BLOCKING SETS IN DESARGUESIAN PROJECTIVE PLANES

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ABSTRACT

Using theorems of Rédei, and of Brouwer and Schrijver, and Jamison, it is proved that a non-trivial blocking set in a desarguesian projective plane of order \( q \) has at least \( q + \sqrt{q} + 1 \) points, if \( q \) is at least 7, odd and not a square and \( q \neq 27 \). Further one can show that non-trivial blocking sets in the desarguesian planes \( \text{PG}(2,11) \) and \( \text{PG}(2,13) \) have at least 18 resp. 21 points, and this is best possible. In addition a nice description of a blocking set of size \( q + q^{-1} + 1 \) in the desarguesian plane \( \text{PG}(2, q^5) \) is given, where \( q \) is some prime power.

Introduction

A blocking set in a linear space is a set \( S \) of points, such that each line intersects \( S \) in at least one point. \( S \) is called non-trivial, if no line is completely contained in \( S \), in the case of a projective plane. In this note we want to derive lower bounds for the cardinality of \( S \).

Two useful theorems

The following construction yields interesting blocking sets in the desarguesian plane \( \text{PG}(2,q) \):
Let \( f: \text{GF}(q) \to \text{GF}(q) \) be any non-linear function. Form a blocking set consisting of
(i) the \( q \) points forming the graph of \( f \) in \( \text{AG}(2,q) \),
(ii) the directions determined by \( f \) on the line at infinity, say \( m \) points.

Example 1. Let \( q = p \) be a prime, \( f(x) = x^{p+1} \). This yields a blocking set of \( \frac{q(q+1)}{2} \) points, which is conjectured to be best possible ([7], see also [6]).
2. Let \( q = q_1^t \) (\( t > 1 \)), then \( \text{GF}(q_1) \) is a subfield of \( \text{GF}(q) \). Let \( f \) be the trace map from \( \text{GF}(q) \) to \( \text{GF}(q_1) \). Then \( S = q + q/q_1 + 1 \). This is also the best known, if \( q_1 \) is chosen maximal (compare [2, 4]).
The following theorem gives lower bounds for \( m \), where \( q = p^n \), \( p \) prime.

Theorem. ([8, p. 237], see also [6]).

\[
m \geq \frac{q-1}{p^n+1} + 1, \quad \text{and} \quad m \geq \frac{p+3}{2} \quad \text{if } n = 1.
\]

Corollary. Let \( X \) be a set of \( q \) points in the desarguesian affine plane of order \( q \), determining \( m \) directions. Then \( m \) satisfies the above inequalities.

Proof. Either \( X \) determines all directions, or there is a parallel class all of whose lines contain exactly 1 point of \( X \).

Received 1 June 1984.

1980 Mathematics Subject Classification 05B25.

Let $S$ be a minimal blocking set; then each point of $S$ is on at least one tangent. Let $p \in S$ be a point on $t$ tangents, call one tangent $l$, and form a blocking set of $AG(2, q) = PG(2, q) \setminus l$ with $|S| - 1 + t - 1$ points in the obvious way.

**Theorem** ([1, 5]). A blocking set of a desarguesian affine plane $AG(2, q)$ has at least $2q - 1$ points.

As a consequence of this, one has that $t \geq 2q + 1 - |S|$ for each point in $S$. Using these two results it is now a trivial exercise to show that a blocking set in the desarguesian plane $PG(2, 11)$ has at last 18 points, and a rather tedious one to prove $|S| \geq 21$ for $PG(2, 13)$.

**Blocking sets in the desarguesian plane $PG(2, q)$**

It is well known, and due to Bruen [2], that $|S| \geq q + \sqrt{q} + 1$, with equality if and only if $q$ is a square and $S$ a Baer-subplane. When $q$ is not a square this bound can be improved.

Let $S$ be a blocking set of size $|S| = q + m$. If $S$ contains an $m$-secant the corollary gives a lower bound for the cardinality of $S$. The next theorem treats the remaining case.

**Theorem.** Let $S$ be a blocking set of size $q + m$ without an $m$-secant. Then

$$|S| \geq q + \sqrt{(2q)} + 1.$$  

**Proof.** Since each line contains at most $m - 1$ points of $S$, it follows that each point is on at most $q - 1$ tangents. Counting incident pairs (tangent, point not in $S$) in two ways, using the second theorem, one gets

$$q(q + m)(q - m + 1) \leq (q^2 - m + 1)(q - 1)$$

or, rewriting,

$$2q \leq (m - 1)^2 + (m - 1)/q,$$  

whence $m \geq \sqrt{(2q)} + 1$.

**Corollary.** Suppose $q$ is odd, not a square, at least 7 and not 27. Let $S$ be an arbitrary non-trivial blocking set of the desarguesian plane $PG(2, q)$. Then

$$|S| \leq q + \sqrt{(2q)} + 1.$$  

**Final remarks**

If $q < 7$ everything is known; if $q = 27$ we only get $|S| \geq 35$; if $q = 2^{2+1}$ one obtains $|S| \geq 2^{2+1} + 2^{2+1}$.

The first paper relating Redéi’s theorem to blocking sets seems to be [3]. We wonder whether Redéi’s theorem can be improved to $m \geq 1 + q/q_1$, where $q_1$ is the order of maximal subfield of GF$(q)$.

**References**


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