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Transition Systems, Metric Spaces and Ready Sets in the Semantics of Uniform Concurrency*

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Transition systems as proposed by Hennessy and Plotkin are defined for a series of three languages featuring concurrency. The first has shuffle and local nondeterminacy, the second synchronization merge and local nondeterminacy, and the third synchronization merge and global nondeterminacy. The languages are all uniform in the sense that the elementary actions are uninterpreted. Throughout, infinite behaviour is taken into account and modelled with infinitary languages in the sense of Nivat. A comparison with denotational semantics is provided. For the first two languages, a linear time model suffices; for the third language a branching time model with processes in the sense of de Bakker and Zucker is described. In the comparison an important role is played by an intermediate semantics in the style of Hoare and Olderog's specification oriented semantics. A variant on the notion of ready set is employed here. Precise statements are given relating the various semantics terms of a number of abstraction operators.

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1. Introduction

Our paper aims at presenting a thorough study of the semantics of a number of concepts in concurrency. We concentrate on parallel composition modelled by shuffle and synchronization merge, local and global nondeterminacy, and deadlocks. Somewhat more specifically, we provide a systematic analysis of these concepts by confronting, for three sample languages, semantic techniques inspired by earlier work due to Hennessy and Plotkin [HP, Pl1, Pl2] proposing an operational approach, De Bakker *et al.* [BBKM, BZ1, BZ2, BZ3] for a denotational one, and the Oxford School [BHR, Ho2, OH, RB] serving—for the purposes of our paper—an intermediate role.

Our operational semantics is based on transition systems [Ke] as employed successfully in [HP, Pl1, Pl2, Mi, BHR, OH, ABKR]; applications in the analysis of proof systems were developed by Apt [Ap1, Ap2]. Compared with previous instances, our transition systems exhibit some new properties: the transitions describe only the *successful steps* of concurrent statements and, moreover, only *finitely many* schematic axioms and rules are needed. A successful step arises from the execution of an elementary action or the synchronization of two matching communications c and $\bar{c}$. Individually these communications fail or deadlock; only the synchronized execution of both succeeds. To model this phenomenon, [Pl2, Mi, BHR] add "virtual transitions" for the individual communications c and $\bar{c}$. However, these transitions violate the idea of deadlock.

In contrast, we follow Apt [Ap2] and formalize only the successful steps. Whereas [Ap2] uses infinitely many schematic transition rules and deals only with iteration, local nondeterminacy, and one level of parallelism, we show that finitely many axioms and rules are sufficient even when dealing with full recursion, global nondeterminacy, and nested parallelism with synchronization. (These points will be discussed in more detail in the body of our paper, see in particular Sections 3.1 and 4.1.)

Throughout the paper, we restrict ourselves to uniform statements: by this we mean an approach at the *schematic* level, leaving the elementary actions uninterpreted and avoiding the introduction of notions such as assignments or states. Many interesting issues arise at this level, and we feel that it is advantageous to keep questions which arise after interpretation for a treatment at a second level (not dealt with in our paper).

We shall study three languages in increasing order of complexity:

$\mathfrak{L}_0$: shuffle (arbitrary interleaving) + local nondeterminacy

$\mathfrak{L}_1$: synchronization merge + local nondeterminacy

$\mathfrak{L}_2$: synchronization merge + global nondeterminacy.

For $\mathfrak{L}_i$ with typical elements s, we shall present transition system T_i of the type discussed above and define an induced operational semantics $\mathcal{O}_i[\![s]\!]$, $i=0, 1, 2$. We shall also define three denotational semantics $\mathcal{D}_i[\![s]\!]$ based, for $i=0, 1$ on the "linear time" (LT) model which employs sets of sequences and, for $i=2$, on the "branching time" (BT) model employing *processes* (commutative trees, with sets rather than multisets of successors for any node, and with certain closure properties) of [BBKM, BZ1, BZ2]. Both our operational and denotational semantics yield languages with finite and *infinite words* (cf. Nivat [Ni]) or *streams* [Br]. In contrast to the operational semantics $\mathcal{O}_i$ we provide the denotational semantics $\mathcal{D}_i$ only for $\mathfrak{L}_i$ restricted to *guarded recursion* (each recursive call has to preceded by some elementary action); we then have an attractive metric setting with unique fixed points for contractive functions based on Banach's fixed point theorem.

Our main question can now be posed: Do we have that

$$\mathcal{O}_i[\![s]\!] = \mathcal{D}_i[\![s]\!]. \tag{1.1}$$

We shall show that (1.1) only holds for $i=0$. For the more sophisticated languages $\mathfrak{L}_i$, $i=1, 2$, we cannot prove (1.1). In fact, we can even show that there exists no denotational $\mathcal{D}_i$ satisfying (1.1), $i=1, 2$. Rather than trying to modify $\mathcal{O}_i$ (thus spoiling its intuitive operational character) we propose to replace (1.1) by

$$\mathcal{O}_i[\![s]\!] = \alpha_i(\mathcal{D}_i[\![s]\!]), \tag{1.2}$$

where α_i, $i=1, 2$, is an *abstraction* operator which forgets some information present in $\mathcal{D}_i[\![s]\!]$. The operator α_1 turns each failing communication into an indication of failure and deletes all subsequent actions. Thus, α_1 is as Milner's restriction [Mi] or ACP's encapsulation operator [BK1]. For $i=2$, α_i is composed of two operators, one which is like α_1 (but now defined for BT objects) and a second one which abstracts the branching structure from the BT object by mapping it onto the set of all its paths. The proof of (1.2) proceeds by introducing a transition based *intermediate* semantics $\mathcal{O}_i^*[\![s]\!]$. For $i=1$ we shall show that $\mathcal{O}_i^*[\![s]\!] = \mathcal{D}_i[\![s]\!]$. Next, we introduce the operator α_1 and show that $\mathcal{O}_1[\![s]\!] = \alpha_1(\mathcal{O}_1^*[\![s]\!])$.

The idea of using an intermediate semantics as a stepping stone in the equivalence proof of two semantics definitions is of course not new. For example, Stoy uses it in [St1, St2]. Also in the area of compiler construction the use of a suitable intermediate language and semantics is common practise. It allows decomposition of the compilation process of high level programs to machine code into smaller steps. What appears to be new is our specific construction and use of intermediate semantics for languages with recursion, parellelism, and nondeterminism.

The case $i=2$ is more involved, because $\mathfrak{L}_1$ has *local*, and $\mathfrak{L}_2$ has *global* nondeterminacy. Consider a choice a or c, where a is some autonomous action and c needs a parallel $\bar{c}$ to communicate. In the case of local nondeterminacy (written as $a \cup c$) both actions may be chosen; in the global nondeterminacy case (written as $a+c$ with "+" as in CCS [Mi]) c is chosen *only* when in some parallel component $\bar{c}$ is ready to execute. Therefore, $\mathfrak{L}_1$ and $\mathfrak{L}_2$ exhibit different deadlock behaviours.

$\mathcal{O}_2$ is based on the transition system T_2 which is a refinement of T_1, embodying a more subtle set of rules to deal with nondeterminacy. The denotational semantics $\mathcal{D}_2$ is as in [BBKM, BZ1, BZ2]. In order to relate $\mathcal{D}_2$ and $\mathcal{O}_2$ we introduce the notion of *readies* and an associated intermediate semantics $\mathcal{O}_2^*$, inspired by ideas described in [BHR, OH, RB].

$\mathcal{O}_2^*$ involves an extension of the LT model with some branching information (though less than the full BT model) which is amenable to a treatment in terms of transitions. Besides the operational $\mathcal{O}_2^*$ we also base an intermediate denotational semantics $\mathcal{D}_2^*$ on the domain of readies. To prove the desired result (1.2) for $\mathfrak{L}_2$, we shall show that $\mathcal{O}_2^*[\![s]\!] = \mathcal{D}_2^*[\![s]\!]$ and then relate $\mathcal{O}_2$ with $\mathcal{O}_2^*$, $\mathcal{D}_2^*$ with $\mathcal{D}_2$, and thus $\mathcal{O}_2$ with $\mathcal{D}_2$ by a careful choice of suitable abstraction operators.

Summarizing, we see as the main contributions of our paper:

1. The three finite transition systems T_i, $i = 0, 1, 2$, formalizing as transitions only the successful steps of concurrent statements.

2. Distinction of local vs global nondeterminacy and associated deadlock behaviour in the transition systems T_1 and T_2 without use of virtual transitions.

3. The systematic treatment of the denotational definitions (for the guarded case) together with the derivation of the relationship $\mathcal{O}_i = \alpha_i \circ \mathcal{D}_i$ (with α_0 as identity).

4. Application of the technique of intermediate semantics, both operational and denotational, for languages with recursion, parallelism, and nondeterminacy, in particular, the construction and use of the intermediate semantics $\mathcal{O}_1^*$, $\mathcal{O}_2^*$, and $\mathcal{D}_2^*$.

The rest of our paper is organized into Sections 2–4 dealing with the languages $\mathfrak{L}_0$–$\mathfrak{L}_2$. For each language $\mathfrak{L}_i$ the corresponding section is divided into four subsections. The first three introduce the transition system T_i, the operational semantics $\mathcal{O}_i$ and the denotational semantics $\mathcal{D}_i$, respectively. Most demanding is the fourth one which settles the relationship between $\mathcal{O}_i$ and $\mathcal{D}_i$ by establishing $\mathcal{O}_i = \alpha_i \circ \mathcal{D}_i$. To avoid repetitions, we elaborate on a different aspect for each $\mathfrak{L}_i$. For $\mathfrak{L}_0$ we concentrate on recursion, for $\mathfrak{L}_1$ on synchronization merge and for $\mathfrak{L}_2$ on the intermediate ready semantics.

Finally, the Appendix summarizes all results in a diagram.

2. The Language $\mathfrak{L}_0$: Shuffle and Local Nondeterminacy

Let A be a finite set of uninterpreted, elementary *actions*, with $a, b \in A$. Let x, y be elements of the set **stmv** of *statement variables* (used in fixed point constructs for recursion). The set $\mathfrak{L}_0$ of (*concurrent*) *statements*, with $s, t \in \mathfrak{L}_0$, is given by the following syntax:

$$s ::= a \mid s_1 ; s_2 \mid s_1 \cup s_2 \mid s_1 \parallel s_2 \mid x \mid \mu x[s].$$

Thus every action $a \in A$ denotes a statement, the one which finishes (successfully terminates) after performing a. $s_1 ; s_2$ denotes (*sequential*) *composition* such that s_2 starts once s_1 has finished. $s_1 \cup s_2$ denotes *nondeterministic choice*, also known as *local nondeterminism* [FHLR]. $s_1 \| s_2$ denotes *concurrent execution* of s_1 and s_2 modelling *shuffle* (arbitrary interleaving) between the actions of s_1 and s_2. $\mu x[s]$ is a recursive statement. For example, with the definitions to be proposed presently, the intended meaning of $\mu x[(a; x) \cup b]$ is the set $a^* \cdot b \cup \{a^\omega\}$, where a^ω is the infinite sequence of a's.

In general, we will restrict attention to *syntactically closed* statements (i.e., those without free statement variables), since only such statements have a meaning under the operational semantics to be defined below. (We will not always state this explicitly.)

2.1. *The Transition System T_0*

A transition describes what a statement s can do as its next step, using the successor relation between the configurations of an imaginary machine or automaton. This concept of a transition dates back to [Ke] and to automata theoretic notions [RS]. Following Hennessy and Plotkin [HP, Pl1], a transition system is a syntax-directed deductive system for proving transitions (see also [Ap1, Ap2, Pl2]). In this section we use this idea for $\mathfrak{L}_0$.

First we have to discuss what form of configurations to use. For fully interpreted languages configurations of the form

$$\langle s, \sigma \rangle \qquad \text{and} \qquad \sigma$$

are common where s is a statement and σ is a state [HP, Pl2, Ap1, Ap2]. We would like to preserve this form also in our present setting of uniform, i.e., uninterpreted languages. The only difference is that here states are not mappings from program variables to values, but words w over the set A of uninterpreted, elementary actions.

More precisely, let $\perp \notin A$. Then the set A^{st} of *words* [Ni] or *streams* [Br], with $u, v, w \in A^{\mathrm{st}}$, is defined as

$$A^{\mathrm{st}} = A^* \cup A^\omega \cup A^* \cdot \{\perp\}.$$

A^{st} includes the set $A^\infty = A^* \cup A^\omega$ of *finite* and *infinite* words or streams over A [Ni], and additionally the set $A^* \cdot \{\perp\}$ of *unfinished* words or streams. Let ε denote the *empty* word and $\leqslant$ denote the *prefix relation* over words. We define *concatenation* $u \cdot v$ as usual for $u \in A^*$ and $v \in A^{\mathrm{st}}$, and we put $u \cdot v = u$ for $u \in A^* \perp \cup A^\omega$, and $v \in A^{\mathrm{st}}$.

Thus in our case *configurations* will be of the form

$$\langle s, w \rangle \qquad \text{and} \qquad w$$

with $s \in \mathfrak{L}_0$ and $w \in A^{\mathrm{st}}$. One advantage of this form is that it need not be changed if an interpretation is added to actions and hence words w. (For details see the operational semantics of a nonuniform, i.e., interpreted language in [BKMOZ].)

Another advantage is that the *transition relation* is now just a binary relation $\rightarrow$ over configurations [Ke]; there is no need to introduce additional labels distinguishing various versions of $\rightarrow$.

Looking at the "classical" transition systems for languages involving concurrency [HP, Pl1, Pl2, Ap1, Ap2, Mi, BHR], labels appear only for languages where communication between parallel components is possible as in CSP [Pl2] or CCS [Mi]. However, even for languages *with* communication, labelled transitions need not occur. For instance, Apt [Ap2] provides a transition system for CSP where labels are not needed. (In fact, Apt uses labels, but states himself [Ap2, p. 201] that these labels are needed only in the completeness proof of a proof system for partial correctness of CSP, not for providing the semantics.) Of course, the decision on the appearance of configurations and whether or not to use labelled transitions is also a matter of taste. Thus following [Mi, BHR] we could have chosen configurations to be simply statements s, but then we would have to collect the labels of successive transitions to yield the final word w. For the reasons just explained we prefer the esent setting.

A transition relation being a binary relation $\rightarrow$ over configurations, a *transition* is now a formula

$$\langle s, w\rangle \rightarrow \langle s', w'\rangle \qquad \text{or} \qquad \langle s, w\rangle \rightarrow w'$$

denoting an element of $\rightarrow$. A *transition system* T is a formal deductive system for proving transitions. Using a self-explanatory notation, axioms have the format $1 \rightarrow 2$, rules have the format $\frac{1 \rightarrow 2}{3 \rightarrow 4}$, expressing that, if we have established that $1 \rightarrow 2$ holds in T, we may infer that $3 \rightarrow 4$ holds in T. More precisely, axioms and rules should be *schematic*, i.e., in their configurations 1, ..., 4 the statement component should be built up from finitely many metavariables $s, s_1, s_2, ..., s', s'_1, s'_2, ...$ ranging over statements, a ranging over actions, and x ranging over statement variables, and analogously for the word component. In an application of an axiom or rule each metavariable can be replaced by any object of the corresponding range, e.g., a metavariable s by any statement of $\mathfrak{L}_0$. For a transition system T, $T \vdash 1 \rightarrow 2$ expresses that transition $1 \rightarrow 2$ is deducible in the system T. Then $1 \rightarrow 2$ is also called a *T-transition*. For a finite sequence $1 \rightarrow 2 \rightarrow \cdots \rightarrow n$ of T-transitions we also write $T \vdash 1 \rightarrow^* n$.

For a compact representation of closely related transitions, we follow [Ap1, Ap2] and allow (in configurations only) the empty statement E. E expresses *successful termination*, i.e., we shall always identify

$$\langle E, w\rangle = w$$

and

$$s = E;s = s;E = E \parallel s = s \parallel E.$$

For example, we can now represent the two transitions

(i) $\langle s, w\rangle \rightarrow \langle s', w'\rangle$,

(ii) $\langle s, w\rangle \rightarrow w'$

by one transition, viz.,

$$\langle s, w\rangle \rightarrow \langle s', w'\rangle,$$

where s' ranges over $\mathfrak{L}_0 \cup \{E\}$. To avoid any confusion, we shall always state explicitly whether a statement can be empty.

We now present a specific transition system T_0 for $\mathfrak{L}_0$. For $w \in A^\omega \cup A^* \cdot \{\perp\}$ and $s \in \mathfrak{L}_0$ we put

$$\langle s, w\rangle \rightarrow w,$$

and for $w \in A^*$ we distinguish the following cases:

(*elementary action*)

$$\langle a, w\rangle \rightarrow w \cdot a$$

(*local nondeterminacy*)

$$\langle s_1 \cup s_2, w\rangle \rightarrow \langle s_1, w\rangle$$

$$\langle s_1 \cup s_2, w\rangle \rightarrow \langle s_2, w\rangle$$

(*recursion*)

$$\langle \mu x[s], w\rangle \rightarrow \langle s[\mu x[s]/x], w\rangle,$$

where, in general, $s[t/x]$ denotes substitution of t for x in s. Thus recursion is described here by syntactic substitution or copying.

(*composition*)

$$\frac{\langle s_1, w\rangle \rightarrow \langle s', w'\rangle}{\langle s_1; s_2, w\rangle \rightarrow \langle s'; s_2, w'\rangle},$$

where $s' \in \mathfrak{L}_0 \cup \{E\}$.

(*shuffle*)

$$\frac{\langle s_1, w\rangle \rightarrow \langle s', w'\rangle}{\langle s_1 \| s_2, w\rangle \rightarrow \langle s' \| s_2, w'\rangle}$$

$$\frac{\langle s_1, w\rangle \rightarrow \langle s', w'\rangle}{\langle s_2 \| s_1, w\rangle \rightarrow \langle s_2 \| s', w'\rangle}$$

where $s' \in \mathfrak{L}_0 \cup \{E\}$.

Note that with $s' = E$ the first shuffle rule amounts to

$$\frac{\langle s_1, w\rangle \to w'}{\langle s_1 \| s_2, w\rangle \to \langle s_2, w'\rangle}.$$

At the beginning of this section we said that a transition describes what a statement can do as its next step. For T_0 this is made precise by the following lemma.

2.1.1. LEMMA (Initial Step). *$T_0 \vdash \langle s, w\rangle \to \langle s', w'\rangle$ iff there exists some $b \in A \cup \{\varepsilon\}$ with $w' = w \cdot b$ and $T_0 \vdash \langle s, \varepsilon\rangle \to \langle s', b\rangle$.*

Proof. By structural induction on s. ▮

2.2. *The Operational Semantics $\mathcal{O}_0$*

By an *operational semantics* we mean here a semantics which is defined with the help of a transition system. As a first example we now introduce an operational semantics $\mathcal{O}_0$ for $\mathfrak{L}_0$. Formally, $\mathcal{O}_0$ is a mapping

$$\mathcal{O}_0 : \mathfrak{L}_0 \to \mathbb{S}$$

with $\mathbb{S} = \mathfrak{P}(A^{st})$ denoting the set of *infinitary languages*, which may contain both finite and infinite words over A.

We first give some definitions.

(1) A *transition sequence* is a (finite or infinite) sequence of T_0-transitions.

(2) *A path* from s is a maximal transition sequence

$$\pi: \langle s_0, w_0\rangle \to \langle s_1, w_1\rangle \to \langle s_2, w_2\rangle \to \cdots,$$

where $s_0 = s$ and $w_0 = \varepsilon$.

(3) The *word* associated with a path π, **word**(π), is defined according to the following three cases:

(a) π is finite, and of the form

$$\langle s_0, w_0\rangle \to \cdots \to \langle s_n, w_n\rangle \to w.$$

Then **word**$(\pi) = w$.

(b) π is infinite:

$$\langle s_0, w_0\rangle \to \cdots \to \langle s_n, w_n\rangle \to \langle s_{n+1}, w_{n+1}\rangle \to \cdots$$

and the sequence $(w_n)_n$ is infinitely often increasing. Then **word**$(\pi) = \sup_n w_n$ (sup w.r.t. the prefix ordering), an infinite word.

(c) π is infinite as in (b), but the sequence $(w_n)_n$ is eventually constant, i.e., for some n, $w_{n+k} = w_n$ for all $k \geqslant 0$.

Then $\mathbf{word}(\pi) = w_n \cdot \perp$.

It is easy to see that these are the only three possibilities for a path in T_0.

We now define for $s \in \mathfrak{L}_0$:

$$\mathcal{O}_0[\![s]\!] = \{\mathbf{word}(\pi) \mid \pi \text{ is a path from } s\}.$$

EXAMPLES.
$$\mathcal{O}_0[\![(a_1; a_2) \| a_3]\!] = \{a_1 a_2 a_3,\ a_1 a_3 a_2,\ a_3 a_1 a_2\},$$
$$\mathcal{O}_0[\![\mu x[(a; x) \cup b]]\!] = a^* \cdot b \cup \{a^\omega\},$$
$$\mathcal{O}_0[\![\mu x[(x; a) \cup b]]\!] = b \cdot a^* \cup \{\perp\}.$$

We conclude with two simple facts about $\mathcal{O}_0$.

2.2.1. LEMMA (Definedness). *$\mathcal{O}_0$ is well defined, i.e., $\mathcal{O}_0[\![s]\!] \neq \varnothing$ for every $s \in \mathfrak{L}_0$.*

Proof. The claim follows from the fact that for each configuration $\langle s, w \rangle$ at least one transition $\langle s, w \rangle \to \langle s', w' \rangle$ exists in T_0. ∎

2.2.2. LEMMA (Prolongation). *If $T_0 \vdash \langle s, \varepsilon \rangle \to^* \langle s', w \rangle$ and $w' \in \mathcal{O}_0[\![s']\!]$, then also $w \cdot w' \in \mathcal{O}_0[\![s]\!]$.*

Proof. By the definition of $\mathcal{O}_0$ and Lemma 2.1.1. ∎

We remark that corresponding lemmas will also hold for the operational semantics to be discussed subsequently.

2.3. *The Denotational Semantics $\mathcal{D}_0$*

The operational semantics $\mathcal{O}_0$ for $\mathfrak{L}_0$ is *global* in the following sense: to determine $\mathcal{O}_0[\![s]\!]$ we first have to explore the T_0-transition sequences for all of s, and only then we can retrieve the result $\mathcal{O}_0[\![s]\!]$. Further, in T_0, and thus in $\mathcal{O}_0$, recursion is dealt with by syntactic copying. We now look for a *denotational semantics* $\mathcal{D}_0$ for $\mathfrak{L}_0$. A denotational semantics should be *compositional* or *homomorphic*, i.e., for every syntactic operator **op** in $\mathfrak{L}_0$ there should be a corresponding semantic operator $\mathbf{op}^{\mathcal{D}_0}$ satisfying

$$\mathcal{D}_0[\![s_1 \mathbin{\mathbf{op}} s_2]\!] = \mathcal{D}_0[\![s_1]\!] \mathbin{\mathbf{op}^{\mathcal{D}_0}} \mathcal{D}_0[\![s_2]\!],$$

and it should tackle recursion semantically with the help of *fixed points*. This of course requires a suitable structure of the underlying semantic domain.

For $\mathcal{D}_0$ we shall use metric spaces (rather than the more customary cpos) as semantic domain. Our approach is based on [BBKM, BZ2]; for general topological notions such as *closedness*, *limits*, *continuity*, and *completeness*, see [Du].

Following [BZ2], $\mathscr{D}_0$ will be defined only for *guarded* statements, a notion which we define below. We must first define the notion of an *exposed* occurrence of a substatement in a given statement.

We now define the notion: an occurrence of a substatement t of s is *exposed* in s. The definition is by induction on the structure of s:

(a) s is *exposed* in s. (More accurately, the unique occurrence of s in s is exposed in s.)

(b) If an occurrence of t is *exposed* in s_1, then (and only then) the corresponding occurrence is also *exposed* in $s_1; s_2$, $s_1 \| s_2$, $s_2 \| s_1$, $s_1 \cup s_2$, $s_2 \cup s_1$, and $\mu x[s_1]$ (and also $s_1 + s_2$ and $s_2 + s_1$, in the case of the language $\mathfrak{L}_2$ of Section 4).

EXAMPLE. In the statement $x; a \cup b; x$, the first occurrence of x is *exposed*, while the second is not.

A statement is now defined to be *guarded* (cf. [Mi] or [Ni]) if for all its recursive substatements $\mu x[t]$, t contains *no exposed occurrences* of x.

EXAMPLES. $\mu x[a; (x \| b)]$ is guarded, but $\mu x[x]$, $\mu y[y \| b]$, and $\mu x[\mu y[x]]$ (as well as statements containing these) are not.

One advantage of the guardedness restriction is that we will be able to invoke Banach's classical fixed point theorem when dealing with recursion.

Let us now introduce the metric domain for $\mathscr{D}_0$. For $u \in A^{st}$ let $u[n]$, $n \geqslant 0$, be the prefix of u of length n if this exists; otherwise $u[n] = u$. E.g., $a_1 a_2 a_3[2] = a_1 a_2$, $a_1 a_2 a_3[5] = a_1 a_2 a_3$. We define a natural (ultra)metric d on A^{st} by putting

$$d(u, v) = 2^{-\max\{n \mid u[n] = v[n]\}}$$

with the understanding that $2^{-\infty} = 0$. For example, $d(a_1 a_2 a_3, a_1 a_2 a_4) = 2^{-2}$, $d(a^n, a^\omega) = 2^{-n}$. We have that (A^{st}, d) is a complete ultrametric space. For $X \subseteq A^{st}$ we put $X[n] = \{u[n] \mid u \in X\}$. A distance $\hat{d}$ on subsets X, Y of A^{st} is defined by

$$\hat{d}(X, Y) = 2^{-\max\{n \mid X[n] = Y[n]\}}.$$

Let $\mathbb{S}_c \subset \mathbb{S}$ denote the collection of all nonempty metrically *closed* subsets of A^{st}. It can be shown that $(\mathbb{S}_c, \hat{d})$ is a complete ultrametric space (see [Ha]), and that $\hat{d}$ coincides with the Hausdorff metric (cf. [Ni]) induced on $\mathbb{S}_c$ by the metric d on A^{st}.

A sequence $\langle X_i \rangle_{i=0}^{\infty}$ of elements of $\mathbb{S}_c$ is a *Cauchy sequence* whenever $\forall \varepsilon > 0$ $\exists N\ \forall n, m \geqslant N[\hat{d}(X_n, X_m) < \varepsilon]$. For $\langle X_i \rangle_i$ a Cauchy sequence, we write $\lim_i X_i$ for its limit (wich belongs to $\mathbb{S}_c$ by the completeness property).

A function ϕ: $(\mathbb{S}_c, \hat{d}) \to (\mathbb{S}_c, \hat{d})$ is called *contracting* whenever, for all X, Y,

$\hat{d}(\phi(X), \phi(Y)) \leqslant \alpha \cdot \hat{d}(X, Y)$ for some real number α with $0 \leqslant \alpha < 1$. A classical theorem due to Banach states that in any complete metric space, a contracting function has a *unique* fixed point obtained as $\lim_i \phi^i(X_0)$ for arbitrary starting point X_0.

We now define the semantic operators $;^{\mathscr{D}_0}$, $\cup^{\mathscr{D}_0}$, and $\|^{\mathscr{D}_0}$ on $\mathbb{S}_c$. (For ease of notation, we skip superscripts $\mathscr{D}_0$ if no confusion arises.)

(a) $X, Y \subseteq A^* \cup A^* \cdot \{\perp\}$. For $X; Y =_{df} X \cdot Y$ (concatenation) and $X \cup Y$ (set-theoretic union) we adopt the usual definitions (including the clause $\perp \cdot u$ for all u). For $X \| Y$ (shuffle or merge) we introduce as auxiliary operator the so-called left-merge $\lfloor\!\lfloor$ (from [BK1]). It permits a particularly simple definition of $\|$ by putting

$$X \| Y = (X \lfloor\!\lfloor Y) \cup (Y \lfloor\!\lfloor X),$$

where $\lfloor\!\lfloor$ is given recursively by $X \lfloor\!\lfloor Y = \bigcup \{u \lfloor\!\lfloor Y \mid u \in X\}$ with $\varepsilon \lfloor\!\lfloor Y = Y$, $(a \cdot u) \lfloor\!\lfloor Y = a \cdot (\{u\} \| Y)$ and $\perp \lfloor\!\lfloor Y = \{\perp\}$.

(b) $X, Y \in \mathbb{S}_c$, where X, Y do not consist of finite words only. Then

$$X \ \mathbf{op} \ Y = \lim_i (X[i] \ \mathbf{op} \ Y[i]),$$

for $\mathbf{op} \in \{;, \cup, \|\}$. In [BZ2] we have shown that this definition is well formed and preserves closed sets, and the operators are *continuous* (assuming finiteness of A, as in [BBKM]).

We now turn to the definition of $\mathscr{D}_0$. We introduce the usual notion of *environment* which is used to store and retrieve meanings of statement variables. Let $\Gamma_0 =$ **stmv** $\to \mathbb{S}_c$ be the set of environments, and let $\gamma \in \Gamma_0$. We write $\gamma' =_{df} \gamma\langle X/x\rangle$ for a *variant* of γ which is like γ but with $\gamma'(x) = X$. We define

$$\mathscr{D}_0\colon \text{guarded } \mathfrak{L}_0 \to (\Gamma_0 \to \mathbb{S}_c)$$

as follows:

1. $\mathscr{D}_0[\![a]\!](\gamma) = \{a\}$
2. $\mathscr{D}_0[\![s_1 \ \mathbf{op} \ s_2]\!](\gamma) = \mathscr{D}_0[\![s_1]\!](\gamma) \ \mathbf{op} \ \mathscr{D}_0[\![s_2]\!](\gamma)$
3. $\mathscr{D}_0[\![x]\!] = \gamma(x)$
4. $\mathscr{D}_0[\![\mu x[s]]\!](\gamma) = \lim_i X_i$, where $X_0 = \{\perp\}$ and $X_{i+1} = \mathscr{D}_0[\![x]\!](\gamma\langle X_i/x\rangle)$.

By the guardedness requirement, each function $\phi = \lambda X \cdot \mathscr{D}_0[\![s]\!](\gamma\langle X/x\rangle)$ is contracting, $\langle X_i \rangle_i$ is a Cauchy sequence, and $\lim_i X_i$ equals the unique fixed point of ϕ [Ni, BBKM, BZ2]. For statements s without free statement variables we write $\mathscr{D}_0[\![s]\!]$ instead of $\mathscr{D}_0[\![s]\!](\gamma)$. Since $\mathscr{D}_0[\![s]\!]$ is a set of (linear) streams, $\mathscr{D}_0$ is called a *linear time semantics* [BBKM]. (Such a semantics may constitute the basis for a linear time temporal logic for $\mathfrak{L}_0$.)

Remark. An order-theoretic approach to the denotational model is also possible ([Br, Me, BMO], see also survey [BKMOZ]), but less convenient for our special purposes. In fact, the order-theoretic approach does not provide a *direct* treatment for the unguarded case either, it seems to require a contractivity argument for uniqueness of fixed points just as well, and, last but not least, as far as we know, it cannot be used as a basis for the branching time semantics used later in Section 4.3. (In [R] an order-theoretic approach is employed and compared with a metric one, but this setting uses an ordering on *forests* rather than one on the *tree-like* structures we are dealing with in branching time semantics.)

2.4. *Relationship between $\mathcal{O}_0$ and $\mathcal{D}_0$*

In this section we will prove:

2.4.1. THEOREM. *$\mathcal{O}_0[\![s]\!] = \mathcal{D}_0[\![s]\!]$ for all (syntactically closed) guarded $s \in \mathfrak{L}_0$.*

The proof of Theorem 2.4.1 is by induction on the structure of s. For the induction argument we need two important facts about $\mathcal{O}_0$ which we develop first. The first fact states that $\mathcal{O}_0$ behaves compositionally over the operators $\mathbf{op} \in \{;, \cup, \|\}$ of $\mathfrak{L}_0$ in the sense or Section 2.3:

$$\mathcal{O}_0[\![s_1 \ \mathbf{op}\ s_2]\!] = \mathcal{O}_0[\![s_1]\!] \ \mathbf{op}^{\mathcal{D}_0}\ \mathcal{O}_0[\![s_2]\!].$$

We shall not give a full proof here, but refer to Section 3 where this result is established in the more general setting of language $\mathfrak{L}_1$.

Instead we concentrate here on the second fact dealing with recursion because its proof carries over to the languages $\mathfrak{L}_1$ and $\mathfrak{L}_2$ virtually without change. We wish to show that

$$\mathcal{O}_0[\![\mu x[t(x)]]\!] = \lim_n \mathcal{O}_0[\![t^{(n)}(\Omega)]\!],$$

where Ω is a certain auxiliary statement and $t^{(n)}(\cdot)$ denotes n-fold substitution (to be explained in the sequel). This proof is quite involved; it requires a number of auxiliary results on the transition system T_0 and the operational semantics $\mathcal{O}_0$.

In the following, we make the *general assumption* that all our statements are (syntactically closed and) *guarded* (unless explicitly stated otherwise). Guardedness comes into our work in two ways:

(1) in proving the technical results below on transition sequences, notably the Basic Lemma (2.4.4), and

(2) more fundamentally: $\mathcal{D}_0[\![s]\!]$ is only defined for guarded s! (On the other hand, $\mathcal{O}_0[\![s]\!]$ is only defined for syntactically closed s.)

Let us now turn to the first fact about $\mathcal{O}_0$.

Compositionality of $\mathscr{C}_0$.

We state (more generally):

2.4.2. THEOREM. (a) $\mathscr{C}_0[\![a]\!] = \{a\}$

(b) $\mathscr{C}_0[\![s_1 \cup s_2]\!] = \mathscr{C}_0[\![s_1]\!] \cup^{\mathscr{S}_0} \mathscr{C}_0[\![s_2]\!]$

(c) $\mathscr{C}_0[\![\mu x[s]]\!] = \mathscr{C}_0[\![s[\mu x[s]/x]]\!]$

(d) $\mathscr{C}_0[\![s_1, s_2]\!] = \mathscr{C}_0[\![s_1]\!] \;;^{\mathscr{S}_0} \mathscr{C}_0[\![s_2]\!]$

(e) $\mathscr{C}_0[\![s_1 \| s_2]\!] = \mathscr{C}_0[\![s_1]\!] \|^{\mathscr{S}_0} \mathscr{C}_0[\![s_2]\!]$.

Proof. (a), (b), and (c) are clear, by considering transition sequences from $\langle a, \varepsilon\rangle$, $\langle s_1 \cup s_2, \varepsilon\rangle$, and $\langle \mu x[s], \varepsilon\rangle$, which must start with the transition rules of elementary action, local nondeterminacy, and recursion, respectively. Part (d) is proved like (e), but more simply, and the proof of (e) is postponed to Section 3 (Lemma 3.4.6), in a more general context. ∎

We now develop a series of auxiliary results leading to the main fact about recursion (Corollary 2.4.16) used in proving Theorem 2.4.1.

Basic Facts about T_0*-Transitions*

Notation. To display all free occurrences of a variable x in a statement s, we can write $s = s(x)$. Then the result of substituting a statement t for all free occurrences of x in s is denoted formally by $s[t/x]$ and informally by $s(t)$.

We also speak of the *context* $s(\cdot)$ of the occurrence(s) of t displayed in $s(t)$.

We indicate a *specific occurrence* of a substatement t of s by *underlying* it: $s(\underline{t})$.

We also speak of the *context* $s(\cdot)$ (or $s(\underline{\cdot})$), meaning that part of the expression $s(t)$ (or $s(\underline{t})$) excluding the displayed occurrence(s) of t.

TYPES OF TRANSITIONS. We must make a closer analysis of T_0-transitions. Since every deduction rule in T_0 has only one premise, every T_0-transition

$$\langle s, w\rangle \to \langle s', w'\rangle \tag{2.2}$$

is deducible from a single axiom: *elementary action*, *nondeterminacy*, or *recursion*, by a sequence of applications of the rules *composition* and *shuffle*.

There may actually be more than one deduction of (2.2). For example, the transition

$$\langle \mu x[x] \| \mu y[y], w\rangle \to \langle \mu x[x] \| \mu y[y], w\rangle$$

has two different deductions, one starting from $\mu x[x]$ and the other from $\mu y[y]$. Notice, however, that in this example the μ-substatements are unguarded. If (according to our general assumption) we restrict our attention to guarded statements, it is not hard to see that every deducible transition has a *unique deduction* (although our results do not really depend on this fact).

According to which axiom was used in its deduction (elementary action, nondeterminacy, or recursion), (2.2) is called (respectively) and *a-transition*, $\cup$*-transition*, or μ*-transition*.

Substatement Involved in a Transition

Any transition

$$\langle s, w \rangle \to \langle s', w' \rangle \tag{2.3}$$

involves some (unique) *occurrence of a substatement* of s. This notion can be defined by induction on the length of the deduction of (2.3).

(i) *Basis.* If (2.3) is an axiom, then it *involves* the occurrence of s shown.

(ii) *Induction step.* If the premise of an instance of one of the rules in T_0 *involves* an occurrence of s, then the conclusion *involves* the corresponding occurrence of s.

For example, in the following form of the shuffle rule:

$$\frac{\langle s_1(\underline{t}), w_1 \rangle \to \langle s_2, w_2 \rangle}{\langle s' \| s_1(\underline{t}), w_1 \rangle \to \langle s' \| s_2, w_2 \rangle},$$

if the premise involves the occurrence of t shown in s_1, then the conclusion involves the *corresponding occurrence* of t shown in $s' \| s_1$.

It is clear that the substatement involved in a transition is the same as the statement on the l.h.s. of the corresponding axiom.

EXAMPLES. (1) $\langle s_1 \| (a; s_2), w \to \langle s_1 \| s_2, wa \rangle$ is an a-transition, involving the occurrence of a shown.

(2) $\langle ((s_1 \cup s_2); s_3) \| s_4, w \rangle \to \langle (s_2; s_3) \| s_4, w \rangle$ is a $\cup$-transition, involving the occurrence of $s_1 \cup s_2$ shown.

(3) $\langle s_1 \| \mu x[s_2(x)], w \rangle \to \langle s_1 \| s_2(\mu x[s_2(x)]), w \rangle$ is a μ-transition, involving the occurrence of $\mu x[s_2(x)]$ shown.

Passive Substatements. We say that a transition

$$\langle s(\underline{t}), w \rangle \to \langle s', w' \rangle \tag{2.4}$$

affects the substatement occurrence $\underline{t}$ if it *involves* some substatement of $\underline{t}$ (perhaps $\underline{t}$ itself). Conversely, $\underline{t}$ is said to be *passive* in (2.4) if it is *not affected* by (2.4). Denote the (unique) statement occurrence involved in (2.4) by $\underline{t}_0$. Then it is easy to see that the following three statements are equivalent:

(i) $\underline{t}$ is passive in (2.4).

(ii) $\underline{t}_0$ is not contained in $\underline{t}$.

(iii) $\underline{t}$ is either disjoint from $\underline{t}_0$, or properly contained in $\underline{t}_0$.

Free Substatements. A substatement occurrence $\underline{t}$ of a statement s is said to be *free* in s if $\underline{t}$ does not contain any free statement variables which are bound in s.

2.4.3. LEMMA (Substitution of Passive Free Substatements). *Given a T_0-transition*

$$\langle s_1, w_1 \rangle \to \langle s_2, w_2 \rangle, \tag{2.5}$$

if s_1 has the form $s'_1(\underline{t})$, where $\underline{t}$ is free *in s_1 and* passive *in the transition, then s_2 can be written in the form $s'_2(t)$ (displaying* 0, 1, *or more occurrences of t), such that for any statement t', there is a* corresponding *T_0-transition*

$$\langle s'_1(\underline{t}'), w_1 \rangle \to \langle s'_2(t'), w_2 \rangle.$$

Proof. By induction on the length of a deduction of (2.5). Briefly, the deduction of the new transition is formed simply by replacing certain occurrences of t by t' in the deduction of (2.5). The details are left to the reader. ∎

Basic Lemma on Transitions

The following basic lemma shows the significance of the guardedness assumption. It enters three times into our working below!: (a) in the proof of Theorem 2.4.10 (via the Decreasing Exposure Lemma 2.4.7 and the Finiteness Lemma 2.4.8); (b) in the proof of Theorem 2.4.11; and (c) in the proof of Lemma 2.4.14 (via Corollary 2.4.13), which in turn is used in Theorem 2.4.15.

2.4.4. (BASIC) LEMMA. *In the transition*

$$\langle s_1, w_1 \rangle \to \langle s_2, w_2 \rangle, \tag{2.6}$$

if a substatement occurrence $\underline{t}$ is not exposed *in s_1, then $\underline{t}$ is* passive *(and so the lemma of the previous subsection applies).*

Proof. By induction on the length of a deduction of (2.6).

Basis. Suppose (2.6) is an *axiom.* Then, since $\underline{t}$ is not exposed in s_1, it cannot be equal to s_1, i.e., it is a proper substatement of s_1. Hence $\underline{t}$ is passive in (2.6) (since by definition only the full statement s_1 is *involved* in an axiom (2.6)).

Induction Step. Consider first the *composition* rule, and take the case

$$\frac{\langle s_1, w_1 \rangle \to \langle s_2, w_2 \rangle}{\langle s_1; s, w_1 \rangle \to \langle s_2; s, w_2 \rangle}.$$

By assumption, $\underline{t}$ is not exposed in $s_1; s$. Hence (by definition) $\underline{t}$ is either in s or (not exposed) in s_1. If $\underline{t}$ is in s, then it is certainly passive in the conclusion. Suppose $\underline{t}$ is (not exposed) in s_1. By induction hypothesis, $\underline{t}$ is passive in the premise (i.e., the substatement of s_1 involved in the premise does not occur in $\underline{t}$). Hence, clearly, $\underline{t}$ is also passive in the conclusion.

The *shuffle* rule is handled similarly. ▮

A useful version of this lemma is given by:

2.4.5. COROLLARY. *If a transition* $\langle s_1, w_1\rangle \to \langle s_2, w_2\rangle$ involves *a substatement occurrence* $\underline{t}$ *in* s_1, *then* $\underline{t}$ *is exposed in* s_1.

Proof. This is a trivial consequence of the Basic Lemma. (It could also easily be proved directly, by induction on the length of a deduction of the transition.) ▮

Passive and Active Successors

Consider a transition $\langle s, w\rangle \to \langle s', w'\rangle$. Let $\mu_0 = \mu x[t_0(x)]$ be a μ-substatement of s, and consider a particular occurrence of μ_0 in s. Then there may be one or more *corresponding occurrences* of μ_0 in s', stemming from this occurrence of μ_0 in s. These are called the *successor*(s) of this occurrence of μ_0 in s.

We do not give a complete formal definition of the notion of successor; consider, as an example, the following form of the rule of *composition*:

$$\frac{\langle s_1, w\rangle \to \langle s', w\rangle}{\langle s_1; s_2(\underline{\mu_0}), w\rangle \to \langle s'; s_2(\underline{\mu_0}), w\rangle}.$$

The displayed occurrence of μ_0 on the r.h.s. is a *successor* of that on the l.h.s.

Most other cases are just as trivial—call these *passive successors*—except for the case that the transition actually involves the occurrence of μ_0 considered:

$$\langle s(\underline{\mu_0}), w\rangle \to \langle s(\underline{t_0(\mu_0)}), w\rangle \tag{2.7}$$

(where, as stated above, $\mu_0 = \mu x[t_0(x)]$).

In this case, each occurrence of μ_0 shown inside the occurrence of t_0 on the r.h.s. of (2.7) is an *active successor* of the occurrence of μ_0 shown on the l.h.s.

The transitive relation generated by the successor relation is called *descendant*; the converse of that is called *ancestor*.

2.4.6. LEMMA (Transitivity of Exposure). *Given a statement* s_1, *containing a substatement occurrence* $\underline{s}_2$, *containing in turn a substatement occurrence* $\underline{s}_3$:

(a) *If* $\underline{s}_3$ *is exposed in* s_2, and $\underline{s}_2$ *is exposed in* s_1, *then* $\underline{s}_3$ *is exposed in* s_1. *However, if* either

(b) $\underline{s}_3$ *is not exposed in* s_2 or (c) $\underline{s}_2$ *is not exposed in* s_1, *then* $\underline{s}_3$ *is not exposed in* s_1.

Proof. In all cases, by induction on the structure of s_1. ▮

Degree of Exposure of a Statement; Decreasing Exposure Lemma

The *degree of exposure* of s, $\mathbf{de}(s)$, is defined to be the number of *exposed occurrences* of μ-substatements of s. We have an important lemma, which uses the guardedness of statements.

2.4.7. LEMMA (Decreasing Exposure). *If $\langle s, w\rangle \to \langle s', w'\rangle$ is a μ-transition, then* $\mathbf{de}(s') < \mathbf{de}(s)$.

Proof. Suppose this transition involves an occurrence of $\mu_0 = \mu x[t_0(x)]$, and put $s = s(\underline{\mu_0})$, displaying this occurrence. Then $s' = s(\underline{t_0(\mu_0)})$. By the Basic Lemma, $\underline{\mu_0}$ is *exposed* in s. However, all its (active) successors are *not exposed* in $t_0(\mu_0)$ (since, by assumption, μ_0 is *guarded*) and hence also not exposed in s' (by Lemma 2.4.6 on transitivity of exposure).

Now consider all other occurrences of μ-substatements in $s(\underline{\mu_0})$. Any occurrence which is contained in the context $s(\underline{\cdot})$ (i.e., *not* in the displayed occurrence of μ_0) has exactly one (passive) successor in $s(\underline{t_0(\mu_0)})$, which is clearly exposed if and only if the original is.

Finally, consider an occurrence of another μ-substatement, say μ_1, within μ_0, i.e., within $t_0(\cdot)$. Now μ_1 may contain x, so we write $\mu_1 = \mu_1(x)$ and $\mu_0 = \mu x[t_0(\underline{\mu_1(x)}, x)]$ and so

$$s = s(\underline{\mu x[t_0(\underline{\mu_1(x)}, x)]}). \tag{2.8}$$

Now $\underline{\mu_1(x)}$ has, in general, *many* (passive) successors in s', which we can write as

$$s' = s(\underline{t_0(\underline{\mu_1(\mu_0)}, \mu x[t_0(\underline{\mu_1(x)}, x)])}). \tag{2.9}$$

The occurrence $\underline{\mu_1(\mu_0)}$ is exposed in (2.9) iff $\underline{\mu_1(x)}$ is exposed in (2.8), that is (in both cases), iff $\underline{\mu_1(x)}$ is exposed in $t_0(\underline{\mu_1(x)}, x)$ (by the lemma on transitivity of exposure, since $\underline{\mu_0}$ is exposed in $s(\underline{\mu_0})$). All the occurrences of $\underline{\mu_1(x)}$ shown in (2.9) are, in any case, not exposed in s', since they are in $\mu_0 = \mu x[t_0(\underline{\mu_1(x)}, x)]$, which is not exposed in $t_0(\mu_0)$ (again, by the assumption that μ_0 is guarded).

Putting all this together yields the result. ∎

The above lemma is used in the Finiteness Lemma in the following subsection.

Non-increasing Transitions and Transition Sequences; Finiteness Lemma

A transition $\langle s, w\rangle \to \langle s', w'\rangle$ is said to be *non-increasing* if $w' = w$, and *increasing* otherwise (i.e., if $w' = w \cdot a$ for some $a \in A$). Similarly, a transition sequence $\langle s, w\rangle \to \cdots \to \langle s', w'\rangle$ is said to be *non-increasing* if $w' = w$.

Clearly, a transition is non-increasing iff it is a μ- or $\cup$-transition (cf. Types of Transitions above), and increasing iff it is an a-transition.

We now give an important lemma, which will be used in the proof of Theorem 2.4.10 (via Corollary 2.4.9).

2.4.8. LEMMA (Finiteness). *Any non-increasing transition sequence is finite. In fact, for any s, there is a positive integer C, depending only on the length of s (as a string of symbols), such that any non-increasing transition sequence of the form*

$$\langle s, w\rangle = \langle s_1, w\rangle \to \cdots \to \langle s_n, w\rangle = \langle s', w\rangle \tag{2.10}$$

(for any s', w) has length n at most C.

Proof. Let l be the length of s, and $d = \mathbf{de}(s)$. Now a non-increasing transition sequence (2.10) can only contain $\cup$-transitions and μ-transitions. This can include at most d μ-transitions, by the Decreasing Exposure Lemma (2.4.7). Also, each $\cup$-transition decreases the length of the statement. Hence (by a crude estimate, since the length of a statement can be at most squared by a μ-transition) (2.10) can include at most l^{2^d} $\cup$-transitions. Hence the length of (2.10) is at most $d + l^{2^l}$, and so (since, trivially, $d \leqslant l$) we can take $C = l + l^{2^l}$. ∎

COUNTEREXAMPLE FOR AN UNGUARDED STATEMENT. Let $s = \mu x[x; a \cup b]$. Starting with $\langle s, \varepsilon\rangle$, we can perform a μ-transition, followed by a $\cup$-transition, k times (for any k), to get

$$\langle s, \varepsilon\rangle \to \cdots \to \langle s; a^k, \varepsilon\rangle,$$

a non-increasing transition sequence of length k.

2.4.9. COROLLARY. *For a given s, w there are only* finitely many *transition sequences of the form*

$$\langle s, w\rangle \to \cdots \to \langle s', w\rangle \to \langle s'', w \cdot a\rangle \tag{2.11}$$

(for any s', s'', a).

Proof. By the Finiteness Lemma, there is a *finite upper bound* to the length of (2.11). Also, at each step there are only *finitely many* possibilities for the next transition (as is clear from an inspection of the transition rules). ∎

COUNTEREXAMPLE FOR AN UNGUARDED STATEMENT. Let (again) $s = \mu x[x; a \cup b]$. For any k, we construct the sequence

$$\begin{aligned}
\langle s, \varepsilon\rangle &\to^* \langle s; a^k, \varepsilon\rangle && \text{(as in counterexample after 2.4.8)}\\
&\to \langle (s; a \cup b); a^k, \varepsilon\rangle && (\mu\text{-transition})\\
&\to \langle b; a^k, \varepsilon\rangle && (\cup\text{-transition})\\
&\to \langle a^k, b\rangle.
\end{aligned}$$

Such sequences are distinct for different k.

Metric Closure

2.4.10. THEOREM. *For any s, $\mathcal{O}_0[\![s]\!]$ is closed (in the metric on A^{st} given in Section 2.3).*

Proof. Let $(u_1, u_2, ...)$ be a CS (Cauchy sequence) of words in $\mathcal{O}_0[\![s]\!]$. Let $u = \lim_n u_n$. We must show: $u \in \mathcal{O}_0[\![s]\!]$.

If u is finite, it is easy to see that $(u_n)_n$ is eventually constant, i.e., $u_n = u$ for n sufficiently large. Hence $u \in \mathcal{O}_0[\![s]\!]$.

So suppose u is infinite. The idea of the proof is to find a subsequence of $(u_n)_n$ such that not only do the *words* converge, but also the *paths* producing them converge (in a suitable metric, to be discussed in 2.4.13) to a path π of s such that $u \in \mathbf{word}(\pi)$, from which the result follows.

(As before, we use the notation $u[n]$ for the initial segment of a word u of length n.)

We proceed inductively. Since $(u_n)_n$ is a CS, for n sufficiently large (say $n \geqslant N_1$) $u_n[1]$ is constant, i.e., u_n begins with the same letter, say a_1 (which is also the first letter of u).

For all n, let π_n be a path from s producing u_n. Consider the first part of π_n, up to the first appearance of a_1 on the r.h.s. of a configuration:

$$\pi_n: \langle s, \varepsilon \rangle \to \cdots \to \langle s_1, a_1 \rangle \to \cdots.$$

By the Corollary (2.4.9) to the Finiteness Lemma, there are only *finitely many* such transition sequences possible. Hence there is a *subsequence* $(u_{n_1}, u_{n_2}, ...)$ of $(u_n)_n$ such that the corresponding π_{n_k} all begin with the *same* transition sequence (up to the first appearance of a_1 on the r.h.s.).

Since $(u_{n_k})_k$ is a CS, for k sufficiently large $u_{n_k}[2]$ is constant, i.e., u_{n_k} begins with the same two letters, say $a_1 a_2$ (which are also the first two letters of u). Again, by the corollary to the Finiteness Lemma, we can get a *subsequence* of (u_{n_k}) such that the corresponding paths all begin in the same way, up to the first appearance of $a_1 a_2$ on the r.h.s.:

$$\langle s, \varepsilon \rangle \to \cdots \to \langle s_1, a_1 \rangle \to \cdots \to \langle s_2, a_1 a_2 \rangle \to \cdots.$$

Continuing in this way, we get, for all k, successive subsequences of $(u_n)_n$ such that the corresponding paths all begin in the same way, up to the first appearance of k letters on the r.h.s., say $a_1 a_2 \cdots a_k$, which are also the first k letters of u. Finally we take the "diagonal sequence," by *piecing together* the initial segments of these paths, to obtain the path

$$\begin{aligned}\pi: \langle s, \varepsilon \rangle \to \cdots \to \langle s_1, a_1 \rangle &\to \cdots \\ \cdots \to \langle s_2, a_1 a_2 \rangle &\to \cdots \\ \cdots \to \langle s_k, a_1 a_2 \cdots a_k \rangle &\to \cdots.\end{aligned}$$

Clearly, $\pi \in \mathbf{paths}(s)$ and $u = a_1 a_2 \cdots a_k \cdots \in \mathbf{word}(\pi)$. ∎

Discussion (metric on the set of paths). We can define a metric $\tilde{d}$ on the set **path**(s) as follows: $\tilde{d}(\pi, \pi') = 2^{-n}$ if n is maximal such that π and π' agree up to the first appearance of a word of length n on each:

$$\langle s, \varepsilon \rangle \to \cdots \to \langle s_n, a_1 \cdots a_n \rangle \to \cdots.$$

(Note: this is *not* equivalent to agreeing up to the first n transitions!)

The proof of Theorem 2.4.10 produces a subsequence of $(u_n)_n$ such that the corresponding *sequence of paths also converges* (in the metric $\tilde{d}$) to a limiting path π, with $u \in \mathbf{word}(\pi)$.

COUNTEREXAMPLE TO THEOREM 2.4.10 FOR AN UNGUARDED STATEMENT. Again, let $s = \mu x[x; a \cup b]$. Then $\mathcal{O}_0[\![s]\!] = b \cdot a^* \cup \{\perp\}$. This set is not closed, since if we take $u_n = b \cdot a^n \in \mathcal{O}_0[\![s]\!]$, then $\lim_n u_n = b \cdot a^\omega \notin \mathcal{O}_0[\![s]\!]$.

Note that the u_n are produced by paths

$$\pi_n\colon \langle s, \varepsilon \rangle \to \cdots \to \langle a^n, b \rangle \qquad \text{(as in counterexample after 2.4.9)}$$

$$\cdots \to b \cdot a^n \qquad \text{(by } n\ a\text{-transitions)}.$$

But the initial parts of these paths, up to the first appearance of b on the r.h.s., are all *different*, so there is no limiting path (in the metric $\tilde{d}$)!

Linking Operational and Syntactic Approximation

Iterated Substitution; *Depth of a μ-Statement in a Path.* From now on, we will concentrate on a specific μ-statement, $\bar{\mu} = \mu x[\bar{t}(x)]$ which, by our general assumption, is syntactically closed and guarded).

We define the *n-fold substitution* in $\bar{t}(x)$ by a sequence of statements $\bar{t}^n(x)$ ($n = 0, 1, 2, \ldots$), where

$$\bar{t}^0(x) = x$$

$$\bar{t}^{n+1}(x) = \bar{t}(\bar{t}^n(x)) \ (= \bar{t}^n(\bar{t}(x))).$$

Since $\bar{\mu}$ is syntactically closed, $\bar{t}(x)$ contains at most x free. However, there may be many occurrences of x in $\bar{t}$ (none of them exposed!). If, for example $\bar{t}(x) = \bar{t}(\underline{x}, \underline{x}, \underline{x})$ (3 occurrences of x), then $\bar{t}^2(x) = \bar{t}(\underline{\bar{t}(\underline{x}, \underline{x}, \underline{x})}, \underline{\bar{t}(\underline{x}, \underline{x}, \underline{x})}, \underline{\bar{t}(\underline{x}, \underline{x}, \underline{x})})$.

We call a transition involving an occurrence of $\bar{\mu}$ a $\bar{\mu}$-transition.

Now consider a path from some statement s_0 containing $\bar{\mu}$:

$$\pi\colon \langle s_0, \varepsilon \rangle \to \langle s_1, w_1 \rangle \to \cdots \to \langle s_n, w_n \rangle \to \cdots.$$

We define the *depth* of an occurrence of $\bar{\mu}$ in s_n (in π), by induction on n.

Basis ($n = 0$). Every occurrence of $\bar{\mu}$ in s_0 has depth 0.

Induction step ($n \to n+1$). Given any occurrence of $\bar{\mu}$ in s_n of depth d, any passive successor (cf. Passive and Active Successors above) of this occurrence also has depth d; all *active successors* have depth $d+1$.

In other words, the depth of an occurrence of $\bar{\mu}$ in π counts the number of $\bar{\mu}$-transitions involving ancestors of that occurrence.

Syntactic Bottom Symbol; Truncation of a Path. As a technical aid, we adjoin the symbol "Ω" to the syntax of $\mathfrak{L}_0$, and the transition rules (actually axioms):

$$\begin{aligned}
(\Omega_1)\colon\ & \langle \Omega; s, w\rangle \to \langle \Omega, w\rangle \\
& \langle \Omega \parallel s, w\rangle \to \langle \Omega, w\rangle \\
& \langle s \parallel \Omega, w\rangle \to \langle \Omega, w\rangle \\
(\Omega_2)\colon\ & \langle \Omega, w\rangle \to w \cdot \perp
\end{aligned}$$

to T_0. We also define $\mathscr{D}_0[\![\Omega]\!](\gamma) = \{\perp\}$. This symbol will not appear in our final result (2.4.1).

We now define the *n-truncation* of a path π (w.r.t. $\bar{\mu}$), $\mathbf{trunc}_n(\pi)$. This is the path π' formed by "truncating π at a depth of n," by

(1) replacing all occurrences of $\bar{\mu}$ in π, of depth n, by Ω, and

(2) replacing the first transition involving an occurrence of $\bar{\mu}$ of depth n:

$$\pi\colon \cdots \to \langle s(\underline{\bar{\mu}}), w\rangle \xrightarrow{①} \langle s(\underline{\bar{t}(\bar{\mu})}), w\rangle \to \cdots$$

by transitions involving Ω:

$$\pi'\colon \cdots \to \langle s(\underline{\Omega}), w\rangle \xrightarrow{②}{}^{*} \langle \Omega, w\rangle \xrightarrow{\Omega_2} w\perp,$$

thus terminating π'.

The transitions in the sequence ② are deduced from instances of axiom (Ω_1) by successive applications of the composition and shuffle rules, paralleling the deduction of ① from an instance of the recursion rule.

Note that step (1) in the construction of $\mathbf{trunc}_n(\pi)$ above has the effect of replacing $\bar{\mu}$-truncations, involving occurrences of $\bar{\mu}$ of depth $n-1$, by "*non-standard μ-transitions*," in which the active successor of $\bar{\mu}$ is not $\bar{t}(\bar{\mu})$ but $\bar{t}(\Omega)$.

Next we give a notation for the word associated with the n-truncation of π:

$$\mathbf{word}_n(\pi) = \mathbf{word}(\mathbf{trunc}_n(\pi))$$

and finally define the *n-approximation* of the operational meaning of s_0:

$$\mathscr{O}_0^{(n)}[\![s_0]\!] = \{\mathbf{word}_n(\pi) \mid \pi \in \mathbf{path}(s_0)\}.$$

The following theorem shows that for $\mathscr{O}_0$, *operational approximation* (via n-truncation) coincides with *syntactic approximation* (via n-fold substitution). This result facilitates the subsequent considerations on metric limits.

2.4.11. THEOREM. $\mathcal{O}_0^{(n)}[\![\bar{\mu}]\!] = \mathcal{O}_0[\![\bar{t}^{(n)}(\Omega)]\!]$ *for* $n = 0, 1, 2, \ldots$.

Proof. We will actually prove, more generally: for any statement $s_0(x)$ (with only x free, and not containing Ω),

$$\mathcal{O}_0^{(n)}[\![s_0(\bar{\mu})]\!] = \mathcal{O}_0[\![s_0(\bar{t}^{(n)}(\Omega))]\!].$$

(1) $\subseteq$ (This is relatively straightforward.): Let $\pi \in \mathbf{path}_n(s_0(\bar{\mu}))$. We must find $\pi' \in \mathbf{path}(s_0(\bar{t}^{(n)}(\Omega)))$ such that $\mathbf{word}(\pi') = \mathbf{word}(\pi)$. Note that each occurrence of $\bar{\mu}$ in π has depth $< n$ (by definition of $\mathbf{path}_n$).
Form π' from π in two steps:

(a) *Replace* each occurrence of $\bar{\mu}$ of depth $d(<n)$ by $\bar{t}^{n-d}(\Omega)$.

(b) Consider a $\bar{\mu}$-transition in π:

$$\pi\colon \cdots \to \langle s(\bar{\mu}), w\rangle \to \langle s(\bar{t}(\bar{\mu})), w\rangle \to \cdots.$$

Actually, s may contain a number (say m) of occurrences of $\bar{\mu}$: $s(\bar{\mu}) = s(\underline{\bar{\mu}}, \underline{\bar{\mu}}, \ldots, \underline{\bar{\mu}})$. Suppose w.l.o.g. that the *first* of these occurrences shown is involved in the $\bar{\mu}$-transition:

$$\begin{aligned}\pi\colon \cdots &\to \langle s(\underline{\bar{\mu}}, \underline{\bar{\mu}}, \ldots, \underline{\bar{\mu}}), w\rangle \\ &\to \langle s(\underline{\bar{t}(\bar{\mu})}, \underline{\bar{\mu}}, \ldots, \underline{\bar{\mu}}), w\rangle \\ &\to \cdots.\end{aligned}$$

Suppose that the m occurrences of $\bar{\mu}$ shown on the l.h.s. of this transition have depths $d_1, \ldots, d_m$ $(<n)$. Then all occurrences of $\bar{\mu}$ in $\underline{\bar{t}(\bar{\mu})}$ have depth $d_1 + 1$ (they are the *active successors* of the first $\underline{\bar{\mu}}$ on the l.h.s.), and the remaining $\underline{\bar{\mu}}$'s on the r.h.s. (still) have depths $d_2, \ldots, d_m$ (they are the *passive successors* of the corresponding $\underline{\bar{\mu}}$'s on the l.h.s.). Then from step (a), π' is so far (putting $e_i = n - d_i$):

$$\begin{aligned}\pi'\colon \cdots &\to \langle s(\underline{\bar{t}^{e_1}(\Omega)}, \underline{\bar{t}^{e_2}(\Omega)}, \ldots, \underline{\bar{t}^{e_m}(\Omega)}), w\rangle \\ &\to \langle s(\underline{\bar{t}(\bar{t}^{e_1-1}(\Omega))}, \underline{\bar{t}^{e_2}(\Omega)}, \ldots, \underline{\bar{t}^{e_m}(\Omega)}), w\rangle \\ &\to \cdots.\end{aligned}$$

Now *collapse* the above "identity transition" into a single configuration

$$\pi'\colon \cdots \to \langle s(\cdots), w\rangle \to \cdots.$$

(2) $\supseteq$ (Trickier, here we use the Basic Lemma, and the assumption that $\bar{\mu}$ is guarded.): Let $\pi' \in \mathbf{path}(s_0(\bar{t}^n(\Omega)))$. We want to find a path $\pi \in \mathbf{path}_n(s_0(\bar{\mu}))$ with the same associated word. Roughly, we replace occurrences of $\bar{t}^e(\Omega)$ $(0 < e \leqslant n)$ in π' by $\bar{\mu}$ (of depth $n - e$, as it turns out). We will construct π step by step from π'. With each configuration $\langle s, w\rangle$ in π' will be associated a finite sequence

$(\underline{\bar{t}^{e_1}(\Omega)}, ..., \underline{\bar{t}^{e_m}(\Omega)})$ $(0 < e_i < n)$ of occurrences of substatements of s. Then π is extended by adjoining a configuration $\langle s', w \rangle$, where s' is formed from s by replacing $\bar{t}^{e_i}(\Omega)$ by $\underline{\bar{\mu}}$ (of depth $n - e_i$). In detail, the construction of π from π' proceeds as follows. It starts in the obvious way (displaying the different occurrences of $\bar{t}^n(\Omega)$ in s_0):

$$\pi' \colon \langle s_0(\bar{t}^n(\Omega), ..., \bar{t}^n(\Omega)), \varepsilon \rangle \to \cdots$$

$$\pi \colon \langle s_0(\underline{\bar{\mu}}, ..., \underline{\bar{\mu}}), \varepsilon \rangle \to \cdots .$$

Now assume (inductively) that π has been constructed from π' up to a certain stage:

$$\pi' \colon \cdots \to \langle s(\underline{\bar{t}^{e_1}(\Omega)}, ..., \underline{\bar{t}^{e_m}(\Omega)}), w \rangle \xrightarrow{①} \cdots$$

$$\pi \colon \cdots \to \langle s(\underline{\bar{\mu}}, ..., \underline{\bar{\mu}}), w \rangle,$$

where $(\underline{\bar{t}^{e_1}(\Omega)}, ..., \underline{\bar{t}^{e_m}(\Omega)})$ is the sequence associated with the configuration in π', and (by assumption) each $\underline{\bar{t}^{e_i}(\Omega)}$ has been replaced in π by an occurrence of $\bar{\mu}$ of depth $n - e_i$ $(1 \leqslant i \leqslant m)$. Now consider the next transition ① in π'. There are two possibilities:

(a) Transition ① does not affect any of the $\underline{t^{e_i}(\Omega)}$ $(i = 1, ..., m)$. Then the construction of π is extended another step in the obvious way.

(b) Transition ① affects one of the $\underline{t^{e_i}(\Omega)}$, say (w.l.o.g.) $\underline{t^{e_1}(\Omega)}$. There are two subcases:

(i) $e_1 > 1$. Now since $\bar{\mu}$ is guarded, the occurrences of x are not exposed in $\bar{t}(x)$, hence the occurrences of $\bar{t}^{e_i - 1}(\Omega)$ are not exposed in $\bar{t}(\bar{t}^{e_1 - 1}(\Omega)) = \bar{t}^{e_1}(\Omega)$, and hence (by the Lemma 2.4.6 on transitivity of exposure) also not in $s(\underline{\bar{t}(\bar{t}^{e_1 - 1}(\Omega))}, ...)$. Hence by the Basic Lemma, they are *passive* in ①, and so, by the Lemma 2.4.3 on the substitution of passive free substatements (note that the $\underline{t^{e_i - 1}(\Omega)}$ are syntactically closed, and hence free in s), ① has the form:

$$\begin{aligned}
\pi' \colon \cdots \longrightarrow\ & \langle s(\underline{\bar{t}^{e_1}(\Omega)}, \underline{\bar{t}^{e_2}(\Omega)}, ..., \underline{\bar{t}^{e_m}(\Omega)}), w \rangle \\
&= \langle s(\underline{\bar{t}(\bar{t}^{e_1 - 1}(\Omega))}, \underline{\bar{t}^{e_2}(\Omega)}, ..., \underline{\bar{t}^{e_m}(\Omega)}), w \rangle \\
&\xrightarrow{①} \langle s(\underline{t'(\bar{t}^{e_1 - 1}(\Omega))}, \underline{\bar{t}^{e_2}(\Omega)}, ..., \underline{\bar{t}^{e_m}(\Omega)}), w \rangle \\
&\longrightarrow \cdots .
\end{aligned}$$

The sequence associated with this last configuration is the sequence of occurrences of $\bar{t}^{e_1 - 1}(\Omega)$ (shown in the context $t'(\cdot)$), followed by $\underline{\bar{t}^{e_2}(\Omega)}, ..., \underline{\bar{t}^{e_m}(\Omega)}$ as before.

Now the construction of π proceeds with a $\bar{\mu}$-transition, followed by a transition *corresponding to* ① (as given by the lemma on the substitution of passive substatements):

$$\begin{aligned}\pi\colon \cdots &\to \langle s(\underline{\bar{\mu}}, \underline{\bar{\mu}}, ..., \underline{\bar{\mu}}), w\rangle \\ &\to \langle s(\underline{\bar{t}(\bar{\mu})}, \underline{\bar{\mu}}, ..., \underline{\bar{\mu}}), w\rangle \\ &\to \langle s(\underline{t'(\bar{\mu})}, \underline{\bar{\mu}}, ..., \underline{\bar{\mu}}), w\rangle.\end{aligned}$$

(ii) $e_1 = 1$. Again, by the Basic Lemma, transition ① has the form:

$$\begin{aligned}\pi'\colon \cdots &\longrightarrow \langle s(\underline{\bar{t}(\Omega)}, \underline{\bar{t}^{e_2}(\Omega)}, ..., \underline{\bar{t}^{e_m}(\Omega)}), w\rangle \\ &\xrightarrow{①} \langle s(\underline{t'(\Omega)}, \underline{\bar{t}^{e_2}(\Omega)}, ..., \underline{\bar{t}^{e_m}(\Omega)}), w\rangle \\ &\longrightarrow \cdots.\end{aligned}$$

The sequence associated with this last configuration is now $(\underline{\bar{t}^{e_2}(\Omega)}, ..., \underline{\bar{t}^{e_m}(\Omega)})$, and the construction on π proceeds with a non-standard $\bar{\mu}$-transition (converting $\bar{\mu}$ to $\bar{t}(\Omega)$: note that this occurrence of $\bar{\mu}$ has depth $n-1$), followed, again, by a transition corresponding to ①:

$$\begin{aligned}\pi\colon \cdots &\to \langle s(\underline{\bar{\mu}}, \underline{\bar{\mu}}, ..., \underline{\bar{\mu}}), w\rangle \\ &\to \langle s(\underline{\bar{t}(\Omega)}, \underline{\bar{\mu}}, ..., \underline{\bar{\mu}}), w\rangle \\ &\to \langle s(\underline{t'(\Omega)}, \underline{\bar{\mu}}, ..., \underline{\bar{\mu}}), w\rangle.\end{aligned}$$

To show that $\pi \in \mathbf{path}_n(s_0(\bar{\mu}))$: notice that Ω is introduced into π (only) from non-standard $\bar{\mu}$-transitions, involving occurrences of $\bar{\mu}$ of depth n. Now we can construct a path from π, such that π is its n-truncation, by:

(1) replacing all non-standard $\bar{\mu}$-transitions by standard $\bar{\mu}$-transitions,
(2) removing all Ω_1-transitions,
(3) replacing the Ω_2-transition (assuming there is one!) by a $\bar{\mu}$-transition, and then continuing the path arbitrarily.

We leave the details to the reader. ∎

Although guardedness was used in this proof (via the Basic Lemma), we cannot find a counterexample to the theorem by dropping this assumption.

Taking Limits

2.4.12. LEMMA. *Consider a path from* $\bar{\mu}$:

$$\langle \bar{\mu}, \varepsilon \rangle \to \cdots \to \langle s, w \rangle \xrightarrow{①} \langle s', w' \rangle \to \cdots$$
$$\cdots \to \langle s'', w'' \rangle \xrightarrow{②} \langle s''', w''' \rangle \to \cdots,$$

where transition ① involves an occurrence of $\bar{\mu}$ *of depth d and transition ② involves an occurrence of a* descendent *of* $\bar{\mu}$ *of depth* $d+1$. *Then* w'' *is* longer *than* w'.

Proof. By the Basic Lemma, only *exposed* occurrences of $\bar{\mu}$ can be involved in a $\bar{\mu}$-transition. Since $\bar{\mu}$ is guarded, no successor of this occurrence of $\bar{\mu}$ in ① is exposed, and, in fact, no descendant of this occurrence of $\bar{\mu}$ is exposed, as long as there are only μ- and $\cup$-transitions (the proof of which is left to the reader).

Hence, before transition ②, there must be at least one a-transition, which will be lengthen the word. ∎

Let us write $|w|$ to denote the length of the word w.

2.4.13. COROLLARY. *If, in a path from* $\bar{\mu}$:

$$\langle \bar{\mu}, \varepsilon \rangle \to \cdots \to \langle s, w \rangle \xrightarrow{①} \langle s', w' \rangle \to \cdots,$$

the transition ① involves an occurrence of $\bar{\mu}$ *of depth d, then* $|w| \geqslant d$.

COUNTEREXAMPLE FOR AN UNGUARDED STATEMENT. Let $s = \mu x[x; a \cup b]$. Taking the sequence described in the counterexample following 2.4.8, with transitions involving μ-statements of arbitrary depth, we remain with the empty word.

2.4.14. LEMMA. *The sequence* $(\mathcal{O}_0^{(n)}[\![\bar{\mu}]\!])_n$ *is a Cauchy sequence in* $(\mathbb{S}_c, \hat{d})$ (*see Section* 2.3).

Proof. This follows from the fact that for all $\pi \in \mathbf{path}(\bar{\mu})$, $\mathbf{word}_n(\pi) \to \mathbf{word}(\pi)$ as $n \to \infty$, *uniformly* in n (i.e., independent of π) in A^{st}. More precisely, by Corollary 2.4.13, for all $\pi \in \mathbf{path}(\bar{\mu})$, n, k:

$$d(\mathbf{word}_n(\pi), \mathbf{word}_{n+k}(\pi)) \leqslant 2^{-n}.$$

Hence for all n, k:

$$\hat{d}(\mathcal{O}_0^{(n)}[\![\bar{\mu}]\!], \mathcal{O}_0^{(n+k)}[\![\bar{\mu}]\!]) \leqslant 2^{-n}. \quad ∎$$

2.4.15. THEOREM. $\mathcal{O}_0[\![\bar{\mu}]\!] = \lim_n \mathcal{O}_0^{(n)}[\![\bar{\mu}]\!]$.

Proof. By Lemma 2.4.14, the limit on the r.h.s. exists. It is equal to (see [Ha])

$$\{\lim_n w_n \mid (w_n)_n \text{ is a CS in } (A^{\mathrm{st}}, d) \text{ and } w_n \in \mathcal{O}_0^{(n)}[\![\bar{\mu}]\!]\}.$$

We will show that each side is a subset of the other:

(1) $\subseteq$: Clear, since for all $\pi \in \mathbf{path}(\bar{\mu})$, $\mathbf{word}(\pi) = \lim_n(\mathbf{word}_n(\pi))$.

(2) $\supseteq$: Let $w = \lim_n w_n$, where $w_n \in \mathcal{O}_0^{(n)}[\![\bar{\mu}]\!]$. For all n, there exists $v_n \in \mathcal{O}_0[\![\bar{\mu}]\!]$ which extends w_n and such that $w = \lim_n v_n$ also. (Take $v_n = \mathbf{word}(\pi)$ for any π such that $w_n = \mathbf{word}_n(\pi)$.) Then also $w = \lim_n v_n$. Since $\mathcal{O}_0[\![\bar{\mu}]\!]$ is closed (by Theorem 2.4.10), $w \in \mathcal{O}_0[\![\bar{\mu}]\!]$. ∎

We can now state the main fact about recursion used in proving Theorem 2.4.1.

2.4.16. Corollary. $\mathcal{O}_0[\![\bar{\mu}]\!] = \lim_n \mathcal{O}_0[\![\bar{t}^n(\Omega)]\!]$.

Proof. By Theorems 2.4.15 and 2.4.11. ∎

Simple Example. Let $\bar{t}(x) = a \cdot x \cup b$, $\bar{\mu} = \mu x[\bar{t}(x)]$. For all n, $\mathcal{O}_0[\![\bar{t}^n(\Omega)]\!] = \mathcal{O}_0^{(n)}[\![\bar{\mu}]\!] = \{a^i b \mid 0 \leqslant i < n\} \cup \{a^n \perp\}$. This is a CS of sets, with limit $a^*b \cup \{a^\omega\}$, which is equal to $\mathcal{O}_0[\![\bar{\mu}]\!]$, as promised by the theorem.

Counterexample for an Unguarded Statement. Let $\bar{t}(x) = x \cdot a \cup b$, $\bar{\mu} = \mu x[\bar{t}(x)]$. For all n, $\mathcal{O}_0[\![\bar{t}^n(\Omega)]\!] = \mathcal{O}_0^{(n)}[\![\bar{\mu}]\!] = \{ba^i \mid 0 \leqslant i < n\} \cup \{\perp\}$. This is again a CS, with limit $ba^* \cup \{ba^\omega, \perp\}$. However this limit is *not* equal to

$$\mathcal{O}_0[\![\bar{\mu}]\!] = ba^* \cup \{\perp\},$$

which is not even a closed set!

Proof of Theorem 2.4.1

Finally, we are ready to prove that

$$\mathcal{O}_0[\![s]\!] = \mathcal{D}_0[\![s]\!].$$

Since we are assuming that s is syntactically closed, we do not display the environment with $\mathcal{D}_0[\![s]\!]$ above. However, in order to prove it, we must prove a more general result, in which s is not necessarily syntactically closed (but still guarded!), namely

$$\mathcal{O}_0[\![s[t_i/x_i]_{i=1}^k]\!] = \mathcal{D}_0[\![s]\!](\gamma\langle X_i/x_i\rangle_{i=1}^k), \tag{2.12}$$

where

(a) $\mathbf{var}(s) \subseteq \{x_1, ..., x_k\}$,

(b) t_i is syntactically closed for $i = 1, ..., k$,

(c) $\mathcal{O}_0[\![t_i]\!] = X_i$ for $i = 1, ..., k$.

The theorem is then (of course) a special case of (2.12) with $k = 0$.

The proof of (2.12) is by induction on the structure of s. All cases are straightforward (using Theorem 2.4.2) except for $s = \mu y[s_0]$ (assuming w.l.o.g. $y \neq x_1, ..., x_k$). Now

$$\begin{aligned}
&\mathcal{O}_0[\![\mu y[s_0][t_i/x_i]_{i=1}^k]\!] \\
&\quad = \mathcal{O}_0[\![\mu y[s_0[t_i/x_i]_{i=1}^k]]\!] \quad \text{(assuming w.l.o.g. no variable clashes)} \\
&\quad = \lim_n \mathcal{O}_0[\![r_n]\!] \quad \text{(by Corollary 2.4.16)},
\end{aligned}$$

where

$$\begin{aligned}
r_0 &= \Omega, \\
r_{n+1} &= s_0[t_i/x_i]_{i=1}^k \, [r_n/y],
\end{aligned}$$

and

$$\mathcal{D}_0[\![\mu y[s_0]]\!](\gamma\langle X_i/x_i\rangle_{i=1}^k) = \lim_n Y_n,$$

where

$$\begin{aligned}
Y_0 &= \{\bot\}, \\
Y_{n+1} &= \mathcal{D}_0[\![s_0]\!](\gamma\langle X_i/x_i\rangle_{i=1}^k \, \langle Y_n/y\rangle).
\end{aligned}$$

So it is sufficient to show

$$\mathcal{O}_0[\![r_n]\!] = Y_n \tag{2.13}$$

for all n, by induction on n.

For $n = 0$, this is clear. Assume (2.13). We must show $\mathcal{O}_0[\![r_{n+1}]\!] = Y_{n+1}$, i.e.,

$$\mathcal{O}_0[\![s_0[t_i/x_i]_{i=1}^k \, [r_n/y]]\!] = \mathcal{D}_0[\![s_0]\!](\gamma\langle X_i/x_i\rangle_{i=1}^k \, \langle Y_n/y\rangle).$$

But this follows by the main induction hypothesis on (2.12), with s_0 replacing s and $k+1$ replacing k, and using (2.13) to establish the $(k+1)$th part of condition (c). ∎

3. The Language $\mathfrak{L}_1$: Synchronization Merge and Local Nondeterminacy

For $\mathfrak{L}_1$ we introduce some structure to the finite alphabet A. Let $C \subseteq A$ be a subset of so-called *communications*. From now on let c range over C, a over $A \backslash C$

and b over A. Similarly to CCS [Mi] or CSP [Ho] we stipulate a bijection $^{-}: C \to C$ with $\bar{\bar{c}} = c$ which for every $c \in C$ yields a *matching communication* $\bar{c}$. There is a special action $\tau \in A \backslash C$ denoting the result of a synchronization of c with $\bar{c}$ [Mi].

As syntax for $s \in \mathfrak{L}_1$ we now give:

$$s::= a \mid c \mid s_1 ; s_2 \mid s_1 \cup s_2 \mid s_1 \| s_2 \mid x \mid \mu x[s].$$

Apart from a distinction between communications and ordinary elementary actions, the syntax of $\mathfrak{L}_1$ agrees with that of $\mathfrak{L}_0$. The difference between $\mathfrak{L}_1$ and $\mathfrak{L}_0$ lies in a more sophisticated interpretation of $s_1 \| s_2$ to be presented in the next subsection.

3.1. *The Transition System T_1*

The intuition about matching communications c and $\bar{c}$ is as follows: execution of c and $\bar{c}$ individually fails or deadlocks; only execution of the parallel composition of both succeeds. In other words c and $\bar{c}$ have to synchronize (see [Ho1]). The result of the synchronization will be denoted by the joint action τ. Thus in the simplest case a synchronization can be described by the transition

$$\langle c \| \bar{c}, w \rangle \to w \cdot \tau.$$

What makes synchronization of matching communications difficult to describe are the synchronization-transitions of the form

$$\langle s_1 \| s_2, w \rangle \to \langle s_1' \| s_2', w \cdot \tau \rangle, \tag{1}$$

where c appears at a suitable position in s_1 and likewise $\bar{c}$ in s_2.

Milner [Mi] and Plotkin [Pl2] solved this difficulty by introducing virtual transitions for c and $\bar{c}$. In our setting we would have

$$\langle c, w \rangle \to w \cdot c \qquad \text{and} \qquad \langle \bar{c}, w \rangle \to w \cdot \bar{c}. \tag{2}$$

(In fact, [Mi] and [Pl2] use labelled transitions, but we may "code" these labels into the words w.) Adding the axioms (2) to the previous transition system T_0, all transitions of the form (1) can now be generated using only one further rule, viz.

$$\frac{\langle s_1, w \rangle \to \langle s_1', w \cdot c \rangle, \langle s_2, w \rangle \to \langle s_2', w \cdot \bar{c} \rangle}{\langle s_1 \| s_2, w \rangle \to \langle s_1' \| s_2', w \cdot \tau \rangle}. \tag{3}$$

This is a simple and elegant solution, but we do have reservations against it because the virtual transitions (2) violate the idea that individual communications deadlock.

In contrast, we follow Apt [Ap2] and formalize only the successful steps of statements arising from the execution of elementary actions on the synchronization of matching communications. In particular, we do not use virtual transitions (2) for the individual communications. Instead the synchronization transitions (1) will be determined directly by induction on the structure of s_1 and s_2. Individual communications c will fail immediately. This will be described by the transition

$$\langle c, w\rangle \rightarrow w \cdot \delta$$

where δ is a special symbol which may appear only at the end of words and thus signals failure or deadlock. (In Section 4 failure of individual communications c will be modelled even stronger, viz. by the absence of any transition!)

Though Apt [Ap2] deals only with local nondeterminism, iteration, and one level parallelism denoted by $[s_1 \| \cdots \| s_n]$, he uses infinitely many transition rules, e.g., one schematic rule

$$\frac{\langle s_i, w\rangle \rightarrow \langle s_i', w'\rangle}{\langle [s_1 \| \cdots \| s_i \| \cdots \| s_n], w\rangle \rightarrow \langle [s_1 \| \cdots \| s_i' \| \cdots \| s_n], w'\rangle}$$

for each $n \geqslant 2$ and $i \in \{1, ..., n\}$ (cf. the notion of schematic rule explained in Section 2.1). Does the omission of virtual transitions necessitate infinitely many rules? The answer is no; we show that finitely many schematic axioms and rules are sufficient even when dealing with full recursion, nested parallelism, and (in Section 4) with global nondeterminism.

Formally, let $\delta \notin A \cup \{\perp\}$ a new element satisfying $\delta \cdot w = \delta$ for all w. The set of streams or words is extended to

$$A^{\mathrm{st}}(\delta) = A^{\mathrm{st}} \cup A^* \cdot \{\delta\}$$

with u, v, w now ranging over $A^{\mathrm{st}}(\delta)$.

In the transition system T_1 (and in subsequent systems) we shall use the notation

$$\begin{array}{c} 1 \rightarrow 1' \\ \hline 2 \rightarrow 2' \\ \vdots \\ n \rightarrow n' \end{array}$$

as shorthand for a number of rules $\frac{1 \rightarrow 1'}{2 \rightarrow 2'}, ..., \frac{1 \rightarrow 1'}{n \rightarrow n'}$.

The system T_1 consists of all axioms and rules of T_0 extended with

$$\langle s, w\rangle \rightarrow w \qquad \text{for} \quad w \in A^{\omega} \cup A^* \cdot \{\delta, \perp\},$$

and for $w \in A^*$ with (*communication*)

$$\langle c, w\rangle \rightarrow w \cdot \delta$$

(an individual communication fails); (*synchronization*)

$$\langle c \| \bar{c}, w\rangle \rightarrow w \cdot \tau;$$

(*synchronization in a context*)

$$\frac{\langle s_1 \| s_2, w\rangle \to \langle s_1' \| s_2', w\tau\rangle}{\begin{array}{c}\langle (s_1; s) \| s_2, w\rangle \to \langle (s_1'; s) \| s_2', w\tau\rangle \\ \langle (s_1; s) \| s_2, w\rangle \to \langle (s_1'; s) \| s_2', w\tau\rangle \\ \langle (s_1 \| s) \| s_2, w\rangle \to \langle (s_1' \| s) \| s_2', w\tau\rangle \\ \langle s_1 \| (s_2; s), w\rangle \to \langle s_1' \| (s_2'; s), w\tau\rangle \\ \langle s_1 \| (s_2 \| s), w\rangle \to \langle s_1' \| (s_2' \| s), w\tau\rangle \\ \langle s_1 \| (s \| s_2), w\rangle \to \langle s_1' \| (s \| s_2'), w\tau\rangle,\end{array}}$$

where s_1' results from s_1 by replacing an occurrence of a communication c by E and s_2' from s_2 by replacing an occurrence of a matching communication $\bar{c}$ by E. In particular, s_1' or s_2' or both may be E.

Let us briefly analyze the system T_1. As for T_0 (cf. Section 2.4, Types of Transitions) every transition rule of T_1 has only one premise. Thus any deduction starts from a unique axiom (Ax) so that the deduced transition will be called an (Ax)-*transition.* For example, for a *communication-transition* the deduction starts from the axiom

$$\langle c, w\rangle \to w \cdot \delta$$

and for a *synchronization-transition* from the axiom

$$\langle c \| \bar{c}, w\rangle \to w \cdot \tau.$$

Consider now a transition of the form

$$\langle s_1 \| s_2, w\rangle \to \langle s_1' \| s_2', w'\rangle \tag{$*$}$$

for $s_1, s_2 \in \mathfrak{L}_2$ and $s_1', s_2' \in \mathfrak{L}_2 \cup \{E\}$. Then $(*)$ is called a *synchronization-transition between s_1 and s_2* if $(*)$ is a synchronization-transition in the above sense and if the condition of the synchronization rule holds; i.e. s_1' results from s_1 by replacing an occurrence of a communication c by E and s_2' from s_2 by replacing an occurrence of a matching communication $\bar{c}$ by E. Otherwise $(*)$ is called a *local transition.*

EXAMPLE. (1) $\langle c \| \bar{c}, w\rangle \to w\tau$ is a synchronization-transition between $s_1 = c$ and $s_2 = \bar{c}$. Note that here $s_1' = s_2' = E$. Thus the E-simplications explained in Section 2.1 yield $\langle E \| E, w\tau\rangle = w\tau$.

(2) $\langle (c; s_1') \| ((c \| \bar{c}); s'), w\rangle \to \langle s_1' \| (c; s'), w\tau\rangle$ is a synchronization-transition between $s_1 = c; s_1'$ and $s_2 = (c \| \bar{c}); s'$. Here $s_2' = c; s'$.

(3) $\langle (c; s_1') \| ((c \| \bar{c}); s'), w\rangle \to \langle (c; s_1') \| s', w\tau\rangle$ is a local transition involving only the second argument $s_2 = (c \| \bar{c}); s'$ of the top-level $\|$ operator.

In general, the synchronization axiom introduces the basic form of a synchronization-transition between two statements whereas successive applications of the synchronization rule generate all others. Finally, we remark that the Initial Step Lemma (2.1.1) stated for T_0 holds also for T_1 but now with $b \in A \cup \{\delta, \varepsilon\}$.

3.2. *The Operational Semantics* $\mathcal{O}_1$

Analogously to $\mathcal{O}_0$ we base an operational semantics $\mathcal{O}_1$ on T_1. $\mathcal{O}_1$ is a mapping $\mathcal{O}_1 \colon \mathfrak{L}_1 \to \mathbb{S}(\delta)$ with $\mathbb{S}(\delta) = \mathfrak{P}(A^{st}(\delta))$, and $\mathcal{O}_1[\![s]\!]$ is defined exactly the same way as $\mathcal{O}_0[\![s]\!]$ in Section 2.2.

EXAMPLES. $\mathcal{O}_1[\![c]\!] = \{\delta\}$, $\mathcal{O}_1[\![c \parallel \bar{c}]\!] = \{\delta, \tau\}$, $\mathcal{O}_1[\![(a; a') \cup (a; c)]\!] = \mathcal{O}_1[\![a; (a' \cup c)]\!] = \{aa', a\,\delta\}$.

Thus under $\mathcal{O}_1$, communications c always create failures—whether or not they can synchronize with a matching communication $\bar{c}$. Also the two statements $(a; a') \cup (a; c)$ and $a; (a' \cup c)$ obtain the same meaning under $\mathcal{O}_1$. This is characteristic of local nondeterminacy $s_1 \cup s_2$ where the choice of s_1 or s_2 is independent of the form of the other component s_2 or s_1, respectively. A more refined treatment will be provided in Section 4. We remark that the Definedness Lemma (2.2.1) and the Prolongation Lemma (2.2.2) of Section 2.2 hold also for $\mathcal{O}_1$. Note also that for $C = \emptyset$ the semantics $\mathcal{O}_1$ coincides with the previous $\mathcal{O}_0$.

Remark 1. It is possible to do away with occurrences of δ in sets $\mathcal{O}_1[\![s]\!]$ in case an alternative for the failure is available. Technically, this is achieved by imposing the axiom

$$\{\delta\} \cup X = X, \qquad X \neq \emptyset. \tag{3.3}$$

In the above example applying the axiom would turn the sets $\{\delta\}$, $\{\delta, \tau\}$, and $\{aa', a\,\delta\}$ into $\{\delta\}$, $\{\tau\}$, and $\{aa'\}$, respectively. (For the latter case we take $\{aa', a\,\delta\} = a \cdot (\{a'\} \cup \{\delta\}) = a \cdot \{a'\} = \{aa'\}$.) One might argue that imposing (3.3) throughout would be more in agreement with the intuitive understanding of communication. The reader is, of course, free to do this throughout Section 3. Our reason for not doing this is that our main result relating $\mathcal{O}_1$ and $\mathcal{D}_1$ does not depend on it. For both $\mathcal{O}_1$ and $\mathcal{D}_1$, (3.3) may or may not be imposed (simultaneously) without affecting the result of Section 3.4.

Remark 2. The elementary action τ plays no special role in either T_1 or $\mathcal{O}_1$ (nor in the definition of $\mathcal{D}_1$ which follows in a moment). Since τ does serve a special purpose in CCS (and many of the papers inspired by it) a comment may be in order here: We have chosen the notation in the axiom $\langle c \parallel \bar{c}, w\rangle \to w\tau$ to, indeed, follow the standard conventions. However, we have preferred not to include into our analysis of $\mathcal{O}_1$ (and $\mathcal{D}_1$) a treatment of the notions of observational equivalence (as in CCS) or abstraction (in the sense of ACP_τ, see [BK2]). Apart from the obvious justification that we do not want to further extend our already long paper (and that a substantial part of the theory of CCS in [Mi] is developed as well *before* issues of abstraction are addressed), let us point out that such τ-abstraction in the LT framework may be imposed, a posteriori, upon both (or none) of the outcomes of $\mathcal{O}_1[\![s]\!]$ and $\mathcal{D}_1[\![s]\!]$ (just as in Remark 1 above). One obtains the desired abstraction by equating τ^n, $n \geq 1$, with ε and τ^ω with $\perp$. Of course, this is no longer so simple for the BT framework, and we refer to Section 4 on $\mathfrak{L}_2$ for a remark on the situation with respect to τ-abstraction in the latter setting.

3.3. *The Denotational Semantics* $\mathcal{D}_1$

This is as in Section 2.3, but extended/modified as shown below: First, we refine the definition of $\|: S_c(\delta) \times \mathbb{S}_c(\delta) \to \mathbb{S}_c(\delta)$ as follows:

1. For $X, Y \subseteq A^* \cup A^* \cdot \{\perp, \delta\}$ we define

$$X \| Y = (X \mathbin{\|\!\!\_} Y) \cup (Y \mathbin{\|\!\!\_} X) \cup (X \mid Y),$$

where

(i) $X \mathbin{\|\!\!\_} Y = \bigcup \{u \mathbin{\|\!\!\_} Y\colon u \in X\}$, $\perp \mathbin{\|\!\!\_} Y = \{\perp\}$, $\delta \mathbin{\|\!\!\_} Y = \{\delta\}$, $\varepsilon \mathbin{\|\!\!\_} Y = Y$, $(b \cdot w) \mathbin{\|\!\!\_} Y = b \cdot (\{w\} \| Y)$.

(ii) $X \mid Y = \bigcup \{u \mid v\colon u \in X,\ v \in Y\}$, where $(c \cdot u_1) \mid (\bar{c} \cdot v_1) = \tau(\{u_1\} \| \{v_1\})$ and $u \mid v = \varnothing$ for u, v not of such a form.

2. For X or Y with infinite words we define

$$X \| Y = \lim_n \,(X(n) \| Y(n)),$$

where $X(n)$, $Y(n)$ are, as before, the sets of all n-prefixes of elements in X and Y. (This definition of $X \| Y$ is from [BK].)

The definition of $\mathcal{D}_1$ is now as follows: Let $\Gamma_1 = \mathbf{stmv} \to \mathbb{S}_c(\delta)$ and let $\gamma \in \Gamma_1$. We define

$$\mathcal{D}_1\colon \text{guarded } \mathfrak{L}_1 \to (\Gamma_1 \to \mathbb{S}_c(\delta))$$

by the clauses

$$\mathcal{D}_1[\![a]\!](\gamma) = \{a\} \qquad \text{for} \quad a \in A \backslash C,$$

$$\mathcal{D}_1[\![c]\!](\gamma) = \{c\} \qquad \text{for} \quad c \in C,$$

$$\mathcal{D}_1[\![s_1 \ \mathbf{op}\ s_2]\!](\gamma) = \mathcal{D}_1[\![s_1]\!](\gamma)\ \mathbf{op}^{\mathcal{D}_1}\ \mathcal{D}_1[\![s_2]\!](\gamma)$$

$$\text{for } \mathbf{op} \in \{\delta, \cup, \|\},\ ;^{\mathcal{D}_1} = \cdot,\ \cup^{\mathcal{D}_1} = \cup,\ \|^{\mathcal{D}_1} = \|,$$

$$\mathcal{D}_1[\![x]\!](\gamma) = \gamma(x),$$

$$\mathcal{D}_1[\![\mu x[s]]\!](\gamma) = \lim_i X_i, \qquad \text{where} \quad X_0 = \{\perp\} \text{ and}$$

$$X_{i+1} = \mathcal{D}_1[\![s]\!](\gamma\langle X_i/x\rangle).$$

Thus, apart from the clause for c, $\mathscr{D}_1$ is as $\mathscr{D}_0$ but for the refinement of $\|^{\mathscr{D}_1}$ with respect to $\|^{\mathfrak{T}_0}$.

3.4. *Relationship between $\mathcal{O}_1$ and $\mathscr{D}_1$*

Here we do *not* simply have that

$$\mathcal{O}_1[\![s]\!] = \mathscr{D}_1[\![s]\!] \tag{3.4}$$

holds for all guarded statements $s \in \mathfrak{L}_1$. As a counterexample take $s = c$. Then $\mathcal{O}_1[\![c]\!] = \{\delta\}$ but $\mathscr{D}_1[\![s]\!] = \{c\}$. Even worse, we can state:

3.4.1. THEOREM. *There does not exist any denotational (implying compositional) semantics $\mathscr{D}$ satisfying* (3.4).

The proof is based on:

3.4.2. LEMMA. *$\mathcal{O}_1$ does not behave compositionally over $\|$; i.e., there exists no "semantic" operator*

$$\|^{\mathscr{S}}\colon \mathbb{S}(\delta) \times \mathbb{S}(\delta) \to \mathbb{S}(\delta)$$

such that

$$\mathcal{O}_1[\![s_1 \| s_2]\!] = \mathcal{O}_1[\![s_1]\!] \|^{\mathscr{S}} \mathcal{O}_1[\![s_2]\!]$$

holds for all (guarded) $s_1, s_2 \in \mathfrak{L}_1$).

Proof. Consider $s_1 = c$ and $s_2 = \bar{c}$ in $\mathfrak{L}_1$. Then $\mathcal{O}_1[\![s_1]\!] = \mathcal{O}_1[\![s_2]\!] = \{\delta\}$. Suppose now that $\|^{\mathscr{S}}$ exists. Then $\{\delta\} = \mathcal{O}_1[\![s_1 \| s_1]\!] = \mathcal{O}_1[\![s_1]\!] \|^{\mathscr{S}} \mathcal{O}_1[\![s_1]\!] = \mathcal{O}_1[\![s_1]\!] \|^{\mathscr{S}} \mathcal{O}_1[\![s_2]\!] = \mathcal{O}_1[\![s_1 \| s_2]\!] = \{\delta, \tau\}$. Contradiction. ∎

We remedy this not by redefining T_1 (which adequately captures the operational intuition for $\mathfrak{L}_1$), but rather by introducing an *abstraction operator* $\alpha_1\colon \mathbb{S}(\delta) \to \mathbb{S}(\delta)$ such that

$$\mathcal{O}_1[\![s]\!] = \alpha_1(\mathscr{D}_1[\![s]\!]) \tag{3.5}$$

holds for guarded $s \in \mathfrak{L}_1$. We take $\alpha_1 = \mathbf{restr}_{\mathbb{S}}$ which for $W \in \mathbb{S}(\delta)$ is defined by

$$\begin{aligned} \mathbf{restr}_{\mathbb{S}}(W) = {} & \{w \mid w \in W \text{ does not contain any } c \in C\} \\ & \cup \{w \cdot \delta \mid \exists c' \in C,\ w' \in A^{\mathrm{st}}(\delta)\colon w \cdot c' \cdot w' \in W \\ & \quad \text{and } w \text{ does not contain any } c \in C\}. \end{aligned}$$

Informally, $\mathbf{restr}_{\mathbb{S}}$ replaces all unsuccessful synchronizations by deadlock. It thus resembles the restriction operator $\cdot \backslash C$ in CCS [Mi].

But how to prove (3.5)? Note that we cannot prove it directly by structural induction on s, because $\alpha_1 = \mathbf{restr}_{\mathbb{S}}$ does not behave compositionally (over $\|$) due to Lemma 3.4.2. Our solution to this problem is to introduce a new *intermediate operational semantics* $\mathcal{O}_1^*$ such that we can show on the one hand

$$\mathcal{O}_1[\![s]\!] = \mathbf{restr}_{\mathbb{S}}(\mathcal{O}_1^*[\![s]\!])$$

by purely operational, i.e., transition based arguments, and on the other hand

$$\mathcal{O}_1^*[\![s]\!] = \mathcal{D}_1[\![s]\!]$$

for guarded s, analogously to $\mathcal{O}_0[\![s]\!] = \mathcal{D}_0[\![s]\!]$ in Section 2.4. Combining these two results we will obtain the desired relationship (3.5).

For $\mathcal{O}_1^*$ we modify the transition system T_1 into a system T_1^* which is the same as T_1 except for the communication axiom which now takes the form of a virtual transition: (*communication**)

$$\langle c, w\rangle \rightarrow w \cdot c.$$

We base $\mathcal{O}_1^*$ on T_1^* as we based $\mathcal{O}_1$ on T_1.

EXAMPLES. $\mathcal{O}_1^*[\![c]\!] = \{c\}$, $\mathcal{O}_1^*[\![c \| \bar{c}]\!] = \{c\bar{c}, \bar{c}c, \tau\}$, $\mathcal{O}_1^*[\![(a; a') \cup (a; c)]\!] = \mathcal{O}_1^*[\![a; (a' \cup c)]\!] = \{aa', ac\}$.

Introducing virtual transitions in T_1^* seems to violate our principles put forward for the transition system T_1 on $\mathfrak{L}_1$. However, T_1^* is only an auxiliary tool to define the intermediate semantics $\mathcal{O}_1^*$ that is used in the proof of $\mathcal{O}_1[\![s]\!] = \alpha_1(\mathcal{D}_1[\![s]\!]$. Such auxiliary tools may use any technical device that is convenient. In fact, as we shall see in Lemma 3.4.6, the above virtual transition is just sufficient to make the corresponding operational semantics $\mathcal{O}_1^*$ behave compositionally over $\|$. This allows us to prove $\mathcal{O}_1^*[\![s]\!] = \mathcal{D}_1[\![s]\!]$ by structural induction on s.

We begin with:

3.4.3. THEOREM. $\mathcal{O}_1[\![s]\!] = \mathbf{restr}_{\mathbb{S}}(\mathcal{O}_1^*[\![s]\!])$ *for every* $s \in \mathfrak{L}_1$.

The proof uses the following lemma which establishes the link between the underlying transition systems T_1 and T_1^*.

3.4.4. LEMMA. *For all* $s \in \mathfrak{L}_1$, $s' \in \mathfrak{L}_1 \cup \{E\}$, *and* $w, w' \in (A \backslash C)^*$:

(i) $T_1 \vdash \langle s, w\rangle \rightarrow \langle s', w'\rangle$ *iff*

$$T_1^* \vdash \langle s, w\rangle \rightarrow \langle s', w'\rangle$$

(ii) $T_1 \vdash \langle s, w\rangle \rightarrow \langle s', w\,\delta\rangle$ *iff*

$$\exists c \in C\colon T_1^* \vdash \langle s, w\rangle \rightarrow \langle s', wc\rangle.$$

Proof. Recall that $\delta \notin A$ and that T_1 and T_1^* differ only in their communication axioms:

$$\langle c, w \rangle \rightarrow w \cdot \delta \tag{3.6}$$

in T_1, and

$$\langle c, w \rangle \rightarrow w \cdot c \tag{3.6*}$$

in T_1^*. Therefore every transition in T_1 which is not a communication-transition, is also a transition in T_1^*, and vice versa. This implies (i). On the other hand, every communication-transition in T_1 corresponds to (another) communication-transition in T_1^* which is obtained by replacing axiom (3.6) by (3.6*) at the root of the proof tree, and otherwise applying exactly the same rules in T_1^* as in T_1. This argument also holds vice versa, thus proving (ii). ∎

With Lemma 3.4.4 we are prepared for the

Proof of Theorem 3.4.3. Observe that both

$$\mathcal{O}_1[\![s]\!], \mathbf{restr}_{\mathbb{S}}(\mathcal{O}_1^*[\![s]\!]) \subseteq (A \backslash C)^* \cup (A \backslash C)^\omega \cup (A \backslash C)^* \cdot \{\perp, \delta\}.$$

Therefore we consider the following cases.

Case 1. $w \in (A \backslash C)^* \cup (A \backslash C)^\omega \cup (A \backslash C)^* \cdot \{\perp\}$. Then as an immediate consequence of Lemma 3.4.4(i) we have

$$w \in \mathcal{O}_1[\![s]\!] \quad \text{iff} \quad w \in \mathcal{O}_1^*[\![s]\!].$$

Case 2. $w\,\delta \in (A \backslash C)^* \cdot \{\delta\}$. Then

$$w\,\delta \in \mathcal{O}_1[\![s]\!]$$

iff $T_1 \vdash \langle s, \varepsilon \rangle \rightarrow^* w\,\delta$, iff $\exists c' \in C,\ s' \in \mathfrak{L}_1 \cup \{E\}$: $T_1^* \vdash \langle s, \varepsilon \rangle \rightarrow \langle s', wc' \rangle$ (by Lemma 3.4.4(ii). Note that the second alternative can arise.) iff

$$\begin{aligned}&(\exists c' \in C\colon T_1^* \vdash \langle s, \varepsilon \rangle \rightarrow^* wc')\\ &\qquad \vee (\exists c' \in C, s' \in \mathfrak{L}_1, w' \in A^* \cup A^w \cup A^* \cdot \{\perp\}:\\ &\qquad\quad T_1^* \vdash \langle s, \varepsilon \rangle \rightarrow^* \langle s', wc \rangle \wedge w' \in \mathcal{O}_1^*[\![s']\!])\end{aligned}$$

(by the Definedness Lemma 2.2.1 which also holds for $\mathcal{O}_1^*$) iff $\exists c' \in C,\ w' \in A^* \cup A^\omega \cup A^* \cdot \{\perp\}$: $wc'w' \in \mathcal{O}_1^*[\![s]\!]$ (by the Prolongation Lemma 2.2.2 which also holds for $\mathcal{O}_1^*$).

Combining Cases 1 and 2 we find

$$\mathcal{O}_1[\![s]\!] = \mathbf{restr}_{\mathbb{S}}(\mathcal{O}_1^*[\![s]\!]),$$

by the definition of $\mathbf{restr}_{\mathbb{S}}$. This proves the theorem. ∎

Next we discuss

3.4.5. THEOREM. $\mathcal{O}_{\mathrm{I}}^*[\![s]\!] = \mathcal{D}_1[\![s]\!]$ *for all (syntactically closed) guarded* $s \in \mathfrak{L}_1$.

Its proof has the same structure as that of "$\mathcal{O}_0[\![s]\!] = \mathcal{D}_0[\![s]\!]$" (Theorem 2.4.1). In fact, Theorems 2.4.10, 2.4.11, and 2.4.15 also hold for $\mathcal{O}_{\mathrm{I}}^*$, $\mathcal{D}_1$, and $\mathfrak{L}_1$ instead of $\mathcal{O}_0$, $\mathcal{D}_0$, and $\mathfrak{L}_0$, with identical proofs. We therefore concentrate here only on the proof that $\mathcal{O}_{\mathrm{I}}^*$ behaves compositionally over $\|$ (thereby completing the proof of Theorem 2.4.2). More precisely, we show

3.4.6. LEMMA. $\mathcal{O}_{\mathrm{I}}^*[\![s_1 \| s_2]\!] = \mathcal{O}_{\mathrm{I}}^*[\![s_1]\!] \|^{\mathfrak{T}_1} \mathcal{O}_{\mathrm{I}}^*[\![s_2]\!]$ *for all* $s_1, s_2 \in \mathfrak{L}_1$.

As an auxiliary tool we need a result recalling Apt's "merging lemma" in [Ap2].

3.4.7. LEMMA (*Synchronization*). $\forall s_1, s_2 \in \mathfrak{L}_1 \forall s_1', s_2' \in \mathfrak{L}_1 \cup \{E\} \forall w, w_1, w_2 \in A^*$:

$$T_{\mathrm{I}}^* \vdash \langle s_1 \| s_2, w\rangle \rightarrow \langle s_1' \| s_2', w\tau\rangle$$

where the considered transition is a synchronization-transition between s_1 *and* s_2 *iff*

$$\exists c \in C: T_{\mathrm{I}}^* \vdash \langle s_1, w_1\rangle \rightarrow \langle s_1', w_1 c\rangle$$

and

$$T_{\mathrm{I}}^* \vdash \langle s_2, w_2\rangle \rightarrow \langle s_2', w_2 \bar{c}\rangle.$$

Proof. By the Initial Step Lemma it suffices to prove the present lemma for $w = w_1 = w_2 = \varepsilon$ only.

"$\Rightarrow$" Suppose $T_{\mathrm{I}}^* \vdash \langle s_1 \| s_2, \varepsilon\rangle \rightarrow \langle s_1' \| s_2', \tau\rangle$ as above. By the assumptions about this transition, its proof in T_{I}^* starts with a synchronization-axiom of the form

$$\langle c \| \bar{c}, \varepsilon\rangle \rightarrow \tau,$$

where c occurs in s_1 and $\bar{c}$ in s_2. By the definition of T_{I}^*, s_1 and s_1' (respec. s_2 and s_2') are obtained from c and E ($\bar{c}$ and E) by successive embeddings in contexts of the form

$$\cdot\,; s, \cdot \| s \qquad \text{and} \qquad s \| \cdot \tag{3.7}$$

for arbitrary statements $s \in \mathfrak{L}_1$ (by the rule "synchronization in a context" of T_{I}^*).

To construct a proof of $\langle s_1, \varepsilon\rangle \rightarrow \langle s_1', c\rangle$ in T_{I}^*, we start with the axiom

$$\langle c, \varepsilon\rangle \rightarrow c$$

in T_{I}^* and then lift this transition to

$$\langle s_1, \varepsilon\rangle \rightarrow \langle s_1', c\rangle$$

by successive applications of the rules of sequential composition and shuffle corresponding to the successive context embedding of c described in (3.7). This proves $T_{\mathrm{I}}^* \vdash \langle s_1, \varepsilon \rangle \to \langle s_1', c \rangle$. Analogously we prove $T_{\mathrm{I}}^* \vdash \langle s_2, \varepsilon \rangle \to \langle s_2', \bar{c} \rangle$.

"$\Leftarrow$" Suppose $T_{\mathrm{I}}^* \vdash \langle s_1, \varepsilon \rangle \to \langle s_1', c \rangle$. Let us analyze the structure of s_1 by investigating the possible proofs in T_{I}^* leading to a transition which produces "c." Clearly such a proof must start with the communication*-axiom

$$\langle c, \varepsilon \rangle \to c,$$

and it can proceed only by applying the rules of sequential composition and shuffle. Thus s_1 has the BNF-syntax

$$s_1 ::= c \mid s_1 ; s \mid s_1 \parallel s \mid s \parallel s_1, \tag{3.8}$$

where s is an arbitrary statement in $\mathfrak{L}_1$. An analogous analysis holds for s_2 in $T_{\mathrm{I}}^* \vdash \langle s_2, \varepsilon \rangle \to \langle s_2', \varepsilon \rangle$.

To show $T_{\mathrm{I}}^* \vdash \langle s_1 \parallel s_2, \varepsilon \rangle \to \langle s_1' \parallel s_2', \tau \rangle$, we start the proof with the synchronization axiom

$$\langle c \parallel \bar{c}, \varepsilon \rangle \to \tau,$$

and complete it by successive applications of the rule for synchronization in a context according to the structure of s_1 and s_2 as determined in (3.8). Note that we may arbitrarily "interleave" the applications concerning s_1 with those concerning s_2. This finally yields the proof of

$$\langle s_1 \parallel s_2, \varepsilon \rangle \to \langle s_1' \parallel s_2', \tau \rangle$$

in T_{I}^*. Now by its construction this transition is a synchronization transition between s_1 and s_2. This finishes the proof of the lemma. ∎

We now turn to the proof of the announced lemma.

3.4.6. LEMMA. *$\mathcal{O}_{\mathrm{I}}^*[\![s_1 \parallel s_2]\!] = \mathcal{O}_{\mathrm{I}}^*[\![s_1]\!] \parallel^{\mathscr{D}_1} \mathcal{O}_{\mathrm{I}}^*[\![s_2]\!]$ for all $s_1, s_2 \in \mathfrak{L}_1$.*

Proof. "$\subseteq$" Let $w \in \mathcal{O}_{\mathrm{I}}^*[\![s_1 \parallel s_2]\!]$, with $w \in A^* \cup A^\omega \cup A^* \cdot \{\bot\}$. (Note that δ's are not present in $\mathcal{O}_{\mathrm{I}}^*$.) Then there exists a finite or infinite transition sequence

$$T_{\mathrm{I}}^* \vdash \langle s_1 \parallel s_2, \varepsilon \rangle = \langle s_0' \parallel s_0'', w_0 \rangle \to \cdots \to \langle s_n' \parallel s_n'', w_n \rangle \to \cdots \tag{3.9}$$

such that s_n', s_n'' may be E, s_n' stems from s_1 and s_n'' from s_2, and the following hold:

(i) if $w \in A^*$ then $\exists n \geqslant 0 : s_n' = s_n'' = E \wedge w = w_n$

(ii) if $w \in A^\omega$ then $w = \sup_n w_n$

(iii) if $w \in A^* \cdot \{\bot\}$ then $\exists n \geqslant 0\ \forall m \geqslant n : w_m = w_n \wedge w = w_n \bot$.

We have to find words $u \in \mathcal{O}_{\mathrm{I}}^*[\![s_1]\!]$ and $v \in \mathcal{O}_{\mathrm{I}}^*[\![s_2]\!]$ with $w \in \{u\} \parallel_{\mathscr{D}_1} \{v\}$. To this end, we first establish the following claim.

CLAIM. *There exist finite or infinite transition sequences*

$$T_1^* \vdash \langle s_1, \varepsilon\rangle = \langle t'_0, u_0\rangle \to \cdots \to \langle t'_k, u_k\rangle \to \cdots,$$
$$T_1^* \vdash \langle s_2, \varepsilon\rangle = \langle t''_0, v_0\rangle \to \cdots \to \langle t''_l, v_l\rangle \to \cdots,$$

such that there are sequences

$$0 \leqslant k_0 \leqslant k_1 \leqslant k_2 \leqslant \cdots,$$
$$0 \leqslant l_0 \leqslant l_1 \leqslant l_2 \leqslant \cdots,$$

with

$$s'_n = t'_{k_n} \quad \text{and} \quad s''_n = t''_{l_n},$$
$$w_n \in \{u_{k_n}\} \parallel_{\mathscr{D}_1} \{v_{l_n}\},$$
$$n \leqslant k_n + l_n, \qquad \max\{k_n, l_n\} \leqslant n$$

for all $n \geqslant 0$.

Proof of the Claim. By induction on $n \geqslant 0$.

Basis. $n = 0$. Clear: choose $k_0 = l_0 = 0$.

Hypothesis. Assume the claim holds for $n \geqslant 0$, i.e., there are transition sequences

$$T_1^* \vdash \langle s_1, \varepsilon\rangle \to \cdots \to \langle t'_{k_n}, u_{k_n}\rangle,$$
$$T_1^* \vdash \langle s_2, \varepsilon\rangle \to \cdots \to \langle t''_{l_n}, v_{l_n}\rangle$$

with $s'_n = t'_{k_n}$, $s''_n = t''_{l_n}$, $w_n \in \{u_{k_n}\} \parallel^{\mathscr{D}_1} \{v_{l_n}\}$, and $\max\{k_n, l_n\} \leqslant n \leqslant k_n + l_n$.

Step. $n \to n+1$. Let us analyze the final transition producing w_{n+1} in (3.9):

$$T_1^* \vdash \langle s'_n \parallel s''_n, w_n\rangle \to \langle s'_{n+1} \parallel s''_{n+1}, w_{n+1}\rangle. \tag{3.10}$$

Note that s'_{n+1} stems from s'_n and s''_{n+1} from s''_n.

Case 1. This is a local transition. Then, say, the first component is affected, i.e.,

$$T_1^* \vdash \langle s'_n, w_n\rangle \to \langle s'_{n+1}, w_{n+1}\rangle \qquad \text{and} \qquad s''_n = s''_{n+1}.$$

(Note that we may have $w_n = w_{n+1}$.) By the Initial Step Lemma, also

$$T_1^* \vdash \langle s'_n, u_{k_n}\rangle \to \langle s'_{n+1}, u_{k_n}\cdot(w_{n+1} - w_n)\rangle.$$

Combining this transition with the hypothesis yields

$$T_1^* \vdash \langle s_1, \varepsilon\rangle \to \cdots \to \langle t'_{k_n}, u_{k_n}\rangle \to \langle s'_{n+1}, u_{k_n}\cdot(w_{n+1} - w_n)\rangle$$

(where, if w' is a word extending w, say $w' = wu$, we define $w' - w$ to be u).

Now we define

$$k_{n+1} = k_n + 1, \qquad l_{n+1} = l_n$$
$$t_{k_{n+1}} = s'_{n+1}, \qquad u_{k_{n+1}} = u_{k_n} \cdot (w_{n+1} - w_n).$$

By the definition of $\|^{\mathscr{D}_1}$,

$$\begin{aligned} w_{n+1} &= w_n \cdot (w_{n+1} - w_n) \\ &\in \{u_{k_n} \cdot (w_{n+1} - w_n)\} \|^{\mathscr{D}_1} \{v_{l_n}\} = \{u_{k_{n+1}}\} \|^{\mathscr{D}_1} \{v_{l_{n+1}}\} \end{aligned}$$

and, of course, $\max\{k_{n+1}, l_{n+1}\} \leqslant n+1 \leqslant k_{n+1} + l_{n+1}$. This proves the claim for $n+1$ in Case 1.

Case 2. (3.10) is a synchronization-transition between s_1 and s_2. Then $w_{n+1} = w_n \tau$ and, by the Synchronization Lemma, there exists some $c \in C$ with

$$T_1^* \vdash \langle s'_n, u_{k_n} \rangle \to \langle s'_{n+1}, u_{k_n} \cdot c \rangle,$$
$$T_1^* \vdash \langle s''_n, v_{k_n} \rangle \to \langle s''_{n+1}, v_{k_n} \cdot \bar{c} \rangle.$$

Combining these transitions with the hypothesis yields

$$T_1^* \vdash \langle s_1, \varepsilon \rangle \to \cdots \to \langle t'_{k_n}, u_{k_n} \rangle \to \langle t'_{n+1}, u_{k_n} \cdot c \rangle,$$
$$T_1^* \vdash \langle s_2, \varepsilon \rangle \to \cdots \to \langle t''_{l_n}, v_{l_n} \rangle \to \langle t''_{n+1}, v_{l_n} \cdot \bar{c} \rangle.$$

Obviously, we define

$$k_{n+1} = k_n + 1, \qquad l_{n+1} = l_n + 1,$$
$$t'_{k_{n+1}} = s'_{n+1}, \qquad t''_{l_{n+1}} = s''_{n+1},$$
$$u_{k_{n+1}} = u_{k_n} \cdot c, \qquad v_{l_{n+1}} = v_{l_n} \cdot \bar{c}.$$

By the definition of $\|^{\mathscr{D}_1}$,

$$w_{n+1} = w_n \tau \in \{u_{k_n} \cdot c\} \|^{\mathscr{D}_1} \{v_{l_n} \cdot \bar{c}\} = \{u_{k_{n+1}}\} \|^{\mathscr{D}_1} \{v_{l_{n+1}}\}$$

and of course $\max\{k_{n+1}, l_{n+1}\} \leqslant n+1 \leqslant k_{n+1} + l_{n+1}$. This proves the claim for $n+1$ also in Case 2.

Hence the claim holds in general.

Using the claim, it is easy to find appropriate words u, v. The construction corresponds to the case analysis (i)–(iii) of w above. For example, we define u as follows:

— if $\exists k \geqslant 0 : s'_k = E$, then $u = u_k \in A^*$,

— if $\forall k \geqslant 0 \; \exists K > k : w_k < w_K$, then $u = \sup_k u_k \in A^\omega$,

— if $\exists k \geqslant 0 \; \forall K \geqslant k : w_k = w_K$, then $u = u_k \perp \in A^* \cdot \{\perp\}$.

Analogously we proceed for v. Clearly,

$$u \in \mathcal{O}_1^*[\![s_1]\!] \qquad \text{and} \qquad v \in \mathcal{O}_1^*[\![s_2]\!].$$

To verify

$$w \in \{u\} \parallel^{\mathcal{S}_1} \{v\} \tag{3.11}$$

we examine the cases (i)–(iii) of w.

In case (i) we have a finite path

$$T_1^* \vdash \langle s_1 \parallel s_2, \varepsilon \rangle \to \cdots \to \langle s_n' \parallel s_n'', w_n \rangle = \langle E \parallel E, w \rangle = w.$$

By the claim and the definition of u, v,

$$T_1^* \vdash \langle s_1, \varepsilon \rangle \to \cdots \to \langle t_{k_n}', u_{k_n} \rangle = \langle E, u_{k_n} \rangle = u,$$
$$T_1^* \vdash \langle s_2, \varepsilon \rangle \to \cdots \to \langle t_{l_n}', v_{l_n} \rangle = \langle E, v_{l_n} \rangle = v,$$

and thus (3.11) as required.

In case (ii) we have an infinite path (3.9) producing infinitely often increasing words w_n. By the claim at least one of the paths of s_1 and s_2, say that of s_1, must also be infinite, producing infinitely often increasing words u_k, yielding an infinite $u = \sup_k u_k$. Now by definition

$$\{u\} \parallel^{\mathcal{S}_1} \{v\} = \lim_n (\{u[n]\} \parallel^{\mathcal{S}_1} \{v[n]\}).$$

Consider now the approximation w_n of w. By the claim,

$$w_n \in \{u_{k_n}\} \parallel^{\mathcal{S}_1} \{v_{l_n}\}.$$

Since $\max\{k_n, l_n\} \leqslant n$, we have

$$u_{k_n} \leqslant u[n] \qquad \text{and} \qquad v_{l_n} \leqslant v[n].$$

Thus $\exists \tilde{w} \in \{u[n]\} \parallel^{\mathcal{S}_1} \{v[n]\}$ with

$$d(w_n, \tilde{w}) \leqslant 2^{-|w_n|}.$$

This shows

$$w \in \lim_n (\{u[n]\} \parallel^{\mathcal{S}_1} \{v[n]\}),$$

and thus proves (3.11).

In case (iii) we have an infinite path

$$T_1^* \vdash \langle s_1 \parallel s_2, \varepsilon \rangle \to \cdots \to \langle s_n' \parallel s_n'', w_n \rangle \to \langle \cdots, w_{n+1} \rangle \to \cdots$$

with $w_n = w_{n+1} = \cdots$ and thus $w = w_n \perp$. By the claim

$$T_1^* \vdash \langle s_1, \varepsilon \rangle \to \cdots \to \langle t'_{k_n}, u_{k_n} \rangle,$$

$$T_1^* \vdash \langle s_2, \varepsilon \rangle \to \cdots \to \langle t''_{l_n}, v_{l_n} \rangle,$$

with $w_n \in \{u_{k_n}\} \|^{\mathscr{D}_1} \{v_{l_n}\}$. Moreover, due to the condition "$n \leqslant k_n + l_n$ for all n" in the claim, at least one of the transition sequences of s_1 (or s_2) can be extended to an infinite one without expanding u_{k_n} (or v_{l_n}). So $u = u_{k_n} \perp$ (or $v = v_{l_n} \perp$). If the other path of s_2 (or s_1) is finite, we may assume w.l.o.g. that $t''_{l_n} = E$ (or $t'_{k_n} = E$). So then we have $v = v_{l_n}$ (or $u = u_{k_n}$). Combining these facts establishes (3.11).

"$\supseteq$" Let $w \in \mathscr{O}_1^*[\![s_1]\!] \|^{\mathscr{D}_1} \mathscr{O}_1^*[\![s_2]\!]$. Then there exist words $u \in \mathscr{O}_1^*[\![s_1]\!]$, $v \in \mathscr{O}_1^*[\![s_2]\!]$ with

$$w \in \{u\} \|^{\mathscr{D}_1} \{v\}.$$

We have to prove

$$w \in \mathscr{O}_1^*[\![s_1 \| s_2]\!].$$

By the definition of $\mathscr{O}_1^*$ there are corresponding finite or infinite transition sequences in T_1^* for u and v:

$$T_1^* \vdash \langle s_1, \varepsilon \rangle = \langle t'_0, u_0 \rangle \to \cdots \to \langle t'_k, u_k \rangle \to \cdots, \tag{3.12}$$

$$T_1^* \vdash \langle s_2, \varepsilon \rangle = \langle t''_0, v_0 \rangle \to \cdots \to \langle t''_l, v_l \rangle \to \cdots, \tag{3.13}$$

where (in case of finite sequences) t'_k and t''_l may be E. Recall that u and v are obtained from (3.12) and (3.13) just as described for w by the cases (i)–(ii) in part "$\subseteq$." We now construct a finite or infinite path

$$T_1^* \vdash \langle s_1 \| s_2, \varepsilon \rangle = \langle s'_0 \| s''_0, w_0 \rangle \to \cdots \to \langle s'_n \| s''_n, w_n \rangle \to \cdots \tag{3.14}$$

which is *maximal* w.r.t.

$$w_n \leqslant w$$

and which moreover satisfies the following properties: there are sequences

$$0 \leqslant k_0 \leqslant k_1 \leqslant \cdots \qquad \text{and} \qquad 0 \leqslant l_0 \leqslant l_1 \leqslant \cdots$$

such that for each $n \geqslant 0$

$$s'_n = t'_{k_n}, \qquad s''_n = t''_{l_n}$$

$$w_n \in \{u_{k_n}\} \|^{\mathscr{D}_1} \{v_{l_n}\},$$

$$\max\{k_n, l_n\} \leqslant n, \qquad n \leqslant k_n + l_n.$$

The construction of (3.14) proceeds by induction on $n \geqslant 0$.

Basis. $n = 0$. Choose $k_0 = l_0 = 0$.

Hypothesis. Assume the construction works already up to $n \geqslant 0$. If the configurations

$$\langle t'_{k_n}, u_{k_n} \rangle \quad \text{and} \quad \langle t''_{l_n}, v_{l_n} \rangle \tag{3.15}$$

in (3.12) and (3.13) are both final ones, i.e., with $t'_{k_n} = t''_{l_n} = E$, the constructed path (3.14) is already maximal because also

$$s'_n \parallel s''_n = E$$

holds. In all other cases (3.14) has to be extended.

Step $n \to n+1$. We analyze the configurations (3.15).

Case 1a. Path (3.12) has a transition $\langle t'_{k_n}, u_{k_n} \rangle \to \langle t'_{k_n+1}, u_{k_n+1} \rangle$ with $u_{k_n} = u_{k_n+1}$. Then we put

$$w_{n+1} = w_n$$

and $k_{n+1} = k_n + 1$, $l_{n+1} = l_n$, $s'_{n+1} = t'_{k_n+1}$, $s''_{n+1} = s''_n$, and add the transition

$$\langle s'_n \parallel s''_n, w_n \rangle \to \langle s'_{n+1} \parallel s''_{n+1}, w_{n+1} \rangle$$

to (3.14).

Case 1b. Symmetric to Case 1a, but with regards to path (3.13).

Case 2a. Path (3.12) has a transition $\langle t'_{k_n}, u_{k_n} \rangle \to \langle t'_{k_n+1}, u_{k_n+1} \rangle$ with $u_{k_n+1} = u_{k_n} \cdot b$, where $b \in A$ and $w_n \cdot b \leqslant w$.

(Note. b can be an elementary action a (including the case $a = \tau$), or a communication c. Also, $w_n \cdot b \leqslant w$ is always true for $b = a$ or $b = \tau$.) Now we put

$$w_{n+1} = w_n \cdot b$$

and $k_n + 1 = k_{n+1}$, $l_{n+1} = l_n$, $s'_{n+1} = t'_{k_n+1}$, $s''_{n+1} = s''_n$, and add the transition

$$\langle s'_n \parallel s''_n, w_n \rangle \to \langle s'_{n+1} \parallel s''_{n+1}, w_{n+1} \rangle$$

to (3.14).

Case 2b. Symmetric to Case 2a, but with regards to path (3.13).

Case 3. Path (3.12) has a transition $\langle t'_{k_n}, u_{k_n} \rangle \to \langle t'_{k_n+1}, u_{k_n+1} \rangle$ with $u_{k_n+1} = u_{k_n} \cdot c$, where $c \in C$, but $w_n \cdot c \nleqslant w$.

Since $w \in \{u\} \parallel_{\mathscr{D}_1} \{v\}$, we conclude that $w_n \cdot \tau \leqslant w$ and that path (3.13) has a transition

$$\langle t'_{l_n}, v_{l_n} \rangle \to \langle t''_{l_n+1}, v_{l_n+1} \rangle$$

with

$$v_{l_n+1} = v_{l_n} \cdot \bar{c}.$$

Then we put

$$w_{n+1} = w_n \cdot \tau$$

and

$$k_{n+1} = k_n + 1, \, l_{n+1} = l_{n+1}, \, s'_{n+1} = t'_{k_n+1}, \, s''_{n+1} = t''_{l_n+1},$$

and add the transition

$$\langle s'_n \| s''_n, w_n \rangle \to \langle s'_{n+1} \| s''_{n+1}, w_{n+1} \rangle$$

to (3.14). This finishes the construction of path (3.14). We now claim that (3.14) yields w according to the definition of $\mathcal{O}_1^*[\![s_1 \| s_2]\!]$. This is clearly true for $w \in A^* \cup A^\omega$ due to the maximality of (3.14) and the conditions "$w_n \in \{u_{k_n}\} \|^{\mathcal{D}_1} \{v_{l_n}\}$ for $n \geqslant 0$" which link up with $w \in \{u\} \|^{\mathcal{D}_1} \{v\}$ analogously to part "$\subseteq$."

If $w \in A^* \cdot \{\perp\}$, then at least one of u or v, say u, is in $A^* \cdot \{\perp\}$ as well. Then path (3.12) is infinite. By the conditions "$\max\{k_n, l_n\} \leqslant n$ for $n \geqslant 0$," also the constructed path (3.14) is infinite. Thus (3.14) yields indeed w in $\mathcal{O}_1^*[\![s_1 \| s_2]\!]$. ∎

This also finishes our argument for Theorem 3.4.5. By combining Theorems 3.4.4 and 3.4.5 we finally obtain our desired result:

3.4.8. THEOREM. *$\mathcal{O}_1[\![s]\!] = \mathbf{restr}_{\mathbb{S}}(\mathcal{D}_1[\![s]\!])$ for every guarded $s \in \mathfrak{L}_1$.*

4. The Language $\mathfrak{L}_2$: Synchronization Merge and Global Nondeterminacy

We assume the same structure of the alphabet A as for $\mathfrak{L}_1$, and the same use of the variables a, b, and c. But the syntax for $s \in \mathfrak{L}_2$ is now given by

$$s ::= a \mid c \mid s_1 ; s_2 \mid s_1 + s_2 \mid s_1 \| s_2 \mid x \mid \mu x[s].$$

The symbol "$+$" denoting global nondeterminacy is taken from CCS [Mi].

Remark. Simultaneous incorporation of "$\cup$" and "$+$" into one language is in principle possible. We prefer not to do this since it firstly conflicts with our aim to clarify the two forms of non-determinism by treating them in an orthogonal setting. Second, we observe that in the operational semantics, no serious complications arise: we may essentially combine the two systems T_1 and T_2. However, the required modifications in the denotational semantics would be somewhat involved since a "linear time version" of the operation of process union would have to be combined with the (normal) set-theoretic union of two processes (cf. Section 4.3).

4.1. *The Transition System T_2*

The essential difference between local nondeterminacy $s_1 \cup s_2$ and global nondeterninacy $s_1 + s_2$ is the treatment of communications in s_1 and s_2. For example, the $\mathfrak{L}_1$-statement

$$a \cup c$$

involving local nondeterminacy may choose "on its own" between a and c. In the transition system T_1 this was formalized by the two transitions

$$\langle a \cup c, w \rangle \to \langle a, w \rangle,$$

$$\langle a \cup c, w \rangle \to \langle c, w \rangle.$$

The first choice leads to successful termination, viz.

$$\langle a, w \rangle \to w \cdot a,$$

whereas the second choice leads to communication failure, in T_1 represented by

$$\langle c, w \rangle \to w \cdot \delta.$$

Contrast this behaviour with that of the $\mathscr{L}_2$-statement

$$a + c$$

involving global nondeterminacy. Here the choice between a and c depends on the context in which $a + c$ is placed. Considered in isolation, only the transition

$$\langle a + c, w \rangle \to w \cdot a$$

should occur. We then say that the first alternative of $a + c$ is selected by the action a. The communication c should not produce anything. Now consider $a + c$ in the context of a parallel composition with the communication $\bar{c}$. Then the selection of a is still possible yielding

$$\langle (a + c) \| \bar{c}, w \rangle \to \langle \bar{c}, w \cdot a \rangle$$

but no further transition. Additionally, however, c can synchronize with the matching communication $\bar{c}$ and lead to successful termination:

$$\langle (a + c) \| \bar{c}, w \rangle \to w \cdot \tau.$$

We say that the second alternative of $a + c$ is selected by the synchronization of c with $\bar{c}$.

This form of global nondeterninacy is typical for languaes like CSP[Ho1], Ada[Ad], and Occam[In]. There the elementary action $\underline{a}$ would correspond to a purely Boolean guard evaluating to true, and the synchronization of c with $\bar{c}$ to matching communication guards in two parallel components. In the uninterpreted setting of uniform concurrency global nondeterminacy was first discussed by Milner [Mi] and later in [BHR, Ho2, OH]. In our approach, we follow [BHR, Ho2, Mi, OH] in that recursive μ-unfolding does not select any alternative. For example, we would like to have

$$\langle \mu x[a] + c, w \rangle \rightarrow \langle a + c, w \rangle.$$

This case does not arise in the original CSP[Ho1], Ada[Ad] or Occam[In] due to syntactic restrictions in these languages.

Obviously, formalizing global nondeterminacy in a transition system is more demanding than formalizing local nondeterminacy. It is here where Milner [Mi], Plotkin [Pl2], Brookes, Hoare, and Roscoe [BHR] and others profit most from the introduction of virtual transitions

$$\langle c, w \rangle \rightarrow w \cdot c$$

for $c \in C$ (cf. Section 3.1). With them global nondeterminacy can be captured by adding to rule (3) discussed in 3.1 a rule of the form

$$\frac{\langle s_1, w \rangle \rightarrow \langle s', w' \rangle}{\begin{array}{c}\langle s_1 + s_2, w \rangle \rightarrow \langle s', w' \rangle \\ \langle s_2 + s_1, w \rangle \rightarrow \langle s', w' \rangle\end{array}}$$

for both actions ($w' = w \cdot a$) and communications ($w' = w \cdot c$) plus some extra rule for μ-unfolding. In addition, they also have transition rules for a syntactical restriction operator that eliminates all virtual transitions that do not contribute to synchronization transitions via rule (3).

But again, we would like to manage *without* virtual transitions and formalize instead only the successful execution steps as in Apt [Ap2]. But is this possible for the combination of synchronization merge and global nondeterminacy? Apt does not consider this case. We give a positive answer by giving a finite transition system T_2 for $\mathfrak{L}_2$. Thus we provide further insight into the issue of local vs global nondeterminacy for transition systems that describe only the successful steps of concurrent statements and need only finitely many schematic axioms rules.

A final difference between our approach and that of Plotkin's [Pl2] consists in our definition of $\mathcal{O}_2$ which collects information from a (finite or infinite) sequence of transitions in a way which has no counterpart in [Pl2].

Formally, T_2 is like T_1 but without the axioms for local nondeterminacy and for communication ($\langle c, w \rangle \rightarrow w\delta$). Instead we have new rules for *global nondeterminacy*:

(μ-*unfolding*)

$$\frac{\langle s_1, w\rangle \to \langle s', w\rangle}{\begin{array}{c}\langle s_1 + s_2, w\rangle \to \langle s' + s_2, w\rangle\\ \langle s_2 + s_1, w\rangle \to \langle s_2 + s', w\rangle\end{array}}$$

Here the word on the r.h.s. of the premise is equal to the word on the l.h.s. ($=w$). This implies that the premise (and hence the conclusion) is a recursion transition.

(*selection by action*)

$$\frac{\langle s_1, w\rangle \to \langle s', wa\rangle}{\begin{array}{c}\langle s_1 + s_2, w\rangle \to \langle s', wa\rangle\\ \langle s_2 + s_1, w\rangle \to \langle s', wa\rangle\end{array}}$$

where s' may be E (and the premise is an elementary action transition or a synchronization transition (in this case $a=\tau$)).

(*selection by synchronization*)

$$\frac{\langle s_1 \| s_2, w\rangle \to \langle s_1' \| s_2', w\tau\rangle}{\begin{array}{c}\langle (s_1 + s) \| s_2, w\rangle \to \langle s_1' \| s_2', w\tau\rangle\\ \langle s + s_1) \| s_2, w\rangle \to \langle s_1' \| s_2', w\tau\rangle\\ \langle s_1 \| (s_2 + s), w\rangle \to \langle s_1' \| s_2', w\tau\rangle\\ \langle s_1 \| (s + s_2), w\rangle \to \langle s_1' \| s_2', w\tau\rangle\end{array}}$$

where s_1' results from s_1 by replacing an occurrence of a communication c by E and s_2' from s_2 by replacing an occurrence of a matching communication $\bar{c}$ by E. This condition is as for the synchronization rule (cf. Section 3.1). Note that the ";" and "$\|$"-context rules for $\|$ remain valid.

4.2. *The Operational Semantics* $\mathcal{O}_2$

$\mathcal{O}_2$ is a mapping $\mathcal{O}_2\colon \mathfrak{L}_2 \to \mathbb{S}(\delta)$ with $\mathbb{S}(\delta) = \mathfrak{P}(A^{st}(\delta))$ as for $\mathfrak{L}_1$. The definition of $\mathcal{O}_2[\![s]\!]$ is as for $\mathcal{O}_0$ and $\mathcal{O}_1$, i.e.,

$$\mathcal{O}_2[\![s]\!] = \{\mathbf{word}(\pi) \mid \pi \text{ is a path from } s\}.$$

However, there is now an additional fourth clause in the definition of $\mathbf{word}(\pi)$, namely:

(d) if π is finite, and of the form

$$\langle s, \varepsilon\rangle = \langle s_0, w_0\rangle \to \cdots \to \langle s_n, w_n\rangle, \text{ with } s_n \neq E$$

where no further transition $\langle s_n, w_n\rangle \to \langle s', w'\rangle$ is deducible in T_2, then $\mathbf{word}(\pi) = w_n \cdot \delta$.

The pair $\langle s_n, w_n \rangle$ in (d) is called a deadlocking configuration. (Such configurations did not exist under T_0 or T_1.) Note that by (d) the Definedness Lemma 2.2.1 remains valid for $\mathcal{O}_2$: $\mathcal{O}_2[\![s]\!] \neq \emptyset$ for all $s \in \mathfrak{L}_2$.

The following examples mark the differences from $\mathcal{O}_1$.

EXAMPLES. $\mathcal{O}_2[\![c]\!] = \{\delta\}$, $\mathcal{O}_2[\![c \parallel \bar{c}]\!] = \{\tau\}$, $\mathcal{O}_2[\![(a; a') + (a; c)]\!] = \{aa', a\delta\}$, $\mathcal{O}_2[\![a; (a' + c)]\!] = \{aa'\}$. (Remember, $\mathcal{O}_1[\![a; (a' \cup c)]\!] = \mathcal{O}_1[\![(a; a') \cup (a; c)]\!] = \{aa', a\delta\}$.)

Because it is important to see the difference between the last two examples, we shall show they are derived:

(i) $\mathcal{O}_2[\![(a; a') + (a; c)]\!] = \{aa', a\delta\}$.

Proof. Note that

$$\langle a; a', \varepsilon \rangle \to \langle a', a \rangle \to aa'$$

and

$$\langle a; c, \varepsilon \rangle \to \langle c, a \rangle$$

are deducible. So by selection by elementary action we obtain also

$$\langle (a; a') + (a; c), \varepsilon \rangle \to aa'$$

and

$$\langle (a; a') + (a; c), \varepsilon \rangle \to \langle c, a \rangle.$$

So, since no further deductions can be made from $\langle c, a \rangle$, we get by the definition of $\mathcal{O}_2$: $\mathcal{O}_2[\![(a; a') + (a; c)]\!] = \{aa', a\delta\}$.

(ii) $\mathcal{O}_2[\![a; (a' + c)]\!] = \{aa'\}$.

Proof. First note that

$$\langle a; (a' + c), \varepsilon \rangle \to \langle a' + c, a \rangle.$$

Since we have that

$$\langle a', a \rangle \to aa',$$

we also have

$$\langle a' + c, a \rangle \to aa',$$

and therefore

$$\langle a; (a' + c), \varepsilon \rangle \to aa'.$$

Since we cannot deduce anything from $\langle c, a \rangle$, aa' is all we can deduce from $\langle a; (a' + c), \varepsilon \rangle$. Consequently, $\mathcal{O}_2[\![a; (a' + c)]\!] = \{aa'\}$.

Thus with global nondeterminacy "+," the statements $s_1 = (a; a') + (a; c)$ and $s_2 = a; (a' + c)$ get different meanings under $\mathcal{O}_2$. This difference can be understood as follows: If s_1 performs the elementary action a, the remaining statement is either the elementary action a' or the communication c. In case of c, a deadlock occurs since no matching communication is available. However, if s_2 performs a, the remaining statement is $a' + c$ which cannot deadlock because the action a' is always possible. Thus communications c create deadlocks only if neither a matching communication $\bar{c}$ nor an alternative elementary action a' is available.

Remark (on the role of τ). Again, in $\mathcal{O}_2$ (or in $\mathcal{D}_2$) we find no special treatment for τ, for the reasons mentioned earlier. In the branching time outcome (as delivered by $\mathcal{D}_2$), one may perform τ-abstraction steps by (repeatedly) applying Milner's τ-laws (first described in [Mi] and studied in many places, see [BK2] for an example). For $\mathcal{O}_2$ this is not so clear, since the τ-laws expect branching time objects to operate upon. We have not studied the question whether it is possible to define a modified version of T_2 which incorporates the effects of the τ-laws in some way. A relevant reference is [vG], where in an operational setting with transitions of the form

$$x \xrightarrow{a} y,$$

with x, y ACP processes, one encounters rules such as

$$\frac{x \to^{a} y,\ y \to^{\tau} z}{x \to^{a} z}.$$

In fact, it may be seen that this rule embodies Milner's third law:

$$a(y + \tau z) = a(y + \tau z) + az.$$

4.3. *The Denotational Semantics* $\mathcal{D}_2$

We follow [BZ1, BZ2, BBKM] in introducing a *branching time* semantics for $\mathfrak{L}_2$. Let, as usual, $\perp \notin A$ and let $A_\perp$ be short for $A \cup \{\perp\}$. Again, we assume a special element τ in A. Let the ultrametric spaces $(\mathbb{P}_n, d_n)$, $n \geqslant 0$, be defined by

$$\mathbb{P}_0 = \mathfrak{P}(A_\perp), \qquad \mathbb{P}_{n+1} = \mathfrak{P}(A_\perp \cup (A \times \mathbb{P}_n)),$$

where $\mathfrak{P}(\cdot)$ denotes all subsets of $(\cdot)$, and the ultrametrics d_n will be defined in a moment. Let $\mathbb{P}_\omega = \bigcup_n P_n$. Elements of $\mathbb{P}_\omega$ are called (finite) processes and typical elements are denoted by $p, q, \ldots$. Processes p in $\mathbb{P}_n$ are often denoted by $p_n, q_n, \ldots$. For $p \in \mathbb{P}_\omega$ we call the least n such that $p \in \mathbb{P}_n$ its *degree*. Note that each process is a *set*; hence, a process has elements for which we use $x, y, \ldots$ (not to be confused with $x, y \in \mathbf{Stmv}$). For each $p(\varepsilon\mathbb{P}_\omega)$ we define its nth projection $p(n)$ as follows:

$$p(n)=\{x(n)\mid x\in p\}, \qquad n=0,1,\ldots$$
$$x(n)=x \text{ if } x\in A_\perp, \qquad n=0,1,\ldots$$
$$[b,p](n)=\begin{cases} b, & n=0\\ [b,p(n-1)], & n=1,2,\ldots.\end{cases}$$

(For easier readability, pair formation in processes is denoted by $[\cdot,\cdot]$.) We can now define d_n by

$$d_0(p_0',p_0'')=\begin{cases}0 & \text{if } \; p_0'=p_0''\\ 1 & \text{if } \; p_0'\neq p_0''\end{cases}$$
$$d_{n+1}(p_{n+1}',p_{n+1}'')=2^{-\sup\{k\mid p_{n+1}'(k)=p_{n+1}''(k)\}}$$

with $2^{-\infty}=0$ as before.

On $\mathbb{P}_\omega$ we define the ultrametric d by putting $d(p,q)=d_n(p,q)$, where $n=\max(\text{degree}(p), \text{degree}(q))$. We now define the set $\mathbb{P}$ of finite and infinite processes as the *completion* of $\mathbb{P}_\omega$ with respect to d. A fundamental result of [BZ2] is that we have the equality (more precisely, the isometry)

$$\mathbb{P}=\mathfrak{P}_{\text{closed}}(A_\perp\cup(A\times\mathbb{P})).$$

Examples of finite elements of $\mathbb{P}$ are $\{[b,\{b_1\}],[b,\{b_2\}]\}$ and $\{[b,\{b_1,b_2\}]\}$. The following trees represent these:

Thus, the branching structure is preserved. An example of an infinite element of $\mathbb{P}$ is the process p which satisfies the equation $p=\{[b_1,p],[b_2,p]\}$. Processes are like commutative trees which have, in addition, sets rather than multisets for successors of nodes and which satisfy a closedness property. An example of a set which is not a process is $\{a,[a,\{a\}],[a,\{[a,\{a\}]\}]\cdots\}$, where this set does not include the infinite branch of a's.

Remark. One might wonder as to the relationship between the domain $\mathbb{P}$ described above and the domains obtained in terms of the familiar bisimulation equivalence [Pa] on (graphs or) trees. We do not have a complete answer to this question, but the following partial result is available (R. van Glabbeek and J. W. Klop, personal communication). Let the alphabet A be finite (as usual in our paper). Consider the set of *process trees*, i.e., the set of all (rooted directed) finite or infinite trees with edges labelled by elements from A and leaves which may be labelled by $\varnothing$ or $\perp$, which satisfy the restriction that each node has at most countable splitting degree. Assume, moreover, a natural definition of closedness for process trees given (details omitted here), and let $\mathscr{T}$ denote the set of all closed process trees. Let $\leftrightarrow$ denote the bisimulation equivalence. We then have (with some abuse of notation).

$$(\mathscr{T}/\leftrightarrow) \simeq \mathfrak{P}_{\text{closed}}(A_\perp \cup A \times (\mathscr{T}/\leftrightarrow)).$$

Further study is necessary for a full understanding of the above-mentioned cardinality restrictions.

The empty set *is* a process and takes the role of δ. Note that in the previous *linear time* (LT) framework $\emptyset$ cannot replace δ since by the definition of concatenation (for *LT*) we have $b \cdot \emptyset = \emptyset$ which is undesirable for an element modelling failure. (An action which fails should not cancel all previous actions!) In the present branching time framework, $\{[b, \emptyset]\}$ is a process which is indeed different from (and irreducible to) $\emptyset$.

The following operations on processes are defined. We first take the case that both processes are finite, and use induction on the degree(s) of the processes concerned:

concatenation $\circ$: $p \circ q = \bigcup \{x \circ q : x \in p\}$, where $\perp \circ q = \perp$,

$$b \circ q = [b, q],\ [b, p'] \circ q = [b, p' \circ q].$$

union $\cup$: $p \cup q$ is the set-theoretic union of p and q.

merge $\|$: $p \| q = (p \mathbin{\|\!\_} q) \cup (q \mathbin{\|\!\_} p) \cup (p \mid q)$, where $p \mathbin{\|\!\_} q = \bigcup \{x \mathbin{\|\!\_} q : x \in p\}$, $\perp \mathbin{\|\!\_} q = \perp$, $b \mathbin{\|\!\_} q = [b, q]$, $[b, p'] \mathbin{\|\!\_} q = [b, p' \| q]$. Moreover, $p \mid q = \bigcup \{x \mid y : x \in p, y \in q\}$, where

$$[c, p'] \mid [\bar{c}, q'] = \{[\tau, p' \| q']\}$$
$$[c, p'] \mid \bar{c} = \{[\tau, p']\}$$
$$c \mid [\bar{c}, q'] = \{[\tau, q']\}$$
$$c \mid \bar{c} = \{\tau\}$$

and $x \mid y = \emptyset$ for x, y not of one of the above four forms.

For p or q infinite we have (since $\mathbb{P}$ is defined by completion of $\mathbb{P}_\omega$) that $p = \lim_n p_n$, $q = \lim_n q_n$, p_n and q_n finite, $n = 0, 1, \ldots$, and we define $p \mathbin{\mathbf{op}} q = \lim_n (p_n \mathbin{\mathbf{op}} q_n)$, where $\mathbf{op} \in \{\cdot, \cup, \|\}$. (By [BZ2], $(p_n \mathbin{\mathbf{op}} q_n)_n$ forms a Cauchy sequence.) It is now straightforward to define $\mathscr{D}_2$: guarded $\mathfrak{L}_2 \to (\Gamma_2 \to \mathbb{P})$, where $\Gamma_2 = \mathbf{Stmv} \to \mathbb{P}$, by following the clauses in the definition of $\mathscr{D}_0$, $\mathscr{D}_1$. Thus, we put

$$\mathscr{D}_2[\![a]\!](\gamma) = \{a\}$$
$$\mathscr{D}_2[\![c]\!](\gamma) = \{c\}$$
$$\mathscr{D}_2[\![s_1 \mathbin{\mathbf{op}} s_2]\!](\gamma) = \mathscr{D}_2[\![s_1]\!](\gamma) \mathbin{\mathbf{op}^{\mathscr{D}_2}} \mathscr{D}_2[\![s_2]\!](\gamma)$$

for $\mathbf{op} \in \{;, +, \|\}$, where $;^{\mathscr{D}_2} = \circ$, $+^{\mathscr{D}_2} = \cup$, $\|^{\mathscr{D}_2} = \|$

$$\mathscr{D}_2[\![x]\!](\gamma) = \gamma(x)$$

$$\mathscr{D}_2[\![\mu x[s]]\!](\gamma) = \lim_i p_i, \qquad \text{where} \quad p_0 = \{\bot\} \text{ and}$$

$$p_{i+1} = \mathscr{D}_2[\![s]\!](\gamma\langle p_i/x\rangle).$$

Mutatis mutandis, the contractivity results for $\mathscr{D}_0$, $\mathscr{D}_1$ hold again.

4.4. *Relationship between $\mathcal{O}_2$ and $\mathscr{D}_2$*

For a suitable abstraction operator α_2 we shall show that

$$\mathcal{O}_2[\![s]\!] = \alpha_2(\mathscr{D}_2[\![s]\!]) \tag{4.2}$$

holds for all guarded $s \in \mathfrak{L}_2$. We define $\alpha_2 \colon \mathbb{P} \to \mathbb{S}(\delta)$ in two steps:

1. First we define a restriction mapping $\mathbf{restr}_{\mathbb{P}} \colon \mathbb{P} \to \mathbb{P}$. For $p \in \mathbb{P}_\omega$ we put inductively (recall that a ranges over $A \backslash C$):

$$\mathbf{restr}_{\mathbb{P}}(p) = \{a \mid a \in p\} \cup \{\bot \mid \bot \in p\}$$

$$\bigcup \{[a, \mathbf{restr}_{\mathbb{P}}(q)] \mid [a, q] \in p\}.$$

For $p \in \mathbb{P} \backslash \mathbb{P}_\omega$ we have $p = \lim_n p_n$, with $p_n \in \mathbb{P}_n$, and we put

$$\mathbf{restr}_{\mathbb{P}}(p) = \lim_n (\mathbf{restr}_{\mathbb{P}}(p_n)).$$

EXAMPLE. Let $p = \mathscr{D}_2[\![(a+c) \| (a'+\bar{c})]\!] = \mathscr{D}_2[\![(a; (a'+\bar{c})) + (c; (a'+\bar{c})) + (a'; (a+c)) + (\bar{c}; (a+c)) + \tau]\!]$. Then $\mathbf{restr}_{\mathbb{P}}(p) = \{[a, \{a'\}], [a', \{a\}], \tau\} = \mathscr{D}_2[\![(a; a') + (a'; a) + \tau]\!]$.

2. Then we define a mapping $\mathbf{streams} \colon \mathbb{P} \to \mathbb{S}_c(\delta)$. For $p \in \mathbb{P}_\omega$ we put inductively (recall that b ranges over A):

$$\mathbf{streams}(p) = \begin{cases} \{b \mid b \in p\} \cup \{\bot \mid \bot \in p\} & \\ \bigcup \{b \cdot \mathbf{streams}(q) \mid [b, q] \in p\} & \text{if } p \neq \varnothing \\ \{\delta\} & \text{if } p = \varnothing. \end{cases}$$

Note that $b \cdot \mathbf{streams}(q)$ itself is a set of streams. For $p \in \mathbb{P} \backslash \mathbb{P}_\omega$ we have $p = \lim_n p_n$, with $p_n \in \mathbb{P}_n$, and we put

$$\mathbf{streams}(p) = \lim_n (\mathbf{streams}(p_n)).$$

Note that "$\lim_n$" above is taken with respect to the metric on $\mathbb{S}_c(\delta)$ [see Section 2.3]. (For a proof that $\mathbf{streams}(p)$ is *closed* in $\mathbb{S}(\delta)$ we refer to [BBKM].)

EXAMPLE. With p as in the previous example we have $\mathbf{streams}(p) = \{aa', a\bar{c}, ca', c\bar{c}, a'a, a'c, \bar{c}a, \bar{c}c, \tau\}$ and $\mathbf{streams}(\mathbf{restr}_{\mathbb{P}}(p)) = \{aa', a'a, \tau\}$.

Finally we put

$$\alpha_2 = \mathbf{streams} \circ \mathbf{restr}_{\mathbb{P}}$$

in (4.2). Similarly to α_1, we cannot prove (4.2) directly by structural induction on s because α_2 does not behave compositionally. Thus again the question arises how to prove (4.2). Note that here things are rather more difficult than with $\mathcal{O}_1\llbracket s\rrbracket = \alpha_1(\mathcal{D}_1\llbracket s\rrbracket)$ because the semantic domains of $\mathcal{O}_1$ and $\mathcal{D}_1$ are quite different: linear streams vs branching processes.

Our solution to this problem is to introduce

— a new intermediate semantic domain $\mathbb{R}$,
— a new intermediate operational semantics $\mathcal{O}_2^*$ on $\mathbb{R}$,
— a new intermediate denotational semantics $\mathcal{D}_2$ on $\mathbb{R}$,

and then prove the following diagram:

$$\begin{tikzcd}
\mathcal{O}_2 & \mathcal{O}_2^* = \mathcal{D}_2^* \arrow[l, "\mathbf{restr}_{\mathbb{R}}"'] & \mathcal{D}_2 \arrow[l, "\mathbf{readies}"'] \arrow[ll, bend left, "{\alpha_2 = \mathbf{streams} \circ \mathbf{restr}_{\mathbb{P}} = \mathbf{restr}_{\mathbb{R}} \circ \mathbf{readies}}"]
\end{tikzcd}$$

where $\mathbf{restr}_{\mathbb{R}}$ and $\mathbf{readies}$ are two further abstraction operators.

The Intermediate Semantic Domain $\mathbb{R}$

We start with the intermediate semantic domain. To motivate its construction, let us first demonstrate that a simple *stream-like* variant of $\mathcal{O}_2$ is not appropriate as intermediate operational semantics $\mathcal{O}_2^*$ here. Indeed, if we base $\mathcal{O}_2^*$—similarly to $\mathcal{O}_1^*$—on a transition system obtained by just adding the axiom

$$\langle c, w\rangle \rightarrow w \cdot c$$

to T_2, we cannot retrieve $\mathcal{O}_2$ from $\mathcal{O}_2^*$. As a counterexample consider the programs $s_1 = (a; c_1) + (a; c_2)$, $s_2 = a; (c_1 + c_2)$ and $s = \bar{c}_1$. Then $\mathcal{O}_2\llbracket s_1 \parallel s\rrbracket = \{a\tau, a\delta\} \neq \{a\tau\} = \mathcal{O}_2\llbracket s_2 \parallel s\rrbracket$, but $\mathcal{O}_2^*\llbracket s_1 \parallel s\rrbracket = \mathcal{O}_2^*\llbracket s_2 \parallel s\rrbracket$. Thus whatever operator α we apply to $\mathcal{O}_2^*\llbracket\cdot\rrbracket$, the results for $s_1 \parallel s$ and $s_2 \parallel s$ will turn out the same. Thus we cannot retrieve $\mathcal{O}_2$ from this $\mathcal{O}_2^*$.

To solve this problem, we introduce for $\mathcal{O}_2^*$ a new semantic domain which, besides streams $w \in A^{st}$, also includes very weak information about the *local branching structure* of a process. This information is called a *ready set* or *deadlock possibility*; it takes the form of a subset X of C, the set of communications, and may appear (locally) after every word $w \in A^*$ of successful actions. Informally, such a set X after w indicates that after w the process is ready for all communications $c \in X$

and that deadlock can be avoided only if some communication $c \in X$ can synchronize with a matching communication $\bar{c}$ in some other parallel component. Thus X can be seen as a "more informative δ." This view is confirmed by the fact that there will be no ready set X after w if the process can do an elementary action $a \in A \backslash C$ and thus avoid deadlock on its own. With some variations this notion of a ready set appears in the work of [BHR, FLP, OH, RB].

Formally, we take $\Delta = \mathfrak{P}(C)$ and define the set of *streams with ready sets* as

$$A^{\mathrm{rd}} = A^{\mathrm{st}} \cup A^*:\Delta$$

where $A^*:\Delta$ denotes the set of all pairs of the form $w:X$ with $w \in A^*$ and $X \in \Delta$. For $X \in \Delta$, let $\bar{X} = \{\bar{c} \mid c \in X\}$. As intermediate domain we take the *ready domain*

$$\mathbb{R} = \mathfrak{P}(A^{\mathrm{rd}}).$$

Just as we did for A^{st} and $A^{\mathrm{st}}(\delta)$, we can define a metric d on A^{rd} and a corresponding metric $\hat{d}$ turns the collection $\mathbb{R}_c \subseteq \mathbb{R}$ of nonempty closed subsets of A^{rd} into a complete metric space $(\mathbb{R}_c, \hat{d})$.

The Intermediate Operational Semantics $\mathcal{O}_2^*$

We now turn to the intermediate operational semantics $\mathcal{O}_2^*$ on $\mathbb{R}$. It is based on the following transition system T_2^* which consists of all axioms and rules of T_2 extended (for $w \in A^*$) by:

(*communication**)

$$\langle c, w \rangle \to w \cdot c$$

(*ready sets cq. deadlock possibilities*)

(i) $$\langle c, w \rangle \to w: \{c\}$$

(ii) $$\frac{\langle s_1, w \rangle \to w: X}{\langle s_1; s_2, w \rangle \to w: X}$$

(iii) $$\frac{\langle s_1, w \rangle \to w: X,\ \langle s_2, w \rangle \to w: Y}{\langle s_1 + s_2, w \rangle \to w: (X \cup Y)}$$

(iv) $$\frac{\langle s_1, w \rangle \to w: X,\ \langle s_2, w \rangle \to w: Y}{\langle s_1 \| s_2, w \rangle \to w: (X \cup Y)}$$

where $X \cap \bar{Y} = \phi$.

Axiom (i) introduces ready sets or deadlock possibilities, and rules (ii)–(iv) propagate them. In particular, rule (iii) says that $s_1 + s_2$ has a deadlock possibility if s_1 and s_2 have, and rule (iv) says that $s_1 \| s_2$ has a deadlock possibility if both s_1 and s_2 have, and no synchronization is possible. The transitions deducible with these axioms and rules are virtual transitions, but they are needed only as a technical device in the proof of $\mathcal{O}_2[\![s]\!] = \alpha_2(\mathcal{D}_2[\![s]\!])$.

Since the rules (iii) and (iv) have two premises, deduction in T_2^* need not start any more from a single axiom. But every deduction of a transition

$$\langle s, w \rangle \to \langle s', w' \rangle$$

or

$$\langle s, w \rangle \to w'$$

or

$$\langle s, w \rangle \to w' : X$$

in T_2^* is such that all its axioms are instances of the same axiom scheme. Thus similarly to Section 2.4 (see Types of Transitions) we may talk of an (Ax)-*transition* if (Ax) is the name of the axiom. Note also that the Initial Step Lemma 2.1.1 remains valid for T_2^*.

The intermediate operational semantics

$$\mathcal{O}_2^* : \mathfrak{L}_2 \to \mathbb{R}$$

is defined in terms of T_2^* just as $\mathcal{O}_2$ was defined in terms of T_2. In particular, for each finite path π of the form

$$\langle s, \varepsilon \rangle = \langle s_0, w_0 \rangle \to \cdots \to \langle s_n, w_n \rangle \to w : X$$

we include $\mathbf{word}(\pi) = w : X$ in $\mathcal{O}_2^*[\![s]\!]$.

EXAMPLES. (i) $\mathcal{O}_2^*[\![a; (a' + c)]\!] = \{aa', ac\}$.

Proof. We explore all transition sequences in T_2^* starting in $\langle a; (a' + c), \varepsilon \rangle$:

(1) $\langle a, \varepsilon \rangle \to a$ (elementary action)

(2) $\langle a; (a' + c), \varepsilon \rangle \to \langle a' + c, a \rangle$ ((1), composition)

(3) $\langle a', a \rangle \to aa'$ (elementary action)

(4) $\langle c, a \rangle \to ac$ (communication)

$$\searrow$$
$$a : \{c\}$$

(5) $\langle a' + c, a \rangle \to aa'$ ((3), (4), global nondeterminacy).

$$\searrow$$
$$ac$$

No more transitions are deducible for $\langle a' + c, a \rangle$.

(6) Thus

$$\begin{array}{lll} \langle a; (a'+c), \varepsilon\rangle \to \langle a'+c, a\rangle & \to & aa' \\ & \searrow & \\ & & ac \end{array}$$

are all transition sequences starting in $\langle a; (a'+c), \varepsilon\rangle$.

This proves the claim. ∎

(ii) $\mathcal{O}_2^*[\![a; a'+a; c]\!] = \{aa', ac, a\colon \{c\}\}$.

Proof. Here we only exhibit all possible transition sequences in T_2^* starting in $\langle a; a'+a; c, \varepsilon\rangle$:

$$\begin{array}{llllll} \langle a; a'+a; c, \varepsilon\rangle & \to & \langle a', a\rangle \to aa' & & & \\ & \searrow & & & & \\ & & \langle c, a\rangle & \to & ac & \\ & & & \searrow & & \\ & & & & a\colon \{c\}. & \blacksquare \end{array}$$

Note that we can prove $\langle a; a'+a; c, \varepsilon\rangle \to \langle c, a\rangle$ and $\langle c, a\rangle \to a\colon \{c\}$, and therefore $\langle a; a'+a; c, \varepsilon\rangle \to^* a\colon \{c\}$. However, we have $\langle a; (a'+c), \varepsilon\rangle \to \langle a'+c, a\rangle$, but we *cannot* prove $\langle a'+c, a\rangle \to a\colon \{c\}$. (By rule (iii) of ready sets this would only be the case if we could prove, besides $\langle c, a\rangle \to a\colon \{c\}$, also $\langle a', a\rangle \to a\colon X$ for some $X \subseteq \{c\}$. Since a' is not a communication and the only possibilities for X are $\emptyset$ and $\{c\}$, this cannot be proved.) Consequently, $\langle a; (a'+c), \varepsilon\rangle \not\to^* a\colon \{c\}$.

The Intermediate Denotational Semantics $\mathcal{D}_2^*$

We start by defining semantic operators $;^{\mathcal{D}_2^*}$, $+^{\mathcal{D}_2^*}$ and $\|^{\mathcal{D}_2^*}$ on $\mathbb{R}_c$. (Again we omit superscripts $\mathcal{D}_2^*$ whenever possible.) Let $W_1, W_2 \in \mathbb{R}_c$, w, $w_1, w_2 \in A^{\text{st}}$, and $u_1, u_2 \in A^{\text{rd}} = A^{\text{st}} \cup A^*\colon \Delta$.

(a) $W_1, W_2 \subseteq A^* \cup A^* \cdot \{\perp\} \cup A^*\colon \Delta$. Then

$$\begin{aligned} W_1; W_2 = {} & \{w_1 \cdot w_2 \mid w_1 \in W_1 \text{ and } w_2 \in W_2\} \\ & \cup \{w_1\colon X \mid w_1\colon X \in W_1\} \\ & \cup \{w_1 \cdot w_2\colon X \mid w_1 \in W_1 \text{ and } w_2\colon X \in W_2\} \end{aligned}$$

$$\begin{aligned} W_1 + W_2 = {} & \{w \mid w \in W_1 \cup W_2\} \\ & \cup \{\varepsilon\colon (X \cup Y) \mid \varepsilon\colon X \in W_1 \quad \text{and} \quad \varepsilon\colon Y \in W_2\} \\ & \cup \{w\colon X \mid w \neq \varepsilon \quad \text{and} \quad w\colon X \in W_1 \cup W_2\} \end{aligned}$$

$$W_1 \| W_2 = (W_1 \mathbin{\|\!\!\_} W_2) \cup (W_2 \mathbin{\|\!\!\_} W_1) \cup (W_1 \mid W_2) \cup (W_1 \# W_2),$$

where $W_1 \mathbin{\|\!\!\_} W_2 = \bigcup\{u_1 \mathbin{\|\!\!\_} W_2 \mid u_1 \in W_1\}$ with $\varepsilon \mathbin{\|\!\!\_} W_2 = W_2$, $(b \cdot w_1) \mathbin{\|\!\!\_} W_2 =$

$b\cdot(\{w_1\} \parallel W_2)$, $(b\cdot w_1 : X) \mathbin{⫡} W_2 = b\cdot(\{w_1 : X\} \parallel W_2)$, $\perp \mathbin{⫡} W_2 = \{\perp\}$, $\varepsilon : X \mathbin{⫡} W = \varnothing$, and $W_1 \mid W_2 = \bigcup \{(u_1 \mid u_2) \mid u_1 \in W_1$ and $u_2 \in W_2\}$ with $(c\cdot u_1) \mid (\bar{c}\cdot u_2) = \tau\cdot(\{u_1\} \parallel \{u_2\})$ and $w_1 \mid w_2 = \varnothing$ for w_1, w_2 not of the above form, and

$$W_1 \mathbin{\#} W_2 = \{\varepsilon : X\cup Y \mid \varepsilon : X \in W_1 \text{ and } \varepsilon : Y \in W_2 \text{ and } X \cap \bar{Y} = \varnothing\}.$$

(b) $W_1, W_2 \in \mathbb{R}_c$ and W_1, W_2 also contain infinite words. Then extend the previous definitions by taking limits in $\mathbb{R}_c$.

Now we define

$$\mathscr{D}_2^* : \text{guarded } \mathfrak{L}_2 \to (\Gamma_2^* \to \mathbb{R}_c)$$

with $\Gamma_2^* = \mathbf{Stmv} \to \mathbb{R}_c$ in the usual way:

1. $\mathscr{D}_2^*[\![a]\!](\gamma) = \{a\}$
2. $\mathscr{D}_2^*[\![c]\!](\gamma) = \{c, \varepsilon : \{c\}\}$
3. $\mathscr{D}_2^*[\![s_1 \ \mathbf{op}\ s_2]\!](\gamma) = \mathscr{D}_2^*[\![s_1]\!](\gamma)\ \mathbf{op}\ \mathscr{D}_2^*[\![s_2]\!](\gamma)$
4. $\mathscr{D}_2^*[\![x]\!](\gamma) = \gamma(x)$
5. $\mathscr{D}_2^*[\![\mu x[s]]\!](\gamma) = \lim_i W_i$, where $W_0 = \{\perp\}$ and $W_{i+1} = \mathscr{D}_2^*[\![s]\!](\gamma\langle W_i/x\rangle)$.

Relating $\mathcal{O}_2$ and $\mathcal{O}_2^*$

The relationship between $\mathcal{O}_2$ and $\mathcal{O}_2^*$ is similar to that between $\mathcal{O}_1$ and $\mathcal{O}_1^*$ in Section 3.4. In fact, we shall prove:

4.4.1. THEOREM. *$\mathcal{O}_2[\![s]\!] = \mathbf{restr}_{\mathbb{R}}(\mathcal{O}_2^*[\![s]\!])$ for every $s \in \mathfrak{L}_2$.*

Here $\mathbf{restr}_{\mathbb{R}} : \mathbb{R} \to \mathbb{S}(\delta)$ is a restriction operator similar to $\mathbf{restr}_{\mathbb{S}} : \mathbb{S}(\delta) \to \mathbb{S}(\delta)$ of Section 3.4. For $W \in \mathbb{R}$ and $w \in A^{st}$ we define

$$\begin{aligned}\mathbf{restr}_{\mathbb{R}}(W) = {} & \{w \mid w \in W \text{ does not contain any } c \in C\} \\ & \cup \{w\cdot\delta \mid \exists X \in \Delta : w : X \in W \text{ and } w \text{ does not contain any } c \in C\}\end{aligned}$$

For Theorem 4.1 we need the following result concerning the transition systems T_2 and T_2^* (compare Lemma 3.4.4).

4.4.2. LEMMA. *For all $s \in \mathfrak{L}_2$, $s' \in \mathfrak{L}_2 \cup \{E\}$ and $w, w' \in (A\backslash C)^*$:*

(i) *$T_2 \vdash \langle s, w\rangle \to \langle s', w'\rangle$ iff*

$$T_2^* \vdash \langle s, w\rangle \to \langle s', w'\rangle$$

(ii) *$\langle s, w\rangle$ is a deadlocking configuration for T_2 iff*

$$\exists X \subseteq C : T_2^* \vdash \langle s, w\rangle \to w : X.$$

Proof. *ad* (i): "$\Rightarrow$" is clear because T_2^* is an extension of T_2. For "$\Leftarrow$" note that, by the assumption $w, w' \in (A \setminus C)^*$, none of the new axioms and rules in T_2^* was used in proving the transition

$$\langle s, w \rangle \rightarrow \langle s', w' \rangle.$$

Hence it can also be proved in T_2.

ad (ii): First we analyze the structure of deadlocking configurations $\langle s, w \rangle$ in T_2: their statements s (with possible subscripts 1 and 2) have the following BNF-syntax:

$$\begin{aligned} s ::= {} & c \text{ for arbitrary } c \in C \mid \\ & s_1 ; t \text{ for arbitrary } t \in \mathfrak{L}_2 \mid s_1 + s_2 \mid \\ & s_1 \| s_2, \text{ where there is no synchronization-transition} \\ & \text{possible between } s_1 \text{ and } s_2. \end{aligned}$$

Thus in a deadlocking configuration $\langle s, w \rangle$ all the initial actions of s are communications and in the case of a shuffle $s_1 \| s_2$ no matching initial communications (leading to a τ-action) can be found in its components s_1 and s_2. We can express this property more precisely by introducing a partial function

$$\mathbf{dead}\colon \mathfrak{L}_2 \xrightarrow{\text{part}} \Delta = \mathfrak{P}(C)$$

such that $\langle s, w \rangle$ is deadlocking iff $\mathbf{dead}(s)$ is defined. Its definition runs as follows:

$$\begin{array}{lll} \text{(i)} & \mathbf{dead}(a) \text{ is undefined}, & \text{for } a \in A \setminus C \\ \text{(ii)} & \mathbf{dead}(c) = \{c\}, & \text{for } c \in C \\ \text{(iii)} & \mathbf{dead}(s_1 ; t) = \mathbf{dead}(s_1) & \\ \text{(iv)} & \mathbf{dead}(s_1 + s_2) = \mathbf{dead}(s_1) \cup \mathbf{dead}(s_2) & \\ \text{(v)} & \mathbf{dead}(s_1 \| s_2) = \begin{cases} \mathbf{dead}(s_1) \cup \mathbf{dead}(s_2), & \text{if } \mathbf{dead}(s_1) \cap \overline{\mathbf{dead}(s_2)} = \varnothing \\ \text{undefined}, & \text{otherwise.} \end{cases} & \end{array}$$

Now we can prove (ii):

$$\langle s, w \rangle \text{ is a deadlocking configuration in } T_2$$

iff $\mathbf{dead}(s)$ is defined (by the analysis above)

iff $\exists X \subseteq C\colon T_2^* \vdash \langle s, w \rangle \rightarrow w : X$ with $X = \mathbf{dead}(s)$ (by the rules (i)–(iv) for ready sets in T_2^*). ∎

Intuitively, Lemma 4.4.2(ii) says that the ready set rules (i)–(iv) of T_2^* are complete for detecting deadlocks. Using Lemma 4.4.6 we can now give the

Proof of Theorem 4.4.1. Let $s \in \mathfrak{L}_2$. Note that

$$\mathcal{O}_2[\![s]\!], \mathbf{restr}_{\mathbb{R}}(\mathcal{O}_2^*[\![s]\!]) \subseteq (A\setminus C)^* \cup (A\setminus C)^\omega \cup (A\setminus C)^* \cdot \{\perp, \delta\}.$$

We distinguish the following cases.

Case 1. $w \in (A\setminus C)^* \cup (A\setminus C)^\omega \cup (A\setminus C)^* \cdot \{\perp\}$. As an immediate consequence of Lemma 4.4.2(i) and the definition of $\mathbf{restr}_{\mathbb{R}}$ we have

$$w \in \mathcal{O}_2[\![s]\!] \quad \text{iff} \quad w \in \mathbf{restr}_{\mathbb{R}}(\mathcal{O}_2^*[\![s]\!]).$$

Case 2. $w\delta \in (A\setminus C)^* \cdot \{\delta\}$. Here we have the chain of equivalences

$$w\delta \in \mathcal{O}_2[\![s]\!]$$

iff $\langle s, w\rangle$ is a deadlocking configuration in T_2
iff $\exists X \in \Delta : T_2^* \vdash \langle s, w\rangle \to w : X$ (by Lemma 4.4.2(ii))
iff $\exists X \in \Delta : w : X \in \mathcal{O}_2^*[\![s]\!]$
iff $w\delta \in \mathbf{restr}_{\mathbb{R}}(\mathcal{O}_2^*[\![s]\!])$. ∎

Relating $\mathcal{D}_2$ and $\mathcal{D}_2^*$

The relationship between $\mathcal{D}_2$ and $\mathcal{D}_2^*$ is given by an abstraction operator **readies**: $\mathbb{P} \to \mathbb{R}_c$. For $p = \{b_1, ..., b_m, [b'_1, q_1], ..., [b'_n, q_n]\} \in \mathbb{P}_\omega$, inductively we put

$$\begin{aligned}\mathbf{readies}(p) = {} & \{b_1, ..., b_m\} \\ & \cup \{b_j \cdot \mathbf{readies}(q_j) \mid j = 1, ..., n\} \\ & \cup \{\varepsilon : X \mid X = \{b_1, ..., b_m, b'_1, ..., b'_n\} \subseteq C\}.\end{aligned}$$

For $p \in \mathbb{P}\setminus\mathbb{P}_\omega$ we have $p = \lim_n p_n$, with $p_n \in \mathbb{P}_n$, and put

$$\mathbf{readies}(p) = \lim_n (\mathbf{readies}(p_n))$$

where "$\lim_n$" is taken (as before) w.r.t. the metric on $\mathbb{R}_c$.

4.4.3. THEOREM. $\mathcal{D}_2^*[\![s]\!] = \mathbf{readies}(\mathcal{D}_2[\![s]\!])$ *for all guarded* $s \in \mathfrak{L}_2$.

The proof follows from:

4.4.4. LEMMA. *The operator* **readies**: $\mathbb{P} \to \mathbb{R}_c$ *is continuous and behaves homomorphically, i.e., for* $\mathbf{op} \in \{+, ;, \|\}$ *and* $p, p' \in \mathbb{P}$,

$$\mathbf{readies}(p \; \mathbf{op}^{\mathcal{D}_2} p') = \mathbf{readies}(p) \; \mathbf{op}^{\mathcal{D}_2^*} \mathbf{readies}(p').$$

Proof. Continuity is established by a variation of standard reasoning as in

[BBKM, BZ2]. For the same reason it suffices to prove the homomorphism property for $p, p' \in \mathbb{P}_\omega$ only. We proceed inductively and assume

$$p = \{b_1, ..., b_m, [b'_1, q_1], ..., [b'_n, q_n]\},$$
$$p' = \{\tilde{b}_1, ..., \tilde{b}_{m'}, [\tilde{b}'_1, q'_1], ..., [\tilde{b}'_{n'}, q'_{n'}]\}$$

with $m, n, m', n' \geqslant 0$.

Case 1 ($\mathbf{op} = +$).

$$\begin{aligned}
&\mathbf{readies}(p +^{\mathscr{D}_2} p') = \mathbf{readies}(p \cup p') \\
&= \{b_1, ..., b_m, \tilde{b}_1, ..., \tilde{b}_m\} \\
&\quad \cup \{b'_i \cdot \mathbf{readies}(q_i) \mid i = 1, ..., n\} \\
&\quad \cup \{\tilde{b}'_j \cdot \mathbf{readies}(q'_j) \mid j = 1, ..., n'\} \\
&\quad \cup \{\varepsilon : (X \cup Y) \mid X = \{b_1, ..., b_m, b'_1, ..., b'_n\} \subseteq C, \\
&\qquad\qquad Y = \{\tilde{b}_1, ..., \tilde{b}_{m'}, \tilde{b}'_1, ..., \tilde{b}'_{n'}\} \subseteq C\} \\
&= \{w \mid w \in \mathbf{readies}(p) \cup \mathbf{readies}(p')\} \\
&\quad \cup \{\varepsilon : (X \cup Y) \mid \varepsilon : X \in \mathbf{readies}(p) \text{ and } \varepsilon : Y \in \mathbf{readies}(p')\} \\
&\quad \cup \{w : X \mid w \neq \varepsilon \quad \text{and} \quad w : X \in \mathbf{readies}(p) \cup \mathbf{readies}(p')\} \\
&= \mathbf{readies}(p) \mathbin{\dot{+}^{\mathscr{D}_2}} \mathbf{readies}(p').
\end{aligned}$$

Case 2 ($\mathbf{op} = ;$).

$$\begin{aligned}
&\mathbf{readies}(p \,;^{\mathscr{D}_2} p') = \mathbf{readies}(p \cdot p') \\
&= \mathbf{readies}(\{[b_1, p'], ..., [b_m, p'], \\
&\qquad\qquad [b'_1, q_1 \cdot p'], ..., [b'_n, q_n \cdot p']\}) \\
&= \{\varepsilon : X \mid X = \{b_1, ..., b_m, b'_1, ..., b'_n\} \subseteq C\} \\
&\quad \cup \{b_i \cdot \mathbf{readies}(p') \mid i = 1, ..., m\} \\
&\quad \cup \{b'_j \cdot \mathbf{readies}(q_j \cdot p') \mid j = 1, ..., n\} \\
&= \{\varepsilon : X \mid \cdots\} \cup \bigcup \{b_i \cdot \mathbf{readies}(p') \mid \cdots\} \\
&\quad \cup \{b'_j \cdot (\mathbf{readies}(q_j) \mathbin{\dot{;}^{\mathscr{D}_2}} \mathbf{readies}(p')) \mid \cdots\} \qquad \text{(by induction)} \\
&= \{\varepsilon : X \mid \cdots\} \cup \bigcup \{b_i \cdot \mathbf{readies}(p') \mid \cdots\} \\
&\quad \cup \{(b'_j \cdot \mathbf{readies}(q_j)) \mathbin{\dot{;}^{\mathscr{D}_2}} \mathbf{readies}(p') \mid \cdots\} \\
&= (\{\varepsilon : X \mid X = \{b_1, ..., b_m, b'_1, ..., b'_n\} \subseteq C\} \\
&\quad \cup \{b_1, ..., b_m\} \\
&\quad \cup \{b'_j \cdot \mathbf{readies}(q_j)\}) \mathbin{\dot{;}^{\mathscr{D}_2}} \mathbf{readies}(p') \\
&= \mathbf{readies}(p) \mathbin{\dot{;}^{\mathscr{D}_2}} \mathbf{readies}(p').
\end{aligned}$$

Case 3 ($\mathbf{op} = \|$). By definition

$$p \parallel p' = (p \mathbin{⫡} p') \cup (p' \mathbin{⫡} p) \cup (p \mid p'),$$

where

$$
\begin{aligned}
p \mathbin{\lfloor\!\lfloor} p' &= \{[b_i, p'] \mid i = 1, ..., m\} \\
&\quad \cup \{[b'_j, q_j \| p'] \mid j = 1, ..., n\}, \\
p' \mathbin{\lfloor\!\lfloor} p &= \{[\bar{b}_k, p] \mid k = 1, ..., m'\} \\
&\quad \cup \{[\bar{b}'_l, q'_l \| p] \mid l = 1, ..., n'\}, \\
p \mid p' &= \{\tau \mid \exists c \in C : c \in \{b_1, ..., b_m\} \quad \text{and} \quad \bar{c} \in \{\bar{b}_1, ..., \bar{b}_{m'}\}\} \\
&\quad \cup \{[\tau, q'_l] \mid \exists c \in C : c \in \{b_1, ..., b_m\} \\
&\qquad\qquad \text{and } \bar{c} = \bar{b}'_l \text{ and } l \in \{1, ..., n'\}\} \\
&\quad \cup \{[\tau, q_j] \mid \exists c \in C : c \in \{\bar{b}_1, ..., \bar{b}_{m'}\} \\
&\qquad\qquad \text{and} \quad \bar{c} = b'_j \quad \text{and} \quad j \in \{1, ..., n\}\} \\
&\quad \cup \{[\tau, q_j \| q'_l] \mid \exists c \in C : c = b'_j \quad \text{and} \quad \bar{c} = \bar{b}'_l \\
&\qquad\qquad \text{and} \quad j \in \{1, ..., n\} \quad \text{and} \quad l \in \{1, ..., n'\}\}.
\end{aligned}
$$

Thus

$$
\begin{aligned}
&\mathbf{readies}(p \| p') \\
&= \{\varepsilon : (X \cup Y) \mid X \cap \bar{Y} = \emptyset, \text{ where} \\
&\qquad\qquad X = \{b_1, ..., b_m, b'_1, ..., b'_n\} \subseteq C, \\
&\qquad\qquad Y = \{\bar{b}_1, ..., \bar{b}_{m'}, \bar{b}'_1, ..., \bar{b}'_{n'} \subseteq C\} \\
&\quad \cup \mathbf{readies}(p \mathbin{\lfloor\!\lfloor} p') \backslash \varepsilon : \Delta \\
&\quad \cup \mathbf{readies}(p' \mathbin{\lfloor\!\lfloor} p) \backslash \varepsilon : \Delta \\
&\quad \cup \mathbf{readies}(p \mid p') \backslash \varepsilon : \Delta \\
&= \mathbf{readies}(p) \mathbin{\#} \mathbf{readies}(p') \\
&\quad \cup \mathbf{readies}(p) \mathbin{\lfloor\!\lfloor} \mathbf{readies}(p') \\
&\quad \cup \mathbf{readies}(p') \mathbin{\lfloor\!\lfloor} \mathbf{readies}(p) \\
&\quad \cup \mathbf{readies}(p) \mid \mathbf{readies}(p') \\
&\quad (\text{by definition of } \mathbf{readies} \text{ and induction}) \\
&= \mathbf{readies}(p) \|^{\mathscr{D}_2^*} \mathbf{readies}(p').
\end{aligned}
$$

Here we must *simultaneously* prove, by induction:

$$
\begin{aligned}
\mathbf{readies}(p \mathbin{\lfloor\!\lfloor} p') \backslash \varepsilon : \Delta &= \mathbf{readies}(p) \mathbin{\lfloor\!\lfloor} \mathbf{readies}(p') \\
\mathbf{readies}(p \mid p') \backslash \varepsilon : \Delta &= \mathbf{readies}(p) \mid \mathbf{readies}(p') \\
\mathbf{readies}(p \mathbin{\#} p') \backslash \varepsilon : \Delta &= \mathbf{readies}(p) \mathbin{\#} \mathbf{readies}(p').
\end{aligned}
$$

The details are left to the reader. ∎

Relating $\mathcal{O}_2^*$ and $\mathscr{D}_2^*$

Here we discuss

4.4.5. THEOREM. $\mathcal{O}_2^*[\![s]\!] = \mathscr{D}_2^*[\![s]\!]$ *for every guarded* $s \in \mathfrak{L}_2$.

Again, its proof follows the structure of that for "$\mathcal{O}_0[\![s]\!] = \mathcal{D}_0[\![s]\!]$" (Theorem 2.1). In particular, Theorems 2.4.10, 2.4.11, and 2.4.15 remain valid with $\mathcal{O}_2^*$, $\mathcal{D}_2^*$, and $\mathfrak{L}_2$ in place of $\mathcal{O}_0$, $\mathcal{D}_0$, and $\mathfrak{L}_0$. Thus it remains to show compositionality of $\mathcal{O}_2^*$, analogously to Theorem 2.4.2, but now involving the ready domain $\mathbb{R}$ and global nondeterminacy "+".

4.4.6. THEOREM. *For* $\mathbf{op} \in \{+, ;, \|\}$ *and* $s_1, s_2 \in \mathfrak{L}$,

$$\mathcal{O}_2^*[\![s_1 \mathbf{\ op\ } s_2]\!] \stackrel{\mathscr{D}_2}{=} \mathcal{O}_2^*[\![s_1]\!] \mathbf{\ op}^{\mathscr{D}_2}\, \mathcal{O}_2^*[\![s_2]\!].$$

Proof. *Case* 1 ($\mathbf{op} = +$). First we state some simple facts about the rule of global nondeterminacy in the transition system T_2^*:

(i) μ-unfolding:

$$T_2^* \vdash \langle s_1 + s_2, \varepsilon\rangle \rightarrow \langle s', \varepsilon\rangle$$

iff

$$\begin{aligned} &\exists s_1' \in \mathfrak{L}_2(s' = s_1' + s_2 \wedge T_2^* \vdash \langle s_1, \varepsilon\rangle \rightarrow \langle s_1', \varepsilon\rangle) \\ \vee\ &\exists s_2' \in \mathfrak{L}_2(s' = s_1 + s_2' \wedge T_2^* \vdash \langle s_2, \varepsilon\rangle \rightarrow \langle s_2', \varepsilon\rangle) \end{aligned}$$

(ii) selection by an action $b \in A$:

$$T_2^* \vdash \langle s_1 + s_2, \varepsilon\rangle \rightarrow \langle s', b\rangle$$

iff

$$\begin{aligned} &(s' \text{ stems from } s_1 \wedge T_2^* \vdash \langle s_1, \varepsilon\rangle \rightarrow \langle s', b\rangle) \\ \vee\ &(s' \text{ stems from } s_2 \wedge T_2^* \vdash \langle s_2, \varepsilon\rangle \rightarrow \langle s', b\rangle) \end{aligned}$$

(iii) ready sets:

$$T_2^* \vdash \langle s_1 + s_2, \varepsilon\rangle \rightarrow \varepsilon : Z$$

iff

$$\begin{aligned} &\exists X, Y \subseteq C : Z = X \cup Y \\ &\wedge T_2^* \vdash \langle s_1, \varepsilon\rangle \rightarrow \varepsilon : X \\ &\wedge T_2^* \vdash \langle s_2, \varepsilon\rangle \rightarrow \varepsilon : Y. \end{aligned}$$

Let us now analyze the possible elements of $\mathcal{O}_2^*[\![s_1 + s_2]\!]$. These are of the form $\varepsilon : Z$ or $b \cdot w$ with $b \in A$ and $w \in A^{\mathrm{rd}} = A^{\mathrm{st}} \cup A^* : \Delta$. (Note that $\varepsilon \notin \mathcal{O}_2^*[\![s]\!]$ for any $s \in \mathfrak{L}_2$.)

Subcase 1.1 $(\varepsilon: Z)$. $(\varepsilon: Z) \in \mathcal{O}_2^*[\![s_1 + s_2]\!]$

iff $T_2^* \vdash \langle s_1 + s_2, \varepsilon \rangle \to^* \varepsilon: Z$

iff $\exists X, Y \subseteq C: Z = X \cup Y \wedge T_2^* \vdash \langle s_1, \varepsilon \rangle \to^* \varepsilon: X$
$\wedge\ T_2^* \vdash \langle s_2, \varepsilon \rangle \to^* \varepsilon: Y$ (by facts (i) and (iii) above)

iff $\exists X, Y \subseteq C: Z = X \cup Y \wedge (\varepsilon: X) \in \mathcal{O}_2^*[\![s_1]\!] \wedge (\varepsilon: Y) \in \mathcal{O}_2^*[\![s_2]\!]$.

Subcase 1.2 $(b \cdot w)$. $b \cdot w \in \mathcal{O}_2^*[\![s_1 + s_2]\!]$

iff $\exists s' \in \mathfrak{L}_2 \cup \{E\}$:

$$T_2^* \vdash \langle s_1 + s_2, \varepsilon \rangle \to^* \langle s', b \rangle \wedge w \in \mathcal{O}_2^*[\![s']\!]$$

(by convention, we put here $\varepsilon \in \mathcal{O}_2^*[\![E]\!]$)

iff $\exists s' \in \mathfrak{L}_2 \cup \{E\}$:

$$(T_2^* \vdash \langle s_1, \varepsilon \rangle \to^* \langle s', b \rangle \wedge w \in \mathcal{O}_2^*[\![s']\!])$$
$$\vee\ (T_2^* \vdash \langle s_2, \varepsilon \rangle \to^* \langle s', b \rangle \wedge w \in \mathcal{O}_2^*[\![s']\!])$$

(by facts (i) and (ii) above)

iff $b \cdot w \in \mathcal{O}_2^*[\![s_1]\!] \vee b \cdot w \in \mathcal{O}_2^*[\![s_2]\!]$.

By the analysis in Subcase 1.1 and 1.2, we finally have

$$\begin{aligned}\mathcal{O}_2^*[\![s_1 + s_2]\!] &= \{\varepsilon: (X \cup Y) \mid \varepsilon: X \in \mathcal{O}_2^*[\![s_1]\!] \wedge \varepsilon: Y \in \mathcal{O}_2^*[\![s_2]\!]\} \\ &\quad \cup \{w \in A^{\mathrm{st}} \mid w \in \mathcal{O}_2^*[\![s_1]\!] \cup \mathcal{O}_2^*[\![s_2]\!]\} \\ &\quad \cup \{w: X \in A^*: \Delta \mid w \neq \varepsilon \wedge w: X \in \mathcal{O}_2^*[\![s_1]\!] \cup \mathcal{O}_2^*[\![s_2]\!]\} \\ &= \mathcal{O}_2^*[\![s_1]\!] +^{\dot{\mathfrak{D}}_2} \mathcal{O}_2^*[\![s_2]\!].\end{aligned}$$

Case 2 (**op** = ;). Straightforward.

Case 3 (**op** = $\|$). First observe that the Synchronization Lemma 3.4.7 also holds for $\mathfrak{L}_2$ and T_2^* instead of $\mathfrak{L}_1$ and T_1^*. Note that the rules for "global nondeterminacy: selection by synchronization" in T_2^* are needed here because the contexts considered under (3.7) and (3.8) in the proof of Lemma 3.4.7 may now contain "+." E.g., in (3.8) we now have

$$s_1 ::= c \mid s_1 ; s \mid s_1 \| s \mid s \| s_1 \mid s_1 + s \mid s + s_1.$$

Using the Synchronization Lemma we can prove, analogously to Lemma 3.4.6,

$$w \in \mathcal{O}_2^*[\![s_1 \| s_2]\!] \quad \text{iff} \quad \exists u \in \mathcal{O}_2^*[\![s_1]\!], v \in \mathcal{O}_2^*[\![s_2]\!] : w \in \{u\} \|^{\dot{\mathcal{D}}_2} \{v\} \tag{4.3}$$

for $w \in A^{\mathrm{st}}$ and $s_1, s_2 \in \mathfrak{L}_2$.

In the process of proving (4.3), we obtain

$$\forall s_1, s_2 \in \mathfrak{L}_2 \forall s'_1, s'_2 \in \mathfrak{L}_2 \cup \{E\} \; \forall w \in A^*:$$
$$T_2^* \vdash \langle s_1 \| s_2, \varepsilon \rangle \to^* \langle s'_1 \| s'_2, w \rangle$$

iff $\exists u, v \in A^*$: (4.4a)

$$\begin{aligned} & T_2^* \vdash \langle s_1, \varepsilon \rangle \to^* \langle s'_1, u \rangle \\ \wedge\; & T_2^* \vdash \langle s_2, \varepsilon \rangle \to^* \langle s'_2, v \rangle \\ \wedge\; & w \in \{u\} \|^{\mathscr{D}_2^{\bullet}} \{v\} \end{aligned}$$

(compare Lemma 3.4.6). Furthermore, we have

$$\begin{aligned} & \forall s \in \mathfrak{L}_2 \forall w : Z \in A^* : \Delta \\ & w : Z \in \mathcal{O}_2^*[\![s]\!] \quad \text{iff} \quad \exists s' \in \mathfrak{L}_2 : T_2^* \vdash \langle s, \varepsilon \rangle \to^* \langle s', w \rangle \\ & \qquad\qquad\qquad\qquad\qquad\qquad \wedge\; T_2^* \vdash \langle s', \varepsilon \rangle \to \varepsilon : Z. \end{aligned}$$

Moreover, we have, as an immediate consequence of the rules for ready sets in T_2^* (4.4.2), especially rule (iv):

$$T_2^* \vdash \langle s_1 \| s_2, \varepsilon \rangle \to \varepsilon : Z$$

iff

$$\begin{aligned} & \exists X, Y \subseteq C : Z = X \cup Y \wedge X \cap \bar{Y} = \varnothing \\ & \quad \wedge\; T_2^* \vdash \langle s_1, \varepsilon \rangle \to \varepsilon : X \\ & \quad \wedge\; T_2^* \vdash \langle s_2, \varepsilon \rangle \to \varepsilon : Y. \end{aligned} \tag{4.4c}$$

Combining (4.4a), (4.4b), and (4.4c) yields

$$w : Z \in \mathcal{O}_2^*[\![s_1 \| s_2]\!]$$

iff

$$\begin{aligned} & \exists u : X \in \mathcal{O}_2^*[\![s_1]\!], v : Y \in \mathcal{O}_2^*[\![s_2]\!] : \\ & w = \{u\} \|^{\mathscr{D}_2^{\bullet}} \{v\} \wedge Z = X \cup Y \wedge X \cap \bar{Y} = \varnothing. \end{aligned} \tag{4.5}$$

With (4.3) and (4.5) we have indeed

$$\mathcal{O}_2^*[\![s_1 \| s_2]\!] = \mathcal{O}_2^*[\![s_1]\!] \|^{\mathscr{D}_2^{\bullet}} \mathcal{O}_2^*[\![s_2]\!].$$

This finishes the proof of Theorem 4.4.6. ∎

With Theorem 4.4.6 also our argument for Theorem 4.4.5 is completed. ∎

Putting It All Together

Before we can prove the desired relationship between $\mathcal{C}_2$ and $\mathcal{D}_2$ (cf. (4.2)), we need one more lemma.

4.4.7. LEMMA. *For every* $p \in \mathbb{P}$,

$$\mathbf{streams}\ (\mathbf{restr}_{\mathbb{P}}(p)) = \mathbf{restr}_{\mathbb{R}}(\mathbf{readies}(p)).$$

Proof. By limit considerations it suffices to prove the equation for $p \in \mathbb{P}_\omega$. We proceed inductively and assume

$$p = \{b_1, ..., b_m, [b'_1, q_1], ..., [b'_n, q_n]\}$$

with $X =_{\text{df}} \{b_1, ..., b_m, b'_1, ..., b'_n\}$. Then the l.h.s. yields

$$\begin{aligned} \mathbf{restr}_{\mathbb{P}}(p) = \{b_i \mid b_i \in p \quad &\text{and} \quad b_i \notin C\} \\ \cup \{[b'_j, \mathbf{restr}_{\mathbb{P}}(q_j)] \mid [b'_j, q_j] \in p \quad &\text{and} \quad b'_j \notin C\}\} \end{aligned}$$

and thus

$$\mathbf{streams}(\mathbf{restr}_{\mathbb{P}}(p))$$
$$= \begin{cases} \{b_i \mid b_i \in p \quad \text{and} \quad b_i \notin C\} & \\ \cup \{b'_j \cdot \mathbf{streams}(\mathbf{restr}_{\mathbb{P}}(q_j)) \mid [b'_j, q_j] \in p \quad \text{and} \quad b'_j \notin C\} & \text{if} \quad X \nsubseteq C \\ \{\delta\} & \text{if} \quad X \subseteq C \end{cases}$$

Now the r.h.s. yields

$$\begin{aligned} \mathbf{readies}(p) = &\{\varepsilon : X \mid X \subseteq C\} \\ &\cup \{b_i \mid b_i \in p\} \\ &\cup \{b'_j \cdot \mathbf{readies}(q_j) \mid [b'_j, q_j] \in p\} \end{aligned}$$

and thus

$$\mathbf{restr}_{\mathbb{R}}(\mathbf{readies}(p))$$
$$= \begin{cases} \{b_i \mid b_i \in p \quad \text{and} \quad b_i \notin C\} & \\ \cup \{b'_j \cdot \mathbf{restr}_{\mathbb{R}}(\mathbf{readies}(q_j)) \mid [b'_j, q_j] \in p \quad \text{and} \quad b'_j \notin C\} & \text{if} \quad X \nsubseteq C \\ \{\delta\} & \text{if} \quad X \subseteq C. \end{cases}$$

By induction, we have l.h.s. = r.h.s. ∎

Now we are prepared for the main result on $\mathfrak{L}_2$:

4.4.8. THEOREM. *$\mathcal{O}_2[\![s]\!] = \alpha_2(\mathcal{D}_2[\![s]\!])$ for all guarded $s \in \mathfrak{L}_2$, where $\alpha_2 =$* **streams** ∘ **restr**$_{\mathbb{P}}$.

Proof. Theorem 4.4.1 states $\mathcal{O}_2[\![s]\!] = \mathbf{restr}_{\mathbb{R}}(\mathcal{O}_2^*[\![s]\!])$ for $s \in \mathfrak{L}_2$, Theorem 4.4.3 states $\mathcal{D}_2^*[\![s]\!] = \mathbf{readies}(\mathcal{D}_2[\![s]\!])$ for guarded $s \in \mathfrak{L}_2$, and Theorem 4.4.5 states $\mathcal{O}_2^*[\![s]\!] = \mathcal{D}_2^*[\![s]\!]$ for guarded $s \in \mathfrak{L}_2$. Thus we obtain

$$\mathcal{O}_2[\![s]\!] = \mathbf{restr}_{\mathbb{R}}(\mathbf{readies}(\mathcal{D}_2[\![s]\!])).$$

Using Lemma 4.4.7 completes the proof of this theorem. ∎

APPENDIX: DIAGRAM OF RESULTS

$\mathfrak{L}$: *Shuffle and Local Nondeterminacy.*

$$\begin{tikzcd} T_0 \arrow[d, dashed] & & \text{guarded } s \\ \mathcal{O}_0[\![s]\!] & = & \mathcal{D}_0[\![s]\!] \end{tikzcd}$$

linear streams

$\mathfrak{L}_1$: *Synchronization Merge and Local Nondeterminacy.*

$$\begin{tikzcd} T_1 \arrow[d, dashed] & T_1^* \arrow[d, dashed] & & \text{guarded } s \\ \mathcal{O}_1[\![s]\!] & \mathcal{O}_1^*[\![s]\!] \arrow[l, "\mathbf{restr}_{\mathbb{S}}"'] & = & \mathcal{D}_1[\![s]\!] \end{tikzcd}$$

linear streams with δ

$\mathfrak{L}_2$: *Synchronization Merge and Global Nondeterminacy.*

$$\begin{tikzcd} T_2 \arrow[d, dashed] & T_2^* \arrow[d, dashed] & & \text{guarded } s \\ \mathcal{O}_2[\![s]\!] & \mathcal{O}_2^*[\![s]\!] = \mathcal{D}_2^*[\![s]\!] \arrow[l, "\mathbf{restr}_{\mathbb{R}}"'] & & \mathcal{D}_2[\![s]\!] \arrow[ll, "\mathbf{readies}"'] \arrow[lll, bend left, "\mathbf{streams} \circ \mathbf{restr}_{\mathbb{P}}"] \end{tikzcd}$$

streams with δ ready domain branching processes

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