

Strongly Regular Graphs and Partial Geometries

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Abstract

We give a survey of recent results concerning construction, uniqueness or nonexistence of strongly regular graphs and partial geometries. Essentially we restrict ourselves to results which did not occur in the well known earlier surveys.

1. Introduction

Since the introduction of strongly regular graphs and partial geometries by Bose [4] several surveys of the theory and of constructions have appeared. For the theory of strongly regular graphs a recent survey is Seidel [63]. For constructions one usually refers to Hubaut [45]. Much of the theory can also be found in Cameron [10] and Cameron and van Lint [14]. A survey on partial geometries describing the state of affairs in 1976 was given by Thas [70]. The main purpose of this paper is to describe briefly what has happened since these earlier surveys appeared.

We assume that the reader is familiar with the theory and with earlier surveys but for the sake of completeness we give several definitions and theorems in this introduction. For details we refer to the literature.

Let X be a finite set. An *association scheme* with d classes is a pair $(X, \{R_0, R_1, \dots, R_d\})$ such that

- (i) $\{R_0, R_1, \dots, R_d\}$ is a partition of $X \times X$,
- (ii) $R_0 = \{(x, x) \mid x \in X\}$,
- (iii) $R_i = R_i^T$ ($i = 0, 1, \dots, d$),
- (iv) there are numbers p_{ij}^k (the *intersection numbers* of the scheme) such that for any pair $(x, y) \in R_k$ we have $p_{ij}^k = |\{z \mid (x, z) \in R_i \text{ and } (z, y) \in R_j\}|$.

We call $n_i := p_{ii}^0$ the *valency* of R_i and put $v := |X| = \sum n_i$.

A *strongly regular graph* is an association scheme with 2 classes. The points of X are the vertices of the graph and $\{x, y\}$ is an edge if $(x, y) \in R_1$. We shall use the notation

$$srg(v, k, \lambda, \mu), \text{ where } v = |X|, k = n_1, \lambda = p_{11}^1, \mu = p_{11}^2.$$

The complement of a $srg(v, k, \lambda, \mu)$ is a

$$srg(v, l := v - k - 1, v - 2k + \mu - 2, v - 2k + \lambda).$$

The $(0, 1)$ adjacency matrix of a srg is defined by $A(x, y) := 1$ if $\{x, y\}$ is an edge, 0 otherwise. By definition we have

$$AJ = kJ, A^2 + (\mu - \lambda)A + (\mu - k)I = \mu J.$$

Most of the necessary conditions for the existence of strongly regular graphs were found by studying the eigenvalues of A and properties of the 3-dimensional algebra \mathbf{A} of linear combinations of I, J , and A (the *Bose-Mesner algebra* of the srg).

Let E_j ($0 \leq j \leq 2$) be a basis of minimal idempotents of \mathbf{A} . This algebra is also closed under componentwise multiplication (denoted by \circ). We define q_{ij}^k by

$$E_i \circ E_j = v^{-1} \sum_{k=0}^2 q_{ij}^k E_k.$$

We point out that Delsarte [26] defined the concept of *duality* for association schemes (and hence for srg 's). A sufficient condition for the existence of a dual is the existence of a regular abelian group of automorphisms. (If a srg has v vertices, valencies k and l , and multiplicities f and g , then the dual, if it exists, has v vertices, valencies f and g , and multiplicities k and l .) Some of the srg 's mentioned below were found using duality.

Necessary Conditions

We list some of the necessary conditions for the existence of a $srg(v, k, \lambda, \mu)$. We exclude trivial graphs (disconnected graphs and their complements), i.e. we assume $0 < \mu < k < v - 1$. From the parameters of the complementary graph we see that $v - 2k + \mu - 2 \geq 0$ is a necessary condition for the existence of a srg . This kills e.g. $(v, k, \lambda, \mu) = (273, 256, 240, 240)$.

Integrality Condition

A has eigenvalue k with multiplicity 1 and two other eigenvalues $r, s (r > s)$ satisfying $x^2 + (\mu - \lambda)x + (\mu - k) = 0$. The multiplicities of these eigenvalues are

$$f = \frac{-k(s+1)(k-s)}{(k+rs)(r-s)} \text{ and } g = \frac{k(r+1)(k-r)}{(k+rs)(r-s)}$$

and these must clearly be integers. In fact if $f \neq g$ then r and s must be integers. The other case (i.e. $f = g$) is called the *half-case*.

A partial converse to the condition on eigenvalues is the following useful theorem.

THEOREM. A regular connected graph is strongly regular if its adjacency matrix has three eigenvalues.

We shall call a parameter set (v, k, λ, μ) *feasible* if the necessary conditions given above are satisfied.

2. Partial Geometries

A *partial geometry* $pg(K, R, T)$ is a partial linear space (X, L) with constant line size K (i.e. $|L| = K$ for $L \in L$), such that each point is on R lines and such that given a line L and a point $x \notin L$ there are exactly T lines through x which meet L . (Thas [70] uses the notation $s + 1 = K, t + 1 = R, \alpha = T$). If $x \in L, y \in L$ we write $x \sim y$ and say that the points are collinear. The *point graph* of a partial geometry has the points as vertices and an edge $\{x, y\}$ iff $x \sim y$. The point graph of a $pg(K, R, T)$ is strongly regular (possibly trivial) with parameters

$$v = K(1 + \frac{(K-1)(R-1)}{T}), k = R(K-1), \lambda = (K-2) + (R-1)(T-1),$$

$$\mu = RT, r = K-1-T, s = -R.$$

It follows that the necessary conditions for the existence of a *srg* yield necessary conditions for the existence of a *pg*.

If a *srg* has parameters such that it could be the point graph of a *pg* then we call the *srg* *pseudo-geometric* and *geometric* if it is indeed the point graph of a *pg*. We shall see that a pseudo-geometric *srg* is not necessarily geometric. Bose [4] proved the following theorem.

THEOREM. If a *srg* is pseudo-geometric corresponding to $pg(K, R, T)$ and if $2K > R(R-1) + T(R+1)(R^2 - 2R + 2)$ then the graph is geometric. (In fact it follows that $T=R$ or $T=R-1$.)

The partial geometries can be divided into four classes:

1. A *pg* with $T = K$ (dually $T = R$) is a $2-(v, K, 1)$ *design* (*dual design*),

2. A pg with $T = R-1$ (dually $T = K-1$) is a *net (transversal design)*.
3. A pg with $T = 1$ is called a *generalized quadrangle*.
4. If $1 < T < \min\{K-1, R-1\}$ then we call the pg *proper*.

We are only interested in generalized quadrangles (notation: $pg(K, R, 1) = GQ(K-1, R-1)$) and in proper partial geometries.

Constructions for $GQ(s, t)$ are known for $(s, t) =$ resp. $(q, 1), (q, q), (q, q^2), (q^2, q^3), (q-1, q+1)$ and the dual sets (here q is a prime power). The proper pg 's described in Thas [70] have

$K = 2^h - 2^m + 1, R = 2^h - 2^{h-m} + 1, T = (2^m - 1)(2^{h-m} - 1), 0 < m < h,$
resp.

$$K = 2^h, R = 2^{h+m} - 2^h + 2^m, T = 2^m - 1.$$

For recent new constructions see Section 11.

In the following table we list the parameters of possible partial geometries with $v < 100$ (excluding designs and nets and their duals); (of a dual pair we list the one with $K \leq R$).

v	k	λ	μ	K	R	T	
15	6	1	3	3	3	1	$GQ(2, 2)$, graph is $T(6)^*$
27	10	1	5	3	5	1	$GQ(2, 4)$, Schläfli graph
28	15	6	10	4	5	2	pg does not exist, cf. Section 11A
40	12	2	4	4	4	1	$GQ(3, 3)$
45	28	15	21	5	7	3	known, cf. Section 11B
64	18	2	6	4	6	1	$GQ(3, 5)$
66	45	28	36	6	9	4	pg does not exist, cf. Section 11B
70	42	23	28	7	7	4	unknown, several graphs known
75	32	10	16	5	8	2	graph unknown, cf. Section 11H
76	21	2	7	4	7	1	$GQ(3, 6)$ does not exist, cf. Section 11C
81	30	9	12	6	6	2	sporadic, cf. Section 11D
85	20	3	5	5	5	1	$GQ(4, 4)$
91	66	45	55	7	11	5	unknown, graph is $T(14)^*$, cf. Section 11B
95	40	12	20	5	10	2	graph unknown
96	35	10	14	6	7	2	graph unknown
96	50	22	30	6	10	3	graph unknown

3. Nonexistence Results

The following six necessary conditions for the existence of a srg will exclude several feasible sets. In each case we give a feasible parameter set which is excluded by that condition but not by any of the other restrictions, showing that these six restrictions are independent.

A. The Half-Case (conference matrices)

If $f = g$ then we must have $v = 4\mu + 1$, $k = 2\mu$, $\lambda = \mu - 1$. If such a srg exists then there is a conference matrix C of order $v + 1$ (cf. Section 7B). It is known (cf. Belevitch [1], van Lint and Seidel [51]) that such a C can only exist if v is the sum of two squares. This kills e. g. $(v, k, \lambda, \mu) = (21, 10, 4, 5)$.

B. The Krein Conditions

It is not difficult to show that all the numbers q_{ij}^k must be nonnegative (not necessarily integers). By explicitly computing these numbers we find the four so-called Krein conditions (Krein [50], Scott [60], Higman [39], Delsarte [26]). Two of these are always satisfied. The two nontrivial restrictions are:

$$\begin{aligned} \text{Krein1: } q_{11}^1 &= \frac{f^2}{v} \left(1 + \frac{r^3}{k^2} + \frac{(-1-r)^3}{l^2} \right) \geq 0, \\ \text{Krein2: } q_{22}^2 &= \frac{g^2}{v} - 1 + \frac{s^3}{k_2} + \frac{(-1-s)^3}{l^2} \geq 0. \end{aligned}$$

These conditions can be written as

$$\begin{aligned} (r+1)(k+r+2rs) &\leq (k+r)(s+1)^2 \\ (s+1)(k+s+2rs) &\leq (k+s)(r+1)^2. \end{aligned}$$

This kills e. g. $(v, k, \lambda, \mu) = (184, 48, 2, 16)$.

C. The Absolute Bound

A suitable linear combination of I, A , and J is the Gram matrix of inner products of a spherical two-distance set in \mathbb{R}^f . Since such a set has at most $\frac{1}{2}f(f+3)$ vectors (cf. Koornwinder [49], Delsarte, Goethals and Seidel [27], Seidel [63]) we have

$$v \leq \frac{1}{2}f(f+3) \text{ and similarly } v \leq \frac{1}{2}g(g+3).$$

This kills e. g. $(v, k, \lambda, \mu) = (50, 21, 4, 12)$.

This bound was improved by Neumaier [58] to

$$v \leq \frac{1}{2}f(f+1) \text{ unless } q_{11}^1 = 0.$$

(Graphs with $q_{11}^1 = 0$ or $q_{22}^2 = 0$ are called Smith graphs.) This bound kills e.g. $(v, k, \lambda, \mu) = (841, 200, 87, 35)$.

D. The Claw Bound

Bose [4] first studied the connection between strongly regular graphs and partial geometries and derived the inequality which we mentioned in Section 2. This idea was extended by Neumaier [57] and after an improvement by Brouwer it obtained the following form:

$$\text{If } \mu \neq s^2, \mu \neq s(s+1) \text{ then } 2(r+1) \leq s(s+1)(\mu+1).$$

The idea of the proof is to show that if r is large, then the srg is the point graph of a partial geometry and then applying the absolute bound and the Krein conditions to the line graph of this partial geometry. This kills e.g. $(v, k, \lambda, \mu) = (2058, 242, 91, 20)$.

E. The Case $\mu = 1$

As was remarked by Bose and Dowling [5], if $\mu = 1$ then the set of neighbours of a point is a union of $(\lambda+1)$ -cliques. Hence $(\lambda+1) | k$. This kills e.g. $(v, k, \lambda, \mu) = (726, 29, 4, 1)$. The total number of $(\lambda+2)$ -cliques is $vk/(\lambda+1)(\lambda+2)$ which must be an integer. This kills e.g. $(v, k, \lambda, \mu) = (209, 16, 3, 1)$.

F. The Case $\mu = 2$

It was observed by Brouwer and Neumaier [8] that if $\mu = 2$ then the set of neighbours of a point carries the structure of a partial linear space of girth at least five. This implies that if $k < \frac{1}{2}\lambda(\lambda+3)$ then $(\lambda+1) | k$. This kills e.g. $(v, k, \lambda, \mu) = (736, 42, 8, 2)$. Slightly more information is available which also kills $(v, k, \lambda, \mu) = (1944, 67, 10, 2)$. Besides the above six necessary conditions we mention a few other results.

G. The Mu-Bound

By combining various bounds Neumaier [57] proved that

$$\mu \leq s^3(2s+3)$$

with equality iff $v = \frac{1}{2}f(f+3)$. This is a very useful bound, but since it follows as a consequence of the bounds mentioned above it cannot lead to new nonexistence results.

H. Sporadic Nonexistence Results

For two graphs nonexistence was shown by ad hoc arguments. The first of these has $(v, k, \lambda, \mu) = (49, 16, 3, 6)$, (Bussemaker et al [9]) and in this case part of the argument is a computer search. The second set is $(v, k, \lambda, \mu) = (57, 14, 1, 4)$, (Wilbrink and Brouwer [73]). The proof depends on counting arguments for subgraphs. More specifically it is shown that both the assumption that the graph contains a 15-coclique and the contrary lead to a contradiction. These two parameter sets were the smallest for which the existence question could not be settled by any of the other conditions. At present the smallest open case is $(v, k, \lambda, \mu) = (65, 32, 15, 16)$ corresponding to a conference matrix of order 66.

4. Restrictions on Subsets

Although we are primarily interested in constructions we shall have occasion to use some results of a more theoretical nature. First of all we have the so called Hoffman bounds on the size of a clique or a coclique in the graph:

THEOREM 1. Let C be a clique in a strongly regular graph with parameters $(v, k, \lambda, \mu, r, s)$. Then

$$|C| \leq 1 + k/(-s)$$

and equality holds iff each point outside C is adjacent to exactly $\mu/(-s)$ points of C .

COROLLARY. Suppose a graph is pseudogeometric with parameters (K, R, T) . If we can find cliques of size $K (= 1 + k/(-s))$ such that each edge is in exactly one such clique then the graph is geometric (and $T = \mu/(-s)$).

THEOREM 2. Let C be a coclique in a strongly regular graph with parameters $(v, k, \lambda, \mu, r, s)$. Then

$$|C| \leq \frac{v \cdot (-s)}{k-s}$$

and equality holds iff each point outside C is adjacent to exactly $-s$ points of C .

Sometimes the Cvetković bound is sharper:

THEOREM 3. The size of a coclique C of a graph G cannot exceed the number of nonnegative (nonpositive) eigenvalues of G .

THEOREM 4. (Haemers [35]). Suppose a coclique C in a strongly regular graph G meets both previous bounds, i. e.,

$$|C| = g = v \cdot (-s)/(k-s).$$

Then the subgraph of G induced by the complement of C is strongly regular.

A second group of results (Cameron, Goethals & Seidel [11]) concerns the case of equality in the absolute bound or the Krein condition. Let us quote only one theorem.

THEOREM 5. Let G be a strongly regular graph with $q_{11}^1 = 0$ or $q_{22}^2 = 0$. Then for every vertex x the subconstituents $\Gamma(x) (= \{y | y \sim x\})$ and $\Delta(x) (= \{y | y \neq x, y \not\sim x\})$ are both strongly regular.

5. Uniqueness Results

A. Triangular Graphs

The graphs with as vertices the pairs from $\{1, 2, \dots, n\}$ joined by an edge if they have an element in common are the triangular graphs $T(n)$. They have as parameters $v = \binom{n}{2}$, $k = 2(n-2)$, $\lambda = n-2$, $\mu = 4$. Graphs with these parameters are unique for $n \neq 8$. For $n = 8$ apart from $T(8)$ there exist exactly three other graphs with the same parameters, the so-called Chang graphs. All four graphs are switching equivalent (cf. Chang [16,17], Connor [22], Hoffman [42], Seidel [63]).

B. Lattice Graphs

The lattice graph $L(n)$ has as vertices the points $(x, y) \in \{1, 2, \dots, n\}^2$, joined by an edge if the first or second coordinates are the same. They have $v = n^2$, $k = 2(n-1)$, $\lambda = n-2$, $\mu = 2$. Graphs with these parameters are unique for all $n \neq 4$. For $n = 4$ there is one graph besides $L(4)$, the so-called Shrikhande graph. These two are switching equivalent (cf. Shrikhande [65]).

C. Small Graphs

All graphs on $v < 25$ vertices are uniquely determined by their parameters except for the parameter set $(16, 6, 2, 2)$ referred to above. Apart from triangular graphs and lattice graphs (and their complements) this set includes the Paley graphs of order 5, 9, 13 and 17 and the Clebsch graph with parameters $(v, k, \lambda, \mu) = (16, 5, 0, 2)$.

D. Sporadic Graphs

For the following parameter sets it is known that a unique graph exists.

v	k	λ	μ	r	s	f	g	name	unique by
16	5	0	2	1	-3	10	5	Clebsch	Seidel [61], Clatworthy [18]
27	10	1	5	1	-5	20	6	Schläfli ($GQ(2,4)$)	Seidel [61]
50	7	0	1	2	-3	28	21	Hoffman-Singleton	Hoffman & Singleton [43]
56	10	0	2	2	-4	35	20	Sims-Gewirtz	Gewirtz [31,32]
77	16	0	4	2	-6	55	21	co sub HS ($S(3,6,22)$)	Brouwer [6]
100	22	0	6	2	-8	77	22	Higman-Sims	Gewirtz [32]
112	30	2	10	2	-10	90	21	sub McL ($GQ(3,9)$)	Cameron, Goethals & Seidel [11]
162	56	10	24	2	-16	140	21	co sub McL	Cameron, Goethals, & Seidel [11]
275	112	30	56	2	-28	252	22	McLaughlin	Goethals & Seidel [34]

E. Geometric Graphs

Cameron, Goethals and Seidel [11] showed that a pseudo-geometric graph corresponding to $GQ(q, q^2)$ is always geometric. This allows one to show that the graph is unique as soon as uniqueness of the generalized quadrangle is known.

F. Regular Two-Graphs

A two-graph is a set Ω of triples from a set X such that every 4-subset of X contains an even number of triples from Ω . For the strong connection between srg 's and two-graphs and the theory of switching we refer to Seidel [62]. The regular two-graphs on 16, 28 and on 276 vertices are unique (Taylor [69], Goethals and Seidel [34]) and each has a transitive group of automorphisms. By isolating a point through switching and then removing it, this shows that the Schläfli and McLaughlin graphs are unique. It also shows that graphs with parameters $(v, k, \lambda, \mu) = (16, 6, 2, 2)$ or $(28, 12, 6, 4)$ or $(276, 135, 78, 54)$ are switching equivalent. For $v = 16$ see Section 5B; for $v = 28$ see Section 5A; for $v = 276$ only one graph is known.

6. Strongly Regular Graphs found by Merging Classes in an Association Scheme

A. The Johnson Scheme

In the Johnson scheme $\binom{n}{3}$ call two triples adjacent if they have exactly one point in common. When $n = 7$ or $n = 10$ this defines a *srg*. (For $n = 7$ this is the line graph of $PG(3,2)$; for $n = 10$ we find parameters $v = 120, k = 63, \lambda = 30, \mu = 36$.)

In the Johnson scheme $\binom{n}{4}$ call two quadruples adjacent if they have either zero or three points in common. When $n = 9$ or $n = 11$ this defines a *srg*. Parameters: $v = 126, k = 25, \lambda = 8, \mu = 4$ resp. $v = 330, k = 63, \lambda = 24, \mu = 9$. In the Johnson scheme $\binom{12}{4}$ call two quadruples adjacent if they have either zero or two points in common. We get a *srg*(495,238,109,119). (These graphs were found by Mathon [55a].)

B. The Hamming Scheme

In the Hamming scheme 4^n call two vectors adjacent if their distance is odd. This yields a *srg* with parameters $v = 2^{2n}$, $k = 2^{2n-1} + \epsilon 2^{n-1}$, $\lambda = \mu = 2^{2n-2} + \epsilon 2^{n-1}$ with $\epsilon = (-1)^{n-1}$. If n is even this graph has the same parameters as the line graph of an orthogonal array (or transversal design) $OA(2^n, 2^{n-1})$, but for $n > 2$ these are nonisomorphic. In all cases these graphs have the parameters of the graphs in Hubaut's family C_{12} .

In the Hamming scheme 3^n call two two distinct vectors i -th associates iff their distance is congruent to $i \pmod{3}$, ($i = 1, 2, 3$). This yields a 3-class association scheme. If n is even then merging any two of the three classes yields a *srg*. Parameters: write $n = 2m$, then $v = 3^{2m}$.

(a) adjacent iff distance $\equiv 0 \pmod{3}$ (but nonzero):

$$\begin{aligned} k &= 3^{2m-1} + (-1)^m \cdot 2 \cdot 3^{m-1} - 1, \\ \lambda &= 3^{2m-2} + (-1)^m \cdot 2 \cdot 3^{m-1} - 2, \\ \mu &= 3^{2m-2} + (-1)^m \cdot 3^{m-1}. \end{aligned}$$

(b) adjacent iff distance $\equiv 1 \pmod{3}$
resp. adjacent iff distance $\equiv 2 \pmod{3}$:

$$\begin{aligned} k &= 3^{2m-1} - (-1)^m \cdot 3^{m-1}, \\ \lambda &= 3^{2m-2}, \\ \mu &= 4 \cdot 3^{2m-2} + (-1)^m \cdot 3^{m-1} - 2 \end{aligned}$$

(cf. Kageyama, Saha and Das [46])

A simpler proof than the one in [46] can be given by using the fact that the eigenvalues of the Hamming scheme are values of Krawtchouk polynomials. Well known relations for these polynomials show that merging leads to a situation with only three different eigenvalues and then the theorem in Section 1 can be applied. (The weight of a vector (mod 3) is the value of the quadratic form $Q(x) = \sum x_i^2$. From this, one immediately sees that the two graphs under *b*) are isomorphic and that the graph *a*) is isomorphic to the appropriate one in Hubaut's family C_{12} .)

C. The Cyclotomic Scheme

Let $q = p^{(e-1)t}$ where p is a prime, t is even, e is a prime > 2 and p is primitive (mod e). Call two elements x, y of the field IF_q j -th associates ($0 < j \leq e$) if $x - y = \alpha^{ei+j}$ for some i , where α is a fixed primitive element of IF_q . Now any single relation of this association scheme gives us a (rank 3) strongly regular graph of negative Latin square type, and the union of u classes gives a *srg* with parameters

$$\begin{aligned} v &= q, \quad k = \frac{q-1}{e} \cdot u, \quad \lambda = (u^2q - 3ue + u^2 - (e-u)(e-2u)q^{\frac{1}{2}})/e^2, \\ \mu &= (u^2q - ue + u^2 + u(e-2u)q^{\frac{1}{2}})/e^2, \\ r &= (-1 + q^{1/2})u/e, \quad s = r - q^{1/2}, \\ f &= (q - 1)(e - u)/e, \quad g = k \end{aligned}$$

(cf. van Lint and Schrijver [52]).

More generally one might take q a prime power, e a divisor of $q-1$ and call two elements x, y of IF_q j -th associates ($0 < j \leq e$) if $x - y = \alpha^{ei+j}$ for some i . Sometimes a union of classes yields a *srg*. Examples where this works:

$$q=81, \quad e=8, \quad v=81, \quad k=30, \quad \lambda=9, \quad \mu=12$$

(vanLint & Schrijver [52], Mesner [56]),

$$q=243, \quad e=11: \quad v=243, \quad k=22, \quad \lambda=1, \quad \mu=2$$

(Berlekamp, vanLint and Seidel [3]),

$$\nu=243, k=110, \lambda=37, \mu=60$$

(Delsarte [26], dual of previous),

$$q=256, e=15, \nu=256, k=68, \lambda=12, \mu=20$$

(Brouwer, unpublished).

D. Other Examples

Let q be a power of two. The group $PSL(2, q)$ acts as a rank $\frac{1}{2}q$ group on the $\frac{1}{2}q(q-1)$ exterior lines of a hyperoval in $PG(2, q)$. The corresponding association scheme has valencies $n_0 = 1$ and $n_i = q+1$ ($1 \leq i \leq \frac{1}{2}q-1$). Under the larger group $P\Sigma L(2, q)$ some of these classes merge, and H. D. L. Hollmann [44] observed that when $q = 16$ one obtains a 3-class association scheme with valencies 1, 17, 34, 68 such that merging relations R_1 and R_2 yields a srg with parameters $\nu = 120, k = 51, \lambda = 18, \mu = 24$. (Most likely this graph is isomorphic to the $q = 4$ member of the Metz family treated in Section 7A.)

$P\Gamma L(2, 8)$ acts 2-transitively (on the Ree unital with parameters $S(2, 4, 28)$) on 28 points. $PGL(2, 8)$ acts as a rank 4 group, giving a 3-class association scheme with parameters $\nu = 28, n_1 = n_2 = n_3 = 9$,

$$p_{ij}^1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 2 & 4 & 2 \\ 0 & 4 & 2 & 3 \\ 0 & 2 & 3 & 4 \end{pmatrix}, p_{jk}^i = p_{j+1, k+1}^{i+1} \text{ for } \{i, j, k\} \subset \{1, 2, 3\}, \text{ (addition mod 3).}$$

The unital is the collection of 4-sets with a proper colouring, i. e., two disjoint edges in each of the three relations. A second association scheme with the same parameters can be obtained from the previous one by 'switching' with respect to a block of the unital; its group of automorphisms is elementary abelian of order 8. These two are the only schemes with these parameters (Hollman [44]). The eigenvalues of these schemes are

$$P = \begin{pmatrix} 1 & 9 & 9 & 9 \\ 1 & \lambda_1 & \lambda_2 & \lambda_3 \\ 1 & \lambda_3 & \lambda_1 & \lambda_2 \\ 1 & \lambda_2 & \lambda_3 & \lambda_1 \end{pmatrix}$$

where the λ_i are the solutions of $\lambda^3 + \lambda^2 - 9\lambda - 1 = 0$ (cf. Mathon [53]).

Now consider the direct product of two such schemes. This is a 15-class association scheme with relations

$$R_{ij} = \{(xx', yy') \mid (x, y) \in R_i \text{ and } (x', y') \in R_j\}$$

and eigenvalues $\mu_i \mu_j$ where $\mu_i (\mu_j)$ runs through the eigenvalues of $R_i (R_j)$. One finds that $R_{11} \cup R_{22} \cup R_{33}$ has eigenvalues $3n^2 = 243$, $k(\lambda_1 + \lambda_2 + \lambda_3) = -9$, $\lambda_1^2 + \lambda_2^2 + \lambda_3^2 = 19$ and $\lambda_1 \lambda_2 + \lambda_1 \lambda_3 + \lambda_2 \lambda_3 = -9$ so that we have found a strongly regular graph.

Similarly, $R_{11} \cup R_{22} \cup R_{33} \cup R_{10} \cup R_{20} \cup R_{30}$ has eigenvalues 270, 18 and -10, and $R_{11} \cup R_{22} \cup R_{33} \cup R_{10} \cup R_{20} \cup R_{30} \cup R_{01} \cup R_{02} \cup R_{03}$ has eigenvalues 297, 17, and -11.

Thus we have found strongly regular graphs with parameters

v	k	λ	μ	r	s	f	g
784	243	82	72	19	-9	243	540
784	270	98	90	18	-10	270	513
784	297	116	110	17	-11	297	486

This construction is due to R. Mathon (private communication) and applies more generally to pseudo-cyclic 3-class association schemes. The graphs produced have the parameters of Latin square graphs and have new parameters only when v is not a prime power. Write $p_{1j}^1 = (1 \ t-r-s-1 \ s \ r)$ and $n_j = t$. Now a pseudo-cyclic association scheme $SPC(r, s, t)$ on $v = 3t + 1$ points produces three strongly regular graphs on v^2 points with $k = 3t^2 + 3it$, $r = 2t + 1 - i$, $s = -t - i$ ($i = 0, 1, 2$). Now Hollmann constructed a $SPC(50, 58, 165)$ on 496 points (see [44]) so that we find new graphs on 246016 vertices.

The symmetric group S_9 acts as a rank 5 group on the 280 partitions of a 9-set into three triples. (The valencies are 1, 27, 36, 54 and 162.) The graph (X, R_4) is strongly regular with parameters (280, 162, 96, 90) (Mathon and Rosa [55b]).

7. Strongly Regular Graphs found using Classical Groups and Geometries

A. A Construction by R. Metz

Let Q be a nonsingular elliptic quadric in $PG(5, q)$. Let p be a point and let Π be a hyperplane such that p is not on Q or on Π . Let $V := \{x \in \Pi \mid px \text{ meets } Q \text{ in two points}\}$. Let $L := \{l \mid l \text{ is a line of } \Pi \text{ such that the plane } \langle p, l \rangle \text{ meets } Q \text{ in two intersecting lines}\}$. Let $I := \{(x, l) \mid x \in l\}$. Then (V, L, I) is a semipartial geometry with parameters $s+1 = q, t+1 = q^2 + 1, \alpha = 2, \mu = 2q(q-1)$. (For a definition of semipartial geometry see Debroey and Thas [25].) The point graph of this geometry is strongly regular with parameters:

$$v = \frac{1}{2}q^2(q^2-1), \quad k = (q-1)(q^2+1), \quad \lambda = (q-1)(q+2), \quad \mu = 2q(q-1),$$

$$r = q-1, \quad s = -(q-1)^2, \quad f = \frac{1}{2}(q-2)(q+1)(q^2+1), \quad g = \frac{1}{2}q(q^2+1).$$

The above description seems to be due to Hirschfeld and Thas; the description given by Metz being the following (for a discussion of the isomorphism of these descriptions, see Debroey [24]): Let Q be a nonsingular quadric in $PG(4, q)$. Let V be the set of hyperplanes meeting Q in an elliptic quadric. If $x, y \in V$ then let $x \sim y$ if the corresponding elliptic quadrics are tangent. (Now lines can be defined as bundles of mutually tangent quadrics.) This representation of $\Omega(5, q)$ is rank 3 for $q = 4, 8$.

B. A Variant by Brouwer and Wilbrink

Let Q be a nonsingular quadric in $PG(4, q)$. Let V be the set of hyperplanes meeting Q in a hyperbolic quadric. If $x, y \in V$ then let $x \sim y$ if the corresponding hyperbolic quadrics are tangent (i.e., if $x \cap y \cap Q$ consists of two intersecting lines). This defines a *srg* with parameters

$$v = \frac{1}{2}q^2(q^2+1), \quad k = (q+1)(q^2-1),$$

$$\lambda = (q-1)(3q+2), \quad \mu = 2q(q+1),$$

$$r = q^2-2q-1, \quad s = -q-1, \quad f = \frac{1}{2}q(q+1)^2,$$

$$g = \frac{1}{2}(q-2)(q+1)(q^2+1).$$

Again one has lines (bundles of mutually tangent quadrics), and $s+1 = q, t+1 = (q+1)^2$. This time one does not obtain a

semipartial geometry, but a structure with the property that given a point p and a line l such that $p \notin l$ there are 0 or 2 or q points on l joined to p . For odd q the complementary graph is pseudo-geometric corresponding to a $pg(\frac{1}{2}(q^2+1), q(q-2), \frac{1}{2}(q-1)^2-1)$, and it is indeed possible to find cliques of size $\frac{1}{2}(q^2+1)$ in this graph: Let E be an elliptic line (i.e., $E \cap Q = \emptyset$) and define $C_E := \{x \in V \mid E^\perp \subset x \text{ or } E \subset x\}$. Then C_E is a clique of size $\frac{1}{2}(q^2+1)$. The number of elliptic lines E is $\frac{1}{2}q^3(q-1)(q^2+1)$ and for $q > 3$ each C_E determines E uniquely, so that we have far too many cliques for a partial geometry, and it turns out to be impossible to choose an appropriate subcollection. For $q = 3$ however, each C_E belongs to ten distinct lines E and we find 27 sets C_E yielding a partial geometry $pg(5, 3, 1)$. Of course this is the well known generalized quadrangle $GQ(4, 2)$, thus illustrating the sporadic isomorphism $PSU(4, 2) \cong \Omega(5, 3)$.

C. Generalization of the Preceding Two Constructions

Wilbrink observed that the preceding two constructions in fact work in all dimensions. Thus: Let Q be a nonsingular quadric in $PG(2m, q)$. Let V be the set of hyperplanes meeting Q in a hyperbolic (elliptic) quadric. Let $\epsilon = +1(-1)$. If $x, y \in V$ then let $x \sim y$ if the corresponding quadrics are tangent (i.e., if $Q \cap x \cap y$ is degenerate). This defines a *srg* with parameters

$$\begin{aligned} v &= \frac{1}{2}q^m(q^m + \epsilon), k = (q^m - \epsilon)(q^{m-1} + \epsilon), \\ \lambda &= 2(q^{2m-2} - 1) + \epsilon q^{m-1}(q-1) \\ \mu &= 2q^{m-1}(q^{m-1} + \epsilon), \\ r, s &= -\epsilon q^{m-1} - 1 \text{ and } \epsilon q^{m-1}(q-2) - 1. \end{aligned}$$

D. Another Construction by Wilbrink

Let Q be a nonsingular quadric in $PG(2m, 5)$. Let V be the set of nonisotropic points x such that x^\perp meets Q in a $\left\{ \begin{array}{l} \text{hyperbolic} \\ \text{elliptic} \end{array} \right\}$ quadric. Let $\epsilon = \pm 1$. If $x, y \in V$ then $x \sim y$ if $x \perp y$. This defines a *srg* with parameters

$$\begin{aligned}
v &= \frac{1}{2}5^m(5^m + \epsilon), k = \frac{1}{2} \cdot 5^{m-1}(5^m - \epsilon), \\
\lambda &= \frac{1}{2} \cdot 5^{m-1}(5^{m-1} + \epsilon), \mu = \frac{1}{2} \cdot 5^{m-1}(5^{m-1} - \epsilon), \\
r, s &= 2\epsilon \cdot 5^{m-1} \text{ and } -\epsilon \cdot 5^{m-1}.
\end{aligned}$$

E. A construction by D. Taylor

Let H be a hermitian form on the projective plane $PG(2, q^2)$, where q is an odd prime power. Let $U = \{x \mid H(x, x) = 0\}$ be the corresponding unital. Then $|U| = q^3 + 1$. One can define a regular two-graph on U by taking the triples $\{x, y, z\}$ for which $-H(x, y)H(y, z)H(z, x)$ is a nonsquare in $GF(q^2)$. Isolating a point ∞ by switching and then deleting it yields a srg with parameters

$$\begin{aligned}
v &= q^3, k = 2\mu = \frac{1}{2}(q-1)(q^2+1), \lambda = \frac{1}{4}(q-1)^3-1, \\
r &= \frac{1}{2}(q-1), s = -\frac{1}{2}(q^2+1), f = (q-1)(q^2+1), g = q(q-1).
\end{aligned}$$

(This is Hubaut's family C_{20} .) Note that collinear triples of the unital are members of the two-graph, so that after switching $\Lambda\{\infty\}$ has become a clique, for each line l passing through ∞ . Thus our graph has a partition into cliques of size q (achieving the Hoffman bound) so that any point outside such a clique has exactly $\frac{1}{2}(q-1)$ neighbours on it. Consequently, if we take the union of any $\frac{1}{2}(q^2+1)$ of these cliques we get a regular subgraph of degree $\frac{1}{4}(q-1)(q^2+3)$, and adding a new point Ω and switching yields a srg with parameters

$$\begin{aligned}
v &= q^3 + 1, k = \frac{1}{2}q(q^2+1), \lambda = \frac{1}{4}(q-1)(q^2+3), \mu = \frac{1}{4}(q+1)(q^2+1), \\
r &= \frac{1}{2}(q-1), s = -\frac{1}{2}(q^2+1), f = (q-1)(q^2+1), g = q^2 - q + 1.
\end{aligned}$$

(It is conceivable that there is another srg in the switching class of this regular two-graph. It would have $k = \frac{1}{2}q^2(q-1), \lambda = \frac{1}{4}(q^2+1)(q-3), \mu = \frac{1}{4}(q-1)(q^2-1)$. For $q = 3$ such a graph does not exist (it violates the absolute bound). For $q = 5$ such a graph was constructed by Goethals (see Section 10B on the Hoffman-Singleton graph). For larger q such graphs are unknown. Note that for $q = 3^{2n+1}$ the Ree groups have a doubly transitive representation on q^3+1 points, yielding a regular two-graph with the

same parameters as the one associated with $P\Gamma U(3, q^2)$ discussed above. This again leads to srg 's on q^3 points (and possibly $q^3 + 1$?).

F. A Construction by W.M. Kantor

Let V be a vector space over $GF(q)$, equipped with a symplectic, orthogonal or unitary geometry. A *spread* of V is a family Σ of maximal totally isotropic or singular subspaces which partitions the set P of totally isotropic or singular points.

Construct a graph with as vertices the set Ω of all hyperplanes of members of Σ , and join two distinct vertices X, Y when $X \cap Y^\perp \neq 0$. This graph is strongly regular and has the same parameters as the graph with vertices P and as edges the pairs of perpendicular points. This construction and variations of it produce large numbers of non-isomorphic strongly regular graphs (see Kantor [46]).

8. Combinatorial Constructions

A. A Construction by Haemers

Let $q = 2^n$ and let K be a hyperoval in $PG(2, q)$ (considered as hyperplane at infinity of $AG(3, q)$). Let V be the set of lines in $AG(3, q)$ having a point of K as their point at infinity, and for $x, y \in V$ set $x \sim y$ if x and y meet (in an affine point). This defines a strongly regular graph with parameters

$$v = q^2(q+2), k = q(q+1), \lambda = \mu = q,$$

$$r = q, s = -q, f = (\frac{1}{2}q+1)(q^2-1), g = \frac{1}{2}q(q+1)^2.$$

(This is Hubaut's family C_{18} .) (In fact, with the pencils of lines as Lines this is a $GQ(q+1, q-1)$.) Now partition the $q+2$ parallel classes of lines in an arbitrary way into $\frac{1}{2}q+1$ pairs: $k = \cup_{i=1}^{q/2+1} \{u_i, v_i\}$. Redefine adjacency so that for each i two lines with direction u_i (resp. v_i) become adjacent iff the plane they span passes through the point v_i (resp. u_i), while no line with direction u_i will be adjacent to a line with direction v_i . All other adjacencies are unchanged. This defines a new srg with parameters

$$v = q^2(q+2), k = q^2 + q - 1, \lambda = q - 2, \mu = q,$$

$$r = q - 1, s = -(q+1), f = \frac{1}{2}(q+2)(q^2+q-1), g = \frac{1}{2}q(q^2+q-1).$$

B. A Construction by Mathon

A symmetric conference matrix of order n is a matrix C with zero diagonal and entries ± 1 elsewhere such that $C = C^T$ and $CC^T = (n-1)I$. A necessary condition for the existence of such a matrix is $n \equiv 2 \pmod{4}$. A *srg* with half-case parameters ($v-1 = 2k = 4\mu$) exists iff there exists a symmetric conference matrix of order $v+1$. (Given such a matrix, normalize it such that the first row and first column contain 0 and +1's only and delete row and column. If the resulting matrix is B then $A := \frac{1}{2}(B+J-I)$ is the adjacency matrix of a *srg* and the process may be reversed.) Symmetric conference matrices of order n are known in the following cases (cf. Wallis et al [72], Appendix C):

- (i) $n = q+1$, q a prime power, $q \equiv 1 \pmod{4}$,
- (ii) $n = (h-1)^2+1$, h the order of a skew Hadamard matrix,
- (iii) $n = (m-1)^2+1$, m the order of a symmetric conference matrix, a any nonnegative integer.

Mathon gave a new recursive construction for symmetric conference matrices proving that if $q = 4t-1$ is a prime power and there exists a symmetric conference matrix of order $q+3$, then there exists a symmetric conference matrix of order $q^2(q+2)+1$. (See Mathon [54] or Seidel and Taylor [64].) This produces e.g. *srg*(45,22,10,11).

C. A construction by Goethals and Seidel

Let N be the V by B incidence matrix of a Steiner system $S(2,K,V)$. Let H be a Hadamard matrix of order $R+1$; normalize it so that its first column contains ones only and delete this column. Call the resulting $R+1$ by R matrix L . (Now $JL = 0$.) Let P be the $(R+1)V$ by B matrix obtained from N by replacing each row by $R+1$ rows: replace the R one entries of a row by the R columns of L and the zero entries of the row by zero columns. Let $PP^T = RI + A$. Then $(A+RI)(A-(V+K-1)I) = 0$ and $AJ = -RJ$. (See [61].) It follows that $\frac{1}{2}(J-I-A)$ is the adjacency matrix of a *srg* with parameters

$$v = (R+1)V, \quad k = \frac{1}{2}(v-1+R), \quad \lambda = \frac{1}{4}(R-1)(V+K+2), \quad \mu = \frac{1}{4}(R+1)(V+K),$$

$$r = \frac{1}{2}(R-1), \quad s = -\frac{1}{2}(V+K), \quad f = v-1-g, \quad g = B = RV/K.$$

(These are the parameters of a pseudo-geometric graph corresponding to $pg(R+1, \frac{1}{2}(V+K), \frac{1}{2}(R+1))$, but e.g. $pg(4,5,2)$ does not exist so that these graphs cannot always be geometrical.)

D. Regular Symmetric Hadamard Matrices with Constant Diagonal

Suppose H is a regular symmetric Hadamard matrix of order n with constant diagonal, i.e., $HH^T = nI$, $H = H^T$, $HJ = aJ$, $h_{ij} \in \{\pm 1\}$, $h_{ii} = \epsilon$ ($1 \leq i, j \leq n$). (Then clearly $n = a^2$.) Then $\frac{1}{2}(J - I - H + \epsilon I)$ is the adjacency matrix of a *srg* with parameters

$$v = n, k = \frac{1}{2}(n - 1 - a + \epsilon), \lambda = \frac{1}{4}(n - 2a) - (1 - \epsilon), \mu = \frac{1}{4}(n - 2a).$$

Obviously, by replacing H by $-H$ if necessary, we may assume $a > 0$. Also, if $n > 1$ then $4|n$ so that we may write $a = 2t$. This yields for the parameters:

$$\begin{aligned} \text{If } \epsilon = +1: v = 4t^2, \quad k = 2t^2 - t, \quad \lambda = \mu = t^2 - t, \\ r, s = \pm t, \quad f = 2t^2 - t, \quad g = 2t^2 + t - 1. \end{aligned}$$

(These are the parameters of a pseudo Latin square graph $L_t(2t)$.)

$$\begin{aligned} \text{If } \epsilon = -1: v = 4t^2, \quad k = 2t^2 - t - 1, \quad \lambda = t^2 - t - 2, \quad \mu = t^2 - t, \\ r = t - 1, \quad s = -t - 1, \quad f = 2t^2 + t, \quad g = 2t^2 - t - 1. \end{aligned}$$

(These are the parameters of a negative Latin square graph $NL_t(2t)$.) (And conversely, graphs with these parameters yield regular symmetric Hadamard matrices with constant diagonal.) In view of the above we are interested in the set of pairs (n, ϵ) for which such a matrix (with positive row sums) exists. Let us call this set *RSHCD*. There is one recursive construction:

$$(m, \delta), (n, \epsilon) \in \text{RSHCD} \rightarrow (mn, \delta\epsilon) \in \text{RSHCD}.$$

The following direct constructions are known:

- (i) $(4, \pm 1), (36, \pm 1) \in \text{RSHCD}$,
- (ii) If there exists a Hadamard matrix of order m then $(m^2, 1) \in \text{RSHCD}$,
- (iii) If both $a-1$ and $a+1$ are odd prime powers then $(a^2, 1) \in \text{RSHCD}$,
- (iv) If $a+1$ is a prime power and there exists a symmetric conference matrix of order a then $(a^2, 1) \in \text{RSHCD}$,
- (v) If there is a set of $t-2$ mutually orthogonal latin squares of order $2t$ (i.e., a transversal design $T[t; 2t]$) then $(4t^2, 1) \in \text{RSHCD}$.
- (iv) Suppose we have a Steiner system $S(2, K, V)$ with $V = K(2K-1)$. If we form the blockgraph and add an isolated point we get a graph in the switching class of a regular two-graph. The

corresponding Hadamard matrix is symmetric with constant diagonal, but not regular (as claimed in [72], p. 454). If moreover this Steiner system is invariant under a regular abelian group of automorphisms (which necessarily has orbits on the blocks of sizes V , V and $2K-1$) then by switching with respect to a block orbit of size V we obtain a *srg* with parameters

$$v = 4K^2, \quad k = V = K(2K-1), \quad \lambda = \mu = K(K-1)$$

showing that $(4K^2, 1) \in RSHCD$. Steiner systems $S(2, K, K(2K-1))$ are known for $K = 3, 5, 6, 7$ or 2^t , but only for $K = 2, 3, 5, 7$ are systems known that have a regular abelian group of automorphisms. Thus we find $(196, 1) \in RSHCD$. [The required switching set also exists when the design is resolvable: take the union of K parallel classes. Resolvable designs are known for $K = 3$ or 2^t .]

(Most of the constructions in this section are due to Goethals & Seidel [33]; for more details see Wallis [72], Section 5.3.)

9. Rank 3 Representations of Finite Groups

The rank 3 permutation representations of the finite classical groups have been determined by Kantor and Liebler [48]. Their main result is the following: (we quote)

“THEOREM 1. Let M be one of the groups $Sp(2m-2, q)$, $\Omega^\pm(2m, q)$, $\Omega(2m-1, q)$ or $SU(m, q)$ for $m \geq 3$ and q a prime power. Let $M \trianglelefteq G$ with $G/Z(M) \leq \text{Aut}(M/Z(M))$. Assume that G acts as a primitive rank 3 permutation group on the set X of cosets of a subgroup K of G . Then (at least) one of the following holds up to conjugacy under $\text{Aut}(M/Z(M))$.

- (i) X is an M -orbit of singular (or isotropic) points.
- (ii) X is an M -orbit of maximal totally singular (or isotropic) subspaces and $M = Sp(4, q), SU(4, q), SU(5, q), \Omega^-(6, q), \Omega^+(8, q)$, or $\Omega^+(10, q)$.
- (iii) X is any M -orbit of nonsingular points and $M = SU(m, 2), \Omega^\pm(2m, 2), \Omega^\pm(2m, 3)$ or $\Omega(2m-1, 3)$.
- (iv) X is either orbit of nonsingular hyperplanes of $M = \Omega(2m-1, 4)$ or $\Omega(2m-1, 8)$ (where $G = \Gamma O(2m-1, 8)$ in the latter case).
- (v) $M = SU(3, 3), K \cap M = PSL(3, 2)$.
- (vi) $M = SU(3, 5), K \cap M = 3.A_7$.
- (vii) $M = SU(4, 3), K \cap M = 4.PSL(3, 4)$.
- (viii) $M = Sp(6, 2), K = G_2(2)$.

- (ix) $M = \Omega(7,3), K \cap M = G_2(3)$.
- (x) $M = SU(6,2), K \cap M = 3.PSU(4,3).2$.

THEOREM 2. Let $M = PSL(n, q) \leq G \leq AutM$. Assume that G acts as a primitive rank 3 permutation group on the set X of cosets of a subgroup K of G . Then (at least) one of the following occurs up to conjugacy under $Aut M$.

- (i) X is the set of lines for $M, n \geq 4$.
- (ii) $M = PSL(2,4) \cong PSL(2,5), |X| = \binom{5}{2}$,
 $M = PSL(2,9) \cong A_6, |X| = \binom{6}{2}$
 $G = PSL(4,2) \cong A_8, |X| = \binom{8}{2}$, or
 $G = P\Gamma L(2,8), |X| = \binom{9}{2}$.
- (iii) $M = PSL(3,4), M \cap K = A_6$.
- (iv) $M = PSL(4,3), M \cap K = PSp(4,3)$.

The reader is cautioned that isomorphisms between, and automorphisms of, the various groups allow many different ways of viewing some of these cases. Specifically, the isomorphisms $PSp(4, q) \cong \Omega(5, q)$, $PSU(4, q) \cong P\Omega^-(6, q)$, $PSL(4, q) \cong P\Omega^+(6, q)$ and $PSU(4, 2) \cong PSp(4, 3)$ lead to numerous representations under (1(i)). Similarly, a triality automorphism of $P\Omega^+(8, q)$ can be applied to (1(iii)) and a polarity automorphism can be applied to (2(i)).”.

Not each of these permutation representations was mentioned by Hubaut, but it seems that no new graphs are obtained. (For the parameters in case 1(iv) and arbitrary q see Section 7 C - Wilbrink’s construction.)

F. Zara [75] was so kind as to supply us with the parameters of two rank 3 representations of the Fischer groups not mentioned by Hubaut:

v	k	l	λ	μ	$\bar{\lambda}$	$\bar{\mu}$	G	G_0
14080	10920	3159	8408	8680	918	648	Fi_{22}	$O^+(7,3)$
137632	109200	28431	86600	86800	6030	5832	Fi_{23}	$(O^+(8,3))' . S_3$

(Here $O^+(n,3)$ is the subgroup of the orthogonal group generated by the reflections $x \rightarrow x + (x, a)a$, where $(a, a) = 1$.)

10. Some Sporadic Graphs

A. The Higman-Sims Family

Almost all sporadic graphs are constructed in some way from the Steiner systems $S(5,6,12)$ or $S(5,8,24)$ or from the binary or ternary Golay code. (In other words, all graphs described in this section can be found somewhere in the Leech lattice.) Let us start with graphs derived from the Steiner system $S(3,6,22)$.

(i) $(v, k, \lambda, \mu) = (77, 16, 0, 4)$.

Take as vertices the blocks of the (unique) design $S(3,6,22)$ and join two vertices by an edge when the corresponding blocks are disjoint. This produces a strongly regular graph with the stated parameters. (It is unique - Brouwer [6].)

(ii) $(v, k, \lambda, \mu) = (56, 10, 0, 2)$ - the Gewirtz graph.

The preceding graph contains a coclique of size 21 (i.e. meeting both the Hoffman bound and the Cvetković bound) namely all blocks containing a fixed point. Deleting this coclique produces a strongly regular graph with the stated parameters. It was first found by C.C. Sims (unpublished) and shown to be unique by Gewirtz [31]. Adding the identity matrix to an adjacency matrix of this graph we obtain a symmetric block design $2-(56, 11, 2)$ (a biplane). (This always works when $\mu = \lambda + 2$; when $\mu = \lambda$ then the graph itself already gives a symmetric block design. We shall soon see an example where the graph is derived from a block design (with polarity).)

(iii) $(v, k, \lambda, \mu) = (100, 22, 0, 6)$ - the Higman-Sims graph.

Take a symbol ∞ , the 22 points and the 77 blocks of $S(3,6,22)$. Join ∞ to the 22 points, join a point to the blocks containing it, and join two blocks when they are disjoint. This produces a strongly regular graph with the stated parameters. Its uniqueness was shown by Gewirtz [32]. The simple group of Higman and Sims is an index two subgroup of its automorphism group.

(iv) $(v, k, \lambda, \mu) = (231, 30, 9, 3)$ - the Cameron graph.

Take as vertices the $\binom{22}{2} = 231$ unordered pairs from the point set of $S(3,6,22)$. Join two vertices when the pairs are disjoint and their union is contained in a block. This produces a graph with the given parameters (cf. [11]). One may introduce lines as triples of pairwise disjoint pairs with as their union a block of the design; the geometry thus obtained is a gamma space in Higman's terminology, i.e. given a point x not on a line L , then x is joined to 0,1 or all points of L (as observed by [8a] and [10a]).

(v) $(v, k, \lambda, \mu) = (176, 49, 12, 14)$.

(Perhaps this graph does not belong in this section - for instance it does not have 2 as an eigenvalue.) Let x_0 and x_1 be two fixed points in the point set of $S(5, 8, 24)$. Construct a symmetric design with as points the blocks containing x_0 but not x_1 , and as blocks the blocks containing x_1 but not x_0 . A point B is incident with a block B' iff $|B \cap B'| \in \{0, 4\}$. It is straightforward to check that this produces a symmetric $2-(176, 50, 14)$ design. (This design was first constructed by G. Higman [40]; this representation is due to Margaret Smith [67].) This design has exactly two polarities, one with 176 and one with 80 fixed points (Sims [66], Brouwer [7]). By using a polarity of the first kind we can write the incidence matrix A as a symmetric matrix with ones on the diagonal, and $A-I$ is the adjacency matrix of the required *srg*. The graph has S_8 as group of automorphisms. Uniqueness is unknown. For a detailed discussion of G. Higman's geometry and its relations to the Hoffman-Singleton graph, see Broué and Enguehard [5a].

B. The Hoffman-Singleton Family

Moore showed that a graph with valency k and girth $2d+1$ can have at most $1+k+k(k-1)+\dots+k(k-1)^{d-1}$ vertices. (A similar bound can be obtained for the case for even girth.) Graphs that meet this bound are called Moore graphs. For $d=2$ we find $v \leq k^2+1$, and one sees immediately that the graph must be strongly regular when equality holds (with $\lambda=0$ and $\mu=1$). The parameter set $(k^2+1, k, 0, 1)$ is admissible only for $k=2, 3, 7$ and 57 and we find for $k=2$ the pentagon, for $k=3$ the Petersen graph $T(5)^*$, for $k=7$ the Hoffman-Singleton graph to be discussed below, while the case $k=57$ is still open.

(i) $(v, k, \lambda, \mu) = (50, 7, 0, 1)$ - the Hoffman-Singleton graph.

Let $X = \{\infty\} \cup \mathbf{P} \cup \mathbf{B}$ be the point set of the Higman-Sims graph as constructed above. Let B_0 be a block not containing ∞ of a design $S(4, 7, 23)$ with point set $\mathbf{P} \cup \{\infty\}$ that has our design (\mathbf{P}, \mathbf{B}) as its derived design at the point ∞ . Then B_0 induces a partition of the point set of the Higman-Sims graph as follows:

$$X = (\{\infty\} \cup B_0 \cup \{B \in \mathbf{B} \mid |B \cap B_0| = 1\}) \cup (\mathbf{P} \setminus B_0 \cup \{B \in \mathbf{B} \mid |B \cap B_0| = 3\}).$$

One can check that :

- (a) The induced subgraph on each of the two sets into which X is partitioned is (isomorphic to) the Hoffman-Singleton graph.
- (b) Any partition of the Higman-Sims graph into two Hoffman-Singleton graphs arises in this way.

(The Hoffman-Singleton graph was first constructed and proved to be unique by Hoffman and Singleton [43]. The observation that the Higman-Sims graph contains two copies of the Hoffman-Singleton graph seems to be due to Sims [66].)

Conversely, one may define the Higman-Sims graph in terms of the Hoffman-Singleton graph: The Hoffman-Singleton graph contains 100 cliques of size 15. Two such cliques intersect in 0, 3, 5, 8, or 15 points. This defines a 4-class association scheme (relations: R_0 : identity, R_1, R_2, R_3, R_4 : intersection of cardinality 8, 5, 3, 0 respectively) with parameters $v = 100, n_0 = 1, n_1 = 15, n_2 = 42, n_3 = 35, n_4 = 7$

$$\begin{aligned}
 (p_{0j}^i) = I_5, (p_{1j}^i) &= \begin{pmatrix} 0 & 15 & 0 & 0 & 0 \\ 1 & 0 & 14 & 0 & 0 \\ 0 & 5 & 0 & 10 & 0 \\ 0 & 0 & 12 & 0 & 3 \\ 0 & 0 & 0 & 15 & 0 \end{pmatrix}, & (p_{2j}^i) &= \begin{pmatrix} 0 & 0 & 42 & 0 & 0 \\ 0 & 14 & 0 & 28 & 0 \\ 1 & 0 & 35 & 0 & 6 \\ 0 & 12 & 0 & 30 & 0 \\ 0 & 0 & 36 & 0 & 6 \end{pmatrix}, \\
 (p_{3j}^i) &= \begin{pmatrix} 0 & 0 & 0 & 35 & 0 \\ 0 & 0 & 28 & 0 & 7 \\ 0 & 10 & 0 & 25 & 0 \\ 1 & 0 & 30 & 0 & 4 \\ 0 & 15 & 0 & 20 & 0 \end{pmatrix}, & (p_{4j}^i) &= \begin{pmatrix} 0 & 0 & 0 & 0 & 7 \\ 0 & 0 & 0 & 7 & 0 \\ 0 & 0 & 6 & 0 & 1 \\ 0 & 3 & 0 & 4 & 0 \\ 1 & 0 & 6 & 0 & 0 \end{pmatrix} \\
 P &= \begin{pmatrix} 1 & 15 & 42 & 35 & 7 \\ 1 & -15 & 42 & -35 & 7 \\ 1 & 0 & -3 & 0 & 2 \\ 1 & 5 & 2 & -5 & -3 \\ 1 & -5 & 2 & 5 & -3 \end{pmatrix}, & Q &= \begin{pmatrix} 1 & 1 & 56 & 21 & 21 \\ 1 & -1 & 0 & 7 & -7 \\ 1 & 1 & -4 & 1 & 1 \\ 1 & -1 & 0 & -3 & 3 \\ 1 & 1 & 16 & -9 & -9 \end{pmatrix}
 \end{aligned}$$

This scheme is imprimitive: (X, R_4) is the disjoint union of two Hoffman-Singleton graphs. The graph (X, R) is a regular thin near octagon, and is uniquely determined by its parameters. Since $(X, R_1 \cup R_4)$ has only three eigenvalues it is strongly regular, and we find the Higman-Sims graph again. One may construct a system of three 'linked' Hoffman-Singleton graphs on 150 vertices by taking the vertices and each of two sets of 50 cliques; join vertices in the obvious way. Now on each of the three sets of 100 points this graph induces a Higman-Sims graph.

For later use we mention two other constructions of the Hoffman-Singleton graph.

(ii) The first construction is due to N. Robertson [59].

Let P_α ($\alpha \in \mathcal{IF}_5$) be five pentagons and let Q_β ($\beta \in \mathcal{IF}_5$) be five

pentagrams. Number the points of the P_α and Q_β with elements of IF_5 such that points in P_α are adjacent iff their labels differ by one, and in Q_β iff their labels differ by two. Draw an edge between point i of P_α and point j of Q_β whenever $j = i + \alpha\beta$. This produces the Hoffman-Singleton graph.

COROLLARY: It is possible to split the point set of the Hoffman-Singleton graph into two halves such that the induced subgraph on each of these halves is the union of five pentagons. The union of two pentagons from different halves induces a Petersen graph. Each pentagon uniquely determines a split so that there are 126 splits. Each split determines 25 Petersen graphs and any two splits determine a unique Petersen graph. In this way we find a $2-(126,6,1)$ design on splits and Petersen graphs: this design turns out to be the classical unital in $PG(2,5^2)$ so that we can identify the automorphism group of the Hoffman-Singleton graph as $P \Sigma U(3,5^2)$. (Benson and Losey [2].) Two split-halves meet in 10 or in 15 points; if we fix a point x and consider the graph on the splits, where two splits are joined if their halves containing x meet in 10 points, then we find a *srg* with parameters $(v,k,\lambda,\mu) = (126,50,13,24)$. Another construction of this graph is given under (vii).

- (iii) The second construction uses the well known fact that it is possible to identify the lines of $PG(3,2)$ with the triples formed out of seven elements in such a way that intersecting lines correspond to triples that have exactly one element in common.

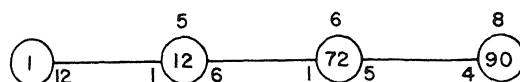
Let C be the set of points and D the set of lines of $PG(3,2)$, so that $|C| = 15$ and $|E| = 35$. Join a line to the points on it and join two lines when the corresponding triples are disjoint. This produces the Hoffman-Singleton graph.

- (iv) $(v,k,\lambda,\mu) = (175,72,20,36)$ and a new partial geometry.

On the 175 edges of the Hoffman-Singleton graph (abbreviated Ho-Si) we have a 3-class association scheme (with relations $R_0 =$ identity, $R_1 =$ intersecting, $R_2 =$ together in a pentagon, $R_3 =$ the remainder), and $\text{Aut}(\text{Ho-Si})$ acts as a rank 4 group. The parameters are

$$(p_{0j}^i) = I_4, (p_{1j}^i) \begin{pmatrix} 0 & 12 & 0 & 0 \\ 1 & 5 & 6 & 0 \\ 0 & 1 & 6 & 5 \\ 0 & 0 & 4 & 8 \end{pmatrix} (p_{2j}^i) = \begin{pmatrix} 0 & 0 & 72 & 0 \\ 0 & 6 & 36 & 30 \\ 1 & 6 & 20 & 45 \\ 0 & 4 & 36 & 32 \end{pmatrix}, (p_{3j}^i) = \begin{pmatrix} 0 & 0 & 0 & 90 \\ 0 & 0 & 30 & 60 \\ 0 & 5 & 45 & 40 \\ 1 & 8 & 32 & 49 \end{pmatrix}.$$

The graph corresponding to R_1 is distance transitive of diameter 3 with diagram



and eigenvalues $12^1, 7^{28}, 2^{21}, (-2)^{125}$; it is the line graph of Ho-Si. (In fact, any distance regular graph with parameters $i(12, 6, 5; 1, 1, 4)$ is necessarily this graph: $\mu = 1$ implies that each edge lies in a unique 7-clique; there are 50 such 7-cliques and one sees immediately that the graph on the 7-cliques is Ho-Si.)

The graph corresponding to R_2 is strongly regular with parameters $(v, k, \lambda, \mu) = (175, 72, 20, 36)$ and eigenvalues $72^1, 2^{153}, (-18)^{21}$.

W.H. Haemers [36] observed that it carries the structure of a partial geometry $\text{pg}(5, 18, 2)$:

We have to find 630 lines of size 5, i.e. 630 5-cliques. But a 5-clique is a set of 5 edges of Ho-Si, pairwise in a pentagon. These 5 edges induce a Petersen subgraph in Ho-Si. Ho-Si contains 525 Petersen graphs and each Petersen graph contains 6 matchings, i.e. our graph contains 3150 5-cliques and we must choose one-fifth of them in such a way that any pentagon determines a unique 'special' Petersen graph. To this end let C be a 15-coclique and write the point set of Ho-Si as $C \cup D$. Observe the following:

- (1) The 35 points in D each have 3 neighbours in C , and we may identify C and D with the points and lines of the projective space $PG(3, 2)$ as above.
- (2) Of the 1260 pentagons, 630 have one point in C and 630 have two points in C . Let $P_i = \{ \text{pentagons with } i \text{ points in } C \}$ ($i = 1, 2$).
- (3) $\text{Aut}(\text{Ho-Si})_C \cong A_7$ acts transitively on each set of 630 pentagons.
- (4) There is a 1-1 map Π , invariant under this group, from P_2 onto P_1 such that each $P_2 \cup \Pi P_2$ induces a Petersen graph P with the property that the three points in $P \cap C$ have a common neighbour. (For: if $P_2 \cap C = \{x, y\}$ then the line $L = xyz$ is a vertex of P_2 , and P_2 together with the edge Lz determines a Petersen graph P . Now $P \cap C = \{x, y, z\} \subset \Gamma(L)$.)
- (5) There are 105 such special Petersen graphs; each is a union $P_1 \cup P_2$ in six ways. Each of them contains 6 matchings, and

thus we find the required 630 lines.

The partial geometry thus constructed has A_7 as its group of automorphisms.

(v) $(v, k, \lambda, \mu) = (630, 85, 20, 10)$.

Given the existence of a $pg(5, 18, 2)$ it follows by duality that $pg(18, 5, 2)$ exists. Its point graph has the indicated parameters.

(vi) $(v, k, \lambda, \mu) = (176, 70, 18, 24)$ and $(v, k, \lambda, \mu) = (176, 90, 38, 54)$.

The graph found under (iv) has $k = 2\mu$, so that adjoining an isolated point produces a regular two-graph on 176 vertices. This two-graph contains strongly regular graphs with the stated two parameter sets in its switching class. In order to construct such graphs we have to indicate suitable switching sets - in this case regular subgraphs of the 175-graph with 70 (resp. 90) points and valency 18 (resp. 38). For the first set there is a natural choice: in the representation of Ho-Si as $C \cup D$ used above there are 70 edges in D , and R_2 has valency 18 on this set of 70 edges. For the second set Haemers indicated the following possibility: Take the union of 18 pairwise disjoint 5-cliques. Since these cliques attain the Hoffman bound, each point outside a clique is adjacent to two points inside, so this union has valency $4 + 17 \cdot 2 = 38$. Such disjoint 5-cliques do exist: in fact from N. Robertson's representation of Ho-Si one sees immediately that there exist 25 pairwise disjoint 5-cliques.

(vii) $(v, k, \lambda, \mu) = (126, 50, 13, 24)$.

If we remove a vertex x and its neighbours from Ho-Si, we get a graph on 42 vertices and 126 edges. J.-M. Geothals observed that the subgraph of the 175-graph (iv) induced on these 126 edges is strongly regular with the stated parameters. The direct verification is somewhat tedious; it is easier to show that the graph has only three eigenvalues. To this end let us quote a very useful theorem from [35]:

THEOREM. Suppose

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{12}^* & A_{22} \end{pmatrix}$$

is hermitean of size n . Suppose A has just two eigenvalues, that is

$$\lambda_1(A) = \dots = \lambda_f(A) > \lambda_{f+1}(A) = \dots = \lambda_n(A)$$

for some f , $1 \leq f < n$. Let n_1 and n_2 be the sizes of A_{11} and A_{22} respectively. Then $\lambda_i(A_{22}) =$

$$\begin{aligned} \lambda_1(A) + \lambda_n(A) - \lambda_{f-i+1}(A_{11}) & \text{ for } \max\{1, f+1-n_1\} \leq i \leq \min\{f, n_2\}, \\ \lambda_1(A) & \text{ for } 1 \leq i \leq f+1-n_1, \\ \lambda_n(A) & \text{ for } f < i \leq n_2. \end{aligned}$$

This theorem can be applied to $(-1, +1, 0)$ adjacency matrices of strong graphs in the switching class of a regular two-graph (i.e., matrices satisfying $(A - \rho_1 I)(A - \rho_2 I) = 0$):

COROLLARY. Suppose some strong graph in the switching class of a regular two-graph with v vertices and eigenvalues ρ_1, ρ_2 ($\rho_1 > \rho_2$) with multiplicities μ_1, μ_2 has a subgraph with $u \geq \mu_1$ vertices and eigenvalues $\lambda_1, \dots, \lambda_u$, where $\lambda_1 = \rho_1$ and $\lambda_2 = \dots = \lambda_{\mu_1}$. Then the subgraph induced on the remaining $v-u$ vertices is a strong graph with two eigenvalues: $\rho_1 + \rho_2 - \lambda_2$ with multiplicity $\mu_1 - 1$ and ρ_2 with multiplicity $v - u - \mu_1 + 1$.

Application: Adding an isolated vertex to the 175 graph and switching with respect to the seven edges on x we get a graph containing Ho-Si in the switching class of the two-graph. The $(-1, +1, 0)$ eigenvalues of the two-graph are $\rho_1 = 35$ and $\rho_2 = -5$ (with $\mu_1 = 22, \mu_2 = 154$) and of Ho-Si 35, 5 and -5 with multiplicities 1, 21 and 28, so that the corollary produces a strong graph on 126 vertices. Observing that it is regular of valency 50 shows that we found the right graph.

C. The McLaughlin Family

(i) $(v, k, \lambda, \mu) = (275, 112, 30, 56)$ - the McLaughlin graph.

Consider the 23 points and 253 blocks of the (unique) Steiner system (X, \mathcal{B}) with parameters $S(4, 7, 23)$. Define a graph on 276 vertices with X as coclique by joining a point to the blocks containing it and joining two blocks whenever they intersect in one point. This defines a (graph in the switching class of a) regular two-graph on 276 points. (For a matrix proof see Seidel [62]; for the interpretation as a set of equiangular lines in the Leech lattice see Taylor [69].) Its automorphism group is Conway .3. Isolating a vertex by switching and then deleting it produces a rank 3 graph on 275 vertices with McLaughlin's simple group as group of automorphisms. Graph and two-graph are unique ([34]).

From the definition it is clear that the 276-graph constructed above contains the Higman-Sims graph (switched w.r.t. one vertex). The $(-1, +1, 0)$ eigenvalues of the two-graph are $\rho_1 = 55$ and $\rho_2 = -5$ with multiplicities $\mu_1 = 23, \mu_2 = 253$, and of the Higman-Sims graph 55, 15, -5 with multiplicities 1, 22, 77 (respectively) so that (by the corollary in B(vii)) we find a strongly regular graph on 176 vertices, with as vertices the blocks of the residual design $S(4, 7, 23) \setminus S(3, 6, 22)$

and as edges the pairs of blocks meeting in one point. The parameters are $(v, k, \lambda, \mu) = (176, 70, 18, 24)$, the same graph as we found above in $B(vi)$. This time we immediately see that its group of automorphisms is the Higman-Sims simple group. The McLaughlin graph is extremal in several respects: it has $g = 22$ and thus satisfies the absolute bound with equality; it has $q_{22}^2 = 0$ so it satisfies the Krein condition with equality; its complement has $s = -3$ and $\mu = 81$ and hence satisfies the μ -bound with equality.

(ii) $(v, k, \lambda, \mu) = (162, 56, 10, 24)$ and $(v, k, \lambda, \mu) = (112, 30, 2, 10)$.

Cameron, Goethals and Seidel [11] proved that $q_{22}^2 = 0$ implies that both subconstituents are strongly regular. Applying this to the previous graph produces two graphs with the stated parameters.

Let us give direct descriptions of these graphs. Under $PSL(3, 4)$ the projective plane $PG(2, 4)$ contains three orbits of Fano subplanes. Take as vertices the points, the lines, and one orbit of Fano subplanes. Then $v = 21 + 21 + 120 = 162$. Let the points and the lines be two 21-cocliques; join a point to a line when they are not incident; join a point to a Fano plane when they are incident; join a line to a Fano plane when it meets it in three points and join two Fano planes when they meet in one point. This produces the unique $srg(162, 56, 10, 24)$.

Let x_0 and x_1 be two fixed points in the point set of $S(4, 7, 23)$. Take as vertices the 112 blocks that contain exactly one of these points. Join two blocks that contain the same fixed point when this is the only point they have in common. Join a block containing x_0 to a block containing x_1 when they have three points in common. This produces the unique $srg(112, 30, 2, 10)$, the point graph of the generalized quadrangle $GQ(3, 9)$. From the construction it is clear that the point set of this graph can be split into two halves such that the graph induced on each half is the Gewirtz graph. Just as we saw for the Hoffman-Singleton graph, it is possible to construct a system of three 'linked' Gewirtz graphs on 168 vertices (starting with three fixed points x_0, x_1 and x_2 in $S(5, 8, 24)$ and taking as vertices those blocks that contain exactly two of these points, joining blocks B and B' iff $|B \cap B'| + 2|B \cap B' \cap \{x_0, x_1, x_2\}| = 6$) such that any two Gewirtz graphs induce the point graph of $GQ(3, 9)$. This graph is the point graph of a partial linear space with lines of size 6.

(iii) $(v, k, \lambda, \mu) = (276, 140, 58, 84)$.

Goethals and Seidel [34] showed (by explicitly indicating a switching set) that the regular two-graph on 276 vertices contains a srg with the stated parameters in its switching class. Uniqueness is unknown.

(iv) $(v, k, \lambda, \mu) = (243, 110, 37, 60)$ - the Delsarte graph.

In the course of their uniqueness proof for the 276-point regular two-graph Goethals and Seidel found that it contained in its switching class a graph with $11K_3$ (the disjoint union of 11 triangles) as induced subgraph, and with each of the remaining vertices having exactly one neighbour on each triangle. From the theorem in B(vii) one finds that the graph on the remaining 243 vertices has $(-1, +1, 0)$ eigenvalues $49^{22}, 22^1, (-5)^{220}$ and hence is the required graph. This graph was first constructed by Delsarte from the ternary Golay code (see below).

D. The Golay Codes and the Hill Cap

(i) $(v, k, \lambda, \mu) = (243, 22, 1, 2)$ - the Berlekamp-vanLint-Seidel graph.

In [3] three distinct constructions are given (one for each author). The easiest is the following: take as vertices the $3^5 = 243$ cosets of the perfect ternary Golay code, and join two vertices when the corresponding cosets have representatives differing by a vector of weight one. This graph has a Delsarte dual (since it has a regular abelian group of automorphisms)-this dual has parameters $(v, k, \lambda, \mu) = (243, 110, 37, 60)$; we saw it earlier as a subgraph of the 276-point regular two-graph.

(ii) $(v, k, \lambda, \mu) = (2048, 276, 44, 36)$ and
 $(v, k, \lambda, \mu) = (2048, 759, 310, 264)$.

Take the $2^{11} = 2048$ cosets of even weight of the extended binary Golay code as vertices, and join two cosets when they have representatives differing by a vector of weight two. This yields a $srg(2048, 276, 44, 36)$. (An equivalent but more complicated description can be found in Hubaut [45]. This graph was first constructed by J.H. Conway and M.S. Smith. Taking all 2^{12} cosets and joining two cosets when they have distance one yields a regular thin near octagon on 4096 vertices in which this graph is the $d = 2$ subgraph on one of the halves of the bipartition.) The Delsarte dual of this graph is a $srg(2048, 759, 310, 264)$. It was first constructed by Goethals and Seidel, who gave the following description: Take the $2^{11} = 2048$ cosets of $\{0, 1\}$ in the extended binary Golay code and join two cosets when they have Hamming distance 8.

(iii) $(v, k, \lambda, \mu) = (1288, 792, 476, 504)$.

The complement of the second graph in (ii) is a $srg(2048, 1288, 792, 840)$. Its first subconstituent is the required graph.

(iv) $(v, k, \lambda, \mu) = (729, 112, 1, 20)$ - the Games graph.

There is a unique 56-cap in $PG(5, 3)$ (that is a set of 56 points such that any line meets it in at most two points), cf. Hill [41]. R. Games observed that if we take as vertices the points of $AG(6, 3)$ and join two

vertices when the line through the points meets the hyperplane at infinity in a point of the cap, then we obtain a $srg(729, 112, 1, 20)$. (cf. [9a], [30].)

11. Recent Results on Partial Geometries

A. $pg(4, 5, 2)$ does not exist.

In [19] F. de Clerck classifies the pseudo-geometric graphs corresponding to a partial geometry with $T = K - 2$. An interesting result is the nonexistence of a $pg(4, 5, 2)$. In Section 5A we observed that there are exactly four non-isomorphic $srg(28, 15, 6, 10)$: the complement of the triangular graph $T(8)$ and the Chang graphs. For each of these one can assume that the srg is geometric and arrive at a contradiction. For the triangular graph the proof is as follows. Let the points be the unordered pairs (x, y) from $\{1, 2, \dots, 8\}$. A line is a partition of $\{1, 2, \dots, 8\}$. W.l.o.g. we can take (12) (34) (56) (78) as a line and in fact we can give all the lines through (12) w.l.o.g. This leaves two possibilities for the line through (13) and (24) and then it turns out to be impossible to give the remaining lines through (13). The other proofs are similar.

B. Partial Geometries from $T(n)^*$

If n is even then $srg\left(\binom{n}{2}, \binom{n-2}{2}, \binom{n-4}{2} \binom{n-3}{2}\right)$ is pseudo-geometric, corresponding to $pg\left(\frac{1}{2}n, n-3, \frac{1}{2}n-2\right)$. We saw above that it is possible to settle the question of existence by hand if $n = 8$. For $n = 10$ a $pg(5, 7, 3)$ is known. It can be constructed as follows. Consider a hyperoval Σ in $PG(2, 8)$. Take the 45 secants of Σ as new points and the points not on Σ as new lines. The new points and lines are the pg . If we number the points of Σ from 1 to 10 then the five secants through a point not on Σ yield a partition of $\{1, 2, \dots, 10\}$ corresponding to our representation above.

It was shown by R. Mathon [55] that there is one other $pg(5, 7, 3)$. (See also [29a].) At present an exhaustive computer search is being carried out by C.W.H. Lam for the next case, i.e. $pg(6, 9, 4)$. If this partial geometry does not exist then it will follow that a possible projective plane of order 10 does not have a hyperoval and hence does not have an extension.

Added in proof: After 183 days of computation the exhaustive search was completed and no pg was found [50a]. Thus, assuming correctness of the programs involved, no $pg(6, 9, 4)$ and no $S(3, 12, 112)$

exists.

C. $pg(4,7,1)$ does not exist

It is not known whether a $srg(76,21,2,7)$ exists or not but Dixmier and Zara [28,29] have shown that a $pg(4,7,1)$ cannot exist. We give a slight modification of their proof. Suppose the partial geometry exists.

- (i) Let $x \not\sim y$, $tr(x,y) := \{z \mid z \sim x, z \sim y\}$,
 $\Delta(x,y) := \{z \mid z \not\sim x, z \not\sim y\}$. Then $|tr(x,y)| = 7, |\Delta(x,y)| = 39$.
- (ii) For $i = 0, 1, \dots, 7$ let K_i be the subset of $\Delta(x,y)$ consisting of points which are collinear with i points in $tr(x,y)$ and let $n_i := |K_i|$.
- (iii) Elementary counting yields $\sum n_i = 39$, $\sum i \cdot n_i = 105$,
 $\sum \binom{i}{2} n_i = 105$.
- (iv) Let p and q be collinear points in Δ such that the line L through p and q does not meet $tr(x,y)$. Then for the other two points a and b on L we have $x \sim a, y \sim b$ and the lines through x and a , resp. y and b meet $tr(x,y)$ in different points. It follows that if $p \in K_i$ then $q \in K_{5-i}$. From this we find that $(7-i)n_i = (2+i)n_{5-i}$ and hence $n_6 = 0$. From (iii) we then find $n_0 = n_5 = n_7 = 0$,
 $n_1 = 4, n_2 = 12, n_3 = 15, n_4 = 8$.
- (v) Let $p \in K_1$ and let $p \sim z \in tr(x,y)$. From (iv) it follows that the other two points on the line pz are both in K_4 . There are six lines through p not meeting $tr(x,y)$ and on each we have a point in K_4 by (iv). Since $n_4 = 8$ we see that each point in K_1 is adjacent to each point in K_4 , contradicting $\mu = 7$.

D. The Sporadic Geometry $pg(6,6,2)$

In [52] van Lint and Schrijver describe a partial geometry with $K = R = 6$, $T = 2$ which is at present one of the two sporadic partial geometries. The corresponding srg is of the type described in Section 5C. The easiest description is as follows. The 81 vertices are the codewords of the $(5,4)$ linear code C over \mathbb{F}_3 with $(1,1,1,1,1)$ as parity check. Two vertices are joined by an edge if their distance is $\equiv 2 \pmod{3}$. The six codewords $(0,0,0,0,0)$, $(1,2,2,2,2)$, $\dots, (2,2,2,2,1)$ form a clique which we choose as a line S . The remaining lines are the translates $S + \underline{c}$ with $\underline{c} \in C$. Recently Cameron and van Lint [15] showed that the automorphism group of this geometry is the semidirect product of this translation group and S_6 .

The proof uses the following elegant description.

Let \bar{C} be the ternary repetition code of length 6. The sum of the coordinates of a vector in a coset of \bar{C} is a constant which we call the type of the coset. Let A_i be the set of cosets of type i . We define a tripartite graph Γ by joining the coset $\bar{C} + \underline{v}$ to the coset $\bar{C} + \underline{v} + \underline{w}$ for each vector \underline{w} of weight 1. Then any vector in A_i has six neighbours in A_{i+1} and six in A_{i+2} . If we take any A_i as point set and A_{i+1} as line set and define incidence by adjacency, we find the $pg(6,6,2)$. In fact we now have three 'linked' copies of this geometry.

It is of interest to observe that Γ is a subgraph of $srg(243,22,1,2)$ constructed by Berlekamp et al. (Section 10D(i)).

E. A New Infinite Sequence: $pg(2^{2n-1}, 2^{2n-1}+1, 2^{2n-2})$

Cohen [21] gave the first description of a $pg(8,9,4)$. It was (and still may be) sporadic. (*Added in proof:* Recently Tonchev [71a] showed the isomorphism of this pg with the dual of the pg described below.) Subsequently, Haemers and van Lint [37] gave a much simpler description of a $pg(9,8,4)$ as follows.

Take as vertices the words of weight 4 or 8 in $(\mathbb{F}_2)^9$ and join these by an edge if their distance is congruent to two (*mod* 4). Among the cliques of size 9 there are the sets corresponding to the rows of $(J-I)_9$,

$$\begin{pmatrix} 0 & \underline{1} & \underline{1} \\ \underline{1}' & (J-I)_4 & 0 \\ \underline{1}' & 0 & (J-1)_4 \end{pmatrix} \text{ and } \begin{pmatrix} J_3 & I_3 & 0 \\ 0 & J_3 & I_3 \\ I_3 & 0 & J_3 \end{pmatrix} \text{ and permutations.}$$

From these cliques $1+63+2 \cdot 28 = 120$ are chosen as lines. In order to do this in a 'natural' way the 9 positions are identified with the points of $PG(1,8)$ and the action of $PSL(2,8)$ is used. For details we refer to [37].

These examples led F. de Clerck, R.H. Dye and J.A. Thas to a third construction and subsequently the discovery of a new infinite sequence, namely the $pg(2^{2n-1}, 2^{2n-1}+1, 2^{2n-2})$. The construction is as follows. Let Q^+ be a hyperbolic quadric in $PG(4n-1,2)$. The set of maximal totally isotropic subspaces of Q^+ is divided into two disjoint families D_1 and D_2 . If H is a projective space of dimension $2n-2$ on Q^+ then Q^+ contains two maximal totally isotropic subspaces through H , one of each family. Together they determine a $2n$ -space which contains a unique hyperplane $M(H)$ through H and not on Q^+ . We observe that $M(H) \setminus H$ has 2^{2n-1} points not on Q^+ . Let Ψ be a spread of Q^+ consisting of elements of D_1 . We define:

$X := \{ \text{points not on } Q^+ \}$, $L := \{ \text{all spaces } M(H) \text{ where } H \text{ is a hyperplane in an element of } \Psi \}$.

We call the elements of L lines and take natural incidence. Then (X, L) is a $pg(2^{2n-1}, 2^{2n-1}+1, 2^{2n-2})$. For a proof we refer to [20].

The idea of using spreads or define srg 's and pg 's was generalized by Kantor (see Section 6E). He also proved that the partial geometry $pg(8,9,4)$ in the infinite sequence is isomorphic to the dual of the one constructed by Haemers and van Lint. They have A_9 as automorphism group!

F. A Possibly Infinite Sequence: $pg(3^{2h+1}, 3^{2h+1}+1, 2 \cdot 3^{2h})$

In [71] J.A. Thas generalized the construction given in 11E above, using hyperbolic quadrics in $PG(4h+3,3)$. Again the construction depends on the existence of spreads. Only for the cases $h = 0$ and $h = 1$ it is known that such spreads exist. The case $h = 0$ leads to a trivial geometry but $h = 1$ gives a new partial geometry $pg(27,28,18)$. At present this is a sporadic example.

G. The sporadic geometry $pg(5,18,2)$

A remarkable sporadic pg was constructed by W. Haemers [35] using the Hoffman-Singleton graph. This graph contains 525 Petersen graphs as induced subgraphs. A special class of 105 of these is defined. Let the edges of the Hoffman-Singleton graph be the points of a geometry and define lines as 1-factors of the special Petersen subgraphs. This produces $pg(5,18,2)$. More details were given in Section 10B.

H. Some interesting open cases.

In 11B we mentioned the $pg(6,9,4)$ connected with the projective plane of order 10. It is expected that by the end of 1982 it will be known whether this pg exists. (It is dead now.)

In the list of block designs in M. Hall's book [38] the first unknown design is number 35: a $2-(46,6,1)$. If this design exists then it produces $pg(9,6,6)$ and $srg(69,20,7,5)$ which are unknown.

The next unknown case is $pg(5,8,2)$ belonging to the unknown $srg(75,32,10,16)$. Since this geometry has eight lines per point and the same number of lines as $pg(9,8,4)$ one might hope to find it by deleting a suitably chosen set of 60 points from $pg(9,8,4)$. In [37] it was shown that this idea does not work.

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