What multigrid and Poisson do to one's image

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1 Introduction

Though the pun in the title is intended, it is not quite fair to Piet Wesseling as he is a person who promoted the development of multigrid to far more complicated equations than the Poisson equation. Instead, the title should be taken more literally as it truly relates to the contents of this note. It is shown that while multigrid is renowned for his efficiency in solving partial differential equations, integral equations and what not, it can also, maybe surprisingly, be used for the multiresolution of images [6, 7]. For the latter, first a second order partial differential operator is applied to an image function followed by a pyramidal decomposition using typical multigrid operators. The case of isotropic homogeneous diffusion (Poisson) provides an example that leads to a linear multiresolution scheme. Under certain conditions the scheme boasts perfect reconstruction.

Piet Wesseling has been a scientific advisor at CWI for many years. Every question put forward to him, by no matter who, is answered courteously and thoughtfully. Occasionally his answers start by the phrase "Ik heb er geen verstand van, maar ..." ("I am not knowledgeable on this, however ..."). Then I always pricked up my ears because I knew that an important remark was at hand. I hope he enjoys this tribute.

2 The multigrid image transform

Firstly we recapitulate particular items as they are used in multigrid methods [1, 2] for the solution of large linear systems of equations arising from discretized (elliptic) partial differential equations. We confine ourselves to the Poisson equation. Secondly we show how to incorporate those items into an image transform to be.

2.1 Recapitulation on Poisson and multigrid

We consider the Poisson equation in two space dimensions

$$Lu \equiv -\Delta u(x) = f(x) \tag{1}$$

on a rectangular domain $\Omega \subset \mathbb{R}^2$ with adiabatic boundary conditions. For discretization we employ a set of rectangular and increasingly coarser grids (vertex-centered):

$$\Omega_n \supset \Omega_{n-1} \supset \ldots \Omega_k \supset \ldots \supset \Omega_0.$$

The *grid at level k* is described as follows:

$$\Omega_k \equiv \{(x_i, y_i) \mid x_i = o_1 + (i-1)h_k, y_i = o_2 + (j-1)h_k\} \tag{2}$$

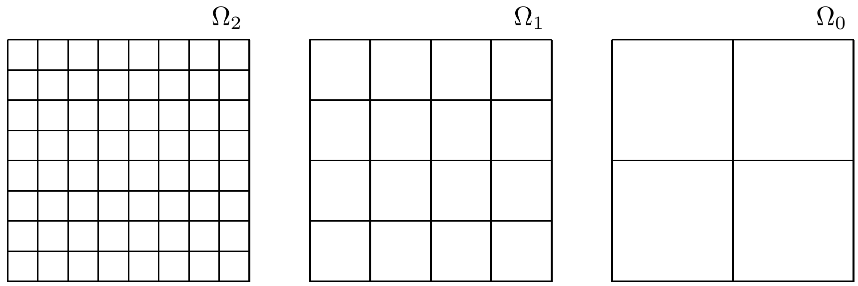

Figure 1: Example sequence of increasingly coarsened grids used in multigrid (vertex-centered)

where (o_1, o_2) is the origin and $h_{k-1} = 2h_k$. See Figure 1 for an example. $S(\Omega_k)$ denotes the linear space of real-valued functions on Ω_k

$$S(\Omega_k) = \{g_k \mid g_k : \Omega_k \to \mathbb{R}\},$$

where $g_k \in S(\Omega_k)$ is called a *grid-function*. We note in advance that images as in Figure 2 relate to such grid-functions as two-dimensional arrays in which each element corresponds to a single *pixel* (with a grey-value assigned to it) in the displayed image. Discretization leads to

$$L_n \overline{u}_n = f_n \tag{3}$$

where

$$L_n \; : \; S(\Omega_n) \to S(\Omega_n) \tag{4}$$

is the discretization of L and $f_n \in S(\Omega_n)$ is the discretization of f. We discretize by means of bilinear finite elements which gives rise to the following 3×3 stencil (or mask) in the interior of the domain:

$$L_n \sim \frac{1}{3} \begin{bmatrix} -1 & -1 & -1 \\ -1 & +8 & -1 \\ -1 & -1 & -1 \end{bmatrix}. \tag{5}$$

We find the solution $\overline{u}_n \in S(\Omega_n)$ of Eq. (3) very efficiently by means of multigrid, operating on all levels $k = 0, \ldots, n$. We opt for the *sawtooth multigrid cycle* (this is not essential but transparent) where at each level k a smoother (like Jacobi, Gauss Seidel, incomplete LU, ...) is applied after the coarse grid correction (CGC) defined shortly. Let u_n be an approximation of $\overline{u}_n$. The CGC at level k (meant to reduce the smooth part of the error) reads:

$$r_k = f_k - L_k u_k; \tag{6}$$
$$r_{k-1} = R_{k-1} r_k; \tag{7}$$
$$\text{solve (approximately) } L_{k-1} e_{k-1} = r_{k-1}; \tag{8}$$
$$\tilde{u}_k = u_k + P_k e_{k-1}. \tag{9}$$

The grid transfer operators mentioned are defined as follows. The *restriction operator*

$$R_{k-1} \; : \; S(\Omega_k) \to S(\Omega_{k-1}), \quad k = n, \ldots, 1 \tag{10}$$

transfers the residual onto the coarser grid, and the *prolongation operator*

$$P_k \; : \; S(\Omega_{k-1}) \to S(\Omega_k), \quad k = 1, \ldots, n \tag{11}$$

interpolates and transfers a correction for the solution towards the finer grid. Here bilinear interpolation is the natural choice for P_k and the restriction is chosen to be the transpose of the prolongation. The operators $L_k,\ k = n-1, \dots 0$ are produced by the *Galerkin coarse grid approximation* which gives rise to the same stencils as in (5), see [5].

2.2 The Poisson multigrid image transform

We are not building yet another Poisson solver in this note. Instead, we do now a bit of reverse thinking. We start of with a greyscale image and consider it to be the solution u_n of Eq. (3). We compute the right-hand side f_n that goes with it (as an image, it reveals edges at Ω_n). Moreover, we compute its representations on the coarser grids:

$$\left\{ \begin{array}{lcl} f_n & = & L_n u_n, \\ f_k & = & R_k f_{k+1}, \quad k = n-1, \dots, 0 \end{array} \right. \tag{12}$$

(the image of f_k reveals edges at Ω_k). This concludes the first step of our multiresolution scheme. The next step requires the *multigrid approximation operator*

$$E_k \ : \ S(\Omega_k) \to S(\Omega_k),\ k = 1, \dots, n \tag{13}$$

which is defined as:

$$E_k \equiv L_k^{-1} - P_k L_{k-1}^{-1} R_{k-1},\ k = 1, \dots, n. \tag{14}$$

This operator used to live as a recluse in the books of Hackbusch [2] and Wesseling [4] and the like, existing solely for the sake of convergence proofs in multigrid theory. Here it finds a new station in life, serving as a *high-pass filter* in the scheme. The multiresolution *decomposition* reads as follows:

$$\left\{ \begin{array}{lcl} a_0 & = & L_0^{-1} f_0, \\ d_k & = & E_k f_k,\ k = 1, \dots, n. \end{array} \right. \tag{15}$$

The gridfunction a_0 is called the *approximation* at level 0 and the d_k are called *details*. The *reconstruction* counterpart reads simply:

$$a_k = P_k a_{k-1} + d_k,\ k = 1, \dots, n. \tag{16}$$

The gridfunction a_k is called the approximation at level k. The reconstruction is a *perfect* one (if we take the outcome of the decomposition as the input for the reconstruction we obtain the original image, $a_n = u_n$). The proof is in [6].

3 The multigrid image transform in action

An image of Piet's face serves as an example, see the top left image of Figure 2. The dimensions read 1537 by 1025, leading to 1575425 pixels. We try and reduce these numbers severely, but would like to retain an acceptable level of image quality. We apply the above transform with 10 levels, producing the approximations $a_0, a_1, \dots, a_9$. Where operations $L_k^{-1} v_k$ on intermediate gridfunctions v_k are required, we employ the multigrid algorithm from [5]. The bottom right image of Figure 2 depicts a_6. Its dimensions read 193 by 129, leading to a mere number of 24897 pixels. For reasons of comparison, we include pictures of plain downsampling (top right) and the approximation by means of the Daubechies 4 wavelet transform (bottom left) on similar coarse grids. If we focus on the hairy parts we observe that plain downsampling yields rather chaotic results, that Daubechies 4 yields a far better result but looks a bit blurred and that multigrid appears the most respectful for Piet's image (small wonder). But then again, I might be biased.

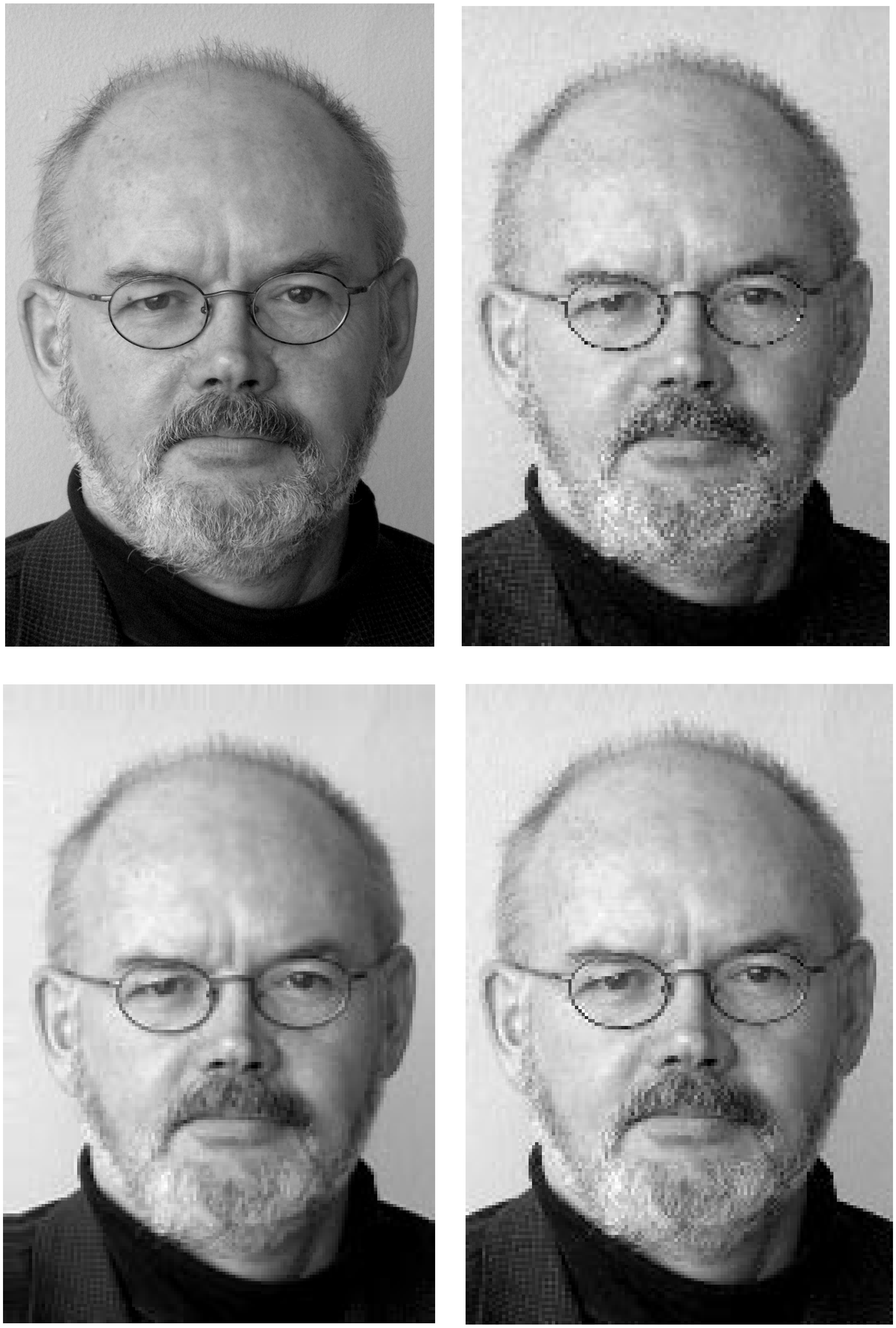

Figure 2: Top left: original image. Top right: plain downsampling, three levels of coarsening. Bottom left: approximation by the Daubechies 4 wavelet transform, three levels of coarsening. Bottom right: approximation by the multigrid image transform, three levels of coarsening.

4 Final remarks

A (linear) multiresolution scheme has been presented, based on multigrid operators and the Poisson equation. When used for the approximation of an image at a coarsened grid it shows little or no blurring. We make a few additional remarks on the scheme. The costs of the scheme appear to be within the same range as wavelets, thanks to multigrid efficiency. Boundary conditions are an issue that can be dealt with neatly. The scheme lends itself to generalization, instead of the Poisson operator we may opt for a more general elliptic partial differential operator to serve special image processing purposes. Particular diffusion tensors of interest can be found in Weickert's book [3] where they are used in the context of time-dependent partial differential equations. Elaboration of the latter remarks is beyond the scope of this note. For more information, see [6, 7].

References

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