A Short Introduction to Exit Problems

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Many phenomena that occur in nature and technology exhibit a stochastic behaviour. When the stochastic element is relevant, it has to be included in the modeling of such phenomena. We shall discuss models with a deterministic component and a small stochastic component. The short term behaviour of these models is determined mainly by the deterministic component, whereas the long term behaviour is influenced considerably by the stochastic component. For the description of the long term behaviour, deterministic stability concepts (stable, neutral equilibrium) are inadequate and have to be replaced by stochastic stability concepts (the expected exit time from a region containing such a deterministic equilibrium). In the study of so-called exit problems we consider a domain in the state space of a stochastic system and try to determine statistical quantities (such as mean exit time, distribution of exit points over the boundary of the domain, etc.) related to leaving this domain. We will treat the exit problems from an asymptotic (in the limit for small noise) point of view.

Note. This paper is dedicated to Prof. H.A. Lauwerier on the occasion of his 65th birthday.

1. Introduction

In this contribution we study some aspects of stochastic dynamical systems that have a deterministic part (referred to as 'the deterministic system') and a small stochastic part consisting of Gaussian white noise (referred to as 'stochastic fluctuations').

In some of these systems, the dynamical characteristics of interest are dominated by the deterministic system, while the stochastic fluctuations are only of secondary importance, in the sense that the omission of the stochastic fluctuations does not essentially alter these characteristics. This is demonstrated, for example, by a 'diffusion *with* the flow', see Figure 1a. Starting at a point in a bounded domain D, the trajectories of the stochastic dynamical system leave the domain D with probability close to one in the same time as the deterministic trajectory through that point. The probability density function defined on the boundary ∂D, describing the point of exit from D of the stochastic dynamical system, is concentrated near the deterministic exit point. Stochastic systems of this type will not be considered here.

In other stochastic dynamical systems, the stochastic fluctuations, though small compared to the deterministic system, are of great importance to the dynamical characteristics of interest. Without stochastic fluctuations these

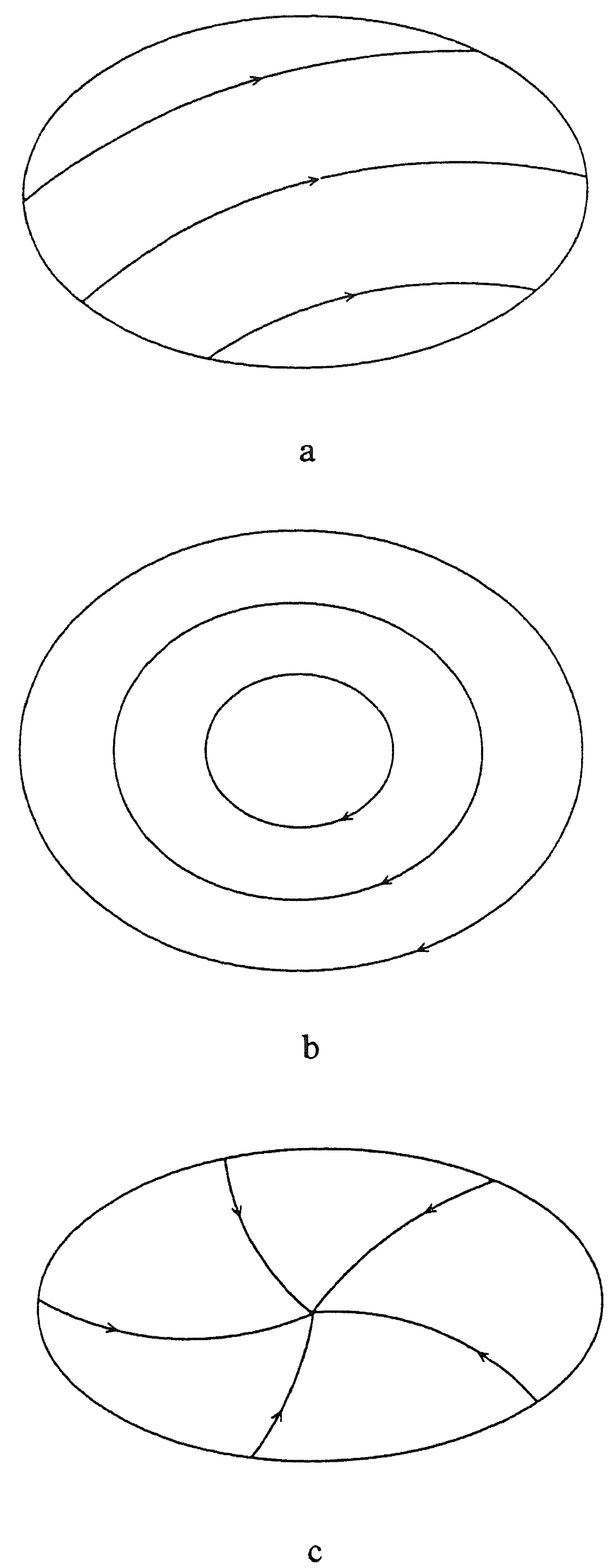

FIGURE 1. Illustration of diffusion (a) with, (b) across and (c) against the flow

characteristics are essentially changed. One such example is a 'diffusion *across* the flow', as depicted in Figure 1b. The deterministic system consists of a center point, surrounded by closed trajectories. Consider a domain D, enclosed by one of these trajectories. In the deterministic system no exit from D can occur, since we follow ceaselessly the closed trajectory through the starting point. In contrast with this fact, in the stochastic dynamical system, i.e. in the deterministic system perturbed by stochastic fluctuations, exit will occur in finite time with probability one. Another such example is a 'diffusion *against* the flow', depicted in Figure 1c. A bounded domain D is entered at its boundary ∂D by deterministic trajectories that converge to an asymptotically stable limit point contained in D. In this deterministic system, if we start at some point in D, we approach the limit point along the trajectory through the starting point. Again, the deterministic system does not allow exit from D, but when this system is perturbed by stochastic fluctuations, exit will happen in finite time with probability one. Although more complicated systems exist that exhibit a similar behaviour, such as attracting limit cycles or strange attractors, etc., notably in higher dimensional domains, we will confine ourselves to systems of the two simple types described here, in particular to systems of the last type.

We will concentrate on a few statistical characteristics related to the problem of exit from a domain, like the expectation value of the time of first exit (which provides a measure for the stability of the stochastic system), and the distribution of exit points over the boundary of the domain.

2. The equations

A stochastic system is frequently described either in terms of a stochastic differential equation (that, as an extension to an ordinary differential equation, contains stochastic terms) [1], or in terms of a Kolmogorov equation. In the former case, an equivalent description in terms of a Kolmogorov equation is often possible. In this section we formulate the forward and backward Kolmogorov equations [19,55], which form the starting point of our analysis.

We consider a stochastic dynamical system that has been defined on the n-dimensional domain D in the state space. Let $v(x,t)dx$ denote the probability that the system is in the infinitesimal subregion $(x,\ x+dx)\in D$ at time t. This function satisfies the forward (Kolmogorov) equation (also called the Fokker-Planck equation)

$$\frac{\partial v}{\partial t} = M_\epsilon v, \quad x \in D, \tag{2.1}$$

where the differential operator M_ϵ is defined by

$$M_\epsilon v \equiv -\sum_{i=1}^{n} \frac{\partial}{\partial x_i}(b_i(x)v) + \frac{\epsilon}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\partial^2}{\partial x_i \partial x_j}(a_{ij}(x)v). \tag{2.2}$$

Equation (2.1) has to be supplemented with the relevant initial and boundary conditions. The first term on the right side of (2.2) represents the deterministic part of the dynamical system, b is called the deterministic or drift vector. The second term on the right side represents the stochastic fluctuations. The matrix a is known as the diffusion matrix and is symmetric positive (semi-) definite. The parameter ϵ, $0<\epsilon\ll 1$, indicates that the stochastic fluctuations are small relative to the deterministic part. When $\epsilon=0$ the stochastic fluctuations are absent and equation (2.1) reduces to the Liouville equation. Then, if the initial position is deterministic, the initial probability density is a delta function, say $\delta(x-x_0)$, and the solution of the Liouville equation corresponds to the solution of the system of ordinary differential equations

$$\frac{dx_i}{dt} = b_i(x), \quad i=1,2,\ldots,n \tag{2.3a}$$

with initial conditions

$$x(0) = x_0. \tag{2.3b}$$

This system is defined as the deterministic system corresponding to the stochastic dynamical system.

In order to determine the distribution of exit points over ∂D as well as the expected time of first exit from D, we use backward (Kolmogorov) equations. Let $p(x,y)\, dS_y$ be the probability of exit at $dS_y \in \partial D$, if we started at $x \in D$ on time $t=0$, i.e. p is the exit density. We define the function $u_s(x)$ as follows:

$$u_s(x) = \int_{\partial D} f(y) p(x,y)\, dS_y, \tag{2.4}$$

where f is a function on ∂D that can be chosen arbitrarily. With f defined as the indicator function

$$f = \begin{cases} 1 & \text{on } \partial_1 D, \text{ where } \partial_1 D \subseteq \partial D, \\ 0 & \text{on } \partial_0 D = \partial D \setminus \partial_1 D, \end{cases} \tag{2.5}$$

$u_s(x)$ is the probability of exit at $\partial_1 D$, given that we started at $x \in D$ on time $t=0$. The function u_s is the solution of the stationary backward equation

$$L_\epsilon u_s = 0, \quad x \in D, \tag{2.6a}$$

subject to the boundary condition

$$u_s = f(x), \quad x \in \partial D, \tag{2.6b}$$

where the differential operator L_ϵ is defined by

$$L_\epsilon u \equiv \sum_{i=1}^{n} b_i(x) \frac{\partial u}{\partial x_i} + \frac{\epsilon}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j}, \tag{2.7}$$

and a and b are the same functions as above.

We consider the time-dependent backward equation

$$\frac{\partial u}{\partial t} = L_\epsilon u \tag{2.8a}$$

as well. With the boundary condition

$$u = f(x), \quad x \in \partial D, \tag{2.8b}$$

where f is the indicator function (2.5), and the initial condition

$$u(x,0) = 0, \quad x \in D, \tag{2.8c}$$

$u(x,t)$ is the probability that exit occurs at $\partial_1 D$ on the time interval $(0,t]$, given that we started at $x \in D$ on time $t=0$.

Let $T(x)$ be the expected time of first exit from D, given that we started at x on time $t=0$:

$$T(x) = \inf\{t \mid x(t) \in \partial D, \ x(0) = x \in D\}. \tag{2.9}$$

The function T is the solution of the boundary value problem:

$$L_\epsilon T = -1, \quad x \in D, \tag{2.10a}$$

$$T = 0, \quad x \in \partial D. \tag{2.10b}$$

Equation (2.10a) is known as the Dynkin equation.

The reader interested in the details of the equations and the corresponding conditions that we have given here, and related material, is referred to the literature [19,55]. In later sections we will be concerned with the asymptotic solution of (2.1), (2.6), (2.8) and (2.10) for small ϵ.

The backward and forward differential operators L_ϵ and M_ϵ defined above are formal adjoints, which means that the following relation holds [48]:

$$\int_D \int (vL_\epsilon u - uM_\epsilon v)\, dx = \int_{\partial D} P \cdot \xi \ dS_x, \tag{2.11}$$

where P is the vector with components

$$P_i = \sum_{j=1}^{n} \frac{\epsilon}{2}\left[a_{ij} v \frac{\partial u}{\partial x_j} - u \frac{\partial (a_{ij} v)}{\partial x_j}\right] + b_i uv, \quad i = 1,2,\ldots,n \tag{2.12}$$

and ξ denotes the outward normal on ∂D.

3. The boundary

In the study of exit problems, the behaviour of the stochastic system at and near the boundary of the domain deserves special attention, since the domain is left via the boundary. For a given stochastic system we must verify whether the boundary can actually be reached from the interior domain.

In many practical situations the type of the boundary is determined by the drift vector and the diffusion matrix. For one-dimensional stochastic systems there is a classification of such boundaries originating from Feller [16]. In a semi-group approach to adjoint forward and backward equations he distinguished the regular, exit, entrance and natural boundaries. In Table 1 we have repeated schematically the boundary classification as it has been described in [54]. The type of boundary depends on the integrability at the boundary point of some of the following integrals:

$$I_1(x) = \exp\left[-\frac{4}{\epsilon}\int^x b(t)/a(t)\, dt\right],$$

$$I_2(x) = \frac{2}{\epsilon a(x)} \exp\left[\frac{4}{\epsilon} \int^{x} b(t)/a(t)\, dt\right], \tag{3.1}$$

$$I_3(x) = I_1(x) \int^{x} I_2(t)\, dt,$$

$$I_4(x) = I_2(x) \int^{x} I_1(x)\, dt.$$

If there are sample paths that hit the boundary in finite time, the boundary is attainable, otherwise it is unattainable. Table 1 indicates that it makes sense to talk of an exit problem only if at least one of the boundaries of the domain (an interval) is a regular or exit boundary (which are the only cases that permit the boundary to be reached in finite time from the interior domain). When we have a regular boundary we should speak of the problem of *first* exit from a domain, since in that case the exited domain can be re-entered and subsequently re-exited. For higher dimensional stochastic systems a similar classification has never been published.

integrable				type of boundary	boundary is attainable	interior is attainable
I_1	I_2	I_3	I_4			
yes	yes			regular	yes	yes
yes	no	yes		exit	yes	no
yes	no	no		natural attracting	no	no
no	yes		yes	entrance	no	yes
no no	yes no		no	natural repelling	no	no

TABLE 1. Boundary classification for one-dimensional stochastic systems. Five boundary types are defined according to the integrability of some of the integrals I_1 to I_4. The last two columns indicate whether the boundary is attainable from the interior domain and whether the interior domain is attainable from the boundary.

In other situations we dispose of a stochastic system, defined by a drift vector and a diffusion matrix on a domain D, and we want to erect a boundary of a desired type at any place in D, thereby restricting the domain to a subdomain D' of D. Examples of such boundaries are absorbing and reflecting boundaries [19]. On reaching an absorbing boundary from the interior domain D', the system is taken apart (is absorbed) so that this domain cannot be entered again, comparable with an exit boundary. At a reflecting boundary no probability can pass, so that exit at this boundary is impossible. With respect to the solution of the forward equation an absorbing boundary implies the boundary condition $v(x,t)=0$, where $x \in \partial D'$, and a reflecting boundary implies the condition $\xi \cdot J(x,t)=0$, where $x \in \partial D'$, ξ is the outward normal on $\partial D'$ and J is the probability current, i.e. the vector with components

$$J_i(x,t) = b_i(x)v - \frac{\epsilon}{2}\sum_{j=1}^{n}\frac{\partial}{\partial x_j}(a_{ij}(x)v), \quad i=1,2,\ldots,n. \tag{3.2}$$

Absorbing and reflecting boundaries can be set up in domains of any dimension.

4. An example of diffusion across the flow

In this section we treat a simple example of diffusion across the flow. Consider an oscillator with a small damping, that is subjected to a stochastic forcing. The damping effect is introduced here since it leads to a realistic model, without essential complications for the analysis that follows. As a consequence of the stochastic effects, the energy of the oscillator can reach a critical level after some time. This critical level is chosen as the energy at which the oscillator breaks down. We will derive an expression for the expected time needed to reach the critical energy level, which is a measure for the stochastic stability of the oscillator. In non-dimensional form, the stochastic differential equation [1] for this problem is [53]

$$\ddot{x} + \epsilon\alpha\dot{x} + x = \sqrt{\epsilon}\,q(x)\xi, \tag{4.1}$$

where x is the deviation from the equilibrium position, the dot denotes differentiation with respect to the time t, ϵ is a small positive parameter, and $\epsilon\alpha$ is a non-negative $O(\epsilon)$ damping constant. The right side of (1) represents a Gaussian white noise process with intensity $\epsilon q^2(x)$. The function q is approximated by the first two terms of its Taylor expansion around $x=0$:

$$q(x) \approx \beta_0 + \beta_1 x. \tag{4.2}$$

The second order differential equation (4.1) can be written as the system of first order equations:

$$\begin{aligned} \frac{dx}{dt} &= \dot{x}, \\ \frac{d\dot{x}}{dt} &= -(\epsilon\alpha\dot{x}+x) + \sqrt{\epsilon}(\beta_0 + \beta_1 x)\xi. \end{aligned} \tag{4.3}$$

The undisturbed ($\epsilon=0$) system (4.1) is an undamped oscillator, whose dynamics are described by closed trajectories around the origin in the $(x,\dot{x})$-phase space. Each trajectory corresponds to an energy level. The energy is larger for orbits farther away from the origin. The effect of a nonzero ϵ is that the trajectories tend to spiral slightly inwards to approach the origin as a consequence of damping if $\alpha \neq 0$, and contain stochastic fluctuations in the $\dot{x}$-direction. The backward equation corresponding to (4.3) reads [19]:

$$\frac{\partial u}{\partial t} = \dot{x}\frac{\partial u}{\partial x} - (\epsilon\alpha\dot{x}+x)\frac{\partial u}{\partial \dot{x}} + \frac{\epsilon}{2}(\beta_0+\beta_1 x)^2\frac{\partial^2 u}{\partial \dot{x}^2}, \tag{4.4}$$

with u defined as in Section 2. This equation is studied asymptotically for small ϵ and on the time scale of $O(\epsilon^{-1})$. With

$$t = \tilde{t}/\epsilon, \quad u = u^0 + \epsilon u^1 + \cdots, \tag{4.5}$$

and the transformations $(x,\dot{x}) \to (r,\theta)$ defined by

$$x = \sqrt{2}r \cos\theta, \quad \dot{x} = \sqrt{2}r \sin\theta, \tag{4.6}$$

we obtain to leading order in ϵ:

$$\frac{\partial u^0}{\partial \theta} = 0, \tag{4.7}$$

implying that u^0 is a function of r and $\tilde{t}$ only. The variable r is the square root of the dimensionless energy of the undisturbed ($\epsilon=0$) system. The non-dimensionalization process can be carried out such that the critical energy corresponds to $r^2=1$, thus $r \in [0,1]$. To the next order in ϵ we obtain an equation in terms of u^0 and u^1. Terms with u^1 vanish by integration of this equation with respect to θ from 0 to 2π and the additional assumption that u^1 is periodic in θ with period 2π. The resulting equation for u^0 reads:

$$\frac{\partial u^0}{\partial \tilde{t}} = (\frac{a_0}{r} + a_2 r)\frac{\partial u^0}{\partial r} + (a_0 + a_1 r^2)\frac{\partial^2 u^0}{\partial r^2}, \tag{4.8a}$$

with

$$a_0 = \beta_0^2/8, \quad a_1 = \beta_1^2/16, \quad a_2 = 3\beta_1^2/16 - \alpha/2. \tag{4.8b}$$

The description to this order in ϵ includes the effects of damping and stochastic fluctuations. If, as a consequence of the latter effect the critical energy $r^2=1$ is reached in finite time with probability one, starting from $r \in [0,1]$, the oscillator is said to be stochastically unstable. In that case, the stability of the oscillator is measured by the expected time of exit from the unit interval at 1.

In the present discussion we only consider the case $|\beta_0|$, $|\beta_1| \gg O(\epsilon^{1/2})$, so that a_0 and a_1 do not vanish in the asymptotics leading to equation (4.8a) and thus appear in this equation indeed. The boundary $r=0$ is then an entrance boundary and at $r=1$ we adopt an absorbing boundary in order to model the breakdown of the oscillator at the critical energy. Thus exit from the unit interval can take place only at $r=1$. Let $u_s(r)$ be the probability of exit at $r=1$, given that we started at r on time $t=0$. The leading order term $u_s^0(r)$ in the expansion of $u_s(r)$ in powers of ϵ is obtained by solving the stationary equation (4.8a) with boundary condition $u_s^0(1)=1$. The only relevant solution (i.e. yielding values $u_s^0(r) \in [0,1]$) is $u_s^0(r) \equiv 1$. There is no freedom to specify an arbitrary boundary condition at $r=0$. We conclude that if we start somewhere on the interval $[0,1]$, exit at $r=1$ will occur with probability one, so that the oscillator is stochastically unstable. Next we consider the expected exit time $T(r)$, starting from a point r. Similar to the time scaling in (4.5) we put $T = \tilde{T}/\epsilon$ and similar to the expansion of u in (4.5) we put $\tilde{T} = \tilde{T}^0 + \epsilon \tilde{T}^1 + \cdots$, so that $T = \tilde{T}^0/\epsilon + \tilde{T}^1 + \cdots$. An approximation for T is found by solving the Dynkin equation

$$-1 = (\frac{a_0}{r} + a_2 r)\frac{\partial \tilde{T}^0}{\partial r} + (a_0 + a_1 r^2)\frac{\partial^2 \tilde{T}^0}{\partial r^2}, \tag{4.9a}$$

with the boundary conditions:

$$\tilde{T}^0(0) \text{ is finite,} \tag{4.9b}$$

$$\tilde{T}^0(1) = 0. \tag{4.9c}$$

For $a_2 \neq a_1$ we find:

$$T(r) \sim \frac{1}{\epsilon(a_1 - a_2)} \int_r^1 \left[\left(\frac{a_1}{a_0} s^2 + 1 \right)^{-\frac{a_2}{2a_1} + \frac{1}{2}} - 1 \right] \frac{1}{s} ds. \tag{4.10}$$

If $a_2 = a_1$, this is substituted into equation (4.9a). Solving the corresponding boundary value problem we find:

$$T(r) \sim \frac{1}{2\epsilon a_1} \int_r^1 \frac{1}{s} \log \left[\frac{a_1}{a_0} s^2 + 1 \right] ds. \tag{4.11}$$

The reader is asked to take notice of the order of magnitude of the results (4.10) and (4.11) in order to compare this with results to be derived later for diffusion against the flow systems. The cases that either β_0 or β_1 are of order $O(\epsilon^{1/2})$ must be treated separately. In the case $\beta_0 = O(\epsilon^{1/2})$ it can be shown that if the damping is larger than a certain value the oscillator is stochastically stable on the time scale under consideration. This means that on this time scale the probability of exit is less than one, in contrast with the result above. A more detailed description of the exit problem for oscillators as described here can be found in [53]. The stochastic stability of oscillators with a different type of damping (as cubic damping) or noise (red, dichotomic, etc.) and with a forcing described by a potential function has been treated in [14]. The asymptotics that we have used in this example to arrive at equation (4.8) are well established and are known under the names of averaging technique [3,31,35,50,56] and adiabatic elimination of fast variables [19].

5. Diffusion against the flow

In this section we discuss the exit problem for systems that are of diffusion against the flow type. First we treat a one-dimensional system, then a multi-dimensional potential system that can be treated with essentially the same means, and we will conclude with more general multi-dimensional systems.

5.1. A one-dimensional system

Consider the stochastic system defined on $[\alpha, \beta]$, where $\alpha < 0$ and $\beta > 0$, with drift coefficient $b(x)$ satisfying

$$b(x) \begin{cases} > 0, & x \in [\alpha, 0), \\ = 0, & x = 0, \\ < 0, & x \in (0, \beta], \end{cases} \tag{5.1.1}$$

so that $x = 0$ is an attractor, and diffusion coefficient $\epsilon a(x)/2$, $0 < \epsilon \ll 1$, with

$$a(x)>0, \quad x \in [\alpha, \beta]. \tag{5.1.1}$$

For this system we will find the functions u_s and T defined in Section 2, asymptotically for small ϵ. The boundary value problem for u_s reads:

$$L_\epsilon u_s \equiv \frac{\epsilon}{2} a(x) \frac{d^2 u_s}{dx^2} + b(x) \frac{du_s}{dx} = 0, \tag{5.1.2a}$$

$$u_s(\alpha) = c_\alpha, \quad u_s(\beta) = c_\beta, \tag{5.1.2b}$$

where c_α and c_β are given constants. The reduced equation (5.1.2a), i.e. equation (5.1.2a) with $\epsilon=0$, is solved by any constant c_0. This solution is valid for x-values bounded away from α and β but not near these points, since the boundary conditions (5.1.2b) cannot be satisfied. We assume that the functions a and b have the Taylor series expansions

$$\begin{aligned} a(x) &= a(\alpha) + a'(\alpha)(x-\alpha) + \cdots && \text{near } x = \alpha, \\ b(x) &= b(\alpha) + b'(\alpha)(x-\alpha) + \cdots && \text{near } x = \alpha, \\ a(x) &= a(\beta) + a'(\beta)(x-\beta) + \cdots && \text{near } x = \beta, \\ b(x) &= b(\beta) + b'(\beta)(x-\beta) + \cdots && \text{near } x = \beta. \end{aligned} \tag{5.1.3}$$

As an abbreviation we use the notation $\tilde{b}(x)=2b(x)/a(x)$. It follows from (5.1.1) that $\tilde{b}(\alpha)>0$ and $\tilde{b}(\beta)<0$. A boundary layer analysis near $x=\alpha$ and $x=\beta$ shows the presence of $O(\epsilon)$ boundary layers near these points. An asymptotic expression for u_s to leading order in ϵ that is uniformly valid on $[\alpha,\beta]$ is given by

$$\begin{aligned} u_s(x) \sim c_0 &+ (c_\alpha - c_0) \exp[-\tilde{b}(\alpha)(x-\alpha)/\epsilon] \\ &+ (c_\beta - c_0) \exp[-\tilde{b}(\beta)(x-\beta)/\epsilon]. \end{aligned} \tag{5.1.4}$$

Note that the constant c_0 is left undetermined by the given asymptotics. To find c_0 we utilize a variational formulation of the boundary value problem (5.1.2), following [23], see also [61]. After multiplication by the factor

$$g(x) = \exp\left[\int_0^x \frac{2b(s) - \epsilon a'(s)}{\epsilon a(s)} ds\right], \tag{5.1.5}$$

equation (5.1.2a) can be written as the Euler equation

$$\frac{dF_{u'_s}}{dx} - F_{u_s} = 0, \tag{5.1.6}$$

with $F=\frac{\epsilon}{4}(u'_s)^2 ag$. Consequently, the solution of (5.1.2) corresponds to an extremal of the functional

$$J[u_s] = \int_\alpha^\beta \frac{\epsilon}{4}(u'_s)^2 ag \, dx, \tag{5.1.7}$$

with respect to functions u_s satisfying the boundary conditions (5.1.2b), see [5,9,46,47]. The expression (5.1.4) for u_s is substituted into the integral in

(5.1.7), and this integral is evaluated asymptotically for small ϵ by the method of Laplace [2,4]. The constant c_0 is determined by the requirement that the corresponding function u_s is an extremal of the functional J thus obtained, that is, by

$$\frac{dJ}{dc_0} = 0. \tag{5.1.8}$$

In addition to (5.1.1) we shall henceforth assume that

$$\tilde{b}'(x)<0, \quad x\in[\alpha, \beta]. \tag{5.1.9}$$

Carrying out the above procedure we then find that the largest contributions to the integral in (5.1.7) are from the neighbourhoods of α and β, and c_0 is given by

$$c_0 = \frac{c_\alpha \tilde{b}(\alpha)\exp[-I(\alpha)/\epsilon]-c_\beta \tilde{b}(\beta)\exp[-I(\beta)/\epsilon]}{\tilde{b}(\alpha)\exp[-I(\alpha)/\epsilon]-\tilde{b}(\beta)\exp[-I(\beta)/\epsilon]}, \tag{5.1.10}$$

where $I(x)$ is defined as:

$$I(x) = -\int_0^x \tilde{b}(s)\, ds, \quad (>0 \text{ for } x\neq 0). \tag{5.1.11}$$

The result (5.1.10) simplifies to:

$$c_0 = \begin{cases} c_\alpha, & \text{if } I(\alpha)<I(\beta), \\ c_\beta, & \text{if } I(\beta)<I(\alpha), \\ \dfrac{c_\alpha \tilde{b}(\alpha)-c_\beta \tilde{b}(\beta)}{\tilde{b}(\alpha)-\tilde{b}(\beta)}, & \text{if } I(\alpha)=I(\beta). \end{cases} \tag{5.1.12}$$

Thus, in the limit $\epsilon\to 0$, if we start outside $O(\epsilon)$-neighbourhoods of the boundaries α and β, exit will occur with probability one at the boundary with the smallest value of I. If $I(\alpha)=I(\beta)$ and if we start outside $O(\epsilon)$-neighbourhoods of the boundaries, the probabilities of exit at α and β are constants with values between zero and one, depending on $\tilde{b}(\alpha)$ and $\tilde{b}(\beta)$. The above asymptotic result is found alternatively by the evaluation for small ϵ of the exact solution of the boundary value problem (5.1.2).

Next we derive an expression for the expected time T of exit from the interval $[\alpha,\beta]$. The function T satisfies the inhomogeneous equation

$$\frac{\epsilon}{2}a(x)\frac{d^2T}{dx^2}+b(x)\frac{dT}{dx} = -1, \tag{5.1.13a}$$

with the conditions

$$T(\alpha) = 0, \quad T(\beta) = 0. \tag{5.1.13b}$$

The approach to this boundary value problem is largely the same as above, the only additional difficulty is the appearance of the inhomogenous term in

(5.1.13a). We anticipate that T is of the form

$$T(x) = c_0(\epsilon)\tau(x), \tag{5.1.14a}$$

where c_0 does not depend on x, and satisfies

$$1/c_0(\epsilon) = o(\epsilon), \quad \epsilon\to 0. \tag{5.1.14b}$$

Expression (5.1.14a) is substituted into (5.1.13), and the corresponding boundary value problem is asymptotically solved to obtain τ. For T we find:

$$T(x) \sim c_0(\epsilon)\left\{1-\exp[-\tilde{b}(\alpha)(x-\alpha)/\epsilon]-\exp[-\tilde{b}(\beta)(x-\beta)/\epsilon]\right\}, \tag{5.1.15}$$

to leading order in ϵ uniformly on $[\alpha,\beta]$. Again the unknown constant c_0 is determined from a variational principle. Equation (5.1.13a) is multiplied by the factor g defined in (5.1.5). The solution of the boundary value problem (5.1.13) then corresponds to an extremal of the functional

$$J[T] = \int_\alpha^\beta \left[\frac{\epsilon}{4}(T')^2 a - T\right] g\, dx\,, \tag{5.1.16}$$

with respect to functions $T(x)$ satisfying the boundary conditions (5.1.13b). This functional is evaluated by substitution of (5.1.15) into (5.1.16), and application of the method of Laplace. The major contributions to the integral in (5.1.16) are from neighbourhoods of α and β and from a neighbourhood of $x=0$. From (5.1.8) it is found that

$$c_0 = \frac{\dfrac{4}{a(0)}\sqrt{\dfrac{2\pi\epsilon}{-\tilde{b}'(0)}}}{\tilde{b}(\alpha)\exp[-I(\alpha)/\epsilon]-\tilde{b}(\beta)\exp[-I(\beta)/\epsilon]}. \tag{5.1.17}$$

This result simplifies to:

$$c_0 = 4\sqrt{\frac{\pi\epsilon}{-b'(0)a(0)}}\cdot\begin{cases}\dfrac{1}{\tilde{b}(\alpha)}\exp[I(\alpha)/\epsilon], & \text{if } I(\alpha)<I(\beta),\\ \dfrac{1}{-\tilde{b}(\beta)}\exp[I(\beta)/\epsilon], & \text{if } I(\beta)<I(\alpha),\\ \dfrac{1}{\tilde{b}(\alpha)-\tilde{b}(\beta)}\exp[I(\beta)/\epsilon], & \text{if } I(\alpha) = I(\beta).\end{cases} \tag{5.1.18}$$

Thus, in the limit $\epsilon\to 0$, if we start outside an $O(\epsilon)$-neighbourhood of the boundaries α and β, the expected exit time equals one of the constants given in (5.1.18), depending on the magnitude of $I(\alpha)$ and $I(\beta)$. Note that in the first order asymptotics to u_s and T, the position of the starting point is of importance only if we start in $O(\epsilon)$-neighbourhoods of α and β.

Other asymptotic approaches to the type of problem we encountered in this subsection can be found in de Groen [24], who used an eigenfunction expansion method, in Jiang Furu [26], who used the two-scale method, and in

Matkowsky & Schuss [41], whose method will be explained further on. A biologically relevant model in which at one of the boundaries of the domain both the drift and diffusion coefficients vanish, linearly with the distance to this boundary, has been treated in [25,59,60].

5.2. Potential systems

The method to determine c_0 in the previous section is based on the fact that with the factor g defined in (5.1.5) the nonself-adjoint backward differential operator L_ϵ turns into a self-adjoint operator, so that consequently variational formulations of the boundary value problems of exit become feasible. In this subsection we shall see that for multi-dimensional stochastic systems a similar factor g exists only for a class of so-called potential systems. Results for these systems will be derived.

We consider an n-dimensional stochastic system with a domain D that contains a deterministic point attractor and that has a boundary ∂D at which the deterministic trajectories enter D. First we study the asymptotic solution of the boundary value problem (2.6). Equation (2.6a) with $\epsilon = 0$ is solved by a constant c_0. It can be shown [19,55] that this solution is valid outside an $O(\epsilon)$-neighbourhood of ∂D. We assume that ∂D is smooth. For points $x \in D$ near ∂D, we introduce $n-1$ new coordinates along ∂D, and the new coordinate $\rho = |x - x'|$, where x' is the projection of x on ∂D. Using the stretching transformation

$$z = \epsilon\rho \tag{5.2.1}$$

we then obtain from (2.6a) the boundary layer equation

$$\frac{1}{2}\bar{a}(x')\frac{\partial^2 u_s}{\partial z^2} + \bar{b}(x')\frac{\partial u_s}{\partial z} = 0, \tag{5.2.2a}$$

with $\bar{a}$ and $\bar{b}$ defined as

$$\bar{a}(x') = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}(x')\xi_i\xi_j, \quad \bar{b}(x') = -\sum_{i=1}^{n} b_i(x')\xi_i, \tag{5.2.2b}$$

where ξ denotes the outward normal on ∂D. Equation (5.2.2a) is solved with the conditions (2.6b) and $\lim_{z\to\infty} u_s = c_0$. In the original variable x we find:

$$u_s(x) \sim c_0 + (f(x') - c_0)\exp[-\tilde{b}(x')|x-x'|/\epsilon], \tag{5.2.3}$$

uniformly on D, where $\tilde{b}(x') = 2\bar{b}(x')/\bar{a}(x')$. We intent to determine the unknown constant c_0 from a variational principle again. In general the backward operator L_ϵ defined in (2.7) is nonself-adjoint. A factor $g(x)$ is sought such that gL_ϵ is self-adjoint. This requirement leads to the following expression:

$$\epsilon\,\frac{\partial \log g}{\partial x_i} = \sum_{j=1}^{n} a_{ij}^{-1}\left[2b_j - \epsilon\sum_{k=1}^{n}\frac{\partial a_{jk}}{\partial x_k}\right] \equiv V_i, \quad 1,2,\ldots,n \tag{5.2.4}$$

where a_{ij}^{-1} denotes the inverse of the diffusion matrix (we assume that this matrix is invertible). A function g satisfying (5.2.4) exists only if the vector field V is irrotational, that is, it can be described by a potential function ϕ as follows:

$$V_i = -\frac{\partial\phi}{\partial x_i}. \tag{5.2.5}$$

Stochastic systems for which (5.2.5) holds are called potential systems. The remaining analysis in this subsection will be restricted to such systems. In order for the vector field V to be irrotational independent of the value of ϵ, we assume in addition that

$$\phi = \phi_0 + \epsilon\phi_1. \tag{5.2.6}$$

From (5.2.4), (5.2.5) and (5.2.6) it follows for g that

$$g(x) = \exp[-\phi_0(x)/\epsilon + \phi_1(x)], \tag{5.2.7a}$$

with:

$$\phi_0(x) = -\int_{x_0}^{x}\sum_{i=1}^{n}\sum_{j=1}^{n}2a_{ij}^{-1}b_j\,dx_i,\quad \phi_1(x) = -\int_{x_0}^{x}\sum_{i=1}^{n}\sum_{j=1}^{n}a_{ij}^{-1}\sum_{k=1}^{n}\frac{\partial a_{jk}}{\partial x_k}dx_i. \tag{5.2.7b}$$

The integrals in (5.2.7b) are functions of x that are independent of the path of integration. The integrals equal zero at the point x_0, which is chosen to coincide with the position of the deterministic equilibrium. Using the relationship (5.2.4) with the matrix a brought to the left side, we find that equation (2.6a) multiplied by g can be written as the Euler equation

$$-F_u + \sum_{i=1}^{n}\frac{\partial}{\partial x_i}F_{u_{x_i}} = 0, \tag{5.2.8a}$$

with F equal to

$$F = \frac{\epsilon}{4}g\sum_{i=1}^{n}\sum_{j=1}^{n}a_{ij}u_{x_i}u_{x_j}. \tag{5.2.8b}$$

In these expressions we suppressed the subscript s of u for the reason of clarity. Thus, the solution of the boundary value problem (2.6) corresponds to an extremal of the functional

$$J[u_s] = \int_D F\,dx, \tag{5.2.9}$$

taken over functions u_s satisfying the boundary condition (2.6b). Expression (5.2.3) for u_s is substituted into the integral in (5.2.9), which subsequently is evaluated for small ϵ by the method of Laplace. To be definite, we assume that the drift vector and the diffusion matrix are such that the major contributions to this integral come from the boundary ∂D. From (5.1.8) we then find:

$$c_0 = \frac{\int_{\partial D}f(y)b(y)\cdot\xi(y)\exp[-\phi_0(y)/\epsilon+\phi_1(y)]\,dS_y}{\int_{\partial D}b(y)\cdot\xi(y)\exp[-\phi_0(y)/\epsilon+\phi_1(y)]\,dS_y}. \tag{5.2.10}$$

Using the definition (2.4) of u_s we write:

$$\lim_{\epsilon\to 0} \int_{\partial D} f(y) \left[p(x,y) - \frac{b(y)\cdot\xi(y) \exp[-\phi_0(y)/\epsilon + \phi_1(y)]}{\int_{\partial D} b(y)\cdot\xi(y) \exp[-\phi_0(y)/\epsilon + \phi_1(y)]\, dS_y} \right] dS_y = 0. \quad (5.2.11)$$

This result indicates that for small ϵ the exit density p is independent of the starting point x, given $|x - x'| \gg O(\epsilon)$, and that this density is sharply peaked near the boundary point(s) with minimal potential ϕ_0. In typical situations, there is a unique y^* such that

$$\phi_0(y) > \phi_0(y^*), \quad \text{for } y \neq y^* \quad y, y^* \in \partial D, \quad (5.2.12)$$

see Figure 2. Then (5.2.11) implies that in the limit $\epsilon \to 0$ the exit density becomes:

$$p(x,y) = \delta(y - y^*), \quad (5.2.13)$$

that is, exit occurs with probability one at y^*. For cases that the minimum of ϕ_0 on ∂D is attained on a set larger than one point, the reader is referred to the literature [41].

An asymptotic expression for the expected time of exit from a region, for systems of the potential type considered above, can be derived as in Subsection 5.1. This is left as an exercise for the reader.

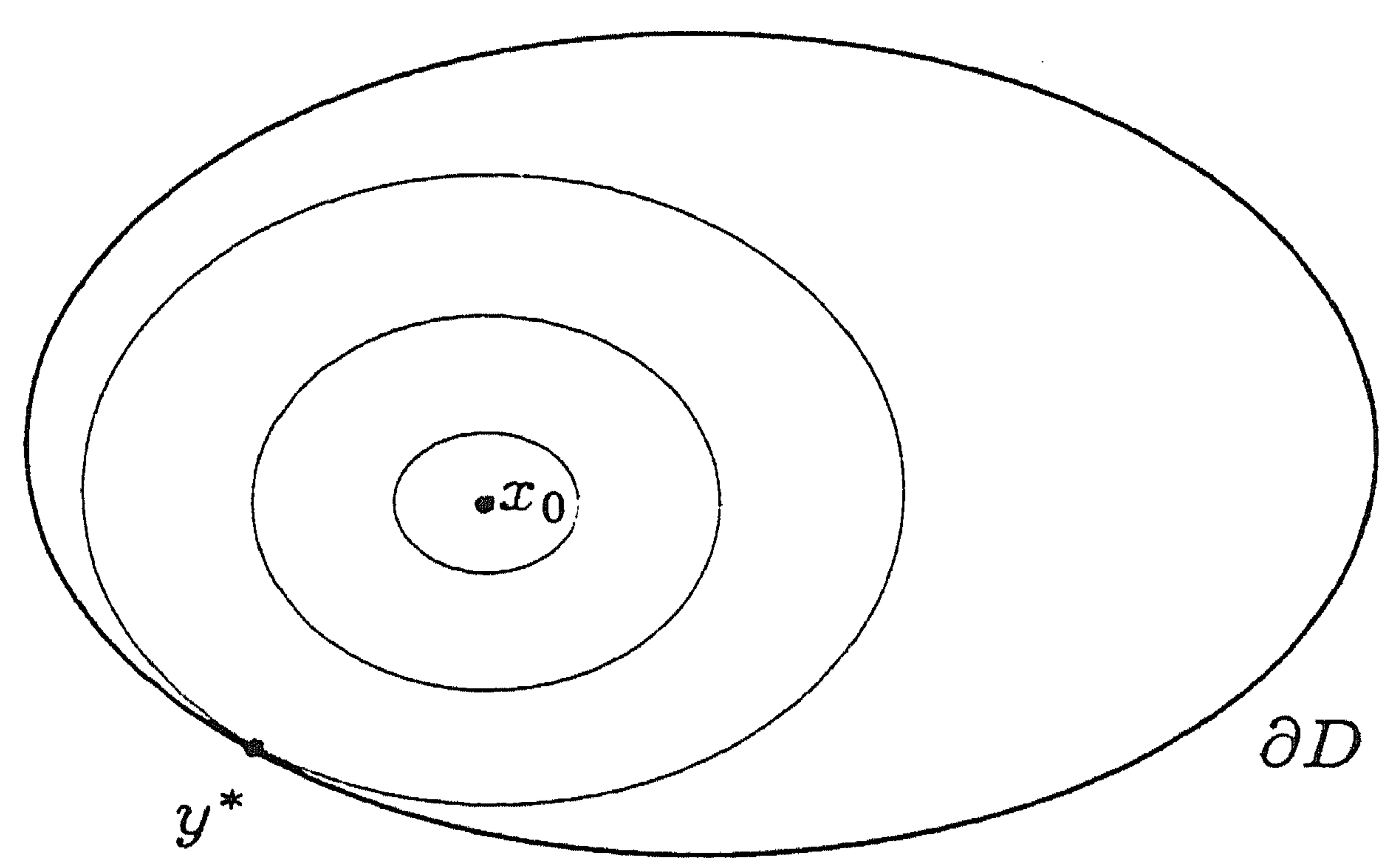

FIGURE 2. Contours on which ϕ_0 attains a constant value. This value is higher for contours farther away from x_0. The lowest value of ϕ_0 on ∂D is attained at y^*.

5.3. More general multi-dimensional systems

As we have seen in Section 5.2, the method to determine c_0 described in Section 5.1 for one-dimensional stochastic systems is applicable to multi-dimensional systems only if they are of a particular potential type. In the present section we give a brief outline of the approach to more general multi-dimensional systems, due to Matkowsky & Schuss [41].

We take over the discussion of Section 5.2 until the determination of the constant c_0. The idea is to determine this constant by the employment of the relationship (2.11) between the backward operator (2.7) and the adjoint forward operator (2.2). To this aim, we first construct a solution of the stationary forward equation (2.1). This is done by means of the WKB-method [37], which assumes that this solution is of the form:

$$v(x) = w(x)\exp[-Q(x)/\epsilon], \tag{5.3.1a}$$

for small ϵ, where

$$Q(x_0) = 0, \quad w(x_0) = 1. \tag{5.3.1b}$$

The condition on w is a normalization. Substitution of (5.3.1a) into (2.1) with the left side set equal to zero yields to leading order in ϵ the eikonal equation

$$\sum_{i=1}^{n} b_i \frac{\partial Q}{\partial x_i} + \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{a_{ij}}{2}\frac{\partial Q}{\partial x_i}\frac{\partial Q}{\partial x_j} = 0, \tag{5.3.2}$$

and to the next order in ϵ the transport equation

$$\sum_{i=1}^{n}\left[\sum_{j=1}^{n} a_{ij}\frac{\partial Q}{\partial x_j} + b_i\right]\frac{\partial w}{\partial x_i} + \sum_{i=1}^{n}\left[\sum_{j=1}^{n}\left\{\frac{a_{ij}}{2}\frac{\partial^2 Q}{\partial x_i \partial x_j} + \frac{\partial a_{ij}}{\partial x_i}\frac{\partial Q}{\partial x_j}\right\} + \frac{\partial b_i}{\partial x_i}\right] w = 0. \tag{5.3.3}$$

The functions Q and w are solutions of equations (5.3.2) and (5.3.3). The relation (2.11) is evaluated with the function v defined by (5.3.1), (5.3.2), (5.3.3) and the expression (5.2.3) for u_s. In the limit $\epsilon \to 0$ we obtain

$$c_0 = \frac{\int_{\partial D} f(y)b(y)\cdot\xi(y)\, w(y)\exp[-Q(y)/\epsilon]\, dS_y}{\int_{\partial D} b(y)\cdot\xi(y)\, w(y)\exp[-Q(y)/\epsilon]\, dS_y}. \tag{5.3.4}$$

Following the argument of the previous subsection, we find that for small ϵ the exit density p is peaked near the boundary point(s) with the lowest value of Q. Thus, the role played by the potential ϕ_0 in Section 5.2 is taken over here by the function Q. The potential ϕ_0 was expressed explicitly in terms of the drift vector and the diffusion matrix by (5.2.7b). Except in some special cases, no such explicit expression exists for Q. In practice this function is obtained through numerical integration of the eikonal equation by the ray method [37]. Such an integration scheme may include the transport equation as well in order to determine w. The method described in the present section is powerful in the sense that it can be applied to a large class of problems in arbitrary dimension. However, the asymptotics to the stationary forward equation (2.1) are not (yet) supported by a solid mathematical background. The asymptotic method described above is similar to an asymptotic method used frequently in geometrical optics and diffraction theory. For the latter method a more or less extensive literature exists, see for example the publications of Keller and coworkers [7,30] and Ludwig [36], up to the more recent work of Duistermaat [13], Maslov [39], Maslov & Fedoriuk [40], etc. For the former method, i.e. the asymptotic method to the partial differential equations related to exit problems, the literature is limited, see for example Cohen & Lewis [8], Ludwig [37] and a more recent paper of Brannan [6].

For the expected exit time the following formula has been derived [41]:

$$T(x) \sim \frac{\sqrt{2\pi\epsilon}\exp[Q(y^*)/\epsilon]}{H_1^{1/2}(x_0)\left[b\cdot\xi w H_2^{-1/2}\right](y^*)}\{1-\exp[-\tilde{b}(x')|x-x'|/\epsilon]\}, \tag{5.3.5a}$$

in which

$$H_1(x_0) = \det\left\{\frac{\partial^2 Q}{\partial x_i \partial x_j}(x_0)\right\}_{i,j=1,2,\ldots,n}, \tag{5.3.5b}$$

$$H_2(y^*) = \det\left\{\frac{\partial^2 Q}{\partial y_i \partial y_j}(y^*)\right\}_{i,j=1,2,\ldots,n-1}, \tag{5.3.5c}$$

where x_0 is the deterministic equilibrium point and y^* is the unique (by assumption, for other cases see the literature) point on ∂D with the lowest value of Q.

Now that we have obtained expressions for the expected exit time for a diffusion across the flow in Section 4 and for diffusions against the flow in Section 5, it is interesting to compare them in their dependence on the small parameter ϵ. For the former type of diffusion this dependence is algebraic, while for the latter it is exponential. Thus, these results express the quantitative difference in stochastic stability between systems of each type, where the diffusions against the flow are the more stable ones (conform intuition, I hope).

In the stochastic systems under consideration the deterministic flow is directed inward at the boundary of the domain. Other systems, in which the deterministic flow at the boundary coincides with the boundary, have been analyzed in [42] and [43]. In the first paper there are no critical points of the deterministic system located on this boundary, whereas in the second paper there are.

In the present paper we studied exit problems using formal asymptotic methods. The same subject has been studied by Ventcel & Freidlin [17,58], Friedman [18] and others from a probabilistic point of view. Rigorous mathematical methods have been used by Day [11,12], Evans & Ishii [15], Kamin [27,28] and others.

The stochastic systems that we considered have a continuous domain. In chemistry, physics, biology and other areas one meets processes with a discontinuous domain, for example birth or birth-death processes. For these processes, asymptotic methods that resemble the method described in this subsection have been presented in [32,33,34,44].

6. Some applications

Exit models have a wide variety of applications. We mention only a few of them. There are applications in population genetics, see for example Crow & Kimura [10], who describe the change in gene frequency of biological populations by means of a stochastic diffusion model. Exit from a domain here corresponds to the fixation of a gene. See also Maruyama [38] and Gillespie [20]. Another application in biology is the description of the dynamics of stochastic populations. In such applications, exit corresponds to extinction of a species. Examples can be found in Goel & Richter-Dyn [21], Ludwig [37], May [45], Nisbet & Gurney [49], Roozen [51,52], Roughgarden [54]. Other applications are in mechanics and reliability theory. Many mechanical systems near equilibrium behave essentially like the diffusion across the flow model or the diffusion against the flow model that have been studied in this paper. The stochastic domain can be chosen as the domain in which the system is known to function properly. Exit corresponds to a break down of the system. The expected exit time is a measure for the reliability of the system. See for example Grasman [22], Katz & Schuss [29], Roozen [53]. For an application of an exit model to the dynamics of the atmospheric circulation, see De Swart & Grasman [57]. The expected exit times predict the lifetimes of alternative circulation types. Other applications of exit models can be found in the literature.

References

1. L. Arnold (1971). *Stochastic Differential Equations: Theory and Applications,* J. Wiley.
2. C.M. Bender, S.A. Orszag (1978). *Advanced Mathematical Methods for Scientists and Engineers,* McGraw-Hill Book Company.
3. A. Bensoussan, J.L. Lions, G. Papanicolaou (1978). Asymptotic analysis for periodic structures. *Studies in Mathematics and its Applications, Vol. 5.*
4. N. Bleistein, R.A. Handelsman (1975). *Asymptotic Expansions of Integrals,* Holt, Rinehart and Winston.
5. O. Bolza (1904). *Lectures on the Calculus of Variations,* Chelsea Publishing Company.
6. J.R. Brannan (1987). *Parabolic Ray Theory with Boundaries,* Dept. of Math. Sc., Clemson University, Clemson, South Carolina.
7. R.N. Buchal, J.B. Keller (1960). Boundary layer problems in diffraction theory. *Comm. Pure Appl. Math. Vol. 13.*
8. J.K. Cohen, R.M. Lewis (1967). A ray method for the asymptotic solution of the diffusion equation. *J. Inst. Math. Appl., 3.*
9. R. Courant, D. Hilbert (1953). *Methods of Mathematical Physics,* Vol. I. Interscience Publishers Inc.
10. J.F. Crow, M. Kimura (1970). *An Introduction to Population Genetics Theory,* Harper and Row.
11. M.V. Day (1982). Exponential leveling for stochastically perturbed dynamical systems. *SIAM J. Math. Anal. Vol. 13, No. 4.*
12. M.V. Day (1986). *Recent Progress on the Small Parameter Exit Problem,*

Dept. of Math. Virginia Polyt. Inst. and St. Univ.
13. J.J. DUISTERMAAT (1974). Oscillatory integrals, Lagrange immersions and unfolding of singularities. *Comm. Pure. Appl. Math., Vol. 27.*
14. M.M. DYGAS, B.J. MATKOWSKY, Z. SCHUSS (1987). *Stochastic Stability of Nonlinear Oscillators,* Technical Report No. 8615, Appl. Math. Northwestern University, Evanston, Illinois.
15. L.C. EVANS, H. ISHII (1985). A PDE approach to some asymptotic problems concerning random differential equations with small noise intensities. *Ann. Inst. Henri Poincaré - Analyse non linéaire - Vol. 2, No. 1.*
16. W. FELLER (1952). The parabolic differential equations and the associated semigroups of transformations. *Ann. of Math. Vol. 55, No. 3.*
17. M.I. FREIDLIN, A.D. WENTZELL (1984). *Random Perturbations of Dynamical Systems,* Springer-Verlag.
18. A. FRIEDMAN (1976). *Stochastic Differential Equations and Applications, Vol. 2,* Academic Press.
19. C.W. GARDINER (1983). *Handbook of Stochastic Methods for Physics, Chemistry and the Natural Sciences,* Springer-Verlag.
20. J.H. GILLESPIE (1985). The interaction of genetic drift and mutation with selection in a fluctuating environment. *Theor. Pop. Biol. 27.*
21. N.S. GOEL, N. RICHTER-DYN (1974). *Stochastic Models in Biology,* Academic Press.
22. J. GRASMAN (1988). Lifetime distributions and stochastic dynamical systems. *Proc. Int. Symp. Probab. and Bayesian Statistics, Innsbruck 1986,* Plenum Press.
23. J. GRASMAN, B.J. MATKOWSKY (1977). A variational approach to singularly perturbed boundary value problems for ordinary and partial differential equations with turning points. *SIAM J. Appl. Math. Vol. 32, No. 3.*
24. P.P.N. DE GROEN (1980). *The Singularly Perturbed Turning Point Problem: a Spectral Approach,* MRC Techn. Summ. Rep. 2140, Univ. of Wisconsin-Madison.
25. F.B. HANSON, C. TIER (1981). An asymptotic solution of the first passage problem for singular diffusion in population biology. *SIAM J. Appl. Math. Vol. 40, No. 1.*
26. JIANG FURU (1980). On the boundary value problems for a class of ordinary differential equations with turning points. *Applied Mathematics and Mechanics* (Engl. Ed., Vol. I, No. 2).
27. S. KAMIN (1978). *Elliptic Perturbation of a First Order Operator with a Singular Point of Attracting Type,* Indiana Univ. Math. J., Vol. 27, No. 6.
28. S. KAMIN (1979). On elliptic singular perturbation problems with turning points. *SIAM J. Math. Anal. Vol. 10, No. 3.*
29. A. KATZ, Z. SCHUSS (1985). Reliability of elastic structures driven by random loads. *SIAM J. Appl. Math. Vol. 45, No. 3.*
30. J.B. KELLER (1958). A geometrical theory of diffraction. *Calculus of Variations and its Applications,* McGraw-Hill.
31. R.Z. KHAS'MINSKII (1966). A limit theorem for the solutions of

differential equations with random right-hand sides. *Theory of Probability and its Applications, Vol. XI, Number 3.*

32. C. KNESSL, M. MANGEL, B.J. MATKOWSKY, Z. SCHUSS, C. TIER (1984). Solution of Kramers-Moyal equations for problems in chemical physics. *J. Chem. Phys. 81 (3).*
33. C. KNESSL, B.J. MATKOWSKY, Z. SCHUSS, C. TIER (1985). An asymptotic theory of large deviations for Markov jump processes. *SIAM J. Appl. Math. Vol. 46, No. 6.*
34. C. KNESSL, B.J. MATKOWSKY, Z. SCHUSS, C. TIER (1986). A singular perturbation approach to first passage times for Markov jump processes. *J. Stat. Phys., Vol. 42, Nos. 1/2.*
35. H.A. KRAMERS (1940). Brownian motion in a field of forces and the diffusion model of chemical reaction. *Physica 7.*
36. D. LUDWIG (1966). Uniform asymptotic expansions at a caustic. *Comm. Pure Appl. Math., Vol. 19.*
37. D. LUDWIG (1975). Persistence of dynamical systems under random perturbations. *SIAM Rev. Vol. 17, No. 4.*
38. T. MARUYAMA (1977). *Stochastic Problems in Population Genetics,* Lecture Notes in Biomathematics, Springer-Verlag.
39. V.P. MASLOW (1972). *Théorie des perturbations et méthodes asymptotiques,* Dunod, Gauthier-Villars.
40. V.P. MASLOW, M.V. FEDORIUK (1981). *Semi-classical Approximation in Quantum Mechanics,* D. Reidel.
41. B.J. MATKOWSKY, Z. SCHUSS (1977). The exit problem for randomly perturbed dynamical systems. *SIAM J. Appl. Math. Vol. 33, No. 2.*
42. B.J. MATKOWSKY, Z. SCHUSS (1982). Diffusion across characteristic boundaries. *SIAM J. Appl. Math. Vol. 42, No. 4.*
43. B.J. MATKOWSKY, Z. SCHUSS, C. TIER (1983). Diffusion across characteristic boundaries with critical points. *SIAM J. Appl. Math. Vol. 43, No. 4.*
44. B.J. MATKOWSKY, Z. SCHUSS, C. KNESSL, C.TIER, M. MANGEL (1984). Asymptotic solution of the Kramers-Moyal equation and first passage times for Markov jump processes. *Phys. Rev. A, Vol. 29, No. 6.*
45. R.M. MAY (1974). *Stability and Complexity in Model Ecosystems,* Princeton Univ. Press, Sec. ed.
46. S.G. MIKHLIN (1962). *Variationsmethoden der mathematischen Physik,* Akad.-Verlag.
47. S.G. MIKHLIN (1965). *The Problem of the Minimum of a Quadratic Functional,* Holden-Day Inc.
48. S.G. MIKHLIN (1967). *Linear Equations of Mathematical Physics,* Holt, Rinehart and Winston.
49. R.M. NISBET, W.S.C. GURNEY (1982). *Modelling Fluctuating Populations,* Wiley-Interscience.
50. G.C. PAPANICOLAOU (1977). Introduction to the asymptotic analysis of stochastic equations. R.C. DIPRIMA (ed.). *Modern Modelling of Continuum Phenomena,* Lectures in Applied Mathematics, Vol. 16, American

Mathematical Society.
51. H. ROOZEN (1987). Equilibrium and extinction in stochastic population dynamics. *Bull. Math. Biol., Vol. 49, No. 6.*
52. H. ROOZEN (1987). An asymptotic solution to a two-dimensional exit problem arising in population dynamics. To appear in *SIAM J. Appl. Math.*
53. H. ROOZEN (1988). *Stochastic Stability of the Loaded Stiff Rod.* Submitted for publication.
54. J. ROUGHGARDEN (1979). *Theory of Population Genetics and Evolutionary Ecology: an Introduction,* Macmillian.
55. Z. SCHUSS (1980). *Theory and Applications of Stochastic Differential Equations,* Wiley.
56. R.L. STRATONOVICH (1963). *Topics in the Theory of Random Noise, Vol. I,* Gordon and Breach.
57. H.E. DE SWART, J. GRASMAN (1987). Effect of stochastic perturbations on a low-order spectral model of the atmospheric circulation. *Tellus, 39A.*
58. A.D. VENTCEL, M.I. FREIDLIN (1970). On small perturbations of dynamical systems. *Uspehi Mat. Nauk 25; Russian Math. Surveys 25.*
59. A.M. WAZWAZ, F.B. HANSON (1986). Matched uniform approximations for a singular boundary point and an interior turning point. *SIAM J. Appl. Math. Vol. 46, No. 6.*
60. A.M. WAZWAZ, F.B. HANSON (1986). Singular boundary resonance with turning point resonance. *SIAM J. Appl. Math. Vol. 46, No. 6.*
61. M. WILLIAMS (1982). Asymptotic exit time distributions. *SIAM J. Appl. Math. Vol. 42, No. 1.*