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A *J*–function for inhomogeneous point processes

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We propose new summary statistics for intensity-reweighted moment stationary point processes, that is, point processes with translation invariant n-point correlation functions for all $n \in \mathbb{N}$, that generalise the well known J-, empty space, and spherical Palm contact distribution functions. We represent these statistics in terms of generating functionals and relate the inhomogeneous J-function to the inhomogeneous reduced second moment function. Extensions to space time and marked point processes are briefly discussed.

Keywords and Phrases: conditional intensity, empty space function, generating functional, J-function, inhomogeneity, intensity-reweighted moment stationarity, marked point process, minus sampling estimator, product density, reduced second moment measure, spatial interaction, spherical Palm contact distribution function.

1 Introduction

The analysis of data in the form of a map of (marked) points often starts with the computation of summary statistics. Some statistics are based on inter-point distances, others on the average number of points in sample regions, or on geometric information. For a survey of the state of the art and a rich source of pointers to the literature, the reader is referred to the recent *Handbook of Spatial Statistics* (Gelfand *et al.*, 2010).

In the exploratory stage, it is usually assumed that the data constitute a realisation of a stationary point process and deviations from a homogeneous Poisson process are studied to suggest a suitable model. Although stationarity is a convenient assumption, especially if – as is often the case – only a single map is available, in many areas of application, though, heterogeneity *is* present. To account for possible non-stationarity, Baddeley, Møller and Waagepetersen (2000) defined a reduced second moment function by considering the random measure obtained from the mapped point pattern by weighting each observed point according to the (estimated) intensity at its location. Gabriel and Diggle (2009) took this idea further into the domain of space time point processes.

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In this article, our aim is to define an extension of the J-function (VAN LIESHOUT and BADDELEY, 1996) that is able to accommodate spatial and/or temporal inhomogeneity. The idea underpinning the J-function is to compare the point pattern around a typical point in the map to that around an arbitrarily chosen origin in space in order to gain insight in the interaction structure of the point process that generated the data. In contrast to the reduced second moment function, the J-function is based on n-point correlation functions of all orders $n \in \mathbb{N}$ and can be expressed in terms of the conditional intensity.

The power of the J-function in hypothesis testing was assessed in CHEN (2003) and THÖNNES and VAN LIESHOUT (1999). Extensions to multivariate point processes were proposed in VAN LIESHOUT and BADDELEY (1999), window based J-functions were suggested in BADDELEY *et al.* (2000) and CHEN (2003). For applications in agriculture, astronomy, forestry and geology, see KERSCHER (1998), KERSCHER *et al.* (1998), KERSCHER *et al.* (1999), STEIN, VAN LIESHOUT and BOOLTINK (2001), FOXALL and BADDELEY (2002), and PAULO (2002).

The plan of this article is as follows. In section 2 we fix notation and recall some basic concepts from stochastic geometry. In section 3 we describe the most important summary statistics that are being used in exploratory analysis of point patterns under the assumption of stationarity. Section 4 introduces the new statistic J_{inhom} and gives representations of it in terms of generating functionals and conditional intensities. It should be stressed that in doing so, two other new summary statistics F_{inhom} and H_{inhom} are introduced that extend the classic empty space and spherical Palm contact distribution functions. Section 5 is devoted to the explicit computation of J_{inhom} for some important classes of point process models. In section 6 we develop a minus sampling estimator and apply it to simulated examples in section 7. The article closes with suggestions for further extensions to space time and marked point processes.

2 Preliminaries and notation

Throughout this article, let X be a simple point process on $\mathbb{R}^d$. A fortiori, X is a random closed set, so we may write $x \in X$ for $x \in \mathbb{R}^d$. Below, we shall often assume the existence of product densities defined in integral terms by

$$\mathbb{E}\left[\sum\nolimits^{\neq}_{x_1,\ldots,x_n \in X} f(x_1,\ldots,x_n)\right] = \int \cdots \int f(x_1,\ldots,x_n)\,\rho^{(n)}(x_1,\ldots,x_n)\,\mathrm{d}x_1 \cdots \mathrm{d}x_n$$

for all measurable functions $f \geq 0$. The superscript $\neq$ indicates that the sum is taken over all n-tuples of distinct points. The non-negative, measurable, permutation invariant function $\rho^{(n)}$ is called the *n-th order product density*. Heuristically speaking, $\rho^{(n)}(x_1,\ldots,x_n)\,\mathrm{d}x_1 \cdots \mathrm{d}x_n$ may be interpreted as the infinitesimal probability of finding points of X at each of $\mathrm{d}x_1,\ldots, \mathrm{d}x_n$. For the special case $n = 1$, $\rho = \rho^{(1)}$ is the *intensity function* of X, which represents the heterogeneity of X since $\rho(x)\,\mathrm{d}x$ can be interpreted as the probability of observing a point at $\mathrm{d}x$. For further details, see for example the textbooks (Stoyan, Kendall and Mecke, 1987; Daley and Vere-Jones, 1988; van Lieshout, 2000; Møller and Waagepetersen, 2004; Illian *et al.*, 2008; Gelfand *et al.*, 2010).

In the physics literature, *n-point correlation functions* tend to be used instead of product densities (Peebles, 1980). They are defined recursively by

$$\xi_1 \equiv 1;$$

$$\frac{\rho^{(n)}(x_1,\ldots,x_n)}{\rho(x_1)\cdots\rho(x_n)} = \sum_{k=1}^{n} \sum_{D_1,\ldots,D_k} \xi_{n(D_1)}(\mathbf{x}_{D_1})\cdots\xi_{n(D_k)}(\mathbf{x}_{D_k}),$$

where the last sum ranges over all partitions $\{D_1,\ldots,D_k\}$ of $\{1,\ldots,n\}$ in k non-empty, disjoint sets, and the $\mathbf{x}_{D_j} = \{x_i : i \in D_j\}$, $j = 1,\ldots,k$, form the corresponding partition of points. Since for a Poisson point process $\xi_n \equiv 0$ for $n > 1$, heuristically speaking n-point correlation functions account for the excess due to n-tuples in comparison to a Poisson point process with the same intensity function. In particular, $\xi_2(x_1,x_2) = \rho^{(2)}(x_1,x_2)/(\rho(x_1)\rho(x_2)) - 1$.

3 Summary statistics

Summary statistics are used by spatial statisticians as tools for exploratory data analysis, testing, and model validation purposes. Popular examples include the *spherical Palm contact distribution function* H, the *empty space function* F, the *reduced second moment function* K and the J-function. More specifically, for a stationary point process X with intensity $\rho > 0$,

$$\begin{cases} F(t) = \mathbb{P}(X \cap B(0,t) \neq \emptyset), \\ H(t) = \mathbb{P}^{!0}(X \cap B(0,t) \neq \emptyset), \\ K(t) = \mathbb{E}^{!0}\left[\sum_{x\in X} 1\{x \in B(0,t)\}\right]/\rho, \\ J(t) = (1 - H(t))/(1 - F(t)), \end{cases} \tag{1}$$

where $B(0,t)$ is the closed ball of radius $t \geq 0$ centred at the origin and $\mathbb{P}^{!0}$ denotes the reduced Palm distribution of X. For further details about these and other summary statistics, see for example, Illian *et al.*, 2008. Note that the J-function is defined only for t such that $F(t) < 1$. Values larger than one indicate inhibition, whereas $J(t) < 1$ suggests clustering, but note the caveats against drawing too strong conclusions in Bedford and van den Berg (1997).

All statistics defined in Equation 1 can be expressed in terms of product densities when they exist. In that case, stationarity implies that the $\rho^{(n)}$ are translation invariant. For example, the K-function can be written as

$$K(t) = \int_{B(0,t)} \frac{\rho^{(2)}(0,x)}{\rho^2}\,\mathrm{d}x = \int_{B(0,t)} (1 + \xi_2(0,x))\,\mathrm{d}x$$

if product densities exist up to order $n=2$. In contrast, the empty space function depends on product densities of all orders (WHITE, 1979),

$$F(t) = -\sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \int_{B(0,t)} \cdots \int_{B(0,t)} \rho^{(n)}(x_1, \ldots, x_n) \mathrm{d}x_1 \cdots \mathrm{d}x_n$$

provided all order product densities exist and the series is absolutely convergent. Similarly,

$$H(t) = -\sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \int_{B(0,t)} \cdots \int_{B(0,t)} \frac{\rho^{(n+1)}(0, x_1, \ldots, x_n)}{\rho} \mathrm{d}x_1 \cdots \mathrm{d}x_n,$$

provided that the series is absolutely convergent. Thence (VAN LIESHOUT, 2006),

$$J(t) = 1 + \sum_{n=1}^{\infty} \frac{(-\rho)^n}{n!} J_n(t)$$

for all $t \geq 0$ for which $F(t) < 1$, where $J_n(t) = \int_{B(0,t)} \cdots \int_{B(0,t)} \xi_{n+1}(0, x_1, \ldots, x_n) \mathrm{d}x_1 \cdots \mathrm{d}x_n$. If product densities of all orders do not exist, one may truncate the series. Indeed, using only product densities up to second-order gives

$$J(t) - 1 \approx -\rho(K(t) - |B(0,t)|),$$

where $|\cdot|$ denotes volume, so the K-function can be seen as a second-order approximation to the J-function.

For non-stationary point processes, the definitions in Equation 1 depend on the choice of origin and adaptations are called for. To this end, BADDELEY *et al.* (2000) introduced the notion of *second-order intensity-reweighted stationarity*. A point process X possesses this property if the random measure

$$\Xi = \sum_{x \in X} \frac{\delta_x}{\rho(x)}$$

is second-order stationary. Here, δ_x denotes the Dirac measure that places a single point at x. If Ξ is stationary, it is also second-order stationary but the converse does not hold. Some examples of second-order intensity-reweighted stationary point processes are Poisson point processes, the random thinning of a stationary point process, and log Gaussian Cox processes driven by a Gaussian random field with a translation invariant covariance function. See also the examples of Cox processes in MØLLER and WAAGEPETERSEN (2007). Cluster processes, as well as more general superposition processes, typically are not second-order intensity-reweighted stationary.

For a second-order intensity-reweighted stationary point process, an inhomogeneous K-function (BADDELEY *et al.* (2000)) can be defined by

$$K_{\mathrm{inhom}}(t) = \frac{1}{|B|} \mathbb{E}\left[\sum_{x,y \in X}^{\neq} \frac{1\{x \in B\} 1\{y \in B(x,t)\}}{\rho(x)\rho(y)}\right]$$

regardless of the choice of bounded Borel set $B \subset \mathbb{R}^d$ with strictly positive volume $|B|$, and using the convention $a/0 = 0$ for $a \geq 0$. Indeed, $K_{\mathrm{inhom}}(t) = \mathcal{K}_{\Xi}(B(0,t) \setminus \{0\})$, where $\mathcal{K}_{\Xi}$ is the reduced second moment measure of the random measure Ξ. See also Definition 4.5 of Møller and Waagepetersen (2004).

Gabriel and Diggle (2009) restrict themselves to point processes X that are simple and have locally finite moment measures of first and second order. Additionally they assume that X has an intensity function ρ that is bounded away from zero and a pair correlation function

$$g(x,y)=g(\|x-y\|)=\frac{\rho^{(2)}(x,y)}{\rho(x)\rho(y)}$$

that depends only on $\|x-y\|$. In this case,

$$K_{\text{inhom}}(t)=\int_{B(0,t)} g(\|z\|)\,\mathrm{d}z.$$

Further discussion can be found in the textbooks, (Illian *et al.* (2008), section 4.10.2) and (Møller and Waagepetersen (2004) section 4.1.2) as well as in Gelfand *et al.* (2010), parts IV–V.

Baddeley *et al.* (2000) briefly discuss how to define empty space and spherical Palm contact distribution functions for inhomogeneous point processes. First, for given $x\in\mathbb{R}^d$ and $t\geq 0$, they propose to determine $r(x,t)$ by solving

$$t=\int_{B(x,r(x,t))}\rho(y)\,\mathrm{d}y,$$

then set

$$F_x(t)=\mathbb{P}(d(x,X)\leq r(x,t)),$$
$$H_x(t)=\mathbb{P}^{!x}(d(x,X)\leq r(x,t)),$$

where $d(x,X)$ denotes the shortest distance from x to a point of X. For Poisson point processes, the above definitions do not depend on x and are both equal to $1-e^{-t}$. The obvious drawback of such an approach is that $r(x,t)$ may be hard to compute in practice. Moreover, the definitions depend on x as well as t. Our goal in the present article is to give an alternative definition of F, H, and J for intensity-reweighted moment stationary point processes based on their representation in terms of product densities that does not depend on the choice of origin and is easy to use in practice.

4 Inhomogeneous *J*-function

Let X be a simple point process on $\mathbb{R}^d$ whose intensity function ρ exists and is bounded away from zero with $\inf_x \rho(x)=\bar{\rho}>0$. Assume that for all $n\in\mathbb{N}$ the n-th order factorial moment measure exists as a locally finite measure and has a Radon–Nikodym derivative $\rho^{(n)}$ with respect to the n-fold product of Lebesgue measure with itself for which the corresponding n-point correlation function ξ_n is translation invariant, that is, $\xi_n(x_1+a,\ldots,x_n+a)=\xi_n(x_1,\ldots,x_n)$ for almost all $x_1,\ldots,x_n\in$

$\mathbb{R}^d$ and all $a \in \mathbb{R}^d$. We shall call such a point process *intensity-reweighted moment stationary*. Note that a fortiori X is second-order intensity-reweighted stationary. Moreover, a stationary point process is also intensity-reweighted moment stationary.

DEFINITION 1. *Let X be an intensity-reweighted moment stationary point process. Set*

$$J_n(t) = \int_{B(0,t)} \cdots \int_{B(0,t)} \xi_{n+1}(0, x_1, \ldots, x_n) \,\mathrm{d}x_1 \cdots \mathrm{d}x_n$$

and define

$$J_{\text{inhom}}(t) = 1 + \sum_{n=1}^{\infty} \frac{(-\bar{\rho})^n}{n!} J_n(t)$$

for all $t \geq 0$ for which the series is absolutely convergent, that is, for which $\limsup_{n\to\infty} \left(\frac{\bar{\rho}^n}{n!} \,|\, J_n(t) \,|\, \right)^{1/n} < 1.$

If X is stationary, $\bar{\rho} = \rho$, so by Proposition 4.2 in VAN LIESHOUT, 2006, $J_{\text{inhom}} \equiv J$ and Definition 1 is in accordance with the definition of the J-function in the stationary case. Like in that case, the series in Definition 1 may be truncated, for example when X is only second-order intensity-reweighted stationary or not all n-point correlation functions exist. For $n = 1$, we obtain

$$J_{\text{inhom}}(t) - 1 \approx -\bar{\rho} \int_{B(0,t)} \xi_2(0, x) \,\mathrm{d}x = -\bar{\rho}(K_{\text{inhom}}(t) - \,|\, B(0,t) \,|\,).$$

Again, broadly speaking, $J_{\text{inhom}}(t) > 1$ indicates inhibition at range t, $J_{\text{inhom}}(t) < 1$ suggests clustering.

In the remainder of this section, we rewrite J_{inhom} in terms of generating functionals and conditional intensities. Recall that for any function $v : \mathbb{R}^d \to [0, 1]$ that is measurable and identically 1 except on some bounded subset of $\mathbb{R}^d$, the generating functional at v is defined as

$$G(v) = \mathbb{E}\left[\prod_{x \in X} v(x)\right],$$

where by convention an empty product is taken to be 1. The distribution of X is determined uniquely by its generating functional (DALEY and VERE-JONES, 2008, Volume II, 9.4.V). The factorial moment measures, provided they exist as locally finite measures, can be derived from the generating functional using its Taylor expansion (DALEY and VERE-JONES, 2008, Volume II, 9.5.VI). Conversely, if product densities of all orders exist, let u be a measurable function with values in [0, 1] that has bounded support. Then

$$G(1-u) = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \int \cdots \int u(x_1) \cdots u(x_n) \rho^{(n)}(x_1, \ldots, x_n) \,\mathrm{d}x_1 \cdots \mathrm{d}x_n,$$

provided the series converges (STOYAN *et al.* 1987, p. 109).

THEOREM 1. *Write, for* $t \geq 0$ *and* $a \in \mathbb{R}^d$,

$$u_t^a(x) = \frac{\bar{\rho} 1\{x \in B(a,t)\}}{\rho(x)}, \quad x \in \mathbb{R}^d,$$

and assume that $\limsup_{n\to\infty} \left(\frac{\bar{\rho}^n}{n!} \int_{B(0,t)} \cdots \int_{B(0,t)} \frac{\rho^{(n)}(x_1,\ldots,x_n)}{\rho(x_1)\cdots\rho(x_n)} \mathrm{d}x_1 \cdots \mathrm{d}x_n \right)^{1/n} < 1$. *Under the assumptions of Definition 1, for almost all* $a \in \mathbb{R}^d$,

$$J_{\text{inhom}}(t) = \frac{G^{!a}(1-u_t^a)}{G(1-u_t^a)} \tag{2}$$

for all $t \geq 0$ *for which the denominator is non-zero, where* $G^{!a}$ *is the generating functional of the reduced Palm distribution* $\mathbb{P}^{!a}$ *at a, G that of* $\mathbb{P}$ *itself.*

Note that the assumption that the intensity function is bounded away from zero implies that the u_t^a take values in $[0,1]$. Hence the generating functionals in Equation 2 are well-defined. Moreover, $G^{!a}(1-u_t^a)$ and $G(1-u_t^a)$ do not depend on the choice of a (cf. the proof below).

For a stationary point process, $u_t^a(x) = 1\{x \in B(a,t)\}$, hence

$$G(1-u_t^a) = \mathbb{P}(X \cap B(a,t) = \emptyset) = 1 - F(t).$$

A similar interpretation holds for the numerator of Equation 2 in terms of the spherical Palm contact distribution function:

$$G^{!a}(1-u_t^a) = \mathbb{P}^{!a}(X \cap B(a,t) = \emptyset) = 1 - H(t).$$

Consequently, one retrieves the classic definition of the J-function given in section 3.

At this point it should be emphasised that the numerator and denominator in the right hand side of Equation 2 generalise the spherical Palm contact distribution function and empty space function to intensity-reweighted moment stationary point processes and may be denoted- H_{inhom} and- F_{inhom} respectively.

PROOF. We begin by showing that

$$\mathbb{E}^{!x}\left[\sum\nolimits_{x_1,\ldots,x_n \in X}^{\neq} \prod_{i=1}^{n} \frac{1\{x_i \in B(x,t)\}}{\rho(x_i)} \right] = \int_{B(0,t)} \cdots \int_{B(0,t)} \frac{\rho^{(n+1)}(0, x_1, \ldots x_n)}{\rho(0)\rho(x_1)\cdots\rho(x_n)} \mathrm{d}x_1 \cdots \mathrm{d}x_n$$

for almost all $x \in \mathbb{R}^d$. To see this, consider the functions

$$f_A(x, X) = \frac{1\{x \in A\}}{\rho(x)} \sum\nolimits_{x_1,\ldots,x_n \in X}^{\neq} \prod_{i=1}^{n} \frac{1\{x_i \in B(x,t)\}}{\rho(x_i)}$$

defined for all bounded Borel sets $A \subset \mathbb{R}^d$ and apply (13.1.5) in Volume II, of Daley and Vere-Jones (2008) and Fubini's theorem to obtain

$$\int_A \mathbb{E}^{!x}\left[\sum\nolimits^{\neq}_{x_1,\ldots,x_n \in X} \prod_{i=1}^{n} \frac{1\{x_i \in B(x,t)\}}{\rho(x_i)}\right] \mathrm{d}x$$
$$= \mathbb{E}\left[\sum\nolimits^{\neq}_{x,x_1,\ldots,x_n} \frac{1\{x \in A\}}{\rho(x)} \prod_{i=1}^{n} \frac{1\{x_i \in B(x,t)\}}{\rho(x_i)}\right].$$

The expectation in the right hand side can be computed in terms of $\rho^{(n+1)}$ and equals

$$\int_A \int_{B(x,t)} \cdots \int_{B(x,t)} \frac{\rho^{(n+1)}(x,x_1,\ldots x_n)}{\rho(x)\rho(x_1)\cdots\rho(x_n)} \mathrm{d}x\, \mathrm{d}x_1 \cdots \mathrm{d}x_n =$$

$$\int_A \int_{B(0,t)} \cdots \int_{B(0,t)} \frac{\rho^{(n+1)}(0,x_1,\ldots x_n)}{\rho(0)\rho(x_1)\cdots\rho(x_n)} \mathrm{d}x\, \mathrm{d}x_1 \cdots \mathrm{d}x_n$$

by the translation invariance of the n-point correlation functions. Hence

$$\mathbb{E}^{!x}\left[\sum\nolimits^{\neq}_{x_1,\ldots,x_n \in X} \prod_{i=1}^{n} \frac{1\{x_i \in B(x,t)\}}{\rho(x_i)}\right]$$

is constant for almost all $x \in \mathbb{R}^d$ and the claim is proved.

Next, as the cardinality of $X \cap B(a,t)$ is almost surely finite,

$$\prod_{x \in X}\left(1 - \frac{\bar{\rho} 1\{x \in B(a,t)\}}{\rho(x)}\right) = 1 + \sum_{n=1}^{\infty} \frac{(-\bar{\rho})^n}{n!} \sum\nolimits^{\neq}_{x_1,\ldots,x_n \in X} \prod_{i=1}^{n} \frac{1\{x_i \in B(a,t)\}}{\rho(x_i)},$$

which can be verified by working out the product in the left-hand side, and the expressions are well-defined under the convention that an empty product takes the value one. Consequently, for almost all a,

$$G^{!a}(1-u_t^a) = 1 + \sum_{n=1}^{\infty} \frac{(-\bar{\rho})^n}{n!} \int_{B(0,t)} \cdots \int_{B(0,t)} \frac{\rho^{(n+1)}(0,x_1,\ldots,x_n)}{\rho(0)\rho(x_1)\cdots\rho(x_n)} \mathrm{d}x_1 \cdots \mathrm{d}x_n \quad (3)$$

provided the power series in the right hand side is absolutely convergent.

By the discussion preceding the statement of the theorem and the fact that X is assumed to be intensity-reweighted moment stationary,

$$G(1-u_t^a) = 1 + \sum_{n=1}^{\infty} \frac{(-\bar{\rho})^n}{n!} \int_{B(0,t)} \cdots \int_{B(0,t)} \frac{\rho^{(n)}(x_1,\ldots,x_n)}{\rho(x_1)\cdots\rho(x_n)} \mathrm{d}x_1 \cdots \mathrm{d}x_n \quad (4)$$

regardless of the choice of a, since the power series in the right hand side is assumed to be absolutely convergent.

Upon recalling the definition of the n-point correlation functions and splitting into terms that do or do not contain the origin, one obtains that the right hand side of Equation 3 is equal to

$$1+\sum_{n=1}^{\infty}\frac{(-\bar{\rho})^n}{n!}\sum_{D\subseteq\{1,\ldots,n\}}J_{n(D)}(t)\sum_{k=1}^{n-n(D)}\sum_{\substack{D_1,\ldots,D_k\neq\emptyset \text{ disjoint}\\ \cup D_j=\{1,\ldots,n\}\setminus D}}I_{n(D_1)}\cdots I_{n(D_k)}$$

(with $\sum_{k=1}^{0}=1$) which in turn can be written as

$$\left[1+\sum_{n=1}^{\infty}\frac{(-\bar{\rho})^n}{n!}J_n(t)\right]\times\left[1+\sum_{m=1}^{\infty}\frac{(-\bar{\rho})^m}{m!}\sum_{k=1}^{m}\sum_{\substack{D_1,\ldots,D_k\neq\emptyset \text{ disjoint}\\ \cup D_j=\{1,\ldots,m\}}}I_{n(D_1)}\cdots I_{n(D_k)}\right] \quad (5)$$

where

$$I_n=\int_{B(0,t)}\cdots\int_{B(0,t)}\xi_n(x_1,\ldots,x_n)\,\mathrm{d}x_1\cdots\mathrm{d}x_n$$

and $n(D)$ denotes the cardinality of the set D. The sum over k in the rightmost term of Equation 5 can be written as

$$\int_{B(0,t)}\cdots\int_{B(0,t)}\frac{\rho^{(m)}(x_1,\ldots,x_m)}{\rho(x_1)\cdots\rho(x_m)}\,\mathrm{d}x_1\cdots\mathrm{d}x_m,$$

hence the second term in Equation 5 is equal to the right hand side of Equation 4. Finally, since both sums in Equation 5 are absolutely convergent, so is Equation 3, an observation that completes the proof.

Next, we focus our attention on *Papangelou conditional intensities* $\lambda(x;X)$, $x\in\mathbb{R}^d$. Assuming they exist, they are defined in integral terms by

$$\mathbb{E}\left[\sum_{x\in X}f(x,X\setminus\{x\})\right]=\int\mathbb{E}^{!x}[f(x,X)]\rho(x)\,\mathrm{d}x=\int\mathbb{E}[f(x,X)\lambda(x;X)]\,\mathrm{d}x$$

for any non-negative measurable function f (Papangelou, 1974), see also Georgii (1976) and Nguyen and Zessin (1979).

Theorem 2. *Assume that X admits a conditional intensity and define the random variable $W_{a,t}(X)=\prod_{x\in X}(1-u_t^a(x))$. Then, under the assumptions of Theorem 1, $\mathbb{E}[W_{a,t}(X)]=0$ implies $\mathbb{E}[\lambda(a;X)W_{a,t}(X)/\rho(a)]=0$, and otherwise for almost all $a\in\mathbb{R}^d$*

$$J_{\text{inhom}}(t)=\mathbb{E}\left[\frac{\lambda(a;X)}{\rho(a)}W_{a,t}(X)\right]/\mathbb{E}W_{a,t}(X),$$

the $W_{a,t}$-weighted expectation of $\lambda(a;X)/\rho(a)$.

As $W_{a,t}(X)=1\{X\cap B(a,t)=\emptyset\}$ when X is stationary, Theorem 2 generalises Theorem 1 in van Lieshout and Baddeley (1996). Note that the expectations in the numerator and denominator of the expression for $J_{\text{inhom}}(t)$ above do not depend on the choice of $a\in\mathbb{R}^d$, cf. Theorem 1 and the proof below. Consequently, $J_{\text{inhom}}(t)\leq 1$

if and only if $\mathrm{cov}\left(\frac{\lambda(a;X)}{\rho(a)}, \mathrm{W}_a(\mathrm{X})\right) \leq 0$ with a similar statement for the opposite inequality sign, cf. van Lieshout and Baddeley (1996), Corollary 1.

Proof. Consider the functions

$$f_A(x, X) = \frac{1\{x \in A\}}{\rho(x)} \prod_{y \in X} \left(1 - \frac{\bar{\rho} 1\{y \in B(x, t)\}}{\rho(y)}\right)$$

defined for all bounded Borel sets $A \subset \mathbb{R}^d$. Arguing as in the proof of Theorem 1 and using the definition of conditional intensities, one obtains

$$\int_A \mathbb{E}^{!x} \left[\prod_{y \in X} \left(1 - \frac{\bar{\rho} 1\{y \in B(x, t)\}}{\rho(y)}\right)\right] \mathrm{d}x$$
$$= \int_A \mathbb{E} \left[\frac{\lambda(x; X)}{\rho(x)} \prod_{y \in X} \left(1 - \frac{\bar{\rho} 1\{y \in B(x, t)\}}{\rho(y)}\right)\right] \mathrm{d}x.$$

Hence,

$$\mathbb{E}^{!x} \left[\prod_{y \in X} \left(1 - \frac{\bar{\rho} 1\{y \in B(x, t)\}}{\rho(y)}\right)\right] = \mathbb{E} \left[\frac{\lambda(x; X)}{\rho(x)} \prod_{y \in X} \left(1 - \frac{\bar{\rho} 1\{y \in B(x, t)\}}{\rho(y)}\right)\right]$$

for almost all $x \in \mathbb{R}^d$. Using the representation of the inhomogeneous J-function given in Theorem 1 completes the proof.

5 Theoretical examples

5.1 Poisson process

Let X be a Poisson point process with intensity function $\rho : \mathbb{R}^d \to \mathbb{R}^+$ that is bounded away from zero. Since $\rho^{(n)}(x_1, \ldots, x_n) = \prod_i \rho(x_i)$, see e.g. Stoyan *et al.* (1987), the n-point correlation functions vanish for $n > 1$, so $J_{\mathrm{inhom}}(t) \equiv 1$ for all $t \geq 0$.

The generating functional of X is $G(1 - u) = \exp[-\int u(x)\rho(x)\mathrm{d}x]$, (Daley and Vere-Jones, 2008, Volume II, p. 60). In particular, for the function $u = u_t^0$ defined in Theorem 1,

$$1 - F_{\mathrm{inhom}}(t) = G(1 - u_t^0) = \exp[-\bar{\rho}\,|\,B(0, t)\,|\,].$$

Since for a Poisson point process $\mathbb{P}^{!0} = \mathbb{P}$, see (Daley and Vere-Jones, 2008, Volume II, p. 281],

$$H_{\mathrm{inhom}}(t) = 1 - G^{!0}(1 - u_t^0) = F_{\mathrm{inhom}}(t).$$

5.2 Location dependent thinning

Let X be a simple, stationary point process on $\mathbb{R}^d$ for which product densities $\rho^{(n)}$ of all orders exist. Let $p : \mathbb{R}^d \to (0,1)$ be a measurable function that is bounded away from zero and consider the thinning of X with retention probability $p(x)$ as in Baddeley *et al.* (2000) or, Volume II, Daley and Vere-Jones (2008) section 11.3. Since the process is simple, the product densities $\rho_{\text{th}}^{(n)}$ of the thinned point process can be expressed in terms of those of X by $\rho_{\text{th}}^{(n)}(x_1, \ldots, x_n) = \rho^{(n)}(x_1, \ldots, x_n) \prod_{i=1}^{n} p(x_i)$. In particular, the intensity function of the thinned point process is $\rho_{\text{th}}(x) = \rho p(x)$, where $\rho > 0$ is the intensity of X. Consequently,

$$\frac{\rho_{\text{th}}^{(n)}(x_1, \ldots, x_n)}{\rho_{\text{th}}(x_1) \cdots \rho_{\text{th}}(x_n)} = \frac{\rho^{(n)}(x_1, \ldots, x_n)}{\rho^n}.$$

Therefore, the n-point correlation functions of the thinned point process coincide with those of the underlying stationary point process X, $\xi_n^{\text{th}}(x_1, \ldots, x_n) = \xi_n(x_1, \ldots, x_n)$, and inherit the property of translation invariance. Hence J_n^{th} is equal to the J_n-function of the underlying point process X. As the intensity function of the thinned point process is bounded from below by $\rho\bar{p}$ where $\bar{p}$ is the infimum of the retention probabilities,

$$J_{\text{inhom}}^{\text{th}}(t) = 1 + \sum_{n=1}^{\infty} \frac{(-\rho\bar{p})^n}{n!} J_n(t)$$

for all $t \geq 0$ for which the series converges. Note that the power series coefficients are identical to those in the power series expansion of the J-function of X.

The generating functional of the thinned point process is $G_{\text{th}}(v) = G(vp + 1 - p)$, where G is the generating functional of X, cf. Daley and Vere-Jones (2008), Volume II, (11.3.2). Hence

$$\begin{aligned} 1 - F_{\text{inhom}}^{\text{th}}(t) &= G_{\text{th}}\left(1 - \frac{\bar{p}}{p(\cdot)} 1\{\cdot \in B(0,t)\}\right) = G(1 - \bar{p} 1\{\cdot \in B(0,t)\}) \\ &= \mathbb{E}[(1 - \bar{p})^{n(X \cap B(0,t))}], \end{aligned}$$

the generating function of the number of points of X that fall in $B(0,t)$ evaluated at $1 - \bar{p}$. As the reduced Palm distribution of the thinned point process coincides with a random location dependent thinning of the reduced Palm distribution of X with retention probabilities given by the function p,

$$1 - H_{\text{inhom}}^{\text{th}}(t) = G_{\text{th}}^{!0}(1 - u_t^0) = \mathbb{E}^{!0}[(1 - \bar{p})^{n(X \cap B(0,t))}],$$

so that under the assumptions of Theorem 1

$$J_{\text{inhom}}^{\text{th}}(t) = \frac{\mathbb{E}^{!0}[(1 - \bar{p})^{n(X \cap B(0,t))}]}{\mathbb{E}[(1 - \bar{p})^{n(X \cap B(0,t))}]}.$$

Note that the assumption of stationarity of the underlying point process X may be weakened to intensity-reweighted moment stationarity.

5.3 Scaling

Let X be a simple point process on $\mathbb{R}^d$ for which product densities $\rho^{(n)}$ of all orders exist. Let $c>0$ be a scalar constant and map the point pattern X to cX. Then all order product densities $\rho^{(n)}_{cX}$ of cX exist and are given by $\rho^{(n)}_{cX}(x_1,\ldots,x_n)=c^{-dn}\rho^{(n)}(x_1/c,\ldots,x_n/c)$. In particular for $n=1$, $\rho_{cX}(x)=c^{-d}\rho(x/c)$. Therefore the n-point correlation functions $\xi^{cX}_n(x_1,\ldots,x_n)=\xi_n(x_1/c,\ldots,x_n/c)$ of cX are invariant under translations if and only if the n-point correlation functions ξ_n of X are, in which case the J_n-functions J^{cX}_n of cX are scaled versions $J^{cX}_n(t)=c^{dn}J_n(t/c)$ of the corresponding functions of X. Furthermore, $\inf_{x\in\mathbb{R}^d}\rho_{cX}(x)=\bar{\rho}c^{-d}$, so the inhomogeneous J-function of cX is

$$J^{cX}_{\text{inhom}}(t)=1+\sum_{n=1}^{\infty}\frac{(-\bar{\rho}c^{-d})^n}{n!}c^{dn}J_n(t/c)=1+\sum_{n=1}^{\infty}\frac{(-\bar{\rho})^n}{n!}J_n(t/c)=J_{\text{inhom}}(t/c),$$

the inhomogeneous J-function of X evaluated at t/c provided the series converges. Note that in contrast to the thinning case, the power series coefficients are not identical to those of the underlying point process X.

The generating functional of the scaled process is given by $G_{cX}(v(\cdot))=G(v(c\cdot))$, where G is the generating functional of X, whence

$$F^{cX}_{\text{inhom}}(t)=F_{\text{inhom}}(t/c).$$

Similarly, noting that $d\mathbb{P}^{!x}_{cX}(\varphi)=d\mathbb{P}^{!x/c}(\varphi/c)$,

$$H^{cX}_{\text{inhom}}(t)=H_{\text{inhom}}(t/c).$$

5.4 Log Gaussian Cox process

Write Q for the distribution of a random measure defined in terms of its Radon–Nikodym derivative Λ with respect to Lebesgue measure. We assume that all moment measures of the random measure exist and are locally finite. Let X be the Cox process directed by the random intensity process Λ, that is, given a realisation $\Lambda=\rho$, X is a Poisson point process with intensity function ρ. It follows from Daley and Vere-Jones (2003), Volume I, section 6.2 that the factorial moment measures of X exist and are equal to the moment measures of the driving random measure. Hence X has product densities $\rho^{(n)}(x_1,\ldots,x_n)=\mathbb{E}[\prod_{i=1}^n\Lambda(x_i)]$. Moreover, the reduced Palm distribution of X at x is the distribution of a Cox process with driving random measure distributed as Q^x, the Palm distribution of the driving measure of X at x (Stoyan *et al.*, 1987 p. 141).

The class of log-Gaussian Cox processes (Møller, Syversveen and Waage-Petersen, 1998) is especially convenient. For models in this class,

$$\Lambda(x)=\exp[Z(x)],$$

where Z is a Gaussian field. We write μ for the mean function and assume the covariance function is of the form $\sigma^2 r$ for fixed $\sigma^2>0$ and correlation function r.

Conditions have to be imposed on these functions in order to make the resulting Cox process well-defined. In particular, the intensity function must be integrable almost surely, and $\Psi_\Lambda(B)=\int_B \Lambda(x)\,\mathrm{d}x$ a finite random variable for all bounded Borel sets $B\subset\mathbb{R}^d$. Moreover, the distribution of the random measure Ψ_Λ must be uniquely determined by that of Z. Sufficient conditions are given in ADLER (1981), Theorem 3.4.1 for zero mean Gaussian processes. Therefore, we additionally assume that the mean function μ is continuous and bounded. Now, if r is translation invariant, X is intensity-reweighted moment stationary and the intensity function $\rho(x)=\exp[\mu(x)+\sigma^2/2]$ is bounded away from zero with infimum $\exp[\sigma^2/2+\inf_{x\in\mathbb{R}^d}\mu(x)]$. Further discussion and examples can be found in MØLLER *et al.* (1998), see also MØLLER and WAAGEPETERSEN (2004).

In order to derive an explicit formula for J_{inhom}, we turn to the generating functional. Recall that a Cox process has a generating functional (DALEY and VERE-JONES, 2003, Volume I, Proposition 6.2.II) defined by $G(v)=\mathbb{E}_Q\exp[-\int(1-v(x))\,\Lambda(x)\,\mathrm{d}x]$. Therefore, for the log-Gaussian Cox process described above,

$$1-F_{\text{inhom}}(t)=G(1-u_t^0)=\mathbb{E}_Z\left[\exp\left[-\bar{\mu}\int_{B(0,t)}e^{Z(x)-\mu(x)}\,\mathrm{d}x\right]\right]$$

where $\bar{\mu}$ denotes $\inf_{x\in\mathbb{R}^d}e^{\mu(x)}$.

The Palm distributions Q^x of a log Gaussian random measure are $\Lambda(x)=e^{Z(x)}$-weighted (MØLLER *et al.*, 1998). Therefore,

$$1-H_{\text{inhom}}(t)=G^{!a}(1-u_t^a)=\mathbb{E}_Z\left[\frac{e^{Z(a)-\mu(a)}}{e^{\sigma^2/2}}\exp\left[-\bar{\mu}\int_{B(a,t)}e^{Z(y)-\mu(y)}\,\mathrm{d}y\right]\right].$$

Since $Y(x)=Z(x)-\mu(x)$, $x\in\mathbb{R}^d$, is a stationary Gaussian process, the above generating functional does not depend on the choice of a. Therefore, under the assumptions of Theorem 1,

$$J_{\text{inhom}}(t)=\frac{\mathbb{E}_Y\left[e^{Y(0)}\exp\left[-\bar{\mu}\int_{B(0,t)}e^{Y(x)}\,\mathrm{d}x\right]\right]}{\mathbb{E}_Y[e^{Y(0)}]\mathbb{E}_Y\left[\exp\left[-\bar{\mu}\int_{B(0,t)}e^{Y(x)}\,\mathrm{d}x\right]\right]}.$$

Note that $J_{\text{inhom}}(t)<1$ if and only if the random variables $e^{Y(0)}$ and $e^{-\bar{\mu}\int_{B(0,t)}e^Y}$ are negatively correlated. The geostatistical models used in practice, for example the one we shall consider in section 7, have a positive, continuously decreasing correlation function. Therefore, by Pitt's theorem (PITT, 1982), the Gaussian fields defined by such correlation functions are associated. Under the further conditions of Theorem 3.4.1. in ADLER (1981), the sample functions $Y(\cdot)$ are almost surely continuous and hence the integral of e^Y over $B(0,t)$ is uniquely defined and the limit of Riemann sums over ever finer partitions of $B(0,t)$. Since Y is associated, $\mathrm{cov}(e^{Y(0)},e^{-c_i\Sigma_i e^{Y(x_i)}})\le 0$ for all finite sums with positive scalar multipliers $c_i>0$. Upon taking the limit, it follows that $J_{\text{inhom}}(t)\le 1$.

6 Estimation

The goal of this section is to develop an estimator for the inhomogeneous J-function of Definition 1. For this purpose, we shall use the representation in terms of generating functionals of Theorem 1 and apply the minus sampling principle outlined in Stoyan *et al.* (1987), p. 127.

Specifically, let Let $W \subset \mathbb{R}^d$ be a compact set with non-empty interior and suppose the point process X is observed in W. For clarity of exposition, we assume that the intensity function ρ is known and bounded away from zero on W with $\bar{\rho} = \inf_{x \in W} \rho(x)$. If ρ is unknown, it can be estimated (for instance using kernel estimation (Berman and Diggle, 1989)) and plugged into the estimators outlined below.

Let $L \subseteq W$ be a finite point grid. Set

$$1 - \widehat{F_{\text{inhom}}}(t) = \frac{\sum_{l_k \in L \cap W_{\ominus t}} \prod_{x \in X \cap B(l_k, t)} \left[1 - \frac{\bar{\rho}}{\rho(x)}\right]}{\# L \cap W_{\ominus t}}, \tag{6}$$

where $W_{\ominus t}$ is the eroded set $\{x \in W : d(x, \partial W) \geq t\} = \{x \in W : B(x,t) \subseteq W\}$. We shall restrict attention to t small enough for $W_{\ominus t}$ to contain points of L. Note that $1 - \widehat{F_{\text{inhom}}}(t)$ is an estimator as for all grid points $l_k \in W_{\ominus t}$ the ball $B(l_k, t)$ is fully contained in W so that no points of $X \setminus W$ are needed for the computation of the product in the numerator of Equation 6. Similarly, set

$$1 - \widehat{H_{\text{inhom}}}(t) = \frac{\sum_{x_k \in X \cap W_{\ominus t}} \prod_{x \in X \setminus \{x_k\} \cap B(x_k, t)} \left[1 - \frac{\bar{\rho}}{\rho(x)}\right]}{\# X \cap W_{\ominus t}}. \tag{7}$$

Compared to Equation 6, the grid points l_k are replaced by the points x_k of $X \cap W_{\ominus t}$. Again, Equation 7 is a function of $X \cap W$ only.

With slight abuse of notation we write, for $t \geq 0$ and $a \in W_{\ominus t}$, $u_t^a(x) = \bar{\rho} 1\{x \in B(a,t)\} / \rho(x)$ for $x \in W$ and zero otherwise. Each u_t^a is measurable, takes values in $[0,1]$ and has bounded support W.

Proposition 1. *Under the assumptions of Theorem 1, the estimator defined by that of Equation 6 is unbiased, that of Equation 7 is ratio-unbiased.*

Proof. Note that

$$G(1 - u_t^{l_k}) = 1 + \sum_{n=1}^{\infty} \frac{(-\bar{\rho})^n}{n!} \int_{B(l_k,t)} \cdots \int_{B(l_k,t)} \frac{\rho^{(n)}(x_1, \ldots, x_n)}{\prod_{i=1}^n \rho(x_i)} \mathrm{d}x_1 \cdots \mathrm{d}x_n,$$

which, becasue of the translation invariance of the integrands, reduces to

$$1 + \sum_{n=1}^{\infty} \frac{(-\bar{\rho})^n}{n!} \int_{B(0,t)} \cdots \int_{B(0,t)} \frac{\rho^{(n)}(x_1, \ldots, x_n)}{\prod_{i=1}^n \rho(x_i)} \mathrm{d}x_1 \cdots \mathrm{d}x_n$$

for all $l_k \in L \cap W_{\ominus t}$. As an aside, full translation invariance of the n-point correlation functions is not needed. Unbiasedness follows.

Next, turn to the numerator of Equation 7. By the definition of Palm distributions and the reduced Campbell–Mecke theorem (STOYAN *et al.*, 1987, p. 107), its expectation can be expressed as

$$\int \int_{W_{\ominus t}} \prod_{y \in \varphi \cap B(x,t)} \left(1 - \frac{\bar{\rho}}{\rho(y)}\right) d\mathbb{P}^{!x}(\varphi)\rho(x)\,\mathrm{d}x.$$

By Fubini and Equation 3, the Palm expectation in the integrand is a constant $G^{!0}(1-u_t^0)$ for almost all $x \in W_{\ominus t}$, hence the expectation of the numerator of Equation 7 equals $G^{!0}(1-u_t^0)\int_{W_{\ominus t}} \rho(x)\,\mathrm{d}x$. As the expectation of the denominator is equal to $\int_{W_{\ominus t}} \rho(x)\,\mathrm{d}x$, the ratio is ratio-unbiased as claimed.

7 Examples

In order to see how $J_{\text{inhom}}(t)$ works in practice, we simulated realisations of three of the models presented in section 5 in the planar unit square. Typical patterns are displayed in the leftmost column of Figure 1. In all three images a smooth intensity gradient can be observed: more points are located near the bottom of the square than near the top. However, the interaction structure seems different. For example, the middle picture contains groups of points that are close together, with large gaps in between the clusters. In the lower picture on the other hand, points seem to avoid being very close together and are more evenly spaced out. In the top picture, both very small and very large interpoint distances occur. In order to quantify the above qualitative remarks, we applied the ideas presented in this article and compared the results to those obtained by a second order analysis. To simulate the patterns and calculate the estimators, the R packages spatstat[1] and Random Fields[2] were used.

Poisson point process

The first example is a heterogeneous Poisson point process with intensity function $\rho(x,y) = 100e^{-y}$. Note that the mean number of points is $100(1-e^{-1}) \approx 60$ per unit area. A realisation is shown in the top left frame in Figure 1. The top middle frame shows (6) (solid line) and (7) (dashed line). Both graphs are close but neither is above the other over the full range of t, in accordance with the fact that for any Poisson point process, $J_{\text{inhom}} \equiv 1$. For comparison, the plug-in minus sampling estimator of K_{inhom} is shown as the solid line in the top right frame. Its graph is close to but consistently smaller than that of the theoretical value πt^2 (dashed line in the top right frame).

Log Gaussian Cox process

The second example is a log Gaussian Cox process. The defining Gaussian random field has exponentially decaying correlation function, unit variance, and mean func-

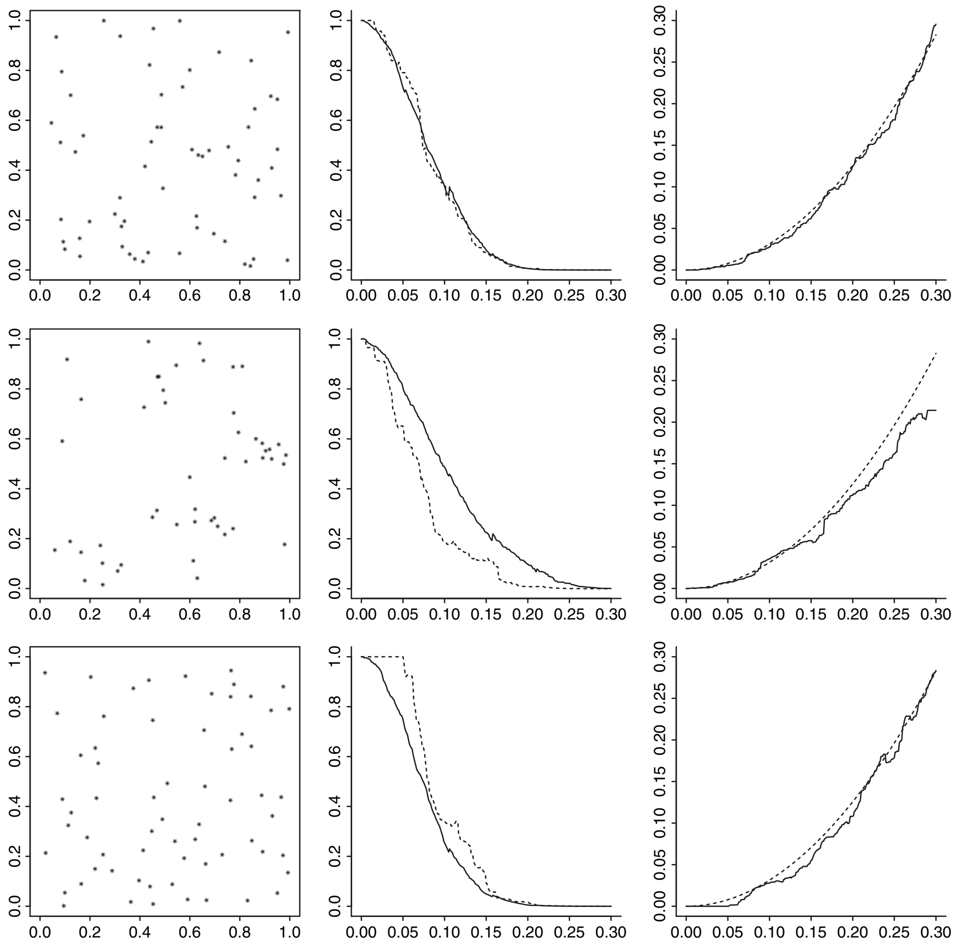

Fig. 1. Each row contains realisations of a point process in the leftmost frame, the graphs of (6) (solid line) and (7) (dashed line) in the middle frame, and the graph of $\widehat{K_{\text{inhom}}}(t)$ (solid line) compared to πt^2 (dashed line) in the rightmost frame. The models are a Poisson point process (top row), a log Gaussian Cox process (middle row) and a thinned hard core process (bottom row).

tion μ satisfying $e^{\mu(x,y)} = 100e^{-y-1/2}$. Note that the intensity function of the Cox process thus defined coincides with that of the Poisson point process discussed above. A realisation is shown in the middle row's leftmost frame in Figure 1. The middle frame in the same row shows (6) (solid line) and (7) (dashed line). Note that the graph of (7) lies well below that of (6), indicative of attraction between points due to the positive correlation of Z after accounting for the inhomogeneity. The plug-in minus sampling estimator of K_{inhom} is shown as the solid line in the rightmost frame in the middle row. In contrast to $\widehat{J_{\text{inhom}}}(t)$, $\widehat{K_{\text{inhom}}}$ indicates no departure from the Poisson hypothesis up till about $t = 0.13$; for larger t, its graph lies below that of the function $t \mapsto \pi t^2$, suggesting repulsion (sic).

Thinned hard core process

The third example is a thinned hard core (Strauss) process defined by its conditional intensity $\beta 1\{d(x, X \setminus \{x\} > R\}$. A realisation for $\beta = 200$, $R = 0.05$ and retention probability $p(x, y) = e^{-y}$ is shown in the bottom left frame in Figure 1. The middle frame in the bottom row shows (6) (solid line) and (7) (dashed line). Note that the hard core distance is clearly reflected in the flat initial segment in the graph of (7), which lies above the graph of (6) up to about $r = 0.2$, indicative of the inhibition between points due to that present in the underlying hard core process after accounting for the inhomogeneity. The plug-in minus sampling estimator of K_{inhom}, shown as the solid line in the bottom right frame, also remains flat for t up to the hard core distance and takes values smaller than that of a Poisson point process up to about $t = 0.2$, thus confirming the picture painted by the J_{inhom}-function approach.

8 Summary and extensions

In this article, we defined three new summary statistics for intensity-reweighted moment stationary point processes, calculated them explicitly for the three representative classes of intensity-reweighted moment stationary point processes presented in Baddeley *et al.* (2000), derived minus sampling estimators and presented simulation examples. The novel statistics can be described in terms of fundamental point process characteristics including product densities, the generating functional and conditional intensity. Our statistics involve product densities of all orders. If one restricts oneself to second-order, the inhomogeneous K-function of Baddeley *et al.* (2000) is retrieved.

Although this article focussed on point processes on $\mathbb{R}^d$, the approach may be extended to space time or marked point processes. First, assume that Y is a simple point process on the product space $\mathbb{R}^d \times \mathbb{R}$ equipped with the supremum distance whose intensity function $\rho(\cdot)$ exists and $\bar{\rho} = \inf_{(x,t)} \rho(x, t) > 0$. Furthermore assume all order factorial moment measures exist as locally finite measures that have Radon–Nikodym derivatives $\rho^{(n)}$ with respect to the n-fold product measure of Lebesgue measure ℓ with itself, $n \in \mathbb{N}$, and the corresponding n-point correlation functions are translation invariant in both components. Then define J_n as in Definition 1. An inhomogeneous space time version of the J-function follows completely analogously to the purely spatial context, and

$$J^{ST}_{\text{inhom}}(t) - 1 \approx -\bar{\rho} \int_{-t}^{t} \int_{\| x \| \leq t} \xi_2((0, 0), (x, s)) \mathrm{d}x \mathrm{d}s,$$

which corresponds to the K^*_{ST}-approach of Gabriel and Diggle (2009). If space and time are scaled differently, see section 5.3, $J^{ST}_{\text{inhom}}(t, s)$ becomes a function of two variables, one for spatial distances, the other for time differences, which is more natural in many applications.

For marked point processes on $\mathbb{R}^d$, one must assume that the n-point correlation functions are translation invariant in the spatial component only. The definition of $J^B_{\text{inhom}}(t)$-functions with respect to mark sets B of positive probability in the spirit of van Lieshout (2006) is then straightforward.

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Notes

1. A. Baddeley and R. Turner (2005) Spatstat: an R package for analyzing spatial point patterns. Journal of Statistical Software 12, 1–42.
2. M. Schlather (2009) Random Fields. Simulation and analysis of random fields. http://CRAN.R-project.org/package=RandomFields.

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