AN INTEGRAL OF PRODUCTS OF ULTRASPHERICAL FUNCTIONS AND A q-EXTENSION

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ABSTRACT

Let $P_n(x)$ and $Q_n(x)$ denote the Legendre polynomial of degree n and the usual second solution to the differential equation, respectively. Din showed that $\int_{-1}^{1} Q_n(x)\,P_m(x)\,P_l(x)\,dx$ vanishes when $|l-m| < n < l+m$, and Askey evaluated the integral for arbitrary integral values of l, m and n. We extend this to the evaluation of $\int_{-1}^{1} D_n^\lambda(x)\,C_m^\lambda(x)\,C_l^\lambda(x)\,(1-x^2)^{2\lambda-1}\,dx$, where $C_n^\lambda(x)$ is the ultraspherical polynomial and $D_n^\lambda(x)$ is the appropriate second solution to the ultraspherical differential equation. A q-extension is found using the continuous q-ultraspherical polynomials of Rogers. Again the integral vanishes when $|l-m| < n < l+m$. It is shown that this vanishing phenomenon holds for quite general orthogonal polynomials. A related integral of the product of three Bessel functions is also evaluated.

1. *Introduction*

Legendre polynomials P_n are orthogonal polynomials of degree n on $(-1, 1)$ with constant weight function and with normalization $P_n(1) = 1$. Corresponding Legendre functions of the second kind Q_n are defined on the cut $(-1, 1)$ by the principal value integral

$$Q_n(x) = \frac{1}{2} \int_{-1}^{1} \frac{P_n(t)}{x-t}\,dt, \qquad -1 < x < 1, \tag{1.1}$$

cf. [**26**, (4.9.12) and (4.62.9)].

In a very surprising paper Din [**9**] showed that

$$\int_{-1}^{1} Q_n(x)\,P_m(x)\,P_l(x)\,dx = 0 \tag{1.2}$$

when $|l-m| < n < l+m$. Part of the surprise was the vanishing and part was the fact that such an attractive result did not seem to have been found before. Askey [**3**] evaluated (1.2) for general integers l, m and n. Besides the trivial result

$$\int_{-1}^{1} Q_n(x)\,P_m(x)\,P_l(x)\,dx = 0 \tag{1.3}$$

when $l+m+n$ is even, he showed that

$$-\int_{-1}^{1} Q_{l+m+1+2j}(x)\,P_m(x)\,P_l(x)\,dx = \int_{-1}^{1} Q_m(x)\,P_{l+m+1+2j}(x)\,P_l(x)\,dx$$
$$= \frac{\Gamma(j+\frac{1}{2})\,\Gamma(j+l+1)\,\Gamma(j+m+1)\,\Gamma(j+l+m+\frac{3}{2})}{2\Gamma(j+1)\,\Gamma(j+l+\frac{3}{2})\,\Gamma(j+m+\frac{3}{2})\,\Gamma(j+l+m+2)} \tag{1.4}$$

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when $j = 0, 1, 2, \ldots$. (The two integrals in (1.4) were not evaluated simultaneously in [3].)

After preprints of an earlier version of the present paper were circulated, J. Boersma in Eindhoven wrote that (1.2)–(1.4) were known. The integral

$$\int_{-1}^{1} Q_n(x)\, P_m(x)\, P_l(x)\, dx$$

was written as a sum by Benthem [7], see Stelling VII attached to [7], and he showed that (1.2) holds. Boersma, in the spring of 1961, summed the series and so obtained (1.4). He did not publish this result, but showed it to the first author in the fall of 1969. In the intervening time Askey forgot this. Boersma also called our attention to a paper of Shabde which contains the same series, see [22].

Formulas (1.2)–(1.4) should be compared with the classical result of Adams and Ferrers (cf. Hsü [15])

$$\int_{-1}^{1} P_n(x)\, P_m(x)\, P_l(x)\, dx = \frac{(\frac{1}{2})_{s-l}\,(\frac{1}{2})_{s-m}\,(\frac{1}{2})_{s-n}\, s!\,2}{(s-l)!\,(s-m)!\,(s-n)!\,(\frac{1}{2})_s(2s+1)} \tag{1.5}$$

and this is zero unless $l+m+n = 2s$ is even and a triangle with sides l, m, n exists, that is, $|l-m| \leqslant n \leqslant l+m$. Here the shifted factorial $(a)_k$ is defined by

$$(a)_k = \Gamma(a+k)/\Gamma(a) = a(a+1)\ldots(a+k-1). \tag{1.6}$$

The fact that the integral in (1.5) vanishes when the triangle condition

$$|l-m| \leqslant n \leqslant l+m$$

fails is a simple general property of orthogonal polynomials. Let p_n be the n-th degree orthogonal polynomial with respect to some positive measure $d\alpha$ on $\mathbb{R}$. Then

$$\int_{-\infty}^{\infty} p_n(x)\, p_m(x)\, p_l(x)\, d\alpha(x) = 0 \tag{1.7}$$

if $|l-m| \leqslant n \leqslant l+m$ fails.

In §3 we shall give proofs of the vanishing in (1.3) and the equality of the two integrals in (1.4) in the framework of general orthogonal polynomials. In order to do that, we need the analogue of Q_n for general orthogonal polynomials p_n. For convenience, assume that the orthogonality measure $d\alpha$ has support within a finite interval $[a, b]$ and that, on $[a, b]$,

$$d\alpha(x) = w(x)\, dx, \tag{1.8}$$

with $w \in C^1((a, b)) \cap L^2([a, b])$. Define a function of the second kind q_n outside the cut by

$$q_n(z) = \int_a^b \frac{p_n(t)}{z-t}\, w(t)\, dt, \qquad z \in \mathbb{C},\ z \notin [a, b], \tag{1.9}$$

and on the cut by

$$q_n(x) = \lim_{y \downarrow 0} \tfrac{1}{2}(q_n(x+iy) + q_n(x-iy)) = ⨍_a^b \frac{p_n(t)}{x-t}\, w(t)\, dt, \quad a < x < b. \tag{1.10}$$

Note that q_n on the cut is the finite Hilbert transform on (a, b) of the function wp_n. Let

$$I(n, m, l) = \int_a^b q_n(x)\, p_m(x)\, p_l(x)\, w(x)\, dx. \tag{1.11}$$

We shall show in §3 that

$$I(n, m, l) = -I(m, n, l) \quad \text{if } l \leqslant m+n. \tag{1.12}$$

For $|l-m| \leqslant n \leqslant l+m$ this implies that

$$I(n, m, l) = 0 \tag{1.13}$$

(a nice complement to (1.7)), while the cases $n > l+m$ and $n < |l-m|$ of $I(n, m, l)$ are related to each other by (1.12). Indeed, (1.12) holds if $n > l+m$, and we have $n > l+m$ if and only if $m < |n-l|$ and $n \geqslant l$.

In the case of Jacobi polynomials, that is, $a = -1$, $b = 1$, $w(x) = (1-x)^\alpha(1+x)^\beta$, $p_n(x) = P_n^{(\alpha,\beta)}(x)$, $p_n(1) = (\alpha+1)_n/n!$, it is convenient to define functions of the second kind as second solutions of the Jacobi differential equation. Thus, outside the cut, one takes

$$Q_n^{(\alpha,\beta)}(z) = \tfrac{1}{2}(z-1)^{-\alpha}(z+1)^{-\beta}q_n(z), \quad z \notin [-1, 1], \tag{1.14}$$

with q_n given by (1.9) (cf. Szegö [**26**, (4.61.4)]). However, our above considerations show that Szegö's definition [**26**, (4.62.9)]

$$Q_n^{(\alpha,\beta)}(x) = \tfrac{1}{2}(Q_n^{(\alpha,\beta)}(x+i0) + Q_n^{(\alpha,\beta)}(x-i0)),$$

is not the most appropriate choice for a Jacobi function of the second kind on the cut. We rather define

$$Q_n^{(\alpha,\beta)}(x) = \tfrac{1}{2}(1-x)^{-\alpha}(1+x)^{-\beta}q_n(x), \quad -1 < x < 1, \tag{1.15}$$

with q_n given by (1.10). Hence, $Q_n^{(\alpha,\beta)}$ on the cut is expressed in terms of $Q_n^{(\alpha,\beta)}$ outside the cut by

$$Q_n^{(\alpha,\beta)}(x) = \tfrac{1}{2}(e^{i\pi\alpha}Q_n^{(\alpha,\beta)}(x+i0) + e^{-i\pi\alpha}Q_n^{(\alpha,\beta)}(x-i0)), \quad -1 < x < 1. \tag{1.16}$$

This function was introduced by Durand [**12**] and was dictated by the natural form of the Nicholson-type integral formula. Earlier, Durand [**11**] had used these functions when $\alpha = \beta$. In this case the polynomials are called ultraspherical polynomials, and they are written in a different way. It is usual to take $a = -1$, $b = 1$, $w(x) = (1-x^2)^{\lambda-\frac{1}{2}}$, $p_n(x) = C_n^\lambda(x)$, $p_n(1) = (2\lambda)_n/n!$, so

$$C_n^\lambda(x) = \frac{(2\lambda)_n}{(\lambda+\frac{1}{2})_n} P_n^{(\lambda-\frac{1}{2},\lambda-\frac{1}{2})}(x). \tag{1.17}$$

For $D_n^\lambda(x)$, the ultraspherical function of the second kind on the cut we follow Durand's convention

$$(1-x^2)^{\lambda-\frac{1}{2}}D_n^\lambda(x) = \pi^{-1}q_n(x) = \pi^{-1}\fint_{-1}^{1} \frac{C_n^\lambda(t)}{x-t}(1-t^2)^{\lambda-\frac{1}{2}}\,dt, \quad -1 < x < 1, \tag{1.18}$$

where $q_n(x)$ is given by (1.10).

Formula (1.7) now holds with the left-hand side replaced by

$$\int_{-1}^{1} C_n^\lambda(x)\,C_m^\lambda(x)\,C_l^\lambda(x)\,(1-x^2)^{\lambda-\frac{1}{2}}\,dx, \tag{1.19}$$

while (1.12) and (1.13) are, in particular, valid with

$$I(n, m, l) = I_\lambda(n, m, l) = \int_{-1}^{1} D_n^\lambda(x)\,C_m^\lambda(x)\,C_l^\lambda(x)\,(1-x^2)^{2\lambda-1}\,dx, \quad \lambda > 0. \tag{1.20}$$

For general l, m, n the integral (1.19) can be evaluated as a quotient of products of gamma functions, as was stated by Dougall [10] and proved by Hsü [15]. In §2 we shall show that $I_\lambda(n, m, l)$ can be evaluated in a similar way. In particular, (1.12) and (1.13) in the ultraspherical case will be obtained in §2 by explicit computations.

Two more degrees of freedom can be added and still have results that are simple enough to be useful. One is to replace $D_n^\lambda(x)(1-x^2)^{\lambda-\frac{1}{2}}$ by $D_n^\mu(x)(1-x^2)^{\mu-\frac{1}{2}}$ in (1.20) and consider the integral

$$I_{\mu,\lambda}(n, m, l) = \int_{-1}^{1} D_n^\mu(x)\, C_m^\lambda(x)\, C_l^\lambda(x)\,(1-x^2)^{\lambda+\mu-1}dx, \quad \lambda > \max\{-\tfrac{1}{2}, -\mu\}. \tag{1.21}$$

When $l = 0$ this integral can be evaluated in terms of gamma functions. In the general case it is a single sum. If one just expands the functions under the integral the resulting series would be a triple sum. Whenever there is a reduction in the number of sums in the evaluation of an integral there is usually something interesting happening. A reduction from a triple sum to a single sum is even more surprising, and in the case when $\mu = \lambda$ the resulting sum can be evaluated as a quotient of gamma functions, and it vanishes when $|l-m| \leqslant n \leqslant l+m$. The remaining degree of freedom that can be added comes from replacing the ultraspherical polynomials by the continuous q-ultraspherical polynomials introduced by Rogers [20], and defining an appropriate function of the second kind.

The integral (1.19) of Dougall arises from the linearization formula

$$C_n^\lambda(x)\, C_m^\lambda(x) = \sum_{l=|n-m|}^{n+m} a(l, m, n)\, C_l^\lambda(x) \tag{1.22}$$

via orthogonality. Surprisingly Rogers [20] had found the coefficients in the q-extension of (1.22) more than twenty years before Dougall [10] independently rediscovered the special case (1.22), in which $q = 1$. This strongly suggests that (1.21) can be extended to the q case, and it can.

Rather than carry out the same derivation twice, in §2 we shall give the details for the continuous q-ultraspherical polynomials, and then state the results for the limiting case of ultraspherical polynomials. By taking a further limit we shall obtain the Hermite polynomial case. In §3 (1.12) and (1.13) will be proved in general. This section can be read independently of §2. Finally, in §4, we shall first derive a formal limit case for Bessel functions of the evaluation of $I_\lambda(n, m, l)$ and next give an independent proof of this limit case.

For the classical orthogonal polynomials there is a strong duality between them as functions of x and as functions of n. This was mentioned by Dougall and he gave a dual result to the evaluation of (1.19). There is also a dual series to the integral (1.20). This was treated by Rahman and Shah [17]. Further series were considered by Rahman and Shah [18] and van Haeringen [28].

2. *Ultraspherical polynomials and their q-extensions*

Take a fixed parameter q, with $|q| < 1$. The q-binomial theorem is

$$\frac{[ax]_\infty}{[x]_\infty} = \sum_{n=0}^{\infty} \frac{[a]_n}{[q]_n} x^n, \qquad |x| < 1, \tag{2.1}$$

where

$$[a]_\infty = \prod_{n=0}^{\infty} (1-aq^n), \tag{2.2}$$

and

$$[a]_n = [a]_\infty/[aq^n]_\infty. \tag{2.3}$$

We shall also use (2.3) when $n < 0$. There are many proofs of the q-binomial theorem (2.1). See Slater [**23**, §3.2.2] for a simple proof. Andrews has recently given a very interesting series of Regional Conference Lectures on q-series. These will be published by the American Mathematical Society. The reader should look at these lectures to get some idea why q-series extensions of classical results can be very important.

The continuous q-ultraspherical polynomials are defined by the generating function

$$\frac{[\beta re^{i\theta}]_\infty [\beta re^{-i\theta}]_\infty}{[re^{i\theta}]_\infty [re^{-i\theta}]_\infty} = \sum_{n=0}^{\infty} C_n(x;\beta\,|\,q)\, r^n, \tag{2.4}$$

$x = \cos\theta$. This is easily seen to be equivalent to the recurrence relation

$$2x(1-\beta q^n)\, C_n(x;\beta\,|\,q) = (1-q^{n+1})\, C_{n+1}(x;\beta\,|\,q) + (1-\beta^2 q^{n-1})\, C_{n-1}(x;\beta\,|\,q). \tag{2.5}$$

The q-binomial theorem and (2.4) imply the explicit formula

$$C_n(\cos\theta;\beta\,|\,q) = \sum_{k=0}^{n} \frac{[\beta]_k [\beta]_{n-k}}{[q]_k [q]_{n-k}} \cos(n-2k)\theta. \tag{2.6}$$

For ordinary ultraspherical polynomials we have

$$C_n^\lambda(\cos\theta) = \sum_{k=0}^{n} \frac{(\lambda)_k (\lambda)_{n-k}}{k!\,(n-k)!} \cos(n-2k)\theta, \tag{2.7}$$

cf. Szegö [**26**, (4.9.19)]. It follows from (2.6), (2.7) that

$$\lim_{q\to 1} C_n(x;q^\lambda\,|\,q) = C_n^\lambda(x). \tag{2.8}$$

Rogers [**20**] extended (2.6) to

$$C_n(x;\beta\,|\,q) = \sum_{k=0}^{[\frac{1}{2}n]} \gamma^k \frac{[\beta\gamma^{-1}]_k [\beta]_{n-k} (1-\gamma q^{n-2k})}{[q]_k [q\gamma]_{n-k} (1-\gamma)} C_{n-2k}(x;\gamma\,|\,q). \tag{2.9}$$

One special case of this is

$$T_n(x) = \frac{1}{2} \sum_{k=0}^{[\frac{1}{2}n]} \gamma^k \frac{[\gamma^{-1}]_k [q]_{n-k} (1-q^n)(1-\gamma q^{n-2k})}{[q]_k [q\gamma]_{n-k} (1-q^{n-k})(1-\gamma)} C_{n-2k}(x;\gamma\,|\,q), \quad n = 1, 2, \ldots, \tag{2.10}$$

since

$$\lim_{\beta\to 1} \frac{(1-q^n)\, C_n(x;\beta\,|\,q)}{2(1-\beta)} = T_n(x), \quad n = 1, 2, \ldots \tag{2.11}$$

with $T_n(\cos\theta) = \cos n\theta$.

When $-1 < \beta, q < 1$ the orthogonality relation is

$$\int_{-1}^{1} C_n(x;\beta\,|\,q)\, C_m(x;\beta\,|\,q)\, w_\beta(x)\, dx = \delta_{m,n}/h_n \tag{2.12}$$

with

$$w_\beta(x) = w_\beta(x\,|\,q) = (1-x^2)^{-\frac{1}{2}} \prod_{k=0}^{\infty} \left[\frac{1-2(2x^2-1)\, q^k + q^{2k}}{1-2(2x^2-1)\, \beta q^k + \beta^2 q^{2k}}\right] \tag{2.13}$$

and

$$h_n = \frac{(1-\beta q^n)\, [q]_n\, [q]_\infty\, [\beta^2]_\infty}{2\pi(1-\beta)\, [\beta^2]_n\, [\beta]_\infty\, [\beta q]_\infty}. \tag{2.14}$$

It is easy to see that

$$\lim_{q\to 1} w_{q^\lambda}(x\,|\,q) = 2^{2\lambda}(1-x^2)^{\lambda-\frac{1}{2}}, \tag{2.15}$$

compatible with (2.12) and (2.8).

See [**4**, **5**, **6**] for various proofs of the orthogonality (2.12). A direct proof of (2.10) using orthogonality is given in [**5**], and then (2.6) and (2.10) are combined to give (2.9). A proof of (2.9) by induction was given by Rogers [**20**].

Once the explicit orthogonality relation has been found, (2.9) can be inverted to obtain

$$w_\beta(x\,|\,q)\,C_n(x;\beta\,|\,q) = \sum_{k=0}^{\infty} a(k,n)\,C_{n+2k}(x;\gamma\,|\,q)\,w_\gamma(x\,|\,q), \tag{2.16}$$

where

$$a(k,n) = \frac{\beta^k[\gamma\beta^{-1}]_k\,[q]_{n+2k}\,[\beta^2]_n\,[\gamma]_{n+k}(1-\gamma q^{n+2k})\,[\gamma^2]_\infty\,[\beta]_\infty\,[\beta q]_\infty}{[q]_k\,[\gamma^2]_{n+2k}\,[q]_n\,[\beta q]_{n+k}(1-\gamma)\,[\gamma]_\infty\,[\gamma q]_\infty\,[\beta^2]_\infty}, \quad |\beta|<1,$$

cf. [**5**, (4.20)].

When $\gamma = q$ this gives, with $|\beta|<1$,

$$w_\beta(\cos\theta\,|\,q)\,C_n(\cos\theta;\beta\,|\,q) = 4\sum_{k=0}^{\infty} b(k,n;\beta)\sin(n+2k+1)\theta, \tag{2.17}$$

where

$$b(k,n;\beta) = \beta^k\,\frac{[q\beta^{-1}]_k\,[\beta^2]_n\,[q]_{n+k}\,[\beta]_\infty\,[\beta q]_\infty}{[q]_k\,[q]_n\,[\beta q]_{n+k}\,[\beta^2]_\infty\,[q]_\infty}. \tag{2.18}$$

Here we have used the fact that

$$w_q(\cos\theta\,|\,q) = (1-\cos^2\theta)^{-\frac{1}{2}}(1-2(2\cos^2\theta-1)+1) = 4\sin\theta.$$

The limit case of (2.17) with $\beta = q^\lambda$ as $q\to 1$ is

$$(\sin\theta)^{2\lambda-1}\,C_n^\lambda(\cos\theta) = \frac{2\Gamma(\lambda+\frac{1}{2})\,(2\lambda)_n}{\Gamma(\lambda+1)\,\Gamma(\frac{1}{2})\,n!}\sum_{k=0}^{\infty}\frac{(1-\lambda)_k\,(n+k)!}{k!(\lambda+1)_{n+k}}\sin(n+2k+1)\theta, \quad \lambda>0, \tag{2.19}$$

see [**25**; **26**, (4.9.22)] for an independent derivation.

We define the q-ultraspherical functions of the second kind, $D_n(x;\beta\,|\,q)$, by (1.10) with $\pi w(x)\,D_n(x) = q_n(x)$, that is,

$$w_\beta(x\,|\,q)\,D_n(x;\beta\,|\,q) = \pi^{-1}\fint_{-1}^{1}\frac{w_\beta(y\,|\,q)\,C_n(y;\beta\,|\,q)}{x-y}\,dy, \quad -1<x<1. \tag{2.20}$$

It follows from (2.17), (2.20) and the properties of the Hilbert transform that the left-hand side of (2.20) can be expanded as the conjugate series of the one in (2.17):

$$w_\beta(\cos\theta\,|\,q)\,D_n(\cos\theta;\beta\,|\,q) = 4\sum_{k=0}^{\infty} b(k,n;\beta)\cos(n+2k+1)\theta, \quad |\beta|<1. \tag{2.21}$$

The formula for $D_n^\lambda(x)$ (cf. (1.18)) corresponding to (2.21), is

$$(\sin\theta)^{2\lambda-1}\,D_n^\lambda(\cos\theta) = \frac{2\Gamma(\lambda+\frac{1}{2})\,(2\lambda)_n}{\Gamma(\lambda+1)\,\Gamma(\frac{1}{2})\,n!}\sum_{k=0}^{\infty}\frac{(1-\lambda)_k(n+k)!}{k!\,(\lambda+1)_{n+k}}\cos(n+2k+1)\theta, \quad \lambda>0. \tag{2.22}$$

Formula (2.22) can also be obtained from [**13**, 3.5(3); **11**, (7)]. It follows from (2.21) and (2.22) that

$$\lim_{q\to 1} D_n(x;q^\lambda\,|\,q) = D_n^\lambda(x), \quad -1<x<1,\ \lambda>0. \tag{2.23}$$

We are now in a position to state the integral that extends (1.21) and the Din–Askey integral (1.2), (1.4):

$$M_{\nu,\beta}(n, m, l\,|\,q) = \int_{-1}^{1} D_n(x;\nu\,|\,q)\, C_m(x;\beta\,|\,q)\, C_l(x;\beta\,|\,q)\, w_\beta(x\,|\,q)\, w_\nu(x\,|\,q)\, dx \quad (2.24)$$

when $|\beta| < 1$, $|\nu| < 1$. This integral vanishes when $l+m+n$ is even, since

$$C_n(-x;\beta\,|\,q) = (-1)^n C_n(x;\beta\,|\,q), \quad (2.25)$$

$$D_n(-x;\beta\,|\,q) = (-1)^{n+1} D_n(x;\beta\,|\,q). \quad (2.26)$$

The strategy for our evaluation of the integral in (2.24) is as follows. First evaluate (2.24) for $l = 0$, and consequently $n+m$ odd, which will be written as

$$I_{\nu,\beta}(n, m) = \int_{-1}^{1} D_n(x;\nu\,|\,q)\, C_m(x;\beta\,|\,q)\, w_\beta(x\,|\,q)\, w_\nu(x\,|\,q)\, dx \quad (2.27)$$

when $|\beta| < 1$, $|\nu| < 1$. This yields a single sum (a very well-poised ${}_6\Phi_5$), which can be summed. Next use Rogers's linearization formula

$$C_m(x;\beta\,|\,q)\, C_l(x;\beta\,|\,q) = \sum_{k=0}^{\min(m,\,l)} a(k, l, m)\, C_{l+m-2k}(x;\beta\,|\,q), \quad (2.28)$$

where

$$a(k, l, m) = \frac{[q]_{l+m-2k}\,[\beta]_k\,[\beta]_{l-k}\,[\beta]_{m-k}\,[\beta^2]_{l+m-k}(1-\beta q^{l+m-2k})}{[\beta^2]_{l+m-2k}\,[q]_k\,[q]_{l-k}\,[q]_{m-k}\,[\beta q]_{l+m-k}(1-\beta)}.$$

Proofs are given in [**8**, **20**]. Substitution of (2.28) into (2.24) and application of the evaluation of (2.27) will again yield a single sum, which can be simplified to a ${}_4\Phi_3$ and which becomes trivial if $\nu = \beta$.

The basic hypergeometric series ${}_{p+1}\Phi_p$ is defined by

$${}_{p+1}\Phi_p\left[\begin{matrix}a_0, \ldots, a_p\\ b_1, \ldots, b_p\end{matrix}; q, x\right] = \sum_{n=0}^{\infty} \frac{[a_0]_n \cdots [a_p]_n}{[q]_n\,[b_1]_n \cdots [b_p]_n}\, x^n. \quad (2.29)$$

It is well poised if $a_0 q = a_1 b_1 = \ldots = a_p b_p$, and very well poised if $a_1 = qb_1$, $a_2 = -a_1$.

Use (2.21) to get

$$I_{\nu,\beta}(n, m) = 4 \sum_{k=0}^{\infty} b(k, n; \nu) \int_{-1}^{1} T_{n+2k+1}(x)\, C_m(x;\beta\,|\,q)\, w_\beta(x\,|\,q)\, dx. \quad (2.30)$$

Now use (2.10) to replace $T_{n+2k+1}(x)$ by a sum of $C_j(x;\beta\,|\,q)$, and then use the orthogonality relation (2.12). The result of this is

$$I_{\nu,\beta}(n, m) = 4\pi \frac{[\nu]_\infty\,[\nu q]_\infty\,[\beta]_\infty\,[\beta q]_\infty}{[q]_\infty^2\,[\nu^2]_\infty\,[\beta^2]_\infty} \cdot \frac{[\nu^2]_n\,[\beta^2]_m}{[\nu q]_n\,[q]_m}$$
$$\cdot \frac{(1-q^{n+1})\,[\beta^{-1}]_{\frac12(n-m+1)}\,[q]_{\frac12(n+m+1)}}{[q]_{\frac12(n-m+1)}\,[\beta q]_{\frac12(n+m+1)}\,(1-q^{\frac12(n+m+1)})}\, \beta^{\frac12(n-m+1)}$$
$$\cdot \sum_{k=0}^{\infty} \frac{[q^{n+1}]_k (1-q^{n+1+2k})\,[q^{\frac12(n+m+1)}]_k\,[q/\nu]_k\,[\beta^{-1}q^{\frac12(n-m+1)}]_k}{[q]_k (1-q^{n+1})\,[q^{\frac12(n-m+3)}]_k\,[\nu q^{n+1}]_k\,[\beta q^{\frac12(n+m+3)}]_k} (\nu\beta)^k.$$

This series can be summed by a result of Jackson [**16**] which sums the very well-poised ${}_6\Phi_5$ (see also [**23**, (3.3.1.3)]):

$$\sum_{k=0}^{\infty} \frac{[a]_k (1-aq^{2k})\,[b]_k\,[c]_k\,[d]_k}{[q]_k (1-a)\,[aq/b]_k\,[aq/c]_k\,[aq/d]_k} \left(\frac{aq}{bcd}\right)^k = \frac{[aq]_\infty\,[aq/bc]_\infty\,[aq/bd]_\infty\,[aq/cd]_\infty}{[aq/b]_\infty\,[aq/c]_\infty\,[aq/d]_\infty\,[aq/bcd]_\infty}.$$

As Jackson remarked [16, p. 104] Rogers found this formula in 1894. The result is

$$I_{\nu,\beta}(n, m) = 4\pi \frac{[\nu]_\infty^2 [\beta]_\infty [\beta q]_\infty [\nu^2]_n [\beta^2]_m}{[\nu^2]_\infty [\beta^2]_\infty [q]_\infty^2 [q]_n [q]_m} \cdot \frac{[\beta^{-1}]_{\frac{1}{2}(n-m+1)} [q]_{\frac{1}{2}(n+m-1)}}{[\nu]_{\frac{1}{2}(n-m+1)} [\beta\nu]_{\frac{1}{2}(n+m+1)}} \beta^{\frac{1}{2}(n-m+1)}. \tag{2.31}$$

Observe that

$$I_{\nu,\beta}(n, m) = -I_{\beta,\nu}(m, n). \tag{2.32}$$

Use of (2.28) and (2.31) in (2.24) leads to

$$\begin{aligned} M_{\nu,\beta}(n, m, l \mid q) = {} & 4\pi \frac{[\nu]_\infty^2 [\beta]_\infty [\beta q]_\infty [\nu^2]_n [\beta]_l [\beta]_m}{[q]_\infty^2 [\nu^2]_\infty [\beta^2]_\infty [q]_n [q]_l [q]_m} \\ & \cdot \frac{[\beta^2]_{l+m} [\beta^{-1}]_{\frac{1}{2}(n+1-l-m)} [q]_{\frac{1}{2}(n+m+l-1)}}{[\beta]_{l+m} [\nu]_{\frac{1}{2}(n+1-l-m)} [\beta\nu]_{\frac{1}{2}(n+m+l+1)}} \beta^{\frac{1}{2}(n+1-l-m)} \\ & \cdot \sum_{k=0}^{\min(l,m)} \frac{[\beta^{-1}q^{-l-m}]_k (1-\beta^{-1}q^{2k-l-m}) [q^{-l}]_k [q^{-m}]_k [\beta^{-1}q^{\frac{1}{2}(n+1-l-m)}]_k}{[q]_k (1-\beta^{-1}q^{-l-m}) [q^{1-m}\beta^{-1}]_k [q^{1-l}\beta^{-1}]_k [q^{\frac{1}{2}(1-l-m-n)}]_k} \\ & \cdot \frac{[\beta]_k [\beta^{-1}\nu^{-1}q^{\frac{1}{2}(1-l-m-n)}]_k}{[\beta^{-2}q^{1-l-m}]_k [\nu q^{\frac{1}{2}(n+1-l-m)}]_k} \left(\frac{\nu q}{\beta}\right)^k. \end{aligned} \tag{2.33}$$

Certain expressions may become indeterminate as 0/0. These are interpreted by treating one of the parameters l, m or n as a continuous parameter and taking a limit as it approaches an integer.

The series is a very well-poised ${}_8\Phi_7$, and so can be transformed to a balanced ${}_4\Phi_3$ series by Watson's transformation [29, 23, 1]. Transforming the ${}_4\Phi_3$ series by Sears's formula [21, (8.3)], also see [6, (1.28)], and simplifying the coefficients we finally obtain (assume $m \geqslant l$ without loss of generality)

$$\begin{aligned} M_{\nu,\beta}(n, m, l \mid q) = {} & 4\pi \frac{[\nu]_\infty [\nu q]_\infty [\beta]_\infty [\beta q]_\infty [\nu^2]_n [\beta^2]_l [\beta^2]_m}{[\nu^2]_\infty [\beta^2]_\infty [q]_\infty [q]_\infty [q]_n [q]_l [q]_m} \\ & \cdot \frac{[\nu]_{\frac{1}{2}(n+m+l+1)} [q]_{\frac{1}{2}(n+m-l-1)} [\beta^{-1}]_{\frac{1}{2}(n-m-l+1)} \beta^{\frac{1}{2}(n-m-l+1)}}{[\beta\nu]_{\frac{1}{2}(n+m+l+1)} [\nu q]_{\frac{1}{2}(n+m-l-1)} [\nu]_{\frac{1}{2}(n-m+l+1)}} \\ & \cdot \sum_{k=0}^{l} [q^{\frac{1}{2}(n-m-l+2k+1)}]_{l-k} \frac{[q^{-l}]_k [q^{-m}]_k [\beta]_k [\beta\nu^{-1}]_k q^k}{[\nu^{-1}q^{\frac{1}{2}(-n-m-l+1)}]_k [\beta^2]_k [q]_k}. \end{aligned} \tag{2.34}$$

In the special case when $\nu = \beta$ the series is equal to $[q^{\frac{1}{2}(n-m-l+1)}]_l$, and so we have, with $m \geqslant l$, $n+m+l$ odd,

$$\begin{aligned} M_{\beta,\beta}(n, m, l \mid q) = {} & 4\pi \frac{[\beta]_\infty^2 [\beta q]_\infty^2 [\beta^2]_n [\beta^2]_l [\beta^2]_m [\beta]_{\frac{1}{2}(n+m+l+1)}}{[\beta^2]_\infty^2 [q]_\infty^2 [q]_n [q]_l [q]_m [\beta^2]_{\frac{1}{2}(n+m+l+1)}} \\ & \cdot \frac{[q]_{\frac{1}{2}(n+m-l-1)} [\beta^{-1}]_{\frac{1}{2}(n-m-l+1)} [q^{\frac{1}{2}(n-m-l+1)}]_l \beta^{\frac{1}{2}(n-m-l+1)}}{[\beta q]_{\frac{1}{2}(n+m-l-1)} [\beta]_{\frac{1}{2}(n-m+l+1)}}. \end{aligned} \tag{2.35}$$

From (2.35) we conclude that

$$M_{\beta,\beta}(n, m, l \mid q) = \begin{cases} -M_{\beta,\beta}(m, n, l \mid q) & \text{if } n > l+m,\ m \geqslant l,\ l+m+n \text{ odd}, \quad (2.36) \\ 0 & \text{if } 0 \leqslant m-l < n < m+l. \quad (2.37) \end{cases}$$

Thus we have a complete evaluation of $M_{\beta,\beta}(n, m, l \mid q)$, and (1.12) and (1.13) are valid for the q-ultraspherical polynomials. It follows from (1.21), (2.8), (2.15), (2.23), (2.24) that

$$I_{\mu,\lambda}(n, m, l) = 2^{-2\lambda-2\mu} \lim_{q \to 1} M_{q^\mu, q^\lambda}(n, m, l \mid q). \tag{2.38}$$

To find this limit replace ν, β by q^μ, q^λ in (2.34) and rewrite it using the q-gamma function [27, 2]

$$\Gamma_q(x) = \frac{[q]_\infty}{[q^x]_\infty}(1-q)^{1-x}, \qquad \lim_{q\to 1}\Gamma_q(x) = \Gamma(x). \tag{2.39}$$

Thus ($m \geqslant l$, $n+l+m$ is odd)

$$\begin{aligned} I_{\mu,\lambda}(n, m, l) &= \frac{\Gamma(\mu+\frac{1}{2})\,\Gamma(\lambda+\frac{1}{2})\,(2\mu)_n\,(2\lambda)_l(2\lambda)_m}{\Gamma(\mu+1)\,\Gamma(\lambda+1)\,n!\,l!\,m!} \\ &\cdot \frac{(\mu)_{\frac{1}{2}(n+m+l+1)}\,(\frac{1}{2}(n+m-l-1)!\,(-\lambda)_{\frac{1}{2}(n-m-l+1)}}{(\lambda+\mu)_{\frac{1}{2}(n+m+l+1)}\,(\mu+1)_{\frac{1}{2}(n+m-l-1)}\,(\mu)_{\frac{1}{2}(n-m+l+1)}} \\ &\cdot \sum_{k=0}^{l} \frac{(\frac{1}{2}(n-m-l+1)+k)_{l-k}\,(-l)_k\,(-m)_k\,(\lambda)_k\,(\lambda-\mu)_k}{(-\mu-\frac{1}{2}(n+m+l-1))_k(2\lambda)_k\,k!} \end{aligned} \tag{2.40}$$

$$\begin{aligned} I_\lambda(n, m, l) &= \frac{[\Gamma(\lambda+\frac{1}{2})]^2\,(2\lambda)_n\,(2\lambda)_l\,(2\lambda)_m}{[\Gamma(\lambda+1)]^2\,n!\,l!\,m!} \\ &\cdot \frac{(\lambda)_{\frac{1}{2}(n+m+l+1)}\,(\frac{1}{2}(n+m-l-1))!\,(-\lambda)_{\frac{1}{2}(n-m-l+1)}\,(\frac{1}{2}(n-m-l+1))_l}{(2\lambda)_{\frac{1}{2}(n+m+l+1)}\,(\lambda+1)_{\frac{1}{2}(n+m-l-1)}\,(\lambda)_{\frac{1}{2}(n-m+l+1)}}. \end{aligned} \tag{2.41}$$

From (2.41) we see that (1.12), (1.13) hold in this case. Also note that (2.41) reduces to Askey's formula (1.4) when we set $\lambda = \frac{1}{2}$. (Recall that $D_n^{\frac{1}{2}}(x) = 2\pi^{-1}Q_n(x)$.)

A further limit can be taken to obtain a result for Hermite polynomials. Define the Hermite polynomials and Hermite functions of the second kind by

$$H_n(x) = (2x)^n\,{}_2F_0(-\tfrac{1}{2}n, \tfrac{1}{2}(1-n);\ -;\ -x^{-2}) \tag{2.42}$$

$$\begin{aligned} 2^{-n}\pi^{\frac{1}{2}}\,G_n(x) &= -\sin(n\tfrac{1}{2}\pi)\,\Gamma(\tfrac{1}{2}(n+1))\,{}_1F_1(-\tfrac{1}{2}n;\tfrac{1}{2};x^2) \\ &\quad + 2x\cos(n\tfrac{1}{2}\pi)\,\Gamma(\tfrac{1}{2}(n+2))\,{}_1F_1(\tfrac{1}{2}(1-n);\ \tfrac{3}{2};\ x^2), \end{aligned} \tag{2.43}$$

see [26, (5.5.4); 11, (47)]. The limits needed are [11, (51)]

$$H_n(x) = n!\lim_{\lambda\to\infty}\lambda^{-\frac{1}{2}n}C_n^\lambda(x\lambda^{-\frac{1}{2}}), \qquad G_n(x) = n!\lim_{\lambda\to\infty}\lambda^{-\frac{1}{2}n}D_n^\lambda(x\lambda^{-\frac{1}{2}}). \tag{2.44}$$

Using these limits in (2.41) gives

$$\int_{-\infty}^{\infty} G_n(x)\,H_m(x)\,H_l(x)\,e^{-2x^2}\,dx = 0 \tag{2.45}$$

when $l+m+n$ is even or when $|m-l| < n < m+l$, and

$$\begin{aligned} &\int_{-\infty}^{\infty} G_{m+l+1+2k}(x)\,H_m(x)\,H_l(x)\,e^{-2x^2}dx \\ &\qquad = -\int_{-\infty}^{\infty} G_m(x)\,H_{m+l+1+2k}(x)\,H_l(x)\,e^{-2x^2}dx \\ &\qquad = (-1)^{k+1}\,2^{k+l+m}(k+1)_m\,(k+l)!, \quad k = 0, 1, \ldots. \end{aligned} \tag{2.46}$$

3. *General orthogonal polynomials*

Let $\{p_n(x)\}$ be a set of polynomials orthogonal with respect to a positive measure $d\alpha(x)$ on the real line having support within a finite interval $[a, b]$. Define a function of the second kind by

$$q_n(z) = \int_a^b \frac{p_n(t)}{z-t}\, d\alpha(t), \qquad z \notin [a, b]. \tag{3.1}$$

Then, by using

$$\frac{1}{z-t} = \frac{1}{z}\left(1+\frac{t}{z}+\ldots+\frac{t^{n-1}}{z^{n-1}}\right)+\frac{t^n}{z^{n+1}}\frac{1}{1-t/z},$$

we obtain

$$q_n(z) = \frac{1}{z^{n+1}}\int_a^b \frac{p_n(t)\, t^n}{1-t/z}\, d\alpha(t).$$

Hence

$$q_n(z) = O(|z|^{-n-1}) \tag{3.2}$$

uniformly as $|z| \to \infty$. Consider

$$C_{n,m,l} = \int_C q_n(z)\, q_m(z)\, p_l(z)\, dz, \tag{3.3}$$

where C is a simple closed curve containing the interval $[a, b]$ in the inside and integration on C is in the counterclockwise direction. Blowing up the contour and using (3.2) gives

$$C_{n,m,l} = 0 \quad \text{if } l \leqslant m+n. \tag{3.4}$$

Now make the further assumption that $d\alpha(t) = w(t)\, dt$, where $w \in L^p([a, b])$, for some p, with $1 < p < \infty$. Then, for $x \in \mathbb{R}$,

$$\lim_{y \to 0^+} q_n(x+iy) = q_n(x+i0), \quad \lim_{y \to 0^+} q_n(x-iy) = q_n(x-i0),$$

in the L^p sense, where $q_n(\cdot+i0)$ and $q_n(\cdot-i0)$ are in $L^p(\mathbb{R})$ and

$$q_n(x+i0)-q_n(x-i0) = -2\pi i p_n(x)\, w(x), \qquad a < x < b, \tag{3.5}$$

if the right-hand side is continuous at x. The right-hand side of (3.5) should be replaced by the average of the right- and left-hand limits when the function has a simple jump at x, and

$$q_n(x+i0)-q_n(x-i0) = 0, \quad \text{when } x < a \text{ or } x > b.$$

For

$$q_n(x+i\varepsilon)-q_n(x-i\varepsilon) = -2i\int_a^b \frac{p_n(t)\, w(t)\, \varepsilon\, dt}{(x-t)^2+\varepsilon^2}$$

and the above conclusions are well known.

Define a function $q_n(x)$ of the second kind on the cut (a, b) by

$$q_n(x) = \tfrac{1}{2}[q_n(x+i0)+q_n(x-i0)]. \tag{3.6}$$

Since $q_n(x) \in L^p$, we assume that $p = 2$ so that $q_n(x)\, q_m(x)$ will be integrable. Contracting the contour C in (3.3) to the cut $[a, b]$ gives

$$C_{n,m,l} = \int_a^b [q_n(x-i0)\, q_m(x-i0) - q_n(x+i0)\, q_m(x+i0)]\, p_l(x)\, dx.$$

Hence, by using (3.5) and (3.6) we obtain

$$C_{n,m,l} = 2\pi i\int_a^b [q_n(x)\, p_m(x)+q_m(x)\, p_n(x)]\, p_l(x)\, w(x)\, dx. \tag{3.7}$$

It follows that

$$I(n, m, l) = \int_a^b q_n(x) p_m(x) p_l(x) w(x)\, dx = \frac{1}{4\pi i}[C_{n,m,l} + C_{n,l,m} - C_{l,m,n}]. \quad (3.8)$$

Hence, because of (3.4), we obtain the following.

THEOREM. *Let w be a square integrable weight function on (a, b), let $p_0, p_1, p_2, \ldots$ be the corresponding orthogonal polynomials and let q_n be defined on (a, b) by* (3.6), (3.1). *Let $I_{n,m,l}$ be defined by* (1.11). *Then* (1.12) *and* (1.13) *are valid.*

Note that (1.13) implies, by orthogonality, that

$$q_n(x) p_m(x) - q_m(x) p_n(x)$$

is a polynomial of degree at most $|n-m|-1$ in x.

A slight extension of (3.7) and (3.4) is

$$\int_a^b [q_n(x; w_1) p_m(x; w_2) w_2(x) + q_m(x; w_2) p_n(x; w_1) w_1(x)]\, x^l\, dx = 0 \quad (3.9)$$

when $l \leqslant m+n$, $w_1 \in L^p([a, b])$, $w_2 \in L^q([a, b])$, $p^{-1}+q^{-1} \leqslant 1$. The proof is the same. The case $l = 0$ of (3.9) explains (2.32).

Formula (3.9) implies that

$$\int_a^b q_n(x; w_1) p_m(x; w_2) p_l(x; w_1) w_2(x)\, dx = -\int_a^b q_m(x; w_2) p_n(x; w_1) p_l(x; w_1) w_1(x)\, dx \quad (3.10)$$

when $l \leqslant m+n$ and w_1, w_2, p, q are as before. Combination of (3.10) and (2.34) makes it possible to evaluate

$$\int_{-1}^1 D_n(x; \nu \mid q)\, C_m(x; \beta \mid q)\, C_l(x; \nu \mid q)\, w_\nu(x; q)\, w_\beta(x; q)\, dx = -M_{\beta,\nu}(m, n, l \mid q) \quad \text{if } l \leqslant m+n. \quad (3.11)$$

The theorem also holds for the Jacobi weight function $w_{\alpha,\beta}(x) = (1-x)^\alpha (1+x)^\beta$, $\alpha, \beta > -\frac{1}{2}$. Formula (3.10) holds for the Jacobi weight functions $w_1 = w_{\alpha,\beta}$, $w_2 = w_{\gamma,\delta}$, $\alpha+\gamma > -1$, $\beta+\delta > -1$. However, explicit evaluation of these integrals in the non-vanishing cases remains open.

4. *Bessel function integrals*

In one of the early treatments of the linearization problem for ultraspherical polynomials Hsü [15] showed how to go from the integral

$$\int_{-1}^1 C_n^\lambda(x)\, C_m^\lambda(x)\, C_l^\lambda(x)\, (1-x^2)^{\lambda-\frac{1}{2}}\, dx$$

to the corresponding integral of Bessel functions:

$$\int_0^\infty J_\nu(at)\, J_\nu(bt)\, J_\nu(ct)\, t^{1-\nu} dt = \begin{cases} 0 & \text{if } a, b, c \text{ are not sides of a triangle,} \\ \dfrac{2^{\nu-1}\Delta^{\nu-\frac{1}{2}}(abc)^{-\nu}}{\sqrt{\pi}\,\Gamma(\nu+\frac{1}{2})} & \text{if } a, b, c \text{ are sides of a triangle of area } \Delta^{\frac{1}{2}}. \end{cases} \quad (4.1)$$

Here $\operatorname{Re} \nu > -\frac{1}{2}$ and

$$\Delta = \tfrac{1}{16}[(b+c)^2-a^2]\,[a^2-(b-c)^2]. \tag{4.2}$$

See [30, p. 411] for another derivation of this result of Sonine.

Hsü used the limit formulas

$$\lim_{n\to\infty} n^{1-2\lambda} C_n^\lambda\left(1-\frac{y}{2n^2}\right) = \frac{\pi^{\frac{1}{2}}}{\Gamma(\lambda)} 2^{\frac{1}{2}-\lambda} y^{\frac{1}{4}(1-2\lambda)} J_{\lambda-\frac{1}{2}}(y^{\frac{1}{2}}) \tag{4.3}$$

(cf. [26, (8.1.1)]),

$$\lim_{n\to\infty} n^{2\lambda-1}\left(1-\left(1-\frac{y}{2n^2}\right)^2\right)^{\lambda-\frac{1}{2}} = y^{\lambda-\frac{1}{2}}, \tag{4.4}$$

where $y\in(0,\infty)$. If one applies (4.3) and (4.4) to the orthogonality relations for the polynomials C_n^λ, with $\lambda = \nu+\frac{1}{2}$, one formally obtains

$$\int_0^\infty \frac{J_\nu((ay)^{\frac{1}{2}})}{(ay)^{\frac{1}{2}\nu}} \frac{J_\nu((by)^{\frac{1}{2}})}{(by)^{\frac{1}{2}\nu}} y^\nu dy = 4b^{-\nu}\delta(a-b), \qquad a, b > 0, \tag{4.5}$$

which is a symbolic way of writing the inversion formula for the Hankel transform, cf. for instance [30, §14.4]. Thus the functions p_a^ν $(a > 0)$, defined by

$$p_a^\nu(t) = (at)^{-\frac{1}{2}\nu} J_\nu((at)^{\frac{1}{2}}), \qquad t > 0, \tag{4.6}$$

form a generalized orthogonal system on $(0,\infty)$ with respect to the measure $t^\nu dt$. We deliberately wrote the Bessel functions in the unusual form (4.6) in order to preserve the full analogy with the orthogonal polynomial case.

In a similar way as Hsü derived (4.1) we can obtain a Bessel function limit case of (2.40), (2.41) by using (4.3), (4.4) and

$$\lim_{n\to\infty} n^{1-2\lambda} D_n^\lambda\left(1-\frac{y}{2n^2}\right) = -\frac{\pi^{\frac{1}{2}}}{\Gamma(\lambda)} 2^{\frac{1}{2}-\lambda} y^{\frac{1}{4}-\frac{1}{2}\lambda} Y_{\lambda-\frac{1}{2}}(y^{\frac{1}{2}}) \tag{4.7}$$

(cf. [19, Chapter V, (55); 11, (41)] for (4.7)). First we shall indicate how the function Y_ν is conceptually related to (4.6). It follows from [30, 13.6(2), 3.7(8)] that

$$\pi^{-1}\int_0^\infty \frac{J_\nu((at)^{\frac{1}{2}})}{(at)^{\frac{1}{2}\nu}} \frac{t^\nu\,dt}{z-t} = -i\frac{H_\nu^{(1)}((az)^{\frac{1}{2}})}{(az)^{\frac{1}{2}\nu}} z^\nu = q_a^\nu(z),$$

$$-1 < \nu < \tfrac{3}{2}, \quad 0 < \arg z < 2\pi, \quad a > 0, \tag{4.8}$$

where the last equality defines the function q_a^ν, analogous to (1.9). Here and in the following it is good to keep in mind the asymptotic expansions as $|z|\to\infty$:

$$\begin{aligned} J_\nu(z) &= e^{-\nu\pi i} J_\nu(ze^{\pi i}) \\ &= (2/\pi)^{\frac{1}{2}} z^{-\frac{1}{2}} \cos(z-\tfrac{1}{2}\nu\pi-\tfrac{1}{4}\pi)(1+O(|z|^{-1})), \quad -\pi < \arg z < \pi, \end{aligned} \tag{4.9}$$

$$Y_\nu(z) = (2/\pi)^{\frac{1}{2}} z^{-\frac{1}{2}} \sin(z-\tfrac{1}{2}\nu\pi-\tfrac{1}{4}\pi)(1+O(|z|^{-1})), \quad -\pi < \arg z < \pi, \tag{4.10}$$

$$H_\nu^{(1)}(z) = (2/\pi)^{\frac{1}{2}} z^{-\frac{1}{2}} e^{i(z-\frac{1}{2}\nu\pi-\frac{1}{4}\pi)}(1+O(|z|^{-1})), \quad -\pi < \arg z < 2\pi, \tag{4.11}$$

cf. [30, 7.2(1), 7.21(1), (2)], while, as $|z|\to 0$, $J_\nu(z)$ is $O(|z|^\nu)$, $Y_\nu(z)$ and $H_\nu^{(1)}(z)$ are $O(|z|^{-|\nu|})$ (or $O(\log|z|^{-1})$ if $\nu = 0$). We shall repeatedly make silent use of these asymptotic formulas in justifying convergence and deformation of integrals.

From (4.8), (4.6) and [30, 3.61(3), 3.7(8)] we derive the following equalities, the first one being a definition:

$$q_a^\nu(x) = \lim_{y \downarrow 0} \tfrac{1}{2}(q_a^\nu(x+iy)+q_a^\nu(x-iy))$$
$$= \frac{Y_\nu((ax)^{\frac{1}{2}})}{(ax)^{\frac{1}{2}\nu}} x^\nu = \frac{1}{\pi} \int_0^\infty p_a^\nu(t) \frac{t^\nu \, dt}{x-t}, \qquad x \in (0, \infty),\ -1 < \nu < \tfrac{3}{2},\ a > 0. \quad (4.12)$$

This is analogous to (1.10). Both (4.12) and (4.7) indicate that the correct analogue of (1.21) is

$$\int_0^\infty q_a^\mu(x) p_b^\nu(x) p_c^\nu(x)\, x^\nu dx$$

or, equivalently,

$$I_{\mu,\nu}^\infty(a, b, c) = \int_0^\infty Y_\mu(ax) J_\nu(bx) J_\nu(cx)\, x^{\mu+1}\, dx, \qquad (4.13)$$

which converges if $a \neq |b \pm c|$, $\mu < \frac{1}{2}$ and $\nu > -1+\max\{0, -\mu\}$. The analogues of (2.19) and (2.22),

$$p_{a^2}^\nu(x^2)\,(x^2)^\nu = \frac{2^{\nu+1}}{\Gamma(\frac{1}{2}-\nu)\pi^{\frac{1}{2}}} \int_a^\infty (t^2-a^2)^{-\nu-\frac{1}{2}} \sin(tx)\, dt, \qquad (4.14)$$

$$q_{a^2}^\nu(x^2) = -\frac{2^{\nu+1}}{\Gamma(\frac{1}{2}-\nu)\pi^{\frac{1}{2}}} \int_a^\infty (t^2-a^2)^{-\nu-\frac{1}{2}} \cos(tx)\, dt \qquad (4.15)$$

(cf. [30, 6.13(3), (4)]), are valid for $-\frac{1}{2} < \nu < \frac{1}{2}$.

In the rest of this section we shall first show that, for $-\frac{1}{2} < \nu < \frac{1}{2}$,

$$I_{\nu,\nu}^\infty(a, b, c) = 0 \quad \text{if } |b-c| < a < b+c, \qquad (4.16)$$

$$I_{\nu,\nu}^\infty(a, b, c) = -I_{\nu,\nu}^\infty(b, a, c) \quad \text{if } c < a+b, \qquad (4.17)$$

by a method analogous to §3. Next we shall evaluate (4.13) with $\mu = \nu$ for all $a, b, c > 0$ by considering it as a (formal) limit of (1.20) and then using (2.41). Since this proof is not rigorous we shall end with a proof completely in terms of Bessel functions.

As suggested by (3.3), consider

$$\int_C q_a^\nu(z)\, q_b^\nu(z)\, p_c^\nu(z)\, dz$$

over a contour C from $-i\eta+\infty$ to $i\eta+\infty$ in the complex plane with cut $[0, \infty)$. Equivalently consider

$$\int_{iy-\infty}^{iy+\infty} H_\nu^{(1)}(az) H_\nu^{(1)}(bz) J_\nu(cz)\, z^{\nu+1}\, dz = 0, \qquad y > 0,\ \nu < \tfrac{1}{2},\ c < a+b, \quad (4.18)$$

where the vanishing follows by letting $y \to \infty$, with use of (4.9), (4.11). Now let $y \downarrow 0$ in (4.18). Then, for $-\frac{1}{2} < \nu < \frac{1}{2}$, $c < a+b$,

$$0 = \int_0^\infty H_\nu^{(1)}(ax) H_\nu^{(1)}(bx) J_\nu(cx)\, x^{\nu+1}\, dx - \int_0^\infty H_\nu^{(2)}(ax) H_\nu^{(2)}(bx) J_\nu(cx)\, x^{\nu+1}\, dx$$
$$= 2i \int_0^\infty (J_\nu(ax)\, Y_\nu(bx) + J_\nu(bx)\, Y_\nu(ax))\, J_\nu(cx)\, x^{\nu+1}\, dx.$$

This is (4.17). By cyclic permutation and by choice of a suitable linear combination, as in (3.8), we now arrive at (4.16).

Next we shall take a limit of the integral (1.21), with $n+m+l$ odd, written in the form

$$I_{\mu,\lambda}(n, m, l) = 2\int_0^{\sqrt{2}} D_n^\mu(1-\tfrac{1}{2}y^2)\, C_m^\lambda(1-\tfrac{1}{2}y^2)\, C_l^\lambda(1-\tfrac{1}{2}y^2)\,(1-(1-\tfrac{1}{2}y^2)^2)^{\lambda+\mu-1}\, y\,dy. \quad (4.19)$$

Fix natural numbers p_1, p_2, p_3, q, put $a = (2p_1-1)/q$, $b = (2p_2-1)/q$, $c = (2p_3-1)/q$, let N run through the set $q, 3q, 5q, \ldots$ and make in (4.19) the substitutions $n = Na$, $m = Nb$, $l = Nc$, $y = N^{-1}x$. Then (4.19) becomes

$$N^{3-2\lambda} I_{\mu,\lambda}(aN, bN, cN) = 2N^{3-4\lambda-2\mu} \cdot \int_0^{N\sqrt{2}} D_{aN}^\mu\left(1-\frac{x^2}{2N^2}\right) C_{bN}^\lambda\left(1-\frac{x^2}{2N^2}\right) C_{cN}^\lambda\left(1-\frac{x^2}{2N^2}\right) \cdot\left(1-\frac{x^2}{4N^2}\right)^{\lambda+\mu-1} x^{2\lambda+2\mu-1}\,dx. \quad (4.20)$$

Write the integral on the right-hand side of (4.20) as an integral over $(0, \infty)$ with the integrand vanishing for $x > N\sqrt{2}$, let $N \to \infty$ and suppose that we may interchange limit and integration. Then, by use of (4.3), (4.7) we should obtain

$$\lim_{N\to\infty} N^{3-2\lambda} I_{\mu,\lambda}(aN, bN, cN) = -\frac{2^{-2\lambda-\mu+\frac{5}{2}}\pi^{\frac{3}{2}}}{\Gamma(\mu)(\Gamma(\lambda))^2}\, a^{\mu-\frac{1}{2}}(bc)^{\lambda-\frac{1}{2}} I^{\infty}_{\mu-\frac{1}{2},\lambda-\frac{1}{2}}(a, b, c) \quad (4.21)$$

with (a, b, c) in a dense subset of $\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+$ as specified earlier. An application of Stirling's formula to (2.41) easily evaluates the left-hand side of (4.21) in the case when $\mu = \lambda$. Finally, by continuity of both sides of (4.21) in a, b, c, we arrive at (4.16), (4.17) together with

$$I^{\infty}_{\nu,\nu}(a, b, c) = \frac{(abc)^\nu}{2^{\nu+1}\Gamma(\frac{1}{2}-\nu)\,\pi^{\frac{1}{2}}(-\Delta)^{\nu+\frac{1}{2}}} \quad \text{if } a > b+c. \quad (4.22)$$

Observe that (4.21) is not the (formal) limit of the full integral (1.20) but rather of the analogous integral associated with Jacobi polynomials and Jacobi functions of the second kind of order $(\lambda-\frac{1}{2}, \frac{1}{2})$, $(\mu-\frac{1}{2}, \frac{1}{2})$, respectively, obtained from the ultraspherical case by quadratic transformation, just as (4.1) is a limit of the linearization formula for Jacobi polynomials $P_n^{(\lambda-\frac{1}{2},-\frac{1}{2})}(x)$, rather than for ultraspherical polynomials.

We now turn to a direct proof of (4.22). Our starting point is the integral [**30**, 13.46(6)]

$$\int_0^\infty K_\mu(at)\, J_\nu(bt)\, J_\nu(ct)\, t^{\mu+1}dt = -\frac{ie^{-\mu\pi i}}{(2\pi)^{\frac{1}{2}}}\frac{a^\mu}{(bc)^{\mu+1}}(X^2-1)^{-\frac{1}{2}\mu-\frac{1}{4}} Q^{\mu+\frac{1}{2}}_{\nu-\frac{1}{2}}(X), \quad (4.23)$$

where $X = (a^2+b^2+c^2)/2bc$; $b, c > 0$; $\operatorname{Re} a > 0$; $\nu > -1+\max\{0, -\mu\}$. Here $Q^{\mu+\frac{1}{2}}_{\nu-\frac{1}{2}}$ is defined as in [**13**, 3.2(5)]. In Watson's formula [**30**, 13.46(6)], Barnes's notation for $Q^{\mu+\frac{1}{2}}_{\nu-\frac{1}{2}}$ is used, cf. [**30**, §5.71, p. 156; **13**, §3.16]. It is interesting to note that Watson's evaluation of (4.23) is completely analogous to our derivation of (2.34). First the case $c = 0$ of (4.23) is derived, that is,

$$\int_0^\infty K_\mu(at)\, J_\nu(bt)\, t^{\mu+\nu+1}dt,$$

which becomes a ${}_2F_1$ that can be elementarily evaluated, and next the left-hand side of (4.23) is written as an integral of the above expressions by writing $J_\nu(bt)\,J_\nu(ct)$ in

(4.23) as an integral over d of functions $J_\nu(dt)$ (Hankel inversion of (4.1) and analogue of (2.28)). Substitution of [**30**, 3.7(8)] and change of a into $ae^{-\frac{1}{2}i\pi}$ in (4.23) yields that

$$\int_0^\infty H_\mu^{(1)}(at)\,J_\nu(bt)\,J_\nu(ct)\,t^{\mu+1}\,dt = -\frac{2^{\frac{1}{2}}e^{-2\mu\pi i}}{\pi^{\frac{3}{2}}}\,\frac{a^\mu}{(bc)^{\mu+1}}\,(X^2-1)^{-\frac{1}{2}\mu-\frac{1}{4}}\,Q_{\nu-\frac{1}{2}}^{\mu+\frac{1}{2}}(X), \tag{4.24}$$

where $X = (-a^2+b^2+c^2)/2bc$; $b, c > 0$, $\operatorname{Im} a > 0$; $\nu > -1+\max\{0, -\mu\}$. If, moreover, $\mu < \frac{1}{2}$ then (4.24) remains valid for $a, b, c > 0$, $a \neq |b \pm c|$. (Use (4.9), (4.11).) By [**30**, 3.6(1)] and (4.13) we see that $I_{\mu,\nu}^\infty$ is the imaginary part of the left-hand side of (4.24). Thus $I_{\mu,\nu}^\infty$ can be expressed in terms of Legendre functions, where one has to distinguish between the cases $X > 1$, $-1 < X < 1$ and $X < -1$, corresponding, respectively, to $a < |b-c|$, $|b-c| < a < b+c$ and $a > b+c$. We leave the explicit computations to the reader (see also [**14**, §3]) and specialize now to the case in which $\mu = \nu$.

By [**13**, 3.2(37)] we get

$$Q_{\nu-\frac{1}{2}}^{\nu+\frac{1}{2}}(z) = e^{i(\nu+\frac{1}{2})\pi}\,2^{\nu-\frac{1}{2}}\,\Gamma(\nu+\tfrac{1}{2})\,(z^2-1)^{-\frac{1}{2}\nu-\frac{1}{4}}, \qquad z \notin (-\infty, 1], \tag{4.25}$$

and (4.2) gives

$$X^2-1 = -4(bc)^{-2}\Delta. \tag{4.26}$$

It follows from (4.24), (4.25), (4.26) that

$$\int_0^\infty H_\nu^{(1)}(at)\,J_\nu(bt)\,J_\nu(ct)\,t^{\nu+1}\,dt = -2^{-\nu-1}\,ie^{-\nu\pi i}\,\pi^{-\frac{3}{2}}\,\Gamma(\nu+\tfrac{1}{2})\,(abc)^\nu(-\Delta)^{-\nu-\frac{1}{2}} \tag{4.27}$$

if $b, c > 0$, $\operatorname{Im} a > 0$, $\nu > -\frac{1}{2}$. If, moreover, $\nu < \frac{1}{2}$, then we may take the limit in (4.27) for a approaching positive real values. If $0 < a < |b-c|$ then (4.27), with $-\frac{1}{2} < \nu < \frac{1}{2}$, remains unchanged. Finally, if we take imaginary parts in (4.27), we obtain

$$I_{\nu,\nu}^\infty(a, b, c) = -\frac{(abc)^\nu}{2^{\nu+1}\Gamma(\frac{1}{2}-\nu)\,\pi^{\frac{1}{2}}(-\Delta)^{\nu+\frac{1}{2}}} \quad \text{if } 0 < a < |b-c|. \tag{4.28}$$

Formulas (4.22) and (4.16) follow in a similar way, but (4.28) is sufficient, since we already had proved (4.16), (4.17) in a different way.

It would be interesting to treat (4.13) by Abel summability to see if the convergence condition at infinity ($\operatorname{Re}\mu < \frac{1}{2}$) can be removed. Probably it can be removed. For the integral (4.1), Szegö [**24**] treated the Gibbs phenomena. This should be done for (4.13).

In an earlier version of this paper we derived (4.28), (4.22), (4.16) by starting with a formula expressing the product $K_\nu(z)\,K_\nu(w)$ as an integral of a single function K_ν. From this we derived a similar integral representation for $H_\nu^{(1)}(x)\,H_\nu^{(2)}(y)$. The result then followed by taking imaginary parts and applying Hankel inversion.

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