BLOCKING SETS IN DESARGUESIAN PROJECTIVE PLANES

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ABSTRACT

Using theorems of Redéi, and of Brouwer and Schrijver, and Jamison, it is proved that a non-trivial blocking set in a desarguesian projective plane of order q has at least $q+\sqrt{(2q)}+1$ points, if q is at least 7, odd and not a square and $q \neq 27$. Further one can show that non-trivial blocking sets in the desarguesian planes PG(2, 11) and PG(2, 13) have at least 18 resp. 21 points, and this is best possible. In addition a nice description of a blocking set of size $q^t+q^{t-1}+1$ in the desarguesian plane PG$(2, q^t)$ is given, where q is some prime power.

Introduction

A blocking set in a linear space is a set S of points, such that each line intersects S in at least one point. S is called *non-trivial*, if no line is completely contained in S, in the case of a projective plane. In this note we want to derive lower bounds for the cardinality of S.

Two useful theorems

The following construction yields interesting blocking sets in the desarguesian plane PG$(2,q)$:
Let f: GF$(q) \to$ GF(q) be any non-linear function. Form a blocking set consisting of
(i) the q points forming the graph of f in AG$(2,q)$,
(ii) the directions determined by f on the line at infinity, say m points.

EXAMPLE 1. Let $q = p$ be a prime, $f(x) = x^{\frac{1}{2}(p+1)}$. This yields a blocking set of $\frac{3}{2}(p+1)$ points, which is conjectured to be best possible ([7], see also [6]).

2. Let $q = q_1^t$ $(t > 1)$, then GF(q_1) is a subfield of GF(q). Let f be the trace map from GF(q) to GF(q_1). Then $S = q+q/q_1+1$. This is also the best known, if q_1 is chosen maximal (compare [**2**, **4**]).

The following theorem gives lower bounds for m, where $q = p^n$, p prime.

THEOREM. ([**8**, p. 237], see also [**6**]).

$$m \geqslant \frac{q-1}{p^{\frac{1}{2}n}+1}+1, \quad \textit{and} \quad m \geqslant \frac{p+3}{2} \quad \textit{if } n=1.$$

COROLLARY. *Let X be a set of q points in the desarguesian affine plane of order q, determining m directions. Then m satisfies the above inequalities.*

Proof. Either X determines all directions, or there is a parallel class all of whose lines contain exactly 1 point of X.

Received 1 June 1984.

1980 *Mathematics Subject Classification* 05B25.

Bull. London. Math. Soc. 18 (1986) 132–134

Let S be a minimal blocking set; then each point of S is on at least one tangent. Let $p \in S$ be a point on t tangents, call one tangent l, and form a blocking set of $\mathrm{AG}(2, q) = \mathrm{PG}(2, q) \backslash l$ with $|S| - 1 + t - 1$ points in the obvious way.

THEOREM ([1, 5]). *A blocking set of a desarguesian affine plane* $\mathrm{AG}(2, q)$ *has at least* $2q - 1$ *points.*

As a consequence of this, one has that $t \geqslant 2q + 1 - |S|$ for each point in S. Using these two results it is now a trivial exercise to show that a blocking set in the desarguesian plane PG(2, 11) has at last 18 points, and a rather tedious one to prove $|S| \geqslant 21$ for PG(2, 13).

Blocking sets in the desarguesian plane PG(2, q)

It is well known, and due to Bruen [2], that $|S| \geqslant q + \sqrt{q} + 1$, with equality if and only if q is a square and S a Baer-subplane. When q is not a square this bound can be improved.

Let S be a blocking set of size $|S| = q + m$. If S contains an m-secant the corollary gives a lower bound for the cardinality of S. The next theorem treats the remaining case.

THEOREM. *Let S be a blocking set of size $q + m$ without an m-secant. Then*

$$|S| \geqslant q + \sqrt{(2q)} + 1.$$

Proof. Since each line contains at most $m - 1$ points of S, it follows that each point is on at most $q - 1$ tangents. Counting incident pairs (tangent, point not in S) in two ways, using the second theorem, one gets

$$q(q+m)(q-m+1) \leqslant (q^2 - m + 1)(q-1)$$

or, rewriting,

$$2q \leqslant (m-1)^2 + (m-1)/q, \quad \text{whence } m \geqslant \sqrt{(2q)} + 1.$$

COROLLARY. *Suppose q is odd, not a square, at least 7 and not 27. Let S be an arbitrary non-trivial blocking set of the desarguesian plane* PG(2, q). *Then* $|S| \leqslant q + \sqrt{(2q)} + 1$.

Final remarks

If $q < 7$ everything is known; if $q = 27$ we only get $|S| \geqslant 35$; if $q = 2^{2t+1}$ one obtains $|S| \geqslant 2^{2t+1} + 2^{t+1}$.

The first paper relating Redéi's theorem to blocking sets seems to be [3]. We wonder whether Redéi's theorem can be improved to $m \geqslant 1 + q/q_1$, where q_1 is the order of maximal subfield of GF(q).

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