A Radon Transform on Circles through the Origin in $\mathbb{R}^2$

M. Zwaan
Centre for Mathematics and Computer Science
P.O. Box 4079, 1009 AB Amsterdam, The Netherlands

We invert a special kind of Radon transform that maps a function to its integrals over circles through the origin.

Note. This paper is dedicated to Prof. H.A. Lauwerier on occasion of his 65th birthday.

1. Introduction

The Radon transform is an integral transform, that maps a function $f: \mathbb{R}^n \to \mathbb{R}$ to the set of its integrals over the hyperplanes of $\mathbb{R}^n$. If the hyperplanes of $\mathbb{R}^n$ are parametrized by a unit normal $\theta \in S^{n-1}$, the unit sphere in $\mathbb{R}^n$, and the distance to the origin $|p|$, then we denote the integral of f over one such plane by $Rf(p,\theta)$ and the map $f \mapsto Rf$ is the Radon transform.

This transform has many practical applications in engineering and medicine ([3,5,7]), but it is also of theoretical interest, with applications to partial differential equations, integral geometry and Lie groups ([1,2]). In a more abstract setting the Radon transform is defined as the transform that maps a function f defined on a differentiable manifold, to its integrals over certain submanifolds.

As a special case of this generalized Radon transform we want to study the Radon transform, denoted by Qf, which maps a function $f: \mathbb{R}^2 \to \mathbb{R}$ onto its integrals over circles through the origin. This transform has been studied by John [7] and Cormack & Quinto [5] for functions $f: \mathbb{R}^n \to \mathbb{R}$. Cormack and Quinto give an inversion formula for this transform by expressing the function f in terms of a (orthonormal) basis of spherical harmonics (in $L_2(S^{n-1})$), $f(p\theta) = \Sigma_l f_l(p) Y_l(\theta)$ ($p \in \mathbb{R}$, $\theta \in S^{n-1}$) and by giving an inversion formula for each component $f_l(p)$. In the case $n=2$ Cormack [6] gives an explicit alternative derivation of an inversion formula for the transform $f \mapsto Qf$.

However, for this interesting case it is possible to exploit the relation between circles and lines through a Möbius transform. In this way a rather elementary and transparent derivation of the inversion formula for Qf can be given. This is the purpose of the paper.

2. Definitions

Define the line $L_{p,\phi} \subset \mathbb{R}^2$ by

$$L_{p,\phi} := \left\{ x \in \mathbb{R}^2 \mid x \cdot \theta - p = 0 ; \theta = (\cos\phi, \sin\phi) \right\}. \tag{2.1}$$

Let $f: \mathbb{R}^2 \to \mathbb{R}$ be a continuous function with compact support. The Radon transform of f is

$$Rf(p,\phi) := \int_{L_{p,\phi}} f(x)dx. \tag{2.2}$$

Notice that $L_{p,\phi} = L_{-p,\phi+\pi}$. In fact, we can represent each line $L_{p,\phi}$ as a point on the cylinder, $(p,\theta) \in \mathbb{R} \times S^1$, with $\theta := (\cos\phi, \sin\phi)$; we can identify the points (p,θ) and $(-p,-\theta)$. This interpretation yields the relation

$$Rf(p,\phi) = Rf(-p,\phi+\pi). \tag{2.3}$$

With the relation between θ and ϕ, we will often write $Rf(p,\theta)$ instead $Rf(p,\phi)$. Let $C_{p',\phi'}$ be the circle through the origin with centre at $(p'/2\cos\phi', p'/2\sin\phi')$ (see Figure 1). The reason for the notation p' and ϕ' will become clear in Section 4.

$$C_{p',\phi'} := \{(x,y) \in \mathbb{R}^2 \mid (x_1 - \frac{p'}{2}\cos\phi')^2 + (x_2 - \frac{p'}{2}\sin\phi')^2 = \frac{(p')^2}{4}\}.$$

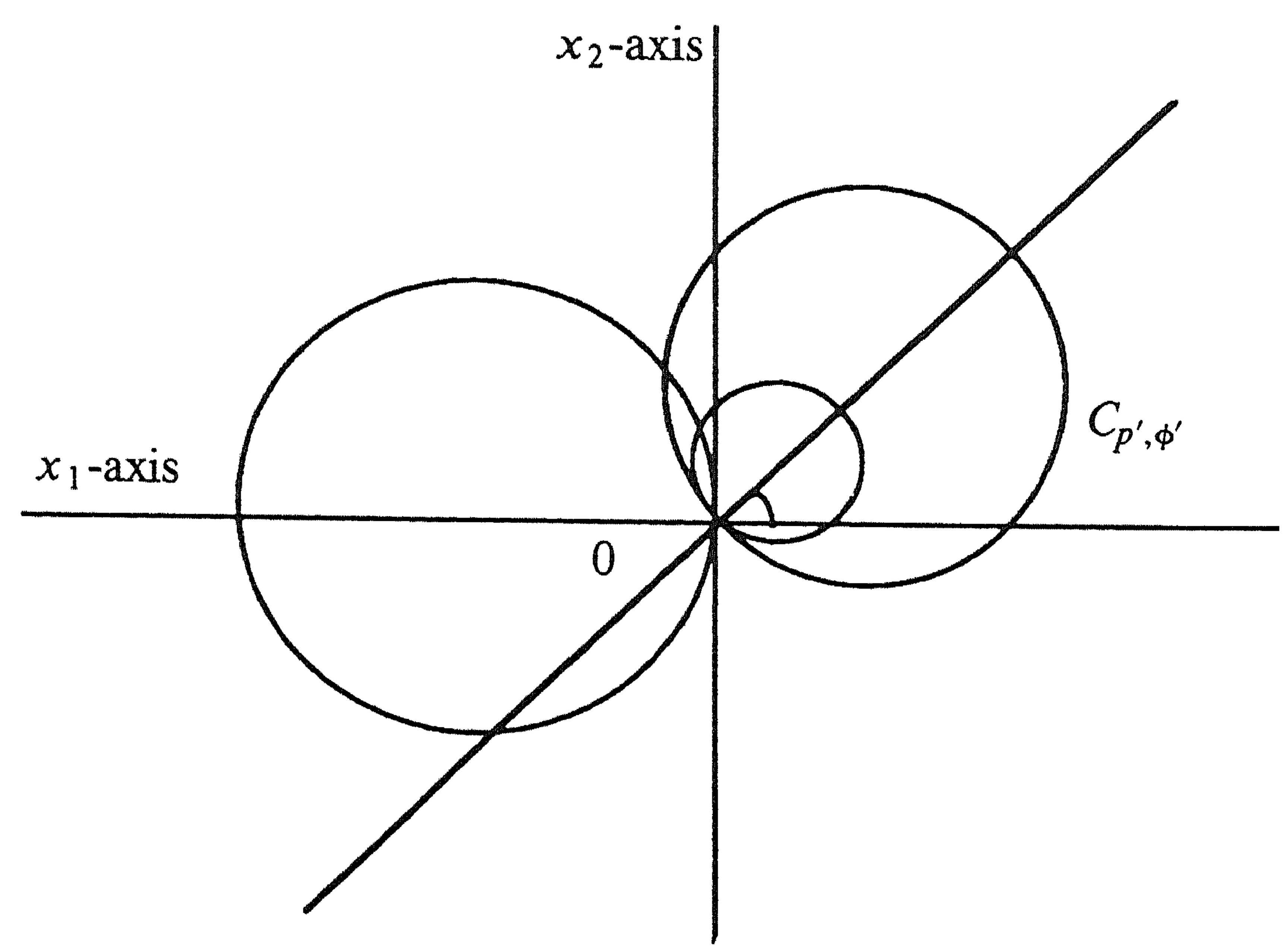


FIGURE 1. Some circles

Define the Radon transform $f \mapsto Qf$ as

$$Qf(p',\phi') := \int_{C_{p',\phi'}} f(x)dx. \tag{2.4}$$

Again we have $C_{p',\phi'} = C_{-p',\phi'+\pi}$, so

$$Qf(p',\phi') = Qf(-p',\phi' + \pi). \tag{2.5}$$

The transform $f \mapsto Rf$, has an inversion formula ([3, p. 21], or [4,8])

$$g(x) = \frac{1}{4\pi^2} \int_0^{2\pi} \int_{-\infty}^{\infty} \frac{\frac{\partial}{\partial p} Rg(p,\phi)}{\theta \cdot x - p} dp d\phi, \tag{2.6}$$

valid for continuous functions with compact support. Here θ is a function of ϕ.

3. Relation between Q and R

Suppose we can find a function $\Phi: \mathbb{R}^2 \to \mathbb{R}^2$, which maps the line $L_{p,\phi}$ onto the circle $C_{p',\phi'}$, which is invertible and continuously differentiable. Then we have

$$\begin{aligned} Qf(p',\phi') &= \int_{C_{p',\phi'}} f(x)\, dx = \int_{\Phi(L_{p,\phi})} f(x)\, dx \\ &= \int_{L_{p,\phi}} J\Phi(x)\, f(\Phi(x))\, dx \\ &= R[\, J\Phi(\cdot)\, f(\Phi\,(\cdot)\,)]\,(p,\phi). \end{aligned} \tag{3.1}$$

Where $J\Phi(x)$ is the Jacobian of Φ in x.

If we use the inversion formula for the transform Rf, (Section 2), then we find an inversion formula for Qf. Now, take $g(x) := J\Phi(x)\, f(\Phi\,(x))$, in formula (2.6), then we obtain.

$$f(\Phi(x)) = \frac{1}{4\pi^2} \left[J\Phi(x)\right]^{-1} \int_0^{2\pi} \int_{-\infty}^{\infty} \frac{\frac{\partial}{\partial p} Qf(p',\phi')}{\theta \cdot x - p} dp d\phi. \tag{3.2}$$

The relation between p', ϕ' and p, ϕ is given below, after choosing Φ.

4. A formula for Φ

We need a function $\Phi: \mathbb{R}^2 \to \mathbb{R}^2$ that maps the family of lines $L_{p,\phi}$, $p \in (-\infty,\infty)$, $\phi \in [0,2\pi)$ onto the family of circles $C_{p',\phi'}$, $p' \in (-\infty,\infty)$, $\phi' \in [0,2\pi)$. A known type of transformation, that maps lines onto circles and vice versa, is the Möbius transformation. Consider the mapping $M: \mathbb{C}_\infty \to \mathbb{C}_\infty$,

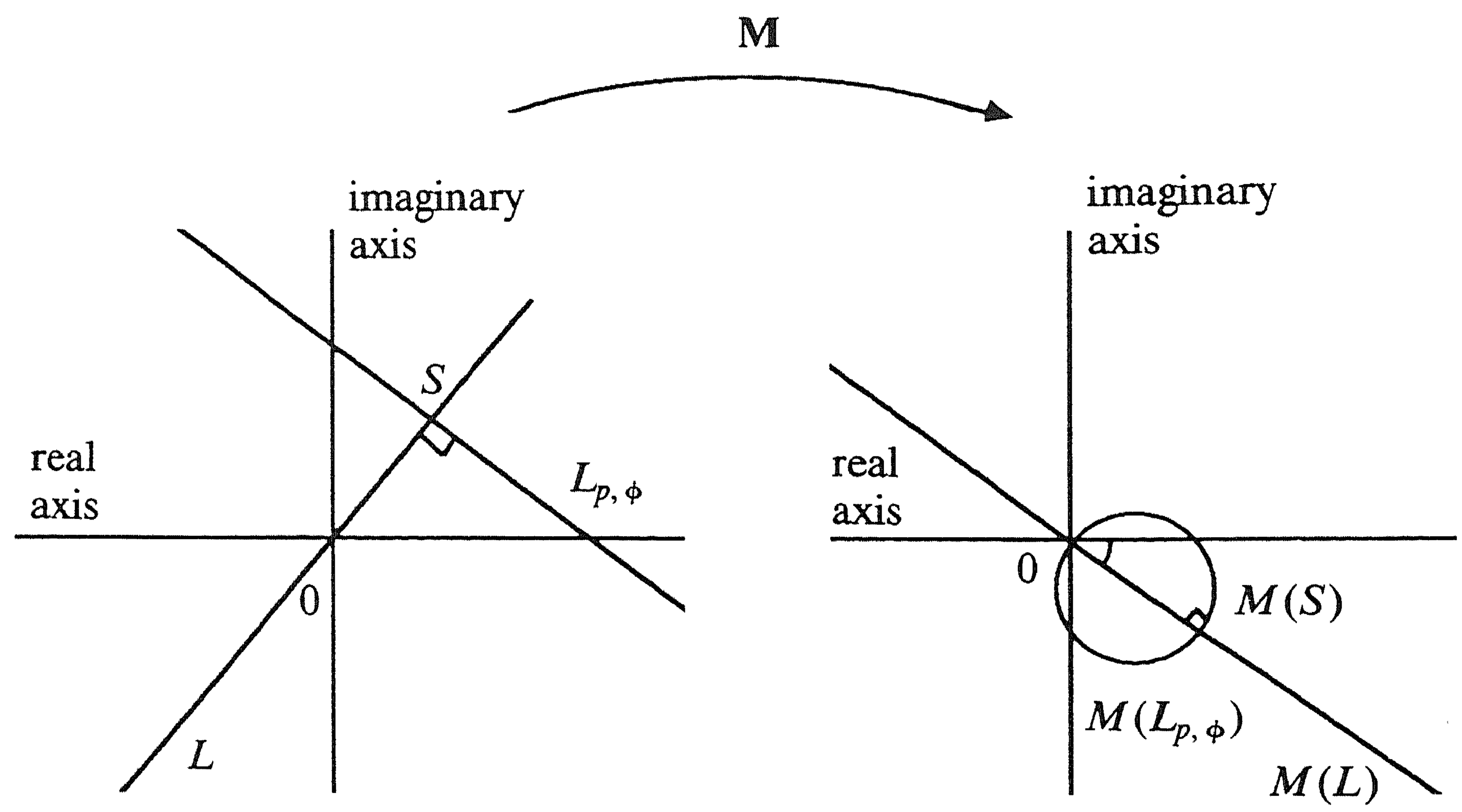
$$M(z) = \frac{1}{z}.$$

($\mathbb{C}_\infty$ is the extended complex plane $\mathbb{C} \cup \{\infty\}$.) M maps a line, not through the origin, onto a circle through the origin, and a line through the origin onto a line through the origin (i.e. a circle with radius ∞). M is a conformal mapping, so it preserves angles. Let S be the point of intersection of the line $L_{p,\phi}$ and of the line L, going through the origin, orthogonal to $L_{p,\phi}$ (Figure 2). $M(L)$ is the line through the origin and $M(S)$. It intersects the circle $M(L_{p,\phi})$

orthogonally at the point $M(S)$, because the line L and $L_{p,\phi}$ are orthogonal in S. So the segment $[O, M(S)]$ determines the circle $M(L_{p,\phi})$, in the sense that we can compute p' and ϕ' if we know $M(S)$ (see Figure 2). The coordinates of S follow directly from the definition: $S = (p\cos\phi, p\sin\phi)$. Write S as a complex number, $S = pe^{i\phi}$, then $M(S) = \frac{e^{-i\phi}}{p}$. So $\phi' = -\phi$ and $p' = 1/p$ (Figure 2).

If we write M in real coordinates, we obtain the desired mapping $\Phi: \mathbb{R}^2 \to \mathbb{R}^2$,

$$\Phi(x) = \left(\frac{x_1}{x_1^2 + x_2^2}, \frac{-x_2}{x_1^2 + x_2^2}\right). \tag{4.1}$$



The Jacobian is

$$J\Phi(x) = \frac{1}{\|x\|^2}. \tag{4.2}$$

With the relations $\phi' = -\phi$ and $p' = 1/p$, where $p \in \mathbb{R}$ and $\phi \in [0, 2\pi]$ we have that the family of lines $\{L_{p,\phi}\}$ is mapped onto the family of circles $\{C_{p',\phi'}\}$.

5. AN INVERSION FORMULA FOR Q

Substituting $y = \Phi(x)$ in (3.2) we obtain

$$f(y) = \frac{1}{4\pi^2 J\Phi(\Phi^{-1}(y))} \int_0^{2\pi} \int_{-\infty}^{\infty} \frac{\frac{\partial}{\partial p} Qf(p', \phi')}{\theta \cdot \Phi^{-1}(y) - p} dp d\phi.$$

Using the transformation of variables

$$p' = \frac{1}{p} \text{ and } \phi' = -\phi, \text{ so } dp = -\frac{1}{p'^2} dp'$$

and $\frac{\partial}{\partial p}Qf(p',\phi') = -p'^2\frac{\partial}{\partial p'}Qf(p',\phi')$ we obtain

$$f(y) = \frac{1}{4\pi^2 J\Phi(\Phi^{-1}(y))}\int_0^{2\pi}\int_{-\infty}^{\infty}\frac{\frac{\partial}{\partial p'}Qf(p',\phi')}{1/p'-\theta'\cdot\Phi^{-1}(y)}dp'd\phi'. \qquad (5.1)$$

Here $\theta'=(\cos\phi', \sin\phi')$. Using (4.1), (4.2) and $\Phi^{-1}(y)=\Phi(y)$. We have, with $y=(y_1,y_2)$

$$f(y) = \frac{1}{4\pi^2}\int_0^{2\pi}\int_{-\infty}^{\infty}\frac{\frac{\partial}{\partial p'}Qf(p',\phi')}{(1/p')(y_1^2+y_2^2)-(y_1\cos\phi'+y_2\sin\phi')}dp'd\phi'. \qquad (5.2)$$

Or with an obvious change of notation

$$f(y) = \frac{1}{4\pi^2}\int_{S^1}\int_{\mathbb{R}}\frac{\frac{\partial}{\partial p'}Qf(p',\theta')}{(1/p')\|y\|^2-y\cdot\theta'}dp'd\theta', \qquad (5.3)$$

where we have used the relation $\theta'=(\cos\phi', \sin\phi')$.

6. The domain of Q

In Section 3 we inverted the Radon transform $f\mapsto Rf$ of a function g of the form

$$g(x) = J\Phi(x)\, f(\Phi(x)).$$

If g is a continuous function with compact support, then the Radon inversion formula holds. So we want to find a region G such that g is continuous on G with $\underline{\text{support in } G}$. Define the annulus around the point m by $A(m,r_1,r_2):= B(m,r_2)\setminus\overline{B(m,r_1)}$, $r_1<r_2$. We know that $\Phi[A(0,\epsilon,1)]= A(0,1,1/\epsilon)$, because $M[A(0,\epsilon,1)]=A(0,1,1/\epsilon)$. So if f has support in $A(0,\epsilon,1)$, then g has support in $A(0,\epsilon,1/\epsilon)$, and vice versa. Thus if f is continuous with support in $G{:}=A(0,1,R)$, then g is continuous with support in $A(0,1/R,1)$ and the Radon inversion formula can be applied.

7. The Ludwig-Helgason conditions

It is not only important to have an inversion formula for a Radon transform, but also to know some of its properties. For example, for $f\mapsto Rf$ there are the so-called Ludwig-Helgason conditions for a function h to satisfy $h(p,\theta)=Rg(p,\theta)$ for some g and $(p,\theta)\in\mathbb{R}\times S^1$. In this section we formulate Ludwig-Helgason conditions for the transform $f\mapsto Qf$.

Recall the symmetry property (2.3) for the Radon transform $f\mapsto Rf$. This property together with the homogeneity property

$$\int_{-\infty}^{\infty}p^k Rg(p,\theta)dp = \pi_k(\theta), \quad \theta\in S^1 \qquad (7.1)$$

where $\pi_k(\theta)$ is a homogeneous polynomial of degree k in θ, are called the

Ludwig-Helgason conditions. These conditions are properties of functions h in the range of R,

$$\text{Im } R := \{h : \mathbb{R}\times S^1 \to \mathbb{R} \mid \exists g : \mathbb{R}^2 \to \mathbb{R},\ \ Rg = h\}.$$

These conditions are of practical importance in e.g. computerized tomography, where the measurements $h(p,\theta) = Rg(p,\theta)$, which may be corrupted by noise, or incomplete, should satisfy (2.3) and (7.1). For $f \mapsto Qf$ we can find analogous formulas, for continuous functions with support in G. We already have the symmetry property (2.5). By substituting $g(x) := J\Phi(x)\, f(\Phi(x))$ in formula (7.1) we obtain

$$\int_{-\infty}^{\infty} p^k Qf(p',\theta')dp = \pi_k(\theta). \tag{7.2}$$

With $p = 1/p'$,

$$\int_{-\infty}^{\infty} (\frac{1}{p'})^{k+2} Qf(p',\theta')dp' = \pi_k(\theta'), \tag{7.3}$$

and $\pi_k(\theta')$ is a homogeneous polynomial of degree k. Formula (7.3) is the analogue of (7.1). Concluding the inversion formula, obtained here, only holds for functions $f : \mathbb{R}^2 \to \mathbb{R}$. For the higher dimensional case an inversion formula is derived by Cormack & Quinto [5]. They consider the components $f_l(p)$ of a function $f \in C^\infty(\mathbb{R}^n)$, given by $f_l(p) := \int_{S^{n-1}} f(p\theta h) Y_l(\theta) dS(\theta)$. Here $p \in \mathbb{R}, \theta \in S^{n-1}$ and Y_l is a normalized spherical harmonic. Cormack and Quinto obtain an inversion formula for each component $f_l(p)$.

For the special case $n=2$, Cormack ([6]) finds an inversion formula for integral transforms of a function over a whole class of families of curves in the plane, the so-called β-curves,

$$p^\beta \cos\{\beta(\theta-\phi)\} = r^\beta;\ \beta>0,\ \ |\theta-\phi| < \frac{\pi}{2\beta}. \tag{7.4}$$

If we fix $\beta=1$, then (7.4) describes a family of circles through the origin, as defined here in Section 2.

The cases $\beta = ½$ and $\beta=2$ correspond to a family of cardioids and a family of one-branched lemniscates respectively. Cormack's inversion formula is, in polar coordinates, where β and m are related,

$$f(p,\phi) = \frac{1}{2\pi^2 p} \int_0^{2\pi} \int_0^{\infty} \frac{\frac{\partial}{\partial p'} Qf(p',\phi')\ U_{m-1}(p/p')}{T_m\ (p/p') - \cos(\phi-\phi')} dp'\, d\phi'.$$

Here $T_k(x)$ and $U_k(x)$ are the Tschebycheff polynomials of the first and the second kind, respectively. If $\beta=1$ then $m=1$, and $U_0(x) \equiv 1$, $T_1(x) = x$. So

$$f(p,\phi) = \frac{1}{2\pi^2 p} \int_0^{2\pi} \int_0^{\infty} \frac{\frac{\partial}{\partial p'} Qf(p',\phi')}{p/p' - \cos(\phi-\phi')} dp' d\phi'.$$

Substituting $p = \|y\|$ and $y = (\|y\| \cos \phi, \|y\| \sin \phi)$ we obtain formula (5.3). So the result obtained here is a special case of the inversion formula in [6], found by Cormack.

REFERENCES

1. I.M. GEL'FAND, M.I. GRAEV, N.YA. VILENKIN (1966). *Generalized Functions Vol 5: Integral Geometry and Representation Theory,* Academic Press, New York.
2. S. HELGASON (1980). *The Radon Transform,* Birkhäuser, Boston.
3. F. NATTERER (1986). *The Mathematics of Computerized Tomography,* John Wiley & Sons, Chichester, New York.
4. Y. NIEVERGELT (1986). Elementary inversion of Radon's Transform. *SIAM Review Vol 28 no. 1.*
5. A.M. CORMACK, E.T. QUINTO (1980). A Radon transform on spheres through the origin in $\mathbb{R}^n$ and applications to the Darboux equation. *Transactions of the AMS Vol 260 no. 2,* 575-581.
6. A.M. CORMACK (1983). Radon's problem, Old and New. *SIAM, AMS Proceedings, Vol. 14, Inverse Problems,* 33-43.
7. F. JOHN (1955). *Plane Waves and Spherical Means with Applications to PDE,* Interscience, New York.
8. N.M. TEMME (1987). The Radon transform; first steps. *CWI Newsletter 15,* 41-46.