Contractibility and NP-Completeness

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ABSTRACT

For a fixed graph H, let H-CON denote the problem of determining whether a given graph is contractible to H. The complexity of H-CON is studied for H belonging to certain classes of graphs, together covering all connected graphs of order at most 4. In particular, H-CON is NP-complete if H is a connected triangle-free graph other than a star. For each connected graph H of order at most 4 other than P_4 and C_4, H-CON is solvable in polynomial time.

1. INTRODUCTION

We use [3] for basic graph theoretic terminology and notations, but speak of vertices and edges instead of points and lines. In describing problems and their complexity, the terminology of [2] is applied.

We recall that an *elementary contraction* of a graph G is obtained by identifying two adjacent vertices u and v, i.e., by the removal of u and v and the addition of a new vertex w adjacent to those vertices to which u or v was adjacent. A graph G is *contractible* to a graph H if H can be obtained from G by a sequence of elementary contractions. In several graph theoretic results, conditions in terms of contractibility to certain graphs occur, e.g., in Wagner's equivalent [5] of Kuratowski's theorem: a graph is planar if and only if it has no subgraph contractible to K_5 or $K_{3,3}$. Numerous examples of such results can also be found in [4], the paper that actually motivated our present research.

Journal of Graph Theory, Vol. 11, No. 1, 71–79 (1987)
 CCC 0364-9024/87/010071-09$04.00

Let the problem CON be defined as follows.

CON.
Instance. Graphs G and H.
Question. Is G contractible to H?

As mentioned in [2], CON is an NP-complete problem. In view of the previous paragraph, it would be interesting to gain an insight into the complexity of the problem that arises from CON if H is fixed to be a specific graph. We are thus led to defining, for a fixed graph H, the problem H-CON.

***H*-CON.**
Instance. Graph G.
Question. Is G contractible to H?

It seems natural to initiate a study of the complexity of H-CON by first considering small graphs H. Furthermore, we restrict attention to connected graphs H. The number of components of a graph is invariant under contractions, and it is easily seen that H-CON is solvable in polynomial time iff, for each component K of H, K-CON is.

After stating preliminary definitions and lemmas in Section 2, we derive, in Sections 3–7, necessary and sufficient conditions for contractibility to each of the connected graphs of order at most 4, except P_4 and C_4. As is easily verified, the conditions can all be checked in polynomial time, so that, if H is one of these graphs, H-CON is solvable in polynomial time. In Section 8 it is shown that P_4-CON and C_4-CON are NP-complete. For the sake of simplicity, only graphs of order at most 4 occur in the titles of Sections 3–8, although some of the complexity results on H-CON are proved for each graph H in some infinite class.

2. PRELIMINARIES

We first develop some additional terminology in order to facilitate discussing contractibility. If G is a graph, then two subsets V_1 and V_2 of $V(G)$ are said to be *close* in G if there is an edge of G joining a vertex of V_1 and one of V_2. Clearly, G is contractible to a graph H with vertex set $V(H) = \{v_1, v_2, \cdots, v_m\}$ iff there exists a partition of $V(G)$ into vertex sets $V_1, V_2, \cdots, V_m$ such that

- the induced subgraph $\langle V_i \rangle$ of G is connected ($i = 1, 2, \cdots, m$);
- V_i and V_j are close in G iff v_i and v_j are adjacent in $H (1 \leqslant i < j \leqslant m)$.

The notion of a block will play an important role in our development. In finding criteria for contractibility to 2-connected graphs, the following obvious lemma will be of much use.

Lemma 1. A graph G is contractible to a 2-connected graph H if and only if G is connected and some block of G is contractible to H.

Another useful and easily proved lemma is the following.

Lemma 2. If G is a 2-connected graph other than a complete graph or a cycle, then G contains two nonadjacent vertices v_1 and v_2 such that $G - \{v_1, v_2\}$ is connected.

3. CONTRACTIBILITY TO K_1, K_2, and K_3

Garey and Johnson's comment [2] on the problem CON is that K_3-CON is solvable in polynomial time. Indeed, a graph G is contractible to K_3 if and only if G is connected and G is not a tree. Clearly, this condition can be checked in polynomial time. Just for the sake of completeness we mention that a graph G is contractible to K_1 iff G is connected and to K_2 iff G is connected and nontrivial.

4. CONTRACTIBILITY TO P_3 and $K_{1,3}$

The following theorem shows that $K_{1,m}$-CON is solvable in polynomial time for all $m \geq 1$.

Theorem 3. A graph G is contractible to $K_{1,m}$ if and only if G is connected and contains an independent set S of m vertices such that $G-S$ is connected.

Proof. If the stated condition is satisfied, then contraction of $G-S$ to a single vertex yields $K_{1,m}$. Hence, the condition is sufficient.

To prove necessity, suppose G is contractible to $K_{1,m}$. Then G is connected, and there exists a partition $\{V_0, V_1, \cdots, V_m\}$ of $V(G)$ such that

(1) for $i = 0, 1, \cdots, m$, $\langle V_i \rangle$ is connected;
(2) for $i < j$, V_i and V_j are close iff $i = 0$.

See Figure 1. For $i = 1, 2, \cdots, m$, let v_i be a vertex of V_i with maximal distance from V_0. Then $\{v_1, v_2, \cdots, v_m\}$ is an independent set of m vertices, whose deletion results in a connected graph. ∎

Criteria for contractibility to P_3 and $K_{1,3}$ are obtained by specializing Theorem 3 to $m = 2$ and $m = 3$, respectively. However, a more explicit characterization of the graphs contractible to P_3 can be found.

Corollary 4. A graph G is contractible to P_3 if and only if G is connected and G is neither a complete graph nor a cycle.

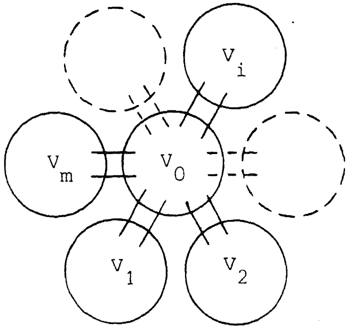

FIGURE 1.

Proof. The necessity of the condition is trivial. To prove sufficiency, let G be a connected graph other than a complete graph or a cycle. If G has a cut vertex, then G is easily shown to be contractible to P_3. If G is 2-connected, then Lemma 2 asserts that G contains two nonadjacent vertices v_1 and v_2 such that $G - \{v_1, v_2\}$ is connected. G is then contractible to P_3 by Theorem 3.

5. CONTRACTIBILITY TO $K_{1,3} + x$

Let $H_{m,n}$ be defined as the graph $K_1 + (mK_1 \cup nK_2)$, so that $K_{1,3} + x = H_{1,1}$. We first obtain a necessary and sufficient condition for contractibility to $H_{m,n}$ within the class of 2-connected graphs.

Theorem 5. A 2-connected graph G is contractible to $H_{m,n}$ if and only if $V(G)$ contains a subset S such that $\langle S \rangle \cong mK_1 \cup nK_2$ and $G-S$ is connected.

Proof. Suppose G is a 2-connected graph satisfying the stated condition. Since the vertices of S all have degree at least 2, contraction of $G-S$ to a single vertex yields $H_{m,n}$.

Now assume that G is 2-connected and contractible to $H_{m,n}$. Then there is a partition $\{V_0, V_1, \cdots, V_{2n+m}\}$ of $V(G)$ such that

(a) for $i = 0, 1, \cdots, 2n + m$, $\langle V_i \rangle$ is connected;
(b) for $i < j$, V_i and V_j are close iff $i = 0$ or $i + 1 = j = 2k \leqslant 2n$.

See Figure 2. For $i = 1, 3, 5, \cdots, 2n - 1$, let $v_i v_{i+1}$ be an edge of $\langle V_i \cup V_{i+1} \rangle$ such that the sum of the distances of the incident vertices from V_0 is maximal; the 2-connectedness of G then implies that $\langle V_0 \cup V_i \cup V_{i+1} \rangle - \{v_i, v_{i+1}\}$ is a connected graph. Furthermore, for $i = 1, 2, \cdots, m$, let v_{2n+i} be a vertex of V_{2n+i} with maximal distance from V_0. Now the set $S = \{v_1, v_2, \cdots, v_{2n+m}\}$ has the required properties. ∎

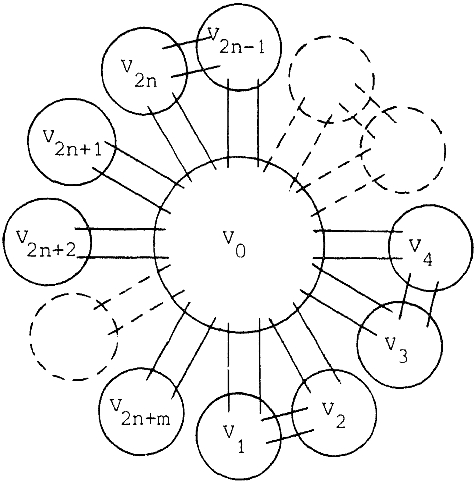

FIGURE 2.

If a graph G is connected, but not 2-connected, then the condition of Theorem 5 is neither necessary nor sufficient for contractibility to $H_{m,n}$. The graph G_1 in Figure 3 is contractible to $H_{m,1}$, but no subset of $V(G_1)$ satisfies the condition of Theorem 5. On the other hand, the subset $V(G_2) - \{v\}$ of $V(G_2)$ satisfies the condition of Theorem 5, whereas G_2 is not contractible to $H_{m,1}$.

With the aid of Theorem 5 it is possible to obtain a necessary and sufficient condition, checkable in polynomial time, for contractibility to $H_{m,n}$ of arbitrary (not necessarily 2-connected) graphs, so that $H_{m,n}$-CON is solvable in polynomial time for arbitrary m and n. However, since this condition looks very nasty when formulated for general m and n, we only give it for $m = n = 1$, in which case it has a simple form. Obviously, a graph G which is connected, but not 2-connected, is contractible to $K_{1,3} + x$ iff at least one block of G is contractible to K_3, or, in other words, iff G is not a tree. Summarizing, we have the following consequence of Theorem 5.

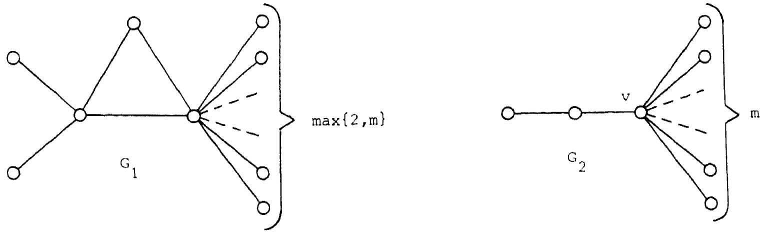

FIGURE 3.

Corollary 6. A graph G is contractible to $K_{1,3} + x$ if and only if either G is connected, has a cut vertex, and is not a tree or G is 2-connected and contains three vertices v_1, v_2, v_3, exactly two of which are adjacent, such that $G-\{v_1, v_2, v_3\}$ is connected.

6. CONTRACTIBILITY TO $K_4 - x$

By Lemma 1, a graph G is contractible to $K_4 - x$ iff G is connected and some block of G is contractible to $K_4 - x$. The blocks of a graph can be found in polynomial time. Hence, in order to show that $(K_4 - x)$-CON is solvable in polynomial time, it suffices to find a polynomial time criterion for contractibility of 2-connected graphs to $K_4 - x$.

Theorem 7. A 2-connected graph G is contractible to $K_4 - x$ if and only if G is neither a complete graph nor a cycle.

Proof. Complete graphs and cycles are not contractible to $K_4 - x$. To prove the converse, assume that G is a 2-connected graph other than a complete graph or a cycle, and let v be a vertex of G of maximal degree. Then $\deg v \geq 3$ and $N(v)$ contains two nonadjacent vertices v_1 and v_2. Let $G_1, G_2, \cdots, G_k$ be the components of $G-\{v, v_1, v_2\}$, and let G' be the graph obtained from G by contracting each of these components to a single vertex. If $k = 1$, then, since G is 2-connected and $\deg v \geq 3$, G' is $K_4 - x$. If $k \geq 2$, then, for some $i \in \{1, 2\}$, $V(G')-\{v, v_1, v_2\}$ contains two vertices v_3 and v_4 such that v_3 is adjacent in G' to v and v_i, while v_4 is adjacent to v_1 and v_2. Contraction of the edge vv_{3-i} now yields a graph G'' in which the vertices of degree at least 2 induce $K_2 + mK_1$, for some $m \geq 2$. Clearly, G'' is contractible to $K_4 - x$. ∎

Note that Corollary 4 is a consequence of Theorem 7 also, since every graph contractible to $K_4 - x$ is contractible to P_3 too.

7. CONTRACTIBILITY TO K_4

For a 2-connected graph G we define the *reduction* $R(G)$ as the graph obtained from G by successively contracting edges incident with vertices of degree 2 until either K_3 or a graph with minimum degree at least 3 results. It is easily seen that $R(G)$ is unique up to isomorphism.

In combination with Lemma 1, the following result implies that K_4-CON is solvable in polynomial time.

Theorem 8. A 2-connected graph G is contractible to K_4 if and only if $R(G)$ is not a triangle.

Proof. Let G be a 2-connected graph. Clearly, G is contractible to K_4 iff $R(G)$ is. Hence, if G is contractible to K_4, then $R(G)$ is not a triangle. Conversely, suppose $R(G)$ is not a triangle, so that $\delta(R(G)) \geq 3$. Dirac [1] proved that every graph with minimum degree at least 3 contains a subdivision of K_4. Obviously, a connected graph with a subdivision of K_4 is contractible to K_4. It follows that $R(G)$, and hence G too, is contractible to K_4. ∎

8. CONTRACTIBILITY TO P_4 AND C_4

We start by showing that P_4-CON is NP-complete. Clearly, P_4-CON is in NP since it is a subproblem of CON. We transform the following problem, which is mentioned in [2] to be NP-complete, to P_4-CON:

Hypergraph 2-Colorability (H2C).
Instance. Hypergraph L with vertex set X and (hyper-) edge set $\boldsymbol{E}$.
Question. Is there a 2-coloring of L, i.e., a partition of X into two subsets X_1 and X_2 such that no edge of $\boldsymbol{E}$ is entirely contained in either X_1 or X_2?

Obviously, H2C remains NP-complete if L is required to satisfy the following additional conditions:

$$|\boldsymbol{E}| \geq 2 \quad \text{and} \quad X \in \boldsymbol{E}. \tag{$*$}$$

From a hypergraph $L = (X, \boldsymbol{E})$ satisfying $(*)$ we construct a graph G_L as follows:

- $V(G_L) = \{v_1, v_2\} \cup X \cup \boldsymbol{E}_1 \cup \boldsymbol{E}_2$ where $\boldsymbol{E}_1$ and $\boldsymbol{E}_2$ are disjoint copies of $\boldsymbol{E}$;
- $N(v_i) = \boldsymbol{E}_i (i = 1, 2)$;
- $\langle X \rangle$ is a complete graph;
- $\langle \boldsymbol{E}_1 \cup \boldsymbol{E}_2 \rangle$ is a complete bipartite graph with maximal independent sets $\boldsymbol{E}_1$ and $\boldsymbol{E}_2$;
- a vertex $u \in X$ is adjacent to a vertex $E \in \boldsymbol{E}_i$ iff $u \cup E (i = 1, 2)$.

An example is depicted in Figure 4. The NP-completeness of P_4-CON is now established by showing that G_L is contractible to P_4 iff L is 2-colorable.

Suppose first there exists a 2-coloring $\{X_1, X_2\}$ of L. Then, in G_L, each vertex of $\boldsymbol{E}_i$ is adjacent to at least one vertex of X_i $(i = 1, 2)$. Since $\langle X_i \rangle$ is complete, it follows that $\langle \boldsymbol{E}_i \cup X_i \rangle$ is connected $(i = 1, 2)$. Now G_L is contractible to P_4 by contracting $\langle \boldsymbol{E}_1 \cup X_1 \rangle$ and $\langle \boldsymbol{E}_2 \cup X_2 \rangle$ to single vertices.

Assume next that G_L is contractible to P_4. Then there is a partition $\{V_1, V_2, V_3, V_4\}$ of $V(G)$ such that

(i) for $1 \leqslant i \leqslant 4$, $\langle V_i \rangle$ is connected;
(ii) for $i < j$, V_i and V_j are close iff $j = i + 1$.

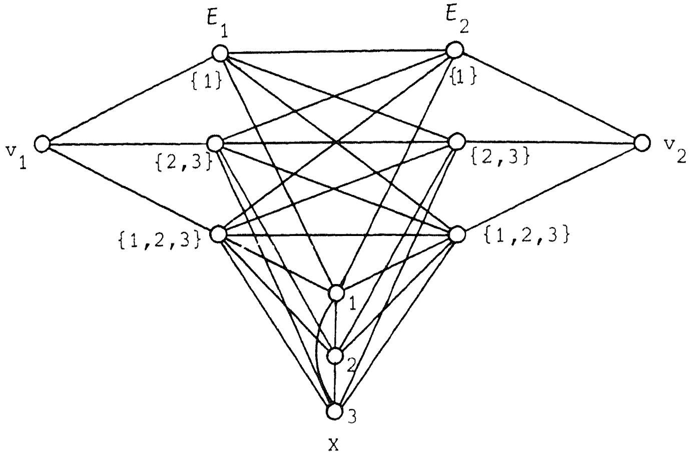

FIGURE 4. The graph G_L in case $X = \{1,2,3\}$, $\boldsymbol{E} = \{\{1\},\{2,3\},\{1,2,3\}\}$.

If $u \in V_1$ and $v \in V_4$, then $d(u,v) \geq 3$. Using (*), it is easily checked that v_1 and v_2 are the only vertices having distance at least 3 in G_L, implying that $|V_1| = |V_4| = 1$ and $V_1 \cup V_4 = \{v_1, v_2\}$. Assume without loss of generality that $V_1 = \{v_1\}$ and $V_4 = \{v_2\}$. Since all vertices of $\boldsymbol{E}_i$ are adjacent to v_i $(i = 1,2)$, it follows that $\boldsymbol{E}_1 \subset V_2$ and $\boldsymbol{E}_2 \subset V_3$. Hence, there are two subsets X_1, X_2 of X with $X_1 \cup X_2 = X$ such that $V_2 = \boldsymbol{E}_1 \cup X_1$ and $V_3 = \boldsymbol{E}_2 \cup X_2$. $\langle V_2 \rangle$ is connected and $\boldsymbol{E}_1$ is an independent set of G_L with $|\boldsymbol{E}_1| \geq 2$, so $X_1 \neq \emptyset$, and every vertex of $\boldsymbol{E}_1$ is adjacent to at least one vertex of X_1. Similarly, $X_2 \neq \emptyset$, and every vertex of $\boldsymbol{E}_2$ has at least one neighbor in X_2. Thus, $\{X_1, X_2\}$ is a 2-coloring of L, completing the proof. ▮

The following more general result, implying that C_4-CON is NP-complete too, can be established in an analogous way.

Theorem 9. If H is a connected triangle-free graph other than a star, then H-CON is NP-complete.

Since the complete proof of Theorem 9 is quite long, we only give an outline. Let H be a connected triangle-free graph, but not a star, and $L = (X, \boldsymbol{E})$ be a hypergraph satisfying (*). Then H contains an edge u_1u_2 with $\deg u_1 \geq 2$ and $\deg u_2 \geq 2$. Obtain a graph G from disjoint copies of $H - \{u_1, u_2\}$ and $G_L - \{v_1, v_2\}$ by joining each vertex of $H - \{u_1, u_2\}$ neighboring u_i in H to all vertices of $\boldsymbol{E}_i$ $(i = 1,2)$. Now L is 2-colorable iff G is contractible to H, implying the result. The major part of the proof consists of showing that, if G is contractible to H, then $\boldsymbol{E}_1 \cup \boldsymbol{E}_2 \cup X$ is the union of exactly two classes of the relevant partition of $V(G)$. ▮

The fact that H-CON turns out to be NP-complete, even for such small graphs H as P_4 and C_4, makes us expect that the class of graphs H for which H-CON is not NP-complete is very limited.

References

[1] G. A. Dirac, In abstrakten Graphen vorhandene vollständige 4-Graphen und ihre Unterteilungen. *Math. Nachr.* **22** (1960) 61–85.

[2] M. R. Garey and D. S. Johnson, *Computers and Intractability,* Freeman, San Francisco (1979).

[3] F. Harary, *Graph Theory,* Addison-Wesley, Reading, MA (1969).

[4] C. Hoede and H. J. Veldman, Contraction theorems in Hamiltonian graph theory. *Discrete Math.* **34** (1981) 61–67.

[5] K. Wagner, Über eine Eigenschaft der ebenen Komplexe. *Math. Ann.* **114** (1937) 570–590.