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GENERALISED FUNCTIONS AS LINEAR FUNCTIONALS ON GENERALIZED FUNCTIONS

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We give a sketch of a rigorous foundation for the model for a symmetrical theory of generalised functions introduced earlier by the second author. On starting with a suitable subspace *PC* of the space *S'* of tempered distributions, we introduce a space *SGF* of "new" generalised functions as a space of linear functionals on *PC*. Both on *PC* and *SGF* we have all the usual operations including a product. On *PC* this product operation is somewhat arbitrary but on *SGF* it is canonical and much nicer. Finally, *PC* and *SGF* are put together into a space *GF* of linear functionals on *SGF*.

1. Introduction

Distribution theory arose out of the need to give a rigorous foundation to objects such as the delta function, which were used before in a heuristic way. In order to apply Fourier techniques, the space S' of tempered distributions was introduced. When S' is compared with other spaces invariant under the Fourier transform like S or $L^2(\mathbb{R})$ then some simple formal properties are missing in the theory of S' like a scalar product $S' \times S' \to \mathbb{C}$ or an ordinary product $S' \times S' \to S'$.These shortcomings are sometimes bothersome in applications of distribution theory in mathematics or physics.

In [7] a symmetrical theory of generalised functions was designed by the second author in order to combine the desirable features of distribution theory and L^2-theory. Here by "symmetrical" we mean that there is no longer a distinction between test functions and distributions, but that a scalar product exists on the space of generalised functions constructed in [7]. Applications of this theory to quantum electrodynamics were given in [8]. While the presentation of the theory in [7] was heuristic, here we give a sketch of a rigorous approach. Proofs are omitted; these will appear in a later paper.

The construction proceeds in several steps. In order to show the similarities and differences with distribution theory the subspace *SGF* of "new" generalised functions is introduced as a space of linear functionals on a suitable subspace *PC* of *S'*, in such a way that it is closed under the usual operators. On *PC* we define a non-associative product following Keller [4], [5], [6]. (This was earlier done in [7], but there the point singularities remained unspecified because of indeterminacy.) On *SGF*, being a bidual of *S*, a canonical product is inherited from *S*. The formal properties of the product on *SGF* are much nicer than on *PC*. There is also a lot of arbitrariness in the choice of the product on *PC*. The paper concludes with a synthesis of *PC* and *SGF* into a space *GF* of linear functionals on *SGF*. The theory of the space *GF*, when viewed as its own dual, may be shown to coincide with the symmetrical theory of generalised functions in [7]. Throughout the paper, "distributions" will be understood in the sense of Schwartz.

2. The Preliminary Class *PC*

Let *S* be the space of rapidly decreasing C^∞-functions on $\mathbb{R}$, equipped with the usual topology. Below we list a number of continuous linear endomorphisms of *S* by their action on elements ϕ of *S*:

(2.1) $$(D\phi)(x) := \frac{d\phi(x)}{dx},$$

(2.2) $$(X\phi)(x) := x\phi(x),$$

(2.3) $$(e^{aD}\phi)(x) := \phi(x+a), \quad a \in \mathbb{R},$$

(2.4) $$(e^{ibX}\phi)(x) := e^{ibx}\phi(x), \quad b \in \mathbb{R},$$

(2.5) $$(S_c\phi)(x) := \phi(cx), \quad c > 0,$$

(2.6) $$(F\phi)(x) := \int_{-\infty}^{\infty} \phi(\xi)e^{-i\xi x}d\xi,$$

(2.7) $$(P\phi)(x) = \check{\phi}(x) := \phi(-x),$$

(2.8) $$(M_\psi\phi)(x) = (\psi\phi)(x) := \psi(x)\phi(x), \quad \psi \in S,$$

(2.9) $$(C_\psi\phi)(x) = (\psi*\phi)(x) := \int_{-\infty}^{\infty} \psi(y)\phi(x-y)dy, \quad \psi \in S.$$

In (2.3) and (2.4) the power series $\Sigma_k\ a^k\ D^k\phi/k!$ and $\Sigma_k(ib)^k\ X^k\ \phi/k!$ do not converge in S for all ϕ, only on a dense subspace of analytic functions.

There are many well-known identities involving the operators defined above. Here we only mention:

(2.10) $DX - XD = I,$

(2.11) $FD = iXF,$

(2.12) $F^2 = 2\pi P,$

(2.13) $F^{-1} = (2\pi)^{-1}PF,$

(2.14) $F(\phi*\psi) = (F\phi)(F\psi),$

(2.15) $D(\phi\psi) = (D\phi)\psi + \phi(D\psi).$

Consider also the integration functional I and the evaluation functional E, both continious on S:

(2.16) $I(\phi) := \int_{-\infty}^{\infty} \phi(\xi)d\xi,$

(2.17) $E(\phi) := \phi(0).$

They satisfy

(2.18) $I(\phi) = E(F\phi),$

(2.19) $I(\phi\psi) = I(F\phi)(F^{-1}\psi).$

Let S' be the space of tempered distributions, i.e. of all continuous linear functionals on S. Generally, if V is a linear space and V' its dual space then we will write $\langle f,\phi\rangle$ for the linear functional $f \in V'$ evaluated at $\phi \in V$. There is an embedding $S \to S'$ such that

(2.20) $\langle\psi,\phi\rangle = \int_{-\infty}^{\infty} \psi(x)\phi(x)dx, \quad \psi, \quad \phi \in S.$

If A is any of the operators defined by (2.1)-(2.9) then there is a unique continuous linear operator A': $S \to S$ such that

(2.21) $\langle A\psi,\phi\rangle = \langle\psi,A'\phi\rangle,\quad \psi,\quad \phi\in S,$

and there is an extension of A to S' (also denoted by A) such that

(2.22) $\langle Af,\phi\rangle = \langle f,A'\phi\rangle,\quad f\in S',\quad \phi\in S.$

For $\alpha\in\mathbb{C}$, $q\in\mathbb{Z}_+$ we define the element $x_+^\alpha(\log x_+)^q$ of S' as a Hadamard finite part:

(2.23) $\langle x_+^\alpha(\log x_+)^q,\phi\rangle := \mathrm{Res}_{\lambda=0}\lambda^{-1}AC\int_0^\infty \phi(x)x^{\alpha+\lambda}(\log x)^q dx,$

where AC means analytic continuation and $\mathrm{Res}_{\lambda=0}$ the residue at $\lambda = 0$. Also:

(2.24) $x_-^\alpha(\log x_-)^q := P(x_+^\alpha(\log x_+)^q),$

(2.25) $\langle\delta^{(k)},\phi\rangle := (-1)^k(D^k\phi)(0) = (-1)^k E(D^k\phi),\quad k\in\mathbb{Z}_+.$

The linear span of the elements $x_\pm^\alpha(\log x_\pm)^q$ and $\delta^{(k)}$ is invariant under D, X, S_c, F, P (see [3] for explicit formulas).

Let the *preliminary class* PC be the smallest linear subspace of S' which contains all elements $x_\pm^\alpha(\log x_\pm)^q$ and $\delta^{(k)}$ and which is invariant under the operators defined by (2.1)-(2.9). We will rather use the following equivalent characterization as a definition:

DEFINITION 2.1. The space PC consists of all finite linear combinations of the elements

(2.26) $e^{aD}\delta^{(k)}\ (k\in\mathbb{Z}_+,\ a\in\mathbb{R}),$

(2.27) $\phi e^{aD}x_\pm^\alpha(\log x_\pm)^k\ (\alpha\in\mathbb{C},\ k\in\mathbb{Z}_+,\ a\in\mathbb{R},\ \phi\in S),$

(2.28) $\phi * e^{ibX}x_\pm^\alpha(\log x_\pm)^k\ (\alpha\in\mathbb{C},\ k\in\mathbb{Z}_+,\ b\in\mathbb{R},\ \phi\in S).$

The class PC defined above is somewhat smaller than the preliminary class in [7]. This is done for convenience, but the results of this paper will remain valid with respect to the larger class.

More structure can be given to PC by using the spaces O_M of multipliers for S and O_C' of convolutors for P, as introduced by Schwartz [9]:

(2.29) $\mathcal{O}_M := \{f \in \mathcal{S}' \mid \phi f \in \mathcal{S} \text{ for all } \phi \in \mathcal{S}\}$,

(2.30) $\mathcal{O}'_C := \{f \in \mathcal{S}' \mid \phi * f \in \mathcal{S} \text{ for all } \phi \in \mathcal{S}\}$.

Note that all elements of $\mathcal{O}_M$ are C^∞-functions and that $\mathcal{O}_M = \mathcal{F}(\mathcal{O}'_C)$, $\mathcal{S}.\mathcal{S}' \subset \mathcal{O}'_C$, $\mathcal{S}*\mathcal{S}' \subset \mathcal{O}_M$. If $f \in \mathcal{O}_M$, $g \in \mathcal{S}'$ then we can define $M_f g = fg \in \mathcal{S}'$ by

(2.31) $\langle fg,\phi\rangle = \langle g,\phi f\rangle, \quad \phi \in \mathcal{S}$,

and if $f \in \mathcal{O}'_C$, $g \in \mathcal{S}'$ then we define $C_f g = f*g \in \mathcal{S}'$ by

(2.32) $\langle f*g,\phi\rangle = \langle g,\phi*\check{f}\rangle, \quad \phi \in \mathcal{S}$.

Now let:

(2.33) $\mathcal{PC}_M := \mathcal{PC} \cap \mathcal{O}_M, \quad \mathcal{PC}_C := \mathcal{PC} \cap \mathcal{O}'_C$.

PROPOSITION 2.2. $\mathcal{PC} = \mathcal{PC}_M + \mathcal{PC}_C$; $\mathcal{PC}_M \cap \mathcal{PC}_C = \mathcal{S}$; $\mathcal{PC}_M$ _is the linear span of the elements given by_ (2.28); $\mathcal{PC}_C$ _is the linear span of the elements given by_ (2.27).

Thus $\mathcal{PC}_C$ consists of piecewise C^∞-functions on $\mathbb{R}$ whith only finitely many singularities around which they have a quite specific asymptotic behaviour apparent from (2.26), (2.27). Furthermore, as $x \to \pm\infty$ they behave as rapidly decreasing C^∞-functions. The space $\mathcal{PC}_M$ can be characterized in a different way as follows:

PROPOSITION 2.3. $f \in \mathcal{PC}_M$ _if and only if_ $f \in C^\infty(\mathbb{R})$ _and, near_ $\pm\infty$, f _is a linear combination of functions_

$$x \mapsto e^{ibx}\,|x|^\alpha\,(\log|x|)^k\,h_\pm(x),$$

where $b \in \mathbb{R}$, $\alpha \in \mathbb{C}$, $k \in \mathbb{Z}_+$ _and_ $h_\pm \in C^\infty(\mathbb{R})$ _with asymptotic expansion of the form_

(2.34) $$h_\pm(x) \sim \sum_{j=0}^{\infty} c_{j,\pm}\,|x|^{-j}, \quad x \to \pm\infty.$$

Here (2.34) _means that for all_ $n,m \in \mathbb{Z}_+$ _we have:_

$$(\frac{d}{dx})^m(h_\pm(x) - \sum_{j=0}^{n} c_{j,\pm}|x|^{-j}) = \mathcal{O}(|x|^{-n-m-1}) \text{ as } x \to \pm\infty.$$

3. A Product on PC

If $f \in PC_M$ then M_f sends both PC_M and PC_C into itself. If $f \in PC_C$, $g \in PC_M$ then we may define $f.g$ as $M_g f$. However, it remains a problem to give a meaning to $f.g$ if both f and g are in PC_C with common singular points. Similarly, we can ask for the meaning of $f*g$ if $f,g \in PC_M$. There have been many attempts in literature to find a reasonable definition for the product of two distributions on suitable subclasses (see for instance the references in [5]). In our opinion, the best definition for our purposes has been given by Keller [4], [5], [6]. We will adapt his approach in order to define the product on PC.

The point of departure is an extension to PC of the evaluation functional E, defined on S by (2.17).

DEFINITION 3.1. *An evaluation functional* E *is a linear functional on* PC *such that* $E(f)=f(0)$ *if* $f \in PC$ *and* f *is continuous at* 0.

For each choice of E we can define an integration functional I on PC by

$$I(f) := E(\mathcal{F}f), \quad f \in PC. \tag{3.1}$$

Then $I(f) = \int_{-\infty}^{\infty} f(\xi)d\xi$ if $f \in PC \cap L^1(\mathbb{R})$. Note that we can fix any evaluation functional E by an arbitrary choice for $E(\delta^{(k)})$ $(k \in \mathbb{Z}_+)$, $E(x_\pm^\alpha(\log x_\pm)^q)$ $(\mathrm{Re}\,\alpha \le 0,\ \alpha \ne 0,\ q \in \mathbb{Z}_+$ or $\alpha = 0,\ 0 < q \in \mathbb{Z}_+)$, $E(x \mapsto \mathrm{sign}(x))$.

The following theorem is closely related to Keller's results, cf. Theorem 4.3 in part II of [6].

THEOREM 3.2. *For each choice of* E *there is a unique bilinear mapping* $(f,g) \to f.g$: $PC \times PC \to PC$ *such that*:

(i) $f.g = M_f g$ *if* $f \in PC_M$, $g \in PC$;

(ii) $f.(\phi g) = \phi(f.g)$ *if* $\phi \in S$, $f, g \in PC$ ((*S*)-*semi-associativity*);

(iii) $I(\mathcal{P}(f.g)) = I(\mathcal{F}f.\mathcal{F}^{-1}g)$ *if* $f,g \in PC$ (*Parseval formula*).

This mapping has the additional properties:

(iv) *If* $f,g \in PC$ *are continuous at* x *then* $f.g$ *is continuous at* x *and* $(f.g)(x) = f(x)g(x)$;

(v) $D(f.g) = (Df).g + f.(Dg)$;
(vi) <u>If</u> $f \in PC_M$, $g \in PC$ <u>then</u> $f.g = g.f = M_{fg}$.

Now we can define a convolution product on PC (again depending on the choice of E) by

$$(3.2) \qquad f*g := F^{-1}(Ff.Fg), \quad f,g \in PC.$$

A large numbers of further remarks can be made:

a) If $f,g \in PC_C$ then $f.g$ as a linear functional on S is given by

$$\langle f.g,\phi\rangle = I((Ff)(F^{-1}\phi*F^{-1}g)) = E(\check{f}*\phi g), \quad \phi \in S.$$

b) If $f,g,h \in PC$ and $h(x) = f(x)g(x)$ at the common regular points x of f and g then $f.g - h$ is a finite linear combination of elements $e^{aD}\delta^{(k)}$, where $k \in \mathbb{Z}_+$, a is a singular point of f or g. Thus, in order to evaluate f.g it is sufficient to compute the coefficients occurring in these finite linear combinations.

c) If $f,g \in PC$ are boundary values in the sense of S' of analytic functions F,G, respectively, on a strip $\{z \in \mathbb{C} \mid 0 < \mathrm{Im}\, z < b\}$ then f.g is the boundary value of FG. If $f,g \in PC$ have support bounded away from $-\infty$ then $f*g$ as defined by (3.2) coincides with the usual convolution product for such distributions.

d) Whatever the choice of E may be, the multiplication on PC can never be associative or commutative. For the nonassociativity this follows by the example in Schwartz [9]:

$$(\delta.x).x^{-1} = 0.x^{-1} = 0 \neq \delta = \delta.1 = \delta.(x.x^{-1}).$$

For the noncommutativity observe that

$$x^{-1}.\delta = -E(x^{-1})\delta \neq E(x^{-1})\delta - \delta' = \delta.x^{-1}.$$

We may always pass to a commutative algebra with new product

$$f\odot g := \tfrac{1}{4}(f.g+(\check{f}.\check{g})^{\vee} + g.f + (\check{g}.\check{f})^{\vee}),$$

which no longer satisfies property (ii) of Theorem 2.2. Note that $f \circledast g = \frac{1}{2}(f.g+g.f)$ if $E(\check{f}) = E(f)$ for all $f \in PC$.

e) The bilinear form $(f,g) \mapsto I(f.g)$ on $PC \times PC$ is nondegenerate for each choice of E. The Hermitian form

$$(f,g) \to \tfrac{1}{2}(I(f.g^*)+I(g^*.f))$$

on $PC \times PC$ can never be positive definite. Indeed, for real-valued $\phi \in S$

$$I((\delta+\phi).(\delta+\phi)) = E(\delta) + 2\phi(0) + \int_{-\infty}^{\infty} \phi(x)^2 dx$$

and, for given E, ϕ can always be chosen such that the right hand side is negative.

f) There is no preferred choice of E. Indeed, starting with a given E, the evaluation functionals $S_c' E$ and $e^{ibX}E$ ($c>0$, $b\in\mathbb{R}$) defined by

$$(S_c'E)(f) := E(S_c f), \quad f \in PC,$$

$$(e^{ibX}E)(f) := E(e^{ibX}f), \quad f \in PC,$$

also satisfy Definition 3.1 and we have

$$(S_c'E)(\log|x|) = E(\log|x|) + \log c,$$

$$(e^{ibX}E)(x^{-1}) = E(x^{-1}) + ib.$$

More generally, we may transform E by multiplication with a smooth function which equals 1 at 0 or by a smooth transformation of the independent variable which leaves 0 fixed. Still we can impose an important restriction on the freedom of choice for E such that this restriction is invariant under all the above-mentioned transformations of E, namely:

(3.3) $$E(\delta^{(k)}) = 0 (k \in \mathbb{Z}_+) \text{ and } E(x_\pm^{\alpha,q}) = 0 (-\alpha \notin \mathbb{Z}_+, \; q \in \mathbb{Z}_+).$$

In particular this will imply that $\delta^{(k)}.\delta^{(\ell)} = 0$ for all $k,\ell \in \mathbb{Z}_+$. From now on we will assume that (3.3) holds.

g) As pointed out by Keller [6], a particular nice choice for E is

$$(3.4) \qquad E(f) := \mathrm{Res}_{\lambda=0}\, \lambda^{-1}\, \mathrm{AC}\; E\Big(f * \frac{|x|^{\lambda-1}}{2\Gamma(\lambda)\cos\frac{1}{2}\pi\lambda}\Big), \quad f \in \mathcal{PC}_C,$$

which is equivalent to the choice for I made in [7]:

$$(3.5) \qquad I(g) := \mathrm{Res}_{\lambda=0}\, \lambda^{-1}\, \mathrm{AC}\; I(|x|^{-\lambda} g), \quad g \in \mathcal{PC}_M.$$

Note that (3.5) is in the spirit of the Hadamard finite part (cf. (2.23)).

4. A Canonical Product on the Dual of $\mathcal{PC}$

In the previous section we introduced a far from canonical product on $\mathcal{PC}$. However, by using a simple extension principle first observed by Arens [1], [2]*) we can define a canonical associative product on a suitable space of linear functionals on $\mathcal{PC}$.

Let V be an algebra over $\mathbb{C}$, V' its algebraic linear dual space and V" its bidual. Then we can define bilinear mappings
$(\phi,f) \mapsto \phi f\colon V \times V' \to V'$,
$(F,f) \mapsto Ff\colon V'' \times V' \to V'$,
$(F,G) \mapsto FG\colon V'' \times V'' \to V''$ as follows:

$$(4.1) \qquad \langle \phi f, \psi\rangle = \langle f, \phi\psi\rangle, \quad f \in V', \quad \phi,\psi \in V,$$

$$(4.2) \qquad \langle Ff, \psi\rangle = \langle F, f\psi\rangle, \quad F \in V'', \quad f \in V', \quad \psi \in V,$$

$$(4.3) \qquad \langle FG, f\rangle = \langle F, Gf\rangle, \quad F,G \in V'', \quad f \in V'.$$

V is naturally embedded in V" and the product on V" restricted to V gives back the original product on V. If the product on V is associative then the same holds on V", but if the product on V is commutative then this is not necessarily true for the product on V" (cf. R. Arens [2]). Of course, the above construction remains true if V' is replaced by a subspace X of V' and V" by a subspace Y of X', provided $V \times X \subset X$, $Y \times X \subset X$, $Y \times Y \subset Y$.

Let us apply this construction to the case that $V = \mathcal{S}$, $X = \mathcal{PC}$. Then

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*)We thank C.B. Huijsmans for providing us these references.

$(\phi,f) \mapsto \phi f\colon \mathcal{S} \times \mathcal{PC} \to \mathcal{PC}$ coincides with the usual action of $\mathcal{S}$ on $\mathcal{PC}$. If $f \in \mathcal{PC}$ then we can define an element F_f of $\mathcal{PC}'$ by

$$(4.4) \qquad \langle F_f, g\rangle = I(f.g), \quad g \in \mathcal{PC}.$$

Of course, the mapping $f \to F_f$ depends on the choice of E. Now it follows from (4.2) that

$$(4.5) \qquad F_f g = f.g, \quad f,g \in \mathcal{PC},$$

and from (4.3) that $F_f F_g$ $(f.g \in \mathcal{PC})$ is the element of $\mathcal{PC}'$ defined by

$$(4.6) \qquad \langle F_f F_g, h\rangle = I(f.(g.h)), \quad h \in \mathcal{PC}.$$

Thus, if $f,g \in \mathcal{PC}$ then

$$(4.7) \qquad \langle F_f F_g - F_{f.g}, h\rangle = I(f.(g.h)-(f.g).h), \quad h \in \mathcal{PC}.$$

The left hand side of (4.7) vanishes whenever f and g are regular on the support of h. In order to describe $F_f F_g - F_{f.g}$ when acting on h with support on some of the singularities of f and g we have to introduce some further elements of $\mathcal{PC}'$: $\eta^{(\alpha,q)}_{a,+}$, $\eta^{(\alpha,q)}_{a,-}$, $\eta^{(\alpha,q)}_{\infty,b}$, $\eta^{(\alpha,q)}_{-\infty,b}$ $(\alpha \in \mathbb{C},\ q \in \mathbb{Z}_+,\ a,b \in \mathbb{R})$, $\theta^{(k)}_a$ $(k \in \mathbb{Z}_+,\ q \in \mathbb{R})$:

$$(4.8) \qquad \langle \eta^{(\alpha,q)}_{a,\pm}, f\rangle := \text{coefficient of } e^{aD} x_\pm^\alpha (\log x_\pm)^q \text{ in asymptotic series of } f \text{ as } \pm(x-a) \downarrow 0,$$

$$(4.9) \qquad \langle \eta^{(\alpha,q)}_{\pm\infty,b}, f\rangle := \text{coefficientof } e^{-ibX} x_\pm^\alpha (\log x_\pm)^q \text{ in asymptotic series of } f \text{ as } x \to \pm\infty,$$

$$(4.10) \qquad \langle \theta^{(k)}_a, f\rangle := \text{coefficient of } \frac{(-1)^k}{k!} e^{aD} \delta^{(k)} \text{ in } f.$$

(The normalisation in (4.8), (4.9) is slightly different from the one in [7].) Now it is clear that $F_f F_g - F_{f.g}$ is a (possibly infinite) linear combination of elements of $\mathcal{PC}'$ of the type (4.8), (4.9), (4.10).

If $F \in \mathcal{PC}'$ and A is one of the operators given by (2.1)-(2.9) then define $AF \in \mathcal{PC}'$ by

(4.11) $\langle AF,f\rangle := \langle F,A'f\rangle, \quad f \in PC,$

where $\langle A'f,\phi\rangle := \langle f,A\phi\rangle$ ($f \in PC$, $\phi \in S$).

DEFINITION 4.1. Let the space SGF of *special generalised functions* consist of all finite linear combinations of the elements

(4.12) $F_f (f \in PC)$

(4.13) $\sum_{p,q=0}^{\infty} c_{p,q}\, \eta_{a,\pm}^{(\alpha-p,q)} \quad (c_{p,q} \in \mathbb{C},\ a \in \mathbb{R},\ \alpha \in \mathbb{C}),$

(4.14) $\sum_{p,q=0}^{\infty} c_{p,q}\, \eta_{\pm\infty,b}^{(\alpha+p,q)} \quad (c_{p,q} \in \mathbb{C},\ b \in \mathbb{R},\ \alpha \in \mathbb{C}),$

(4.15) $\sum_{k=0}^{\infty} c_k\, \theta_q^{(k)} \quad (c_k \in C,\ a \in \mathbb{R}).$

Note that an infinite sum like (4.13), when tested against an element of PC, yields only finitely many nonzero terms.

THEOREM 4.2.

a) SGF is invariant under the operators inherited from (2.1)-(2.9).
b) $SGF \times PC \subset PC$ with product defined by (4.2).
c) $SGF \times SGF \subset SGF$ with product defined by (4.3).
d) The product on SGF is associative.

It might seem from Definition 4.1 that the definition of SGF depends on the choice of E. However, we can define another embedding $f \to G_f$ of PC in PC', not depending on E, as follows. If f has no singularities on [a,b] except possibly at one interior point c then put

(4.16) $\langle G_f,g) := \mathrm{Res}_{\lambda=0}\, \lambda^{-1}\, AC \langle g,|x-c|^{\lambda} f\rangle,$

whenever $g \in PC$ with support inside [a,b]. (Note that $\langle g,|x-c|^{\lambda} f\rangle$ is well-defined for $\mathrm{Re}\,\lambda$ sufficiently large because g is a distribution of finite order.) Similarly, if f has no singularities at finite points in $[a,\infty)$ then put

(4.17) $\langle G_f,g\rangle := \mathrm{Res}_{\lambda=0}\, \lambda^{-1}\, AC \langle g,|x|^{-\lambda} f\rangle$

whenever $g \in PC$ with support inside $[a,\infty)$. The definition of G_f as $x \to -\infty$ is analogous to (4.17). Now, for each $f \in PC$, $F_f - G_f$ is a finite linear combination of elements of the form (4.13), (4.14), (4.15) and $\langle F_f - G_f, h\rangle = 0$ if $h \in PC$ with support outside the singularities of f.

Let the mapping $F \to f_F$ of *SGF* onto *PC* be defined by

$$(4.18) \qquad \langle f_F, \phi\rangle = \langle F, \phi\rangle, \quad \phi \in S,$$

where at the right hand side ϕ is considered as an element of *PC*. This mapping sends both F_f and G_f back to f and if satisfies

$$(4.19) \qquad f_{F_f F_g} = f.g, \quad f.g \in PC.$$

Summarizing, we see that *SGF* is a much nicer algebra than *PC*. The reason is that *SGF* has much more elements with point support ((4.13),(4.14),(4.15)) than *PC* (only (2.26)). These new elements admit enough freedom to carry information in order to have a product which is associative, behaves nicely under dilatation, and so on.

There is one final step to be made in order to get the full picture of [7]. In [7] the elements of *PC* and *SGF* live together in one bigger algebra of g e n e r a l i s e d f u n c t i o n s which we denote here by *GF*. We might achieve this in our present approach by applying the construction of the beginning of this section once more, such that the algebra now equals *PC* with product obtained by a choice of E. Then we can realize both *PC* and *SGF* as subalgebras of the dual of *SGF*: *PC* by putting $\langle f,F\rangle := \langle F,f\rangle$ if $f \in PC$, $F \in SGF$, and *SGF* by putting $\langle F,G\rangle := \langle FG,1\rangle$ if $F,G \in SGF$. The details, in particular a minimal choice of *GF* as a subspace of *SGF'*, have yet to be worked out.

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