DISAGREEMENT GRAPH FOR MULTI-COLOURED POLYGONAL MARKOV FIELDS

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ABSTRACT

In this paper, we give background information about multi-coloured polygonal Markov fields and their dynamic representation. Also we work out the details of the disagreement graph for multi-coloured polygonal Markov fields.

Index Terms— Arak process, dynamic representation, disagreement loop, disagreement graph, image segmentation, multi-coloured polygonal Markov fields.

1. INTRODUCTION

Segmentation is one of the major tasks for image analysis. Segmentation partitions the image into homogeneous regions [1]. There are numerous methods for segmentation. Material on morphological methods can be found in Vincent [2], chapter 10 in Green et al. [3] Bayesian models. We should also cite the textbook by Winkler [4].

Polygonal field models were considered by Clifford and Middleton [5], and a Metropolis-Hastings style sampler was developed by Nicholls and Clifford [6] with an application to an image reconstruction problem. In 1982, Arak constructed Markov field on the plane which takes two colours [7]. This construction was generalized to random fields with any finite number of colours by Arak and Surgailis in 1989 [8]. Arak et al. discussed the different techniques to construct Markov random graphs with I, V, T and Y-shaped vertices [9].

Kluszczynski et al. revisited image segmentation by polygonal Markov fields in [10] (also [11]) and implemented novel simulation methods. Unlike pixel based Markov random fields, polygonal Markov fields are able to capture global aspects of the image. which provides computational efficiency. The dynamic representation which was introduced also in [7] became useful to obtain important properties of polygonal Markov fields. The disagreement loop concept was proposed in [12] which arose from the dynamic representation. It provides more efficient simulation for polygonal Markov fields.

1.1. Multi-coloured polygonal Markov field

Let $D \subseteq \mathbb{R}^2$ be a bounded convex and open domain and $\mu(dl)$ be finite, non-atomic measure on set $\mathfrak{L}_D$ of all lines l in $\mathbb{R}^2$ which intersect D, and let $J := \{1, \ldots, k\}$ be the set of admissible colours for a fixed $k \geq 2$. For any collection $(l)_n = (l_1, \ldots, l_n)$, $l_1, \ldots, l_n \in \mathfrak{L}_D$, the set $\Gamma_D(l)_n$ consists of all functions $\omega : D \to J$ such that

a. $\partial\omega \subset \bigcup_{n=0}^{\infty} l_i \cap D$, where $\partial\omega$ is the set of discontinuity points of ω.

b. For any line l_i, the intersection $l_i \cap D$ $(i = 1, \ldots, n)$ consists of a single positive length interval and some isolated points $l_i \cap l_j$ $(i \neq j)$ [8].

Define Γ_D as all polygonal configurations on D by

$$\Gamma_D = \bigcup_{n=0}^{\infty} \bigcup_{(l)_n} \Gamma_D(l)_n.$$

The notation $\hat{:}$ is used for coloured polygonal configurations. If there is no $\hat{:}$ symbol, that indicates lines of polygonal configurations.

$\hat{\Gamma}_D$ denotes the family of admissible coloured polygonal configurations in D and $\partial\omega$ corresponds to γ, the planar graph in $D \cup \partial D$. The notation $\hat{\gamma}$ represents the resulting coloured configurations of D by using the set J which satisfy the following conditions:

- Interior vertices of γ may have degree two (V-node) or degree three (T-node) or degree four (X-node).
- All border vertices have degree one.
- Subregions which share the same edge of D cannot be assigned the same colours.
- For each straight line $l \subseteq \mathbb{R}^2$ the intersection $l \cap \gamma$ consists of at most one interval of non-zero length and possibly of some isolated points.

The process $\hat{A}_D$ is defined as;

$$P(\hat{A}_D \in \varepsilon) = \frac{\mathbb{E} \sum_{\hat{\gamma} \in \hat{\Gamma}_D(\Lambda_D) \cap \varepsilon} \exp[-\Phi(\hat{\gamma})]}{\mathbb{E} \sum_{\hat{\gamma} \in \hat{\Gamma}_D(\Lambda_D)} \exp[-\Phi(\hat{\gamma})]}, \quad (1)$$

where

$$\Phi(\hat{\gamma}) := -N_V(\gamma) \log \alpha_V - N_T(\gamma) \log [(k-1)\alpha_T]$$
$$-N_X(\gamma) \log [(k-1)\alpha_X] + |E(\gamma)| \log (k-1) + 2\xi l(\gamma)$$

and the parameters

$$\alpha_X := 1 - \alpha_V, \qquad \alpha_T := \frac{1}{2}(1 - \frac{k-2}{k-1}\alpha_X),$$

$$\xi := \frac{\alpha_V}{k-1} + \frac{(k-2)\alpha_T}{k-1}.$$

$\Phi(\hat{\gamma})$ is the energy function of coloured configuration $\hat{\gamma}$, $l(\gamma)$ is the total length of γ, $E(\gamma)$ is the set of all edges of γ, $N_V(\gamma), N_X(\gamma), N_T(\gamma)$ denote the numbers of $V-$, $X-$, $T-$ nodes respectively and Λ_D is the restriction of the Poisson line process Λ to D. The notation $\hat{\Gamma}_D(\Lambda_D)$ represents random set of coloured polygonal configurations consist of Λ_D which hit D. For further information about the Poisson line process, see chapter 8 Stoyan et al. [13] and for the specific choice of the parameters, we refer to [8] and [10].

This process has very important properties which were shown in [8]. It is isotropic, exactly solvable, consistent and it has a spatial Markov property.

1.2. Dynamic representation

The dynamic representation is an equivalent description of the polygonal Markov field which interprets D as two dimensional *time-space* by using one dimensional evolution of particle system. This representation was introduced in [7] for $k = 2$, $\alpha_V = 1$ and Arak showed that this construction coincides in distribution with the Arak process which is the spacial case of polygonal Markov fields with V-vertices only, two colours and symmetric transition probabilities. Below, we will give the dynamic representation [8] for the specific polygonal Markov field which was considered in [10].

Describe the open convex domain D as a set of *time-space* points $(t, y) \in D$, where t, y denote the time coordinate and spatial coordinate of a particle respectively. These freely moving particles separate D into different coloured regions. The coordinates of the new birth sites are chosen as follows;

- Interior birth sites form a homogeneous Poisson Process with rate $\frac{\pi\alpha_V}{k-1}$.
- Border birth sites form a homogeneous Poisson Process with intensity measure

$$\kappa(B) = \mathbb{E}\text{card}\{l \in \Lambda, \text{in}(l, D) \in B\}, B \subseteq \partial D.$$

For any straight line l which is not parallel to the spatial axis and that is contained in D, $\text{in}(l, D)$ denotes the entry point of line l in D.

Interior birth sites consist of two particles which evolve with initial velocities $v' < v''$ from joint distribution;

$$\theta(dv', dv'') = \pi^{-1}|v' - v''|(1 + v'^2)^{-\frac{3}{2}}(1 + v''^2)^{-\frac{3}{2}}dv'dv''.$$

Colours are chosen with equal probability, under the condition that none of two regions which share the same line segment have the same colours.

The particles evolve with respect to the following rules;

R1 Except for the critical moments, particles move with constant velocity v.

R2 The changes in velocity v_t of a particle in time is given by a pure-jump Markov process as

$$P(v_{t+dt} \in du | v_t = v) = \frac{\alpha_V}{k-1}q(v, du)dt$$

for the transition kernel

$$q(v, du) := |u - v|(1 + u^2)^{-\frac{3}{2}}du$$

R3 A particle with velocity v splits into two particles with probability

$$\frac{k-2}{k-1}\alpha_T q(v, du)dt$$

one of which preserves velocity v, the second particle moves with velocity $u \in du$. The colour between the particles is chosen from the set J excluding the upper and lower colour of the original particle.

R4 When two particles collide;

R4a If the colours above and below the collision points which are $i, j \in J$ respectively, are the same then;

- with probability α_V both colliding particles die,

- with probability $\alpha_X = 1 - \alpha_V$ both particles survive and continue their trajectories. The colour for the interior region after collision is chosen uniformly from the set $J\backslash\{i\} = J\backslash\{j\}$.

R4b If the colours above and below the collision points which are $i, j \in J$ respectively, are different then;

- with probability α_T the upper particle dies and the lower particle survives,

- with probability α_T the lower particle dies and the upper particle survives,

- with probability $(k-2)\alpha_X/(k-1) = 1-2\alpha_T$ both colliding particles survive and continue their trajectories. The colour for the interior area after collision is chosen uniformly from the set $J\backslash\{i, j\}$.

R5 Upon touching the boundary ∂D, the particle dies.

2. DISAGREEMENT GRAPH

Disagreement graph is a concept which arose from the dynamic representation of polygonal Markov fields. Kluszczynski et al. [10] developed a simulated annealing algorithm for image segmentation by multi-colour polygonal Markov fields. For that purpose, they introduced continuous time random dynamics on $\hat{\Gamma}_D$. For their properties, see Theorem 1 in [10]. The disagreement graph has an important role in this image segmentation algorithm. In every step of iteration, the disagreement graph determines the changes in the polygonal configuration. Hence, while a global optimization technique is used, local changes may also be considered.

Consider removing a birth package x from the coloured polygonal configuration $\hat{\gamma}$ to obtain $\hat{\gamma} \ominus x$. A birth package is carried by each birth site and it contains all randomness of the resulting particles. The symmetric difference $\Delta^{\ominus}[x; \gamma] := \gamma \triangle [\gamma \ominus x]$ gives us the disagreement graph which is described in more detail below.

Let p_1 be one of the killed particles. The edges along the trajectories of killed particles are labelled with a minus. If some minus labelled edge p_1 collides with the boundary of D, the disagreement path terminates. But it may continue on another minus, plus or colour changed path which are results of removing the birth site. Note that a plus path represents an edge which does not exist in $\hat{\gamma}$ but in $\hat{\gamma} \ominus x$. A colour changed edge is an edge that separates regions of which at least one is assigned different colour in $\hat{\gamma} \ominus x$ compared to $\hat{\gamma}$.

The killed particle p_1 would have hit an edge of $\hat{\gamma}$, had it survived. If p_1 would have died in $\hat{\gamma}$ and the collision point is a V-node, the minus path terminates but the other collided particle continues as a plus path or colour changed edge.

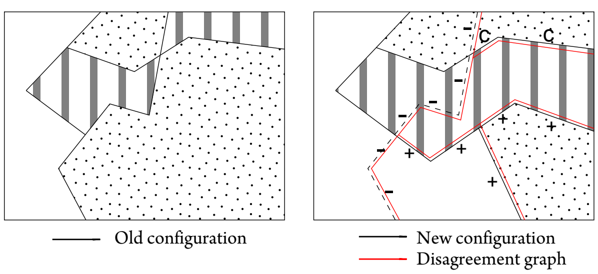

Fig. 1. Disagreement graph after removing a birth package.

If p_1 survives in $\hat{\gamma}$ and the collision point is a T-node, the minus path continues and the other collided particle continues on its trajectory as a plus path or a colour changed path.

Collisions between plus labelled edges only lead to plus labelled edges and the disagreement graph follows these paths or it terminates if particles die in collision. This collision and colouring happen according to **R4**.

Collisions between a plus and a minus edge always result in an X-node as both particles continue without interaction.

At some moment, a plus particle may collide with an edge of γ and **R4** is applied. If both particles die, the disagreement graph follows the trajectory of the old particle and is assigned with a minus. If only plus particle survives, the disagreement graph follows the trajectory of the old particle which is labelled as a minus after collision and the trajectory of plus labelled particle. If only plus particle dies, the old particle continues its trajectory as colour changed edge. If both particles survive, the disagreement path follows the trajectory of plus labelled edge and the trajectory of the old particle which is assigned with a colour change after collision.

If there is a collision between colour changed edges, **R4** has to be applied. First assume that the colours above and below the collision point coincide in the old coloured graph $\hat{\gamma}$. If the collision node is of V-type, the disagreement graph terminates in the case of colours above and below collision point remain unchanged in $\hat{\gamma} \ominus x$. If the colours above and below the collision point are different in $\hat{\gamma} \ominus x$, with probability α_T the disagreement graph follows the trajectory of one colour changed edge or with probability $1 - 2\alpha_T$ the disagreement graph follows the trajectory of both colour changed edges and the colour between these edges will be chosen according to **R4**. If the collision node is of X-type, it will also be of X-type in $\hat{\gamma} \ominus x$, but the trajectory of disagreement graph depends on colours above and below particles after collision. The disagreement graph may terminate or follow one colour changed particle or continue along both particles as colour changed.

Next assume that the colours above and below the collision point do not coincide in the old coloured graph $\hat{\gamma}$. If the collision node is of T-type and the colours above and below the collision coincide in $\hat{\gamma} \ominus x$, with probability α_V both particles die and disagreement graph continue along the trajectory of the survived particle as a minus (see fig. 2). With probability α_X both particles survive and the colour of the interior angle is chosen uniformly excluding the colour above and below the collision point and one obtains a plus path and a path whose status depending on the corresponding edge in $\hat{\gamma}$. If the collision node is of T-type and the colours above and below collision point are still different in $\hat{\gamma} \ominus x$, either disagreement graph terminates or continues along the the trajectory of the survived particle as a colour changed depending on colours above and below of collision point. If the collision node is X-type, it will be also X-type in $\hat{\gamma} \ominus x$, and the trajectory of disagreement graph can be found similar to the same colour case.

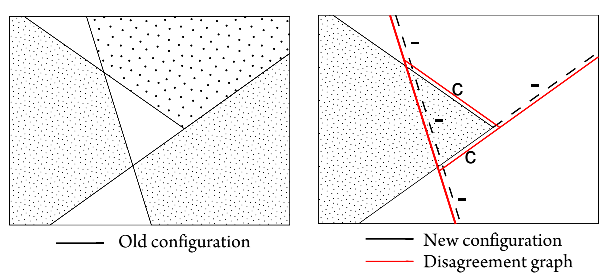

Fig. 2. Collision of colour changed edges while colours above and below collision point do not coincide in $\hat{\gamma}$ but in $\hat{\gamma} \ominus x$.

Next assume there is a collision between a plus labelled edge and a colour changed edge. We need to apply **R4** again. If both particles die, the disagreement graph follows the trajectory of the colour changed particle and is assigned with a minus. If only the plus labelled edge survives, the disagreement graph follows the trajectory of the colour changed particle as a minus path after collision and the trajectory of the plus labelled particle. If only the plus labelled particle dies, the colour changed edge continues its trajectory and the disagreement graph continues along that trajectory. If both particles survive, the disagreement path follows the trajectory of the plus labelled edge, and possibly the trajectory of the colour changed edge.

The collision of a colour changed edge with an edge common to $\hat{\gamma}$ and $\hat{\gamma} \ominus x$ is analogous to the previous case.

If we consider adding a birth site x to the coloured polygonal configuration $\hat{\gamma}$, the resulting configuration will be $\hat{\gamma} \oplus x$. Similarly, the symmetric difference ($\Delta^{\oplus}[x;\gamma] := \gamma \triangle [\gamma \oplus x]$) of $\hat{\gamma}$ and $\hat{\gamma} \oplus x$ describes the disagreement graph. The emitted particles from this birth sites, as well as velocity updates of particle or splits are labelled as plus paths. The minus path represents the edge which exists in $\hat{\gamma}$ but in $\hat{\gamma} \ominus x$.

In this case, particles follow the same collision and evolution rules as in the removing birth site. However, we get the dual picture for disagreement graph. (E.g. if a plus edge hits an edge of the old configuration and both die, the disagreement graph continues with a minus, along the trajectory of the old particle. The plus path terminates.) This concludes our discussion of the disagreement graph.

3. CONCLUSION AND FUTURE WORK

Currently, we are working on implementing the disagreement graph for polygonal Markov fields in C++ and looking forward to use multi-coloured polygonal Markov field for image segmentation. Similar methods as in Kluszczynski et al. [10], considering the image segmentation problem as statistical estimation may give satisfactory results also for the multi-coloured case.

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