that they commute with the operators $T_j^i(\lambda)$.

In an actual 'integrable' quantum system such as the Heisenberg chain, the trace of $T(\lambda)$

(3.5.4) $t(\lambda) = \mathrm{Tr}(T(\lambda)) = \sum_{i=1}^{n} T_i^i(\lambda)$

is the Hamiltonian operator of which it is desired to calculate the eigenvalues and eigenvectors (as functions of $\lambda$). Now take the trace of (3.5.1) to find that (if $R(\lambda, \mu)$ is invertible (which can be assured))

(3.5.5) $t(\lambda)\, t(\mu) = t(\mu)\, t(\lambda)$

so that the $t(\lambda)$ are a commuting family of operators. Actually in concrete situations such as the Heisenberg chains, (3.5.1) implies a good deal more and forms the basis for a procedure to obtain eigenvalues and eigenvectors which goes by the name 'algebraic Bethe Ansatz' which does yield a full set of eigenvectors and eigenvalues in many cases though it has not been proved that it will always work.

Thus it is interesting to find many examples of matrices of operators $T(\lambda)$ for which there is an $R$-matrix such that (3.5.1) holds. For the $R$-matrix in question this has to do with the Yang Baxter equation. Indeed view $R$, which is an $n^2 \times n^2$ matrix, as giving an isomorphism

$$R : \mathbb{C}^n \otimes \mathbb{C}^n \to \mathbb{C}^n \otimes \mathbb{C}^n$$

and also as giving the corresponding isomorphism

$$R : V^n \otimes V^n \to V^n \otimes V^n$$

where $V$ is the space on which the $T_j^i(\lambda)$ act, and where on each copy of

$$V^n \otimes V^n = V^{n^2}$$

(the direct sum of $n^2$ copies of $V$) each $R_{cd}^{ab}$ acts as a scalar. Let $R(12) = R \otimes id$: $V^n \otimes V^n \otimes V^n \to V^n \otimes V^n \otimes V^n$ and $R(23) = id \otimes R$: $V^n \otimes V^n \otimes V^n \to V^n \otimes V^n \otimes V^n$. Then

$$R(12)\, R(23)\, R(12)\, (T(\lambda) \times T(\mu) \times T(v))\, R(12)^{-1}\, R(23)^{-1}\, R(12)^{-1} =$$

$$R(12)\, R(23)\, (T(\mu) \times T(\lambda) \times T(\nu))\, R(23)^{-1}\, R(12)^{-1} =$$

$$R(12)\, (T(\mu) \times T(\nu) \times T(\lambda))\, R(12)^{-1} =$$

$$T(\nu) \times T(\mu) \times T(\lambda)$$

and also

$$R(23)\, R(12)\, R(23)\, (T(\lambda) \times T(\mu) \times T(\nu))\, R(23)^{-1}\, R(12)^{-1}\, R(23)^{-1} =$$

$$R(23)\, R(12)\, (T(\lambda) \times T(\nu) \times T(\mu))\, R(12)^{-1}\, R(23)^{-1} =$$

$$R(23)\, (T(\nu) \times T(\lambda) \times T(\mu))\, R(23)^{-1} =$$

$$T(\nu) \times T(\mu) \times T(\lambda).$$

Thus $(R(12)\,R(23)\,R(12))\,(R(23)\,R(12)\,R(23))^{-1}$ commutes with $T(\lambda) \times T(\mu) \times T(\nu)$. So generically it is a polynomial in $T(\lambda) \times T(\mu) \times T(\nu)$ and being a matrix of scalars it therefore must be a multiple of the identity. The factor involved often can be seen to be one because (as a rule) one requires $R(\lambda, \mu) = R(\lambda - \mu) = Id$ if $\lambda = \mu$. Thus 'morally' $R$ has to satisfy the Yang Baxter equation

(3.5.6) $R(12)\,R(23)\,R(12) = R(23)\,R(12)\,R(23)$.

Now suppose we have a Hopf algebra $H$, together with an element $\check{R} \in H \otimes H$ which satisfies the following identity

(3.5.7) $\mu'(a)\,\check{R} = \check{R}\mu(a), \quad a \in H$

where $\mu' : H \longrightarrow H \otimes H$ is equal to $H \xrightarrow{\mu} H \otimes H \xrightarrow{\tau} H \otimes H$.

Let $\rho_\lambda : H \longrightarrow M_n(\mathbb{C})$ be a family of representations of $H$ *as an algebra.* Each entry of the matrix $\rho_\lambda(a)$ is linear as a function of $a \in H$ and hence defines an element $T^i_j(\lambda) \in H^*$, the dual Hopf algebra of $H$. Recall that the multiplication in $H^*$ is defined by the comultiplication on $H$ via the identity

$$\langle TS, a\rangle = \langle T \otimes S, \mu(a)\rangle, \quad T, S \in H^*,\ a \in H.$$

It follows immediately that

$$\begin{array}{rcl} (\rho_\lambda \otimes \rho_{\lambda'})\mu(a) & = & (T(\lambda) \times T(\lambda'))(a) \\ (\rho_\lambda \otimes \rho_{\lambda'})\mu'(a) & = & (T(\lambda') \times T(\lambda))(a). \end{array}$$

(Note that we are in a slightly more general situation in that $T^i_j(\lambda)$ need not be operators but are simply elements of the abstract algebra $H^*$.)

Now let

$$R(\lambda, \lambda') = (\rho_\lambda \otimes \rho_{\lambda'})(\check{R}) \in M_{n^2}(\mathbb{C}).$$

Then it follows immediately from (3.5.6) that

(3.5.8) $(T(\lambda') \times T(\lambda))\,R(\lambda, \lambda') = R(\lambda, \lambda')(T(\lambda) \times T(\lambda'))$

so that we find (a more abstract version) of the Fundamental Commutation Relations (3.5.1).

Indeed let $a \in H$ and $\mu(a) = \sum_i a'_i \otimes a''_i$, then

$$(T(\lambda) \times T(\lambda'))(a) = \begin{pmatrix} (t^1_1(\lambda)t^1_1(\lambda'))(a) \ldots \\ \vdots \end{pmatrix}$$

$$= \begin{pmatrix} \sum_i t^1_1(\lambda)(a'_i)\, t^1_1(\lambda')(a''_i) \ \ldots \\ \vdots \end{pmatrix}$$

$$= \begin{pmatrix} t_1^1(\lambda) \otimes t_1^1(\lambda') \, (\sum_i a_1' \otimes a_i'') \, \ldots \\ \vdots \end{pmatrix}$$

$$= (\rho_\lambda \otimes \rho_{\lambda'})\mu(a)$$

and similarly $(T(\lambda') \times T(\lambda))a = (\rho_{\lambda'} \otimes \rho_\lambda) \, \mu(a) = (\rho_\lambda \otimes \rho_{\lambda'}) \, \mu'(a)$.

The inverse is also true. Indeed let $R$ be any $n^2 \times n^2$ of scalars. Consider the matrix function Hopf algebra $H_n$

$$H_n = k\langle t_j^i : i, j = 1, \ldots, n\rangle$$

where the $t_j^i$ are $n^2$ noncommuting (but associative) variables. The comultiplication is given by (the 'matrix comultiplication')

$$t_j^i \mapsto t_k^i \otimes t_j^k \quad \text{(summation convention)}$$

and the counit is given by the (Kronecker index)

$$t_j^i \mapsto \delta_j^i.$$

It is easy to verify that this is indeed a Hopf algebra. Let $T$ be the matrix

$$T = \begin{pmatrix} t_1^1 & \ldots & t_n^1 \\ \vdots & & \vdots \\ t_1^n & \ldots & t_n^n \end{pmatrix}$$

and define

$$T_1 = T \times I_n, \quad T_2 = I_n \times T$$

where $I_n$ is the $n \times n$ identity matrix. Now consider the $n^2 \times n^2$ matrix

(3.5.9) $RT_1T_2 - T_2T_1R.$

If the entries of $R$ are labelled as indicated in the $n = 2$ case below

$$R = \begin{pmatrix} R_{11}^{11} & R_{12}^{11} & R_{21}^{11} & R_{22}^{11} \\ R_{11}^{12} & R_{12}^{12} & R_{21}^{12} & R_{22}^{12} \\ R_{11}^{21} & R_{12}^{21} & R_{21}^{21} & R_{22}^{21} \\ R_{11}^{22} & R_{12}^{22} & R_{21}^{22} & R_{22}^{22} \end{pmatrix}$$

then the entries of (3.5.9) are equal to (summation convention).

(3.5.10) $R_{i_1 i_2}^{ab} \, t_c^{i_1} \, t_d^{i_2} - R_{cd}^{j_1 j_2} \, t_{j_2}^b \, t_{j_1}^a$

Let $I(R)$ be the ideal in the algebra $H_n$ generated by the elements (3.5.10).

3.5.11 LEMMA. $I(R)$ is a Hopf-ideal, i.e. $\mu(I(R)) \subset H_n \otimes I(R) + I(R) \otimes H_n$ and $\epsilon(I(R)) = 0$.

Thus there is an induced structure of a Hopf algebra on $H_n(R) = H_n/I(R)$.

3.5.12 Example.

$$R = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -q^{-1} & 0 \\ 0 & -q & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The relations turn out to be

$$\begin{aligned} t^1_2 t^1_1 = q t^1_1 t^1_2, \quad & t^2_1 t^1_1 = q t^1_1 t^2_1,\ t^2_2 t^1_2 = q t^1_2 t^2_2,\ t^2_2 t^2_1 = q t^2_1 t^2_2 \\ & t^1_2 t^2_1 = t^2_1 t^1_2 \\ t^2_2 t^1_1 - \quad & t^1_1 t^2_2 = (q - q^{-1}) t^1_2 t^2_1. \end{aligned}$$

This is the well-known quantum matrix algebra $M_q(2)$ from which the quantum group $SL_q(2)$ is obtained by imposing the extra 'quantum determinant = 1' relation $t^1_1 t^2_2 - q t^1_2 t^2_1 = 1$.

3.5.13 Example.

$$R = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

This yields a Hopf algebra, which, as an algebra, has four generators $u_+, u_-$, $v_+, v_-$ and the relations are

$$u_+ v_+ = v_+ u_+, u_- v_- = v_- u_-, \ u_+ v_- = -v_- u_+, v_- u_+ = -u_+ v_-,$$

$$u_+ u_- = -u_- u_+, v_+ v_- = -v_- v_+$$

Both these examples have no zero divisors in $H_n(R)$. However as a rule zero divisors will occur in $H_n(R)$.

There is nothing particularly unique about the $R$ matrix defining a given $H_n(R)$. In particular if $R_0(n)$ is the $n^2 \times n^2$ matrix

$$R_0(n)^{ab}_{cd} = \delta^a_d \delta^b_c$$

then $R$ and $R + \lambda R_0(n)$ define the same $H_n(R)$. This can be used to make $R$ an invertible matrix.

Let $H^0_n$ be the dual Hopf algebra of $H_n$ and $H_n(R)^0 \subset H^0_n$ the dual Hopf algebra of $H_n(R)$.

Consider the element

$$(3.5.14) \quad \check{R} = R^{i_1 i_2}_{j_1 j_2} e^{j_1}_{i_1} \otimes e^{j_2}_{i_2}$$

in $H^0_n$. Here, quite generally, $e^{j_1 \dots j_r}_{i_1 \dots i_r}$ is the element of $H^0_n$ defined by

$$e^{j_1 \dots j_r}_{i_1 \dots i_r}(t^{i'_1}_{j'_1} \dots t^{i'_s}_{j'_s}) = \delta^s_r \delta^{i_1}_{i'_1} \delta^{i_2}_{i'_2} \dots \delta^{i_r}_{i'_r} \delta^{j_1}_{j'_1} \dots \delta^{j_r}_{j'_r}.$$

(Note that this is indeed in $H_n^0 \subset H_n^*$.)

3.5.15 LEMMA. $\check{R} \in H_n(R)^0 \otimes H_n(R)^0 \subset H_n^0 \otimes H_n^0$.

(trivial, because the relations defining $H_n(R)$ are of degree 2)

3.5.16 PROPOSITION. The element $\check{R} \in H_n(R)^0 \otimes H_n(R)^0$ satisfies

$$\check{R}\mu(a) = \mu'(a)\check{R}$$

for all $a \in H_n(R)^0$.

Putting all this together we see that the search for Fundamental Commutation Relations (3.5.1) amounts to the systematic study of the Hopf algebra $H_n(R)$ (and their duals $H_n(R)^0$) and the representation theory of the $H_n(R)^0$ (as algebras).

## 4 MORE ON HOPF AND BIALGEBRAS IN MATHEMATICS

This section contains brief, almost telegraphic, sections devoted to some (by no means all) other parts of mathematics where Hopf and bialgebras play a significant role.

### *4.1 Niceness theorems*

*4.1.1 Freeness over sub Hopf algebras.* Lagrange's theorem for subgroups $H$ of a finite group $G$ says that the order of $G$ is a multiple of the order $H$. At the level of functions on $G$ and $H$ this translates to (the more precise statement) that the algebra Func($G$) is free as a module over its subalgebra Func($H$).

The situation is that of a Hopf algebra $B$ containing a sub Hopf algebra $A$ and in this more general setting there is a whole slew of theorems saying that as an $A$-module $B$ is free [24,25,26].

*4.1.2 Hopf-Borel structure theorems.* Hopf algebras first arose in work of Heinz Hopf who studied the cohomology algebra of Lie groups and more generally $H$-spaces $G$ (topological spaces with a homotopy associative multiplication, a homotopy inverse, and a homotopy unit). Actually Hopf worked with the homology coalgebra. There is a graded commutative algebra structure on $H^*(G;K)$. This means that $H^*(G) \cong \oplus_{k=0}^{n} H^k(G)$ where $n = \dim(G)$, that the multiplication is grade preserving, i.e. if $x \in H^p(G)$, $y \in H^q(G)$, then $xy \in H^{p+q}(G)$, and finally that the multiplication is graded commutative i.e. $xy = (-1)^{pq}yx$ for $x$ and $y$ as before. The $H$-space structure on $G$, i.e. the multiplication map $G \times G \to G$ induces an algebra homomorphism $H^*(G;K) \to H^*(G;K) \otimes H^*(G;K)$ and $H^*(G;K)$ becomes a Hopf algebra. The graded commutative Hopf algebra structure is a very powerfull structure. So much so that these algebras are completely known; they are tensor products of Hopf algebras which as algebras have one generator and these algebras are very simple ones. They are of the form $K[x]/(x^n)$ where for char$(K) \neq 2$, $n$ is two if $x$ has odd degree and $n$ is a power of char$(K)$

or $\infty$ if $n$ is even, and for $\mathrm{char}(K) = 2$, $n$ is $\infty$ or a power of 2. Cf [22,17] for details.

Some of the terminology in Hopf algebra theory still reflects the topological origin. Thus the property 'connected' of a Hopf algebra generalizes a property of the $H^*(G; K)$ when $G$ is a connected $H$-space.

*4.2 The algebra of representations of the symmetric groups*

Let $S_n$ be the symmetric group of permutations on $n$ letters and let $R(S_n)$ be its algebra of complex representations. I.e. the elements of $R(S_n)$ are the integral linear combinations of the irreducible complex representations of $S_n$. Consider the free abelian group $\oplus_{n=0}^{\infty} R(S_n)$, $R(S_0) = \mathbb{Z}$. We are going to define a Hopf algebra structure on $\otimes_{n=0}^{\infty} R(S_n)$. First the multiplication. Let $\rho$ and $\sigma$ be respectively a representation of $S_n$ and of $S_m$ in $V$ and $W$ respectively. Taking the tensor product gives a representation of $S_n \times S_m$ in $V \otimes W$. Consider $S_n \times S_m$ as a subgroup of $S_{n+m}$ in the natural way and obtain a representation of $S_{n+m}$ by inducing $\rho \otimes \sigma$ up to $S_{n+m}$. This defines the multiplication, i.e.

$$\rho\sigma = \mathrm{Ind}_{S_n \times S_m}^{S_{n+m}} (\rho \otimes \sigma).$$

For the comultiplication we use restriction. Let $\rho$ be a representation of $S_n$. For every $p, q \in \{0, 1, \ldots\}$, $p + q = n$. Consider the restriction of $\rho$ to $S_p \times S_q$ to obtain an element in $R(S_p \times S_q) = R(S_p) \otimes R(S_q)$. The comultiplication is now defined by

$$\mu(\rho) = \sum_{p+q=n} \mathrm{Res}_{S_p \times S_q}^{S_n} (\rho).$$

The theorem is that all this (plus an antipode) defines a Hopf algebra structure (over $\mathbb{Z}$) on $\oplus_{n=0}^{\infty} R(S_n)$.

There is more. This Hopf algebra can be explicitly described. Consider the commutative ring of polynomials in infinitely many variables $c_1, c_2, \ldots$ over $\mathbb{Z}$.

$$U = \mathbb{Z}\,[c_1, c_2, \ldots, c_n].$$

The coalgebra structure is given by $(c_0 = 1)$

$$c_n \;\mapsto\; \sum_{p+q=n} c_p \otimes c_q.$$

The next theorem is that $U$ and $\oplus R(S_n)$ are isomorphic as Hopf algebras. (One of the possible isomorphisms lets $c_n$ correspond with the trivial representation of $S_n$.)

That they are isomorphic as algebras (or modules) is (in one form or another) an old well-known story. But this is of limited use because there are very many automorphisms of $U$ as an algebra. The situation is different at the Hopf algebra level, [20]:

$$\mathrm{Aut}_{\mathrm{Hopf}}(U) \;\simeq\; \mathbb{Z}/(2) \otimes \mathbb{Z}/(2).$$

Thus there are precisely four 'natural' isomorphisms $U \simeq \oplus R(S_n)$. It is an interesting fact (and a compliment to the natural taste of the giants who developed the representation theory of the $S_n$) that precisely these isomorphisms (or at least three of them) have been selected in the past.

As a matter of fact $U$ and $\oplus R(S_n)$ have even more structure. For $\oplus R(S_n)$ we did not at all use that the $R(S_n)$ themselves are also rings. This defines a second multiplication on $\oplus R(S_n)$ which is distributive over the first one, thus making $\oplus R(S_n)$ a ring object in the category of coalgebras (over $\mathbb{Z}$). In the setting of $U$ this second multiplication has also turned up before in combinatorial symmetric functions settings. If this additional bit of structure is also taken into account the automorphism group becomes $\{\mathrm{id}\}$.

There is still more. There is a notion of positivity on $\oplus R(S_n)$: the actual (not virtual) representations are positive. And the various structure maps res-pect positivity. There is also an inner product: $\langle \rho, \sigma \rangle$ counts the number of irreducible representations that $\rho$ and $\sigma$ have in common. Finally the multiplication $m$ and comultiplication $\mu$ are adjoint to each other

$$\langle \rho, \sigma\tau \rangle = \langle \mu(\rho), \sigma \otimes \tau \rangle$$

which is the same as Frobenius reciprocity. This situation has been axiomatized [38] as a $PSH$ algebra (which stands for Positive Selfadjoint Hopf algebra). Under mild assumptions they can be classified. There is one on one generator, viz. the Hopf algebra $U$ described above, and apart from the grading all are tensor products of $U$'s.

Even this does not exhaust the structures of $U$. It is also a boring object in the category of rings; it has the structure of a $\lambda$-ring; it is selfdual; in the category of rings it defines a $\lambda$-ring valued functor $R \mapsto \mathrm{Ring}(U, R)$ which means that $U$ has a second level $\lambda$ and co-$\lambda$ structure; ....

I doubt, that, essentially, there is more then one object like this. Zelevinsky [38] proves this for $PSH$ algebras, but a similar uniqueness should hold without positivity and the assumption that there is an adapted $\mathbb{Z}$-basis, assumptions which are not always natural.

The above and the niceness theorems of 4.1 illustrate a still very poorly understood (if at all) metamathematical observation (or rather two): objects with several different but compatible structures tend to be nice; universal objects - $U$ has several universality and freeness properties - tend to be very nice. Here 'nice' is undefined. What is meant is something like a situation where there are, say, seven independent agreable properties and if two or three hold the others all follow. For some more details on the things touched upon in this section see [18,20,38]; not everything I mentioned has been written up.

## 5 CODA

In the above I have indicated some areas in mathematics and physics where Hopf algebras and bialgebras play a role. I have by no means mentioned all. For instance the role of Hopf algebras in combinatorics. Decompositions, hence coalgebras, are a natural way of life in combinatorics. It has been said, with some justice, that the so-called umbral calculus is simply the study of the Hopf algebra $k[X]$, $X \mapsto 1 \otimes X + X \otimes 1$. See [16] for an inspiring account of possible roles for Hopf algebra and coalgebras in combinatorics; a gread deal remains to be done.

Quite generally the study of Hopf algebras and their applications has essentially only just started. Things already look good though.

REFERENCES


1. EIICHI ABE, *Hopf algebras*, Cambridge Univ. Press, 1977, 284 pp.
2. G.E. ANDREWS, *q-Series: their development and applications in analysis, number theory, combinatorics, physics, and computer algebra*, Amer. Math. Soc., 1986.
3. DAVID B. BENSON, *Bialgebras: some foundations for distributed and concurrent computation*, 1988.
4. DAVID B. BENSON, *The shuffle bialgebra.* In: Proc. 3rd workshop on the mathematical foundations of programming language semantics, Springer, 1988, 616-637.
5. ARMAND BOREL, *Sur la cohomologie des espaces fibrés principaux et des espaces homogènes de groupes de Lie compacts*, Ann. Math., **57**: 1(1953), 115-207.
6. L. CERLIENCO, G. NICOLETTI, F. PIRAS, *Coalgebre e calcolo umbrale*, Sem. Mat. e Fis., (1984), 79-100.
7. V.G. DRINFEL'D, *Quantum groups.* In: Proc. Int. Cong. of Math., Berkeley, Amer. Math. Soc., 1986, 798-820.
8. H. EXTON, *q-Hypergeometric functions and applications*, Ellis Horwood, 1983.
9. LUDWIG FADDEEV, *Integrable models in 1+1 dimensional quantum field theory.* In: Les Houches, session 39, 1982, North Holland, 1984, 563-608.
10. G. GASPER, M. RAHMAN, *Basic hypergeometric series*, Cambridge Univ. Press, 1989.
11. MURRAY GERSTENHABER, S.D. SHACK, *Quantum groups as deformations of Hopf algebras*, 1989.
12. M. HAZEWINKEL, *Formal groups and applications*, Acad. Press, 1978.
13. M. HAZEWINKEL, M. GERSTENHABER (ed.), *Deformations of algebras and applications*, Reidel, 1988.
14. MICHIO JIMBO, *A q-analogue of U(gl(N+1)), Hecke algebra and the Yang Baxter equation*, 1985.
15. MICHIO JIMBO, *A q-difference analogue of* $U(q)$ *and the Yang Baxter equation*, Lett. Math. Phys., **10** (1985), 63-69.
16. S.A. JONI, G.-C. ROTA, *Coalgebras and bialgebras in combinatorics.* In: ROBERT MORRIS (ed.), Umbral calculus and Hopf algebras, Amer. Math. Soc., 1982, 1-47.
17. R.M. KANE, *The homology of Hopf spaces*, North Holland, 1988.
18. D. KNUTSON, *$\lambda$-Rings and the representation theory of the symmetric group*, Springer, 1973.
19. TOM H. KOORNWINDER, *Orthogonal polynomials in connection with quantum groups.* In: P. NEVAI (ed.), Proc. NATO ASI on Orthogonal Polynomials, Columbus, Ohio, 1989, KAP, 1990, 257-292.

20. A. LIULEVICIUS, *Arrows, symmetries and representation rings*, J. pure and appl. Algebra, **19** (1980), 259-273.
21. W. MILLER, *Lie theory and special functions*, Acad. Press, 1968.
22. JOHN W. MILNOR, JOHN C. MOORE, *On the structure of Hopf algebras*, Ann. Math., **81**: 2 (1965), 211-264.
23. WARREN D. NICHOLS, MOSS SWEEDLER, *Hopf algebras and combinatorics.* In: ROBERT MORRIS (ed.), Umbral calculus and Hopf algebras, Amer. Math. Soc., 1982, 49-84.
24. WARREN D. NICHOLS, M. BETTINA ZOELLER, *Finite-dimensional Hopf algebras are free over grouplike subalgebras*, J. pure appl. Algebra, **56** (1989), 51-57.
25. WARREN D. NICHOLS, M. BETTINA ZOELLER, *Freeness of infinite dimensional Hopf algebras over grouplike subalgebras*, Comm. Algebra, **17**: 2(1989), 413-422.
26. DAVID E. RADFORD, *Pointed Hopf algebras are free over Hopf subalgebras*, J. Algebra, **45** (1977), 266-273.
27. M.A. SEMENOV-TIAN-SHANSKY, *Group theoretical aspects of completely integrable systems.* In: H.-D. DOEBNER and T.D. PALEV (ed.), Twistor geometry and non-linear systems, Springer, 1980.
28. M.A. SEMENOV-TIAN-SHANSKY, *What is a classical R-matrix*, Func. Anal. and Appl., **17** (1983), 259-272.
29. M.A. SEMENOV-TIAN-SHANSKY, *Dressing transformations and Poisson group actions*, Publ. RIMS, **21** (1985), 1237-1260.
30. D.J. SMIT, *Quantum groups and algebraic geometry in conformal field theory*, 1989.
31. MOSS E. SWEEDLER, *Hopf algebras*, Benjamin, 1969, 336 pp.
32. N. YA. VILENKIN, *Special functions and theory of group representations*, Nauka, 1965.
33. A. WAWRZYNCZYK, *Group representations and special functions*, Reidel - PWN, 1984.
34. E.P. WIGNER, *Application of group theory to the special functions of mathematical physics*, Princeton, unpublished notes. later prepared for publication by Talman
35. S.L. WORONOWICZ, *Compact matrix pseudogroups*, Comm. Math. Phys., **111** (1987), 613-665.
36. S.L. WORONOWICZ, *Twisted SU(2) group. An example of noncommutative differential calculus*, Publ. Res. Inst. Math. Sci., **23** (1987), 117-181.
37. V.E. ZAKHAROV, *Integration of nonlinear equations of mathematical physics by the method of the inverse scattering problem. II*, Funk. Anal. i Pril., **13**: 3(1979), 13-22.
38. ANDREY V. ZELEVINSKY, *Representations of finite classical groups*, Springer, 1981, 184 pp.