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Numerical Integration of Retarded Differential Equations with Periodic Solutions

H. ARNDT, P.J. VAN DER HOUWEN, B.P. SOMMEIJER

ABSTRACT: It is the purpose of this paper to show that the minimax versions of linear multistep methods, originally derived for ordinary differential equations with a periodic solution, are also suitable for the integration of retarded differential equations possessing a periodic solution. Especially for this type of equations it is extremely useful to have methods yielding highly accurate results for relatively large time steps h. We consider several examples of first-order and second-order equations with constant and state-dependent delay and compare the numerical results with that of the conventional methods.

1. INTRODUCTION

We consider the following initial value problem for retarded differential equations

$$\begin{aligned} &y'(t) = f(t, y(t), z(t)) \quad \text{for} \quad t \in [0, b] \\ &y(t) = \psi(t) \quad \text{for} \quad t \in [-s, 0] \\ &\quad \text{with} \quad z(t) = y(t - \tau(t, y(t))). \end{aligned} \tag{1.1}$$

Here ψ is the initial function, $\psi : [-s, 0] \to \mathbb{R}^m$, the function f is defined in an open subset $\Omega \subset \mathbb{R} \times \mathbb{R}^m \times \mathbb{R}^m$, such that

$$(0, \psi(0), \psi(t)) \in \Omega \qquad \text{for all} \qquad t \in [-s, 0],$$

and the (state-dependent) delay τ is a real function defined on an open subset Ω^* of $\mathbb{R} \times \mathbb{R}^m$, where Ω^* is the projection of Ω on its first $m + 1$ components, and τ is nonnegative and bounded on Ω^*, $0 \leq \tau(t, y) \leq \tau_0$.

If f, ψ, τ are continuous there exists a solution of the initial value problem (1.1) on an interval $[0, b]$; if, in addition, $f(t, y, z)$ is Lipschitz-continuous with respect to y and z and $\tau(t, y)$ is Lipschitz-continuous with respect to y we have uniqueness and continuous dependence on the data; see Driver [3] and Hale [5].

We will study linear multistep methods for the numerical integration of (1.1) in the special case where it is known that (1.1) possesses a periodic solution. We will consider Adams-Moulton and Milne-Simpson methods. Both families of methods will be applied in conventional form and in the so-called minimax form as described in [6]. The minimax versions take into account the periodicity of the solution and are rather effective in the case of ordinary differential equations.

Since a rich source of periodic problems is formed by initial value problems involving retarded differential equations of second order (cf. El'sgol'ts and Norkin[4, p.187]), we also consider linear multistep methods for second-order equations; in particular, we will investigate the conventional and the minimax version of the symmetric method of Lambert and Watson [7] (this family of methods was specifically designed for the integration of periodic problems).

Summarizing, it is the purpose of this paper to show that the minimax versions of linear multistep methods, originally derived for ordinary differential equations with a periodic solution, are also suitable for the integration of retarded differential equations possessing a periodic solution. Especially for this type of equations it is extremely useful to have methods yielding highly accurate results for relatively large time steps h; this is because delay equations require the storage of $y-$ (or $f-$) vectors, the number of which is roughly equal to $\max_{t,y(t)} \tau(t, y(t))/h \leq \tau_0/h$.

2. Linear Multistep Methods

We consider the application of linear multistep (LM) methods known from ordinary differential equations ($ODEs$) to retarded problems. We will discuss the case of first-order $ODEs$, but the case of second-order $ODEs$ can be dealt with in an analogous manner. Generally, the application of an LM method can be accomplished by the following algorithm (for convenience let us first consider the case of a constant delay $\tau > 0$):

Choose a grid $0 = t_0 < t_1 < t_2 < ... < t_N = b$ with constant stepsize $h = t_j - t_{j-1} < \tau$. Let (ρ, σ) be a linear multistep method of order p for ordinary differential equations with characteristic polynomials

$$\varrho(\varsigma) = \sum_{j=0}^{k} a_j \varsigma^j, \qquad \sigma(\varsigma) = \sum_{j=0}^{k} b_j \varsigma^j.$$

Assume that we have starting values $y_0, y_1, ..., y_{k-1}$ as approximations for $y(t_0), y(t_1), ..., y(t_{k-1})$. Then if $y_0, y_1, ..., y_{n+k-1}$ are known compute y_{n+k} from

$$\rho(E)y_n = h\sigma(E)f_n$$

$$\textit{with} \qquad f_n := f(t_n, y_n, z_n)$$

$$\textit{and} \quad z_n := \begin{cases} \psi(t_n - \tau), & \text{if} \quad t_n - \tau \leq 0, \\ u(t_n - \tau), & \text{if} \quad t_n - \tau > 0. \end{cases}$$

Here the function u is a suitable interpolate of the known values, e.g.

$$u(t_j) = y_j, \quad u'(t_j) = f_j, \quad j = 0, 1, ..., n + k - 1;$$

most often piecewise polynomial interpolation of fixed order q (degree $q - 1$) for suitable q is used. The same arguments apply for nonconstant delay τ; for state dependent delay we use $\tau(t_n, y_n)$ as an approximation of $\tau(t_n, y(t_n))$.

Some inherent difficulties may occur: In general the solution y of (1.1) is differentiable only once on $[0, b]$ even if the data are analytic as can be seen from the simple example

$$\begin{aligned} y'(t) &= y(t-1) \quad \text{for} \quad t \geq 0, \\ y(t) &= 1 \qquad\qquad \text{for} \quad t \leq 0, \end{aligned}$$

the solution of which possesses so called jump discontinuities at the natural numbers. In this example the solution gets smoother with increasing t because of the constant delay but this is not the case in general. If $y^{(j)}(t_-) \neq y^{(j)}(t_+)$ and $y^{(j-1)}(t_-) = y^{(j-1)}(t_+)$ then y has a jump discontinuity of order j at t. If all jump discontinuities of order p or less belong to the grid and the method is started again after each of these discontinuities the order of the resulting method is $\min(p, q)$. Observe that in this case one generally has to use a nonequidistant grid.

A second difficulty arises if τ is very small. On the one hand the jump discontinuities in this case lie near together which may reduce the stepsize drastically, on the other hand if $\tau < h$ then the interpolation formula is implicit with respect to the unknown value y_{n+k}.

Last but not least note that in general the exact solution y is only continuous at t_0. Therefore one cannot use the known values $y_{-k+1}, y_{-k+2}, ..., y_0$ of the initial function as starting values of the multistep method, that is $y_{-j} = y(-jh) = \psi(-jh), \quad j = 0, 1, ..., k - 1$. The starting values can be computed e.g. with a onestep method or with LM methods that increase the order.

3. Periodic Solutions

We call a solution y of (1.1) periodic with period $T > 0$ and frequency $\omega = 2\pi/T$ on the interval $[0, b]$ if

$$y(t+T) = y(t) \quad \text{for all} \quad t \in [0, b-T].$$

In general the solution is not periodic on the interval $[-s, b]$. For example consider the problem

$$\begin{aligned} y'(t) &= (y(t) - \sin t) \cdot g(t, y(t), y(t-\tau)) + \cos t, \quad t \geq 0 \\ y(t) &= \psi(t), \quad t \leq 0 \end{aligned}$$

where g is arbitrary. For all initial functions ψ with $\psi(0) = 0$ the function $y(t) = \sin t$ is a periodic solution of the problem on $[0, \infty)$ but in general y is not periodic on $[-s, \infty)$.

If f, ψ and τ are p-times differentiable then the solution y of (1.1) is at least $(p+1)$-times differentiable between jump discontinuities. Because of the bounded delay the solution will be globally $(p+1)$-times differentiable for large enough t. If in addition y is periodic on $[0, b]$ then for sufficiently large b we can conclude that y is $(p+1)$-times differentiable on the whole interval $[0, b]$. Consequently we need not obey the jump discontinuities for periodic solutions of the initial value problem and may use a constant stepsize - at least for tests.

4. Predictor - Corrector Methods

Suppose one decides to solve the implicit relations, arising in the application of an LM method, by a predictor-corrector method. Then we are faced with the problem of choosing the order of the predictor. In the case of non state-dependent delays one could use a predictor of order $p-1$, when the corrector is of order p, because the problem can be handled similar to ordinary differential equations. In problems with state-dependent delays τ it is better to use a predictor with the same order as the corrector because a good approximation y^*_{n+k} is needed for the computation of the retarded argument

$$t_{n+k} - \tau(t_{n+k}, y^*_{n+k})$$

which is used in the corrector step.

5. Interpolation

We want to give some more detailed comments to the type of interpolation procedure. Assume that an Adams method for ordinary differential equations is used,

$$y_{n+k} - y_{n+k-1} = h\sum_{j=0}^{k} b_j f_{n+j}.$$

Such formulas are constructed by approximating the integrand in the Volterra equation

$$y(t_{n+k}) - y(t_{n+k-1}) = \int_{t_{n+k-1}}^{t_{n+k}} f(s, y(s))ds$$

by a polynomial P_{n+k} that interpolates at t_{n+j} the value $f_{n+j}, j = 0, 1, ..., k$, that is

$$y_{n+k} - y_{n+k-1} = \int_{t_{n+k-1}}^{t_{n+k}} P_{n+k}(s)ds = h\sum_{j=0}^{k} b_j f_{n+j}.$$

If in the case of retarded differential equations an approximation for the value $y(t^*)$ with $t^* = t_{n+k} - \tau(t_{n+k}, y(t_{n+k}))$ is needed and we have $t_{l-1} \leq t^* < t_l$ for some $1 \leq l < n+k$, it is very natural to take

$$y(t^*) \approx y_{l-1} + \int_{t_{l-1}}^{t^*} P_l(s)ds$$

where P_l interpolates the values f_{l-j} at $t_{l-j}, j = 0, 1, ..., k$, see Bock, Schlöder [2]. One can show that with these formulas not only the local integration error can be controlled but the local interpolation error as well, cf. Arndt [1].

Another possibility for the interpolation procedure is given by the above mentioned Hermite-interpolation at points $t_{l\pm j}$ for certain j such that t^* lies nearly in the middle of these points. These formulas come along with fewer grid points and therefore theoretically lead to a smaller error.

6. Minimax Methods

The minimax modification of a linear multistep method (ρ, σ) for an $m-th$ order ODE is defined by the equations (cf. [6], [8])

$$\varphi_m(i\nu^{(l)}) = 0, \quad l = 1, 2, ..., r,$$

(6.1)
$$\varphi_m(z) := \rho(e^z) - z^m \sigma(e^z),$$

$$\nu^{(l)} := \left[\frac{1}{2}\left(\bar{\nu}^2 + \underline{\nu}^2\right) + \frac{1}{2}\left(\bar{\nu}^2 - \underline{\nu}^2\right) cos\left(\frac{2l-1}{2r}\pi\right)\right]^{\frac{1}{2}}.$$

Here, $[\underline{\omega}, \bar{\omega}] =: h^{-1}[\underline{\nu}, \bar{\nu}]$ is an estimate of the inverval of dominant frequencies in the exact solution of the ODE. The value of r is determined by the number of free coefficients in the polynomials ρ and σ. Generally, the system (6.1) represents a (linear) system with complex coefficients so that, in order to obtain real-valued coefficients, we should have $2r$ free coefficients in (ρ, σ). In the special case of symmetric methods (i.e., $\rho(\eta) = \eta^k \rho(\eta^{-1})$ and $\sigma(\eta) = \eta^k \sigma(\eta^{-1})$), the system (6.1) has a real coefficient matrix, so that we need only r free parameters in (ρ, σ).

We conclude this section by deriving a relation for the truncation error in the case of a retarded differential equation with periodic solution. Assuming the localizing assumption to be satisfied ($y(t_j) = y_j, j = 0, ..., n$), we may write

$$y(t) = u(t) + I(t, h),$$

where $u(t)$ is the interpolating function introduced in Section 2, and $I(t, h)$ denotes the interpolation error. The truncation error at t_{n+k} is given by

$$\begin{aligned} T_{n+k} &:= \rho(E)\, y(t_n) - h^m \sigma(E)\, f(t_n, y(t_n), u(t_n - \tau)) \\ &\approx \rho(E)\, y(t_n) - h^m \sigma(E)\left[f(t_n, y(t_n), y(t_n - \tau)) - \frac{\partial f}{\partial z} I(t_n, h)\right]. \end{aligned}$$

Recalling the definition of $\varphi_m(z)$ we find

(6.2)
$$T_{n+k} \approx \left[\varphi_m(h\frac{d}{dt})y(t) + h^m \sigma(\exp(h\frac{d}{dt}))\frac{\partial f}{\partial z} I(t, h)\right]_{t=t_n},$$

where m is the order of the differential equation. If φ_m corresponds to a minimax method adapted to the periodic solution $y(t)$, than the truncation error is dominated by the second term containing the interpolation error $I(t, h)$. From this we conclude that high accuracies can be expected provided that we use interpolation of sufficiently high order.

7. Numerical Experiments

In this section we want to demonstrate the performance of the minimax modification of linear multistep methods both in PECE mode and when using Newton iteration. All methods tested are of order $p = 6$. When applied in PECE mode we used (for first-order equations) the Adams-Bashforth method of order 6 (AB_6) as predictor and the Adams-Moulton method of order 6 (AM_6) or the Milne-Simpson method of order 6 (MS_6) as corrector. In the case of the minimax-modification of the PECE method, both the predictor and corrector were modified. In the case of second-order equations, we applied the 4-step symmetric method of Lambert and Watson of order 6 (LW_6) (cf.[7, p.198]). In all experiments, interpolation polynomials of degree 9 (i.e. of order 10) were employed. The abscissas used are: $t_{l-9}, t_{l-8}, ..., t_l$, where l is determined by $t_{l-1} \leq t^* < t_l$ and t^* is the retarded argument (cf. Section 5).

In the tables of results given in the following subsections, the accuracy is measured by the number of correct digits in the numerical solution at the end point t_N, i.e., by

$$sd := -\log_{10}(|\, y_N - y(t_N)\,|).$$

7.1 Delay Equations of First Order

First we consider an example possessing a constant delay:

$$\begin{aligned} y'(t) &= y(t) + y(t-\pi) + 3\cos t + 5\sin t, \quad t \in [0, 10], \\ y(t) &= 3\sin t - 5\cos t, \quad t \leq 0, \end{aligned} \tag{7.1}$$

with exact solution $y(t) = 3\sin t - 5\cos t$.

In Tables 7.1 and 7.2 we list respectively the accuracies of the conventional and minimax methods, obtained for several values of the step length h.

h	$AB_6 - AM_6$ (PECE)	AM_6 (Newton)	$AB_6 - MS_6$ (PECE)	MS_6 (Newton)
2/5	-0.1	0.1	0.1	0.4
1/5	1.8	1.4	1.9	1.8
1/10	3.6	3.0	5.7	3.4
1/20	4.9	4.7	5.5	5.1
1/40	6.6	6.5	7.0	6.9

Table 7.1

sd-values for problem (7.1) using conventional methods

h	$AB_6 - AM_6$ (PECE)	AM_6 (Newton)	$AB_6 - MS_6$ (PECE)	MS_6 (Newton)
2/5	4.6	3.0	4.8	3.0
1/5	6.5	6.0	6.6	5.8
1/10	8.3	7.7	11.3	8.3

Table 7.2

sd-values for problem (7.1) using minimax methods with $[\underline{\omega}, \overline{\omega}] = [0.95, 1.05]$

In this example, the choice of the corrector (AM or MS) is of minor importance whereas the way in which the corrector has been solved (either PECE-mode or Newton iteration) is more crucial. However, the improvement obtained by the minimax versions is easily recognized. It should be noted that the additional effort required by the minimax methods is almost negligible.

Mention should be made of the fact that, for this example, $\lambda := \partial f(t,y,z)/\partial y$ ist positive. As the principle root of the characteristic equation approximates $e^{\lambda h}$ for $h \to 0$ we must reckon with amplification of roundoff errors. For this example, in which $\lambda = 1$ and the endpoint of integration equals 10, the accumulated amplification can be as bad as $(e^h)^{\frac{10}{h}} = e^{10} \simeq 2 \cdot 10^4$ (for small h). Hence, in requiring a result which is accurate in say n digits, we should use a machine which performs the calculations in at least $n + 4$ digits.

In our second example we consider a state-dependent delay term:

$$(7.2)\qquad \begin{aligned} y'(t) &= \omega \cdot \cot(g(t)) \cdot y(t) - \frac{\omega}{\sin(g(t))} y(t - \tau(t, y(t))), \quad t \in [0, 10] \\ y(t) &= \sin(\omega t), \quad t \le 0 \end{aligned}$$

$$\text{with} \quad \tau(t,y) := \tfrac{1}{\omega}(2 + \tfrac{1}{5}e^y)$$
$$\text{and} \quad g(t) := w\tau(t, \sin(\omega t))$$

which has the exact solution $y(t) = \sin(\omega t)$.

We applied the various methods for different values of the frequency ω. In the minimax versions we employed the frequency interval $[\underline{\omega}, \overline{\omega}] = [0.95\omega, 1.05\omega]$. The results can be found in Tables 7.3 an 7.4.

ω	h	$AB_6 - AM_6$ (PECE)	AM_6 (Newton)	$AB_6 - MS_6$ (PECE)	MS_6 (Newton)
	2/5	3.7	4.3	3.9	3.3
1	1/5	5.4	5.7	6.7	6.1
	1/10	7.2	7.4	8.0	7.2
	1/20	9.1	9.2	9.2	9.8
	1/10	3.5	3.8	4.6	0.2
3	1/20	5.5	5.9	2.6	1.2
	1/40	7.9	7.7	3.0	1.1

Table 7.3

sd-values for problem (7.2) using conventional methods

ω	h	$AB_6 - AM_6$ (PECE)	AM_6 (Newton)	$AB_6 - MS_6$ (PECE)	MS_6 (Newton)
	2/5	8.4	5.7	8.6	6.2
1	1/5	10.1	9.3	11.5	10.9
	1/10	11.9	11.8	12.6	11.7
3	1/10	8.2	6.0	8.6	2.7
	1/20	10.2	9.4	7.3	5.0

Table 7.4

sd-values for problem (7.2) using minimax methods with $[\underline{\omega}, \overline{\omega}] = [0.95\omega, 1.05\omega]$

The results for $\omega = 1$ give rise to the same conclusions as in the previous example. However, both in the conventional as well as in the minimax version, the Adams-Moulton method is superior to the Milne-Simpson method as the frequency ω increases. This is due to the better stability properties of the Adams-type methods.

Finally, we consider the influence of an inaccurate estimate of the frequency. For $\omega = 3$ we obtain in case of the AM_6-minimax method the following results

$\underline{\omega}$	$\overline{\omega}$	sd-value for h=1/10
2.85	3.15	6.0 (see Table 7.4)
2.5	3.5	5.9
2.0	4.0	5.2
2.0	2.5	4.9
2.5	3.0	6.0
3.0	3.5	6.1
3.5	4.0	4.6

7.2 Delay Equations of Second Order

Our first example is the second-order equivalent of problem (7.1):

$$
\begin{aligned}
&y''(t) = -y(t) - y(t - \frac{3\pi}{2}) + 3\cos t + 5\sin t, \quad t \in [0, 10]\,,\\
&y(t) = 3\sin t - 5\cos t, \qquad t \le 0,\\
&y'(t) = 3\cos t + 5\sin t, \qquad t \le 0
\end{aligned} \tag{7.3}
$$

with exact solution $y(t) = 3\sin t - 5\cos t$. Table 7.5 shows the results for the Lambert-Watson method and for its minimax variant using the frequency interval $[0.95, 1.05]$. In these tests the implicit relations were solved using Newton's method. Again, a substantial gain in accuracy i· obtained.

h	LW_6 *(conventional)*	LW_6 *(minimax)*
2/5	4.1	5.3
1/5	6.0	7.9
1/10	7.8	11.1

Table 7.5

As a second example we consider a Bessel-type equation involving a state-dependent delay:

$$(7.4)\qquad \begin{aligned} y''(t) + (100 + \tfrac{1}{4t^2})y(t) + y(t-1-y^2(t)) &= g(t), && t \in [3,10] \\ y(t) &= t^{\frac{1}{2}} J_0(10t), && t \le 3 \\ y'(t) &= \tfrac{1}{2} t^{-\frac{1}{2}} \left[J_0(10t) - 20tJ_1(10t) \right], && t \le 3 \end{aligned}$$

where J_0 and J_1 are the Bessel functions of first and second kind, respectively. The inhomogeneous term $g(t)$ is chosen in such a way that we have the almost periodic solution

$$y(t) = t^{\frac{1}{2}} J_0(10t).$$

The results can be found in Table 7.6. Obviously, the frequency is approximately equal to 10; hence, the minimax method was applied using the frequency interval $[9.9, 10.1]$.

h	LW_6 $(conventional)$	LW_6 $(minimax)$
1/10	1.7	4.0
1/20	3.7	6.4
1/40	5.5	10.1

Table 7.6

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H. Arndt, Institut für Angewandte Mathematik, Wegelerstr. 6, D-5300 Bonn

P.J. van der Houwen and B.P. Sommeijer, Centre for Mathematics and Computer Science, Kruislaan 413, NL-1098 SJ Amsterdam