# Fully Abstract Models for Concurrent Languages

VRIJE UNIVERSITEIT

# Fully Abstract Models for Concurrent Languages

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EIICHI HORITA

geboren te Ishikawa Prefecture, Japan

Promotor : prof.dr J.W. de Bakker
Copromotor : dr J.J.M.M. Rutten
Referent : prof.dr P. Panangaden

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# Introduction

## 1.1 Full Abstractness Problem

In this thesis, we treat two kinds of semantic models for concurrent programming languages: *denotational semantics* in the sense of Scott and Strachey (cf. [Mos 90]) and *structural operational semantics* (*SOS*) in the style of Plotkin (cf. [Plo 81]). (In this thesis, we use the terms "model", "semantics", and "semantic model" as synonyms; we mean by a *model* a *function* from the set of programs of a (programming) language to a mathematically defined set, called a *semantic domain.*)

Here we mean, by *denotational models*, those models for a programming language which are constructed compositionally, i.e., in terms of semantic interpretations of syntactic constructs of the language, and which give the meaning of a recursive statement as the *limit* of its finite approximations in an appropriate semantic domain with some notion of *convergence.*

On the other hand, we mean, by *operational models*, all those semantics which are defined in terms of a *labeled transition system* (*LTS*), including, e.g., all semantics of van Glabbeek's *linear time-branching time spectrum*, which consists of *bisimulation semantics*, *trace semantics*, and nine other models lying in between the two (cf. Chapter 1 of [Gla 90]). Operational models in this sense are defined in two steps: First, we define a LTS in terms of some axioms and rules for inferring transitions among programs, and then we can obtain various operational semantics by abstracting from some unwanted details involved in the LTS. We call this approach to program semantics *two-level operational semantics*, following the terminology of De Nicola (cf. § 3.0 of [DeN 85a]), and refer to those semantic models which are obtained from LTS's by some abstraction as *high-level operational models.*

In many cases, the most interesting characteristic of programs (or more generally, software or hardware systems) can be represented by a high-level operational model $\mathcal{O}$ (for example, the set of possible finite sequences of events a program may perform can be characterized operationally by a kind of *trace model*). Such an operational semantics $\mathcal{O}$ serves as a touchstone for assessing various (denotational/operational) semantics, even though $\mathcal{O}$ itself may have some crucial defects such as the lack of *compositionality*. A *fully abstract model* is the one assessed as the highest in terms of such a touchstone, in the sense explained below.

The *full abstractness problem* for programming languages was first raised by Milner ([Mil 75]). In general, a fully abstract model $\mathcal{D}$ w.r.t. $\mathcal{O}$ is the most abstract of those models which are correct w.r.t. $\mathcal{O}$ and compositional, and thus, $\mathcal{D}$ is the most desirable model from the following point of view:

In general, the high-level operational model $\mathcal{O}$ represents only the essence of program behaviors, and may be too abstract to be compositional. (Note that we do not claim here that operational models in general are too abstract or lack compositionality; see Remark 1.1 below.) Compositionality, in turn, is needed for the *modular* definition of program meanings, i.e., in order to define the meaning of a composite statement in terms of the meanings of its components. It is also needed to make it possible for two equivalent systems $A$ and $B$, to substitute $A$ for $B$ within a composite system, without affecting the overall meaning. Thus some *extra* information needs to be involved to construct a compositional model. However, it is desirable for the extra information to be *minimum* so as not to bring about inessential details. The fully abstract model meets these requirements. As we will see such fully abstract models can be constructed both denotationally and operationally.

**Remark 1.1** There is a great diversity of operational models as mentioned above: Actually some of them are very concrete, and some are compositional, as we will see in the later chapters of this thesis. Note also that we do not claim that operational models are in general more abstract than denotational ones: Some operational models may be less abstract than a given denotational model $\tilde{\mathcal{D}}$ and some other ones may be more abstract than $\tilde{\mathcal{D}}$, with the others being incomparable with $\tilde{\mathcal{D}}$. For an illustration of this situation see Figure 3.1, where we may suppose that $\tilde{\mathcal{D}}$ is the fully abstract model in the figure, and various models in the figure are defined operationally, with some of them being compositional. ∎

## 1.2 Approaches to the Problem

We take two approaches to constructing a fully abstract model for languages with recursion: The first approach is taken in Chapters 4, 6, 7, 8, where we construct a fully abstract model *denotationally*, i.e., construct the model in terms of semantic interpretations of syntactic constructs, giving the meaning of a recursive statement as the *limit* of its approximations in a certain semantic domain; we use two kinds of mathematical structures for semantic domains: In Chapters 4, 7, 8, we employ *complete metric spaces* as semantic domains for the fully abstract (denotational) models; in Chapter 6, we employ a *complete partial order*. The second approach is taken in Chapter 5, where we construct a fully abstract model *operationally*, i.e., in terms of a *LTS* in the style of *SOS*.

## 1.3 Overview of the Thesis

The thesis consists of three parts: Part I composed of Chapters 1–3 gives an introduction and preliminaries to the thesis; the main body of the thesis consists of Parts II and III.

### 1.3.1 Overview of the Languages

We treat six languages, $\mathcal{L}_0$ to $\mathcal{L}_5$, in the thesis. The first three, $\mathcal{L}_0$ to $\mathcal{L}_2$, are what we like to call *uniform languages*; they are uniform in that the elementary actions are uninterpreted symbols from some alphabet; the other three, $\mathcal{L}_3$ to $\mathcal{L}_5$, are *nonuniform* in that the elementary actions are (primarily) *assignments* of values to variables and the meaning of atomic statements generally depends on the current state.

First, uniform languages are treated in Chapter 3 and Part II composed of Chapters 4–6; next nonuniform languages are treated in Part III composed of Chapters 7–8. All the languages have a certain kind of *atomic actions*, *alternative choice*, and *sequential* and *parallel* composition; in addition, all of them except $\mathcal{L}_0$ have a certain form of *guarded recursion*. Guardedness is a syntactic restriction reminiscent of Greibach normal form for context-free grammars; it is imposed to ensure contractivity (of operations corresponding to the bodies of recursive programs). An overview of the languages treated in the present thesis is given in Table 1.1. A comparison of the expressive power of the languages is given in Figure 1.1.

The extension from $\mathcal{L}_0$ to $\mathcal{L}_2$ and the one from $\mathcal{L}_3$ to $\mathcal{L}_5$ are motivated by practical interest, as well as by theoretical interest. In the extension from $\mathcal{L}_0$ to $\mathcal{L}_1$, a construct for *communication* (or synchronization of complementary actions) is added so that cooperation among processes is expressible and possible; in the extension from $\mathcal{L}_1$ to $\mathcal{L}_2$, we add *parameterization* and *value-passing*, which are crucial for programs involving *data* in addition to a definite number of events. Also, nonuniform languages, such as $\mathcal{L}_3$–$\mathcal{L}_5$ treated in Part III, are important from the practical point of view, since many algorithms are naturally expressed in this type of languages, and many of practical languages are of this type. (*C* and *Ada* are examples of such languages.) The motive for the extension from $\mathcal{L}_3$ to $\mathcal{L}_4$ is the addition of value passing as in the extension of $\mathcal{L}_1$ to $\mathcal{L}_2$; in the extension from $\mathcal{L}_4$ to $\mathcal{L}_5$, we add *parameterization* and *locality*; of these new features, *parameterization* is crucial for programs with data as mentioned above, and *locality* is very useful for modular design/development of large systems/programs. (For a discussion about the importance of data and nonuniform languages in the context of process algebras, see [Pon 92].)

The emphasis in the thesis is on semantic definitions—rather than on pragmatic use—of language concepts; however, several example programs are given for demonstrating the expressive power of the languages $\mathcal{L}_i$ $(i = 1, \ldots, 5)$ (see Figures 4.1, 4.3, 8.1, 8.2). (Of the example programs, the ones written in the uniform languages $\mathcal{L}_i$ $(i = 1, 2)$ have been executed by means of the CCS interpreter

Table 1.1: Summary of the Languages

| Type of Language | | Chapter/ Section | Form of Recursion | Features |
|---|---|---|---|---|
| Uniform | $\mathcal{L}_0$ | Chap. 3 | | |
| | $\mathcal{L}_1$ | § 4.2 | Declaration | Communication, Recursion |
| | $\mathcal{L}_{\bar{1}}$ | Chap. 5, Chap. 6 | $\mu$-notation | |
| | $\mathcal{L}_2$ | § 4.3 | Declaration | Parameterization, Value-Passing, Recursion |
| Nonuniform | $\mathcal{L}_3$ | § 7.3 | Declaration | Recursion |
| | $\mathcal{L}_4$ | § 7.4 | Declaration | Value-Passing, Recursion |
| | $\mathcal{L}_5$ | Chap. 8 | $\mu$-notation | Parameterization, Value-Passing, Locality, Recursion |

reported in [Hor 88] and [Hor 89], to check that they run as intended.)

There are two approaches to introducing *recursion* to languages: One is to use some form of *$\mu$-notation* (in the form of, say, "$(\mu X.\ S(X))$"), and the other is to use *declarations*. Each approach has its advantages and disadvantages:

Using $\mu$-notation has the advantage that the resulting language is simple in the sense that it has only one syntactic category, i.e., that of *statements*. The $\mu$-notation, however, entails so-called *nested recursion* (in the form of, say, "$(\mu X_0.\ (\mu X_1.\ S(X_0, X_1)))$"), which may make the definition of denotational semantics complicated.

The advantages and disadvantages of using declarations contrast with the ones of $\mu$-notation: It has the disadvantage that the resulting language is complex in the sense that it has at least two syntactic categories, i.e., those of *statements* and *declarations*. The definition of denotational semantics, however, may be made simpler because of the absence of nested recursion.

We take both approaches in the thesis (see Table 1.1), because $\mu$-notation is convenient for the proof of full abstractness in some situations, while using declarations is sufficient in others. In fact, there is a minor variant, which we denote by $\mathcal{L}_{\bar{1}}$, of the language $\mathcal{L}_1$: The language $\mathcal{L}_1$ treated in § 4.2 utilizes *declarations*, and the variant $\mathcal{L}_{\bar{1}}$ treated in Chapters 5, 6 utilizes *$\mu$-notation* (the interested reader may want to compare the technicalities of § 4.2 with those of Chapter 6). We usually identify $\mathcal{L}_1$ and $\mathcal{L}_{\bar{1}}$ when we discuss the expressive power of languages, because the difference between the two is inessential in this respect.

$$\begin{array}{llll} \mathcal{L}_0 & & \mathcal{L}_3 & \\ \downarrow & +\ \text{Communication, Recursion} & \downarrow & +\ \text{Value-Passing} \\ \mathcal{L}_1 & \xrightarrow{\quad < \quad} & \mathcal{L}_4 & \\ \downarrow & +\ \text{Parameterization, Value-Passing} & \downarrow & +\ \text{Parameterization, Locality} \\ \mathcal{L}_2 & \xrightarrow{\quad < \quad} & \mathcal{L}_5 & \end{array}$$

Figure 1.1: Comparison of Expressive Power of Languages

Below, we give an overview of the following chapters.

### 1.3.2 Overview of the Chapters

**Part I. Introduction and Preliminaries**

**Chapter 2. Mathematical Preliminaries:** Basic notions concerning complete metric spaces (cms's) and cpo's are explained. Then, the relationship between cms's and cpo's is discussed, which leads to the following result:

> *if a chain in the space of* closed *sets of sequences is also a Cauchy sequence, then the lub of the chain coincides with its metric limit.*

This is an extension of a similar result for the space of *compact* sets of sequences; this extension is necessary in Chapter 6, because we need to treat closed sets of sequences there, rather than compact ones.

**Chapter 3. Introduction to Fully Abstract Models:** This chapter presents general definitions concerning the full abstractness problem, providing a fully abstract model for a very simple concurrent language $\mathcal{L}_0$, which has *atomic actions*, *alternative choice*, and *sequential* and *parallel* composition, but does not have communication (or *synchronization* of complementary actions). Even for this simple language, the linear operational model involving both termination and deadlock is not compositional, and the fully abstract model is given so that it will serve as an illustrative example of constructing a fully abstract model.

**Part II. Models for Uniform Languages**

**Chapter 4. Full Abstractness of Metric Semantics for Communicating Processes with Value-Passing:** Two uniform languages $\mathcal{L}_1$ and $\mathcal{L}_2$ are treated. The first language $\mathcal{L}_1$ has *communication* (or *synchronization* of complementary actions) and recursion in addition to all constructs of $\mathcal{L}_0$. The second language $\mathcal{L}_2$ has *parameterization* and *value passing* in addition. For each language $\mathcal{L}_i$ $(i = 1, 2)$

a denotational model $\mathcal{M}_i$ is constructed on the basis of a complete metric space, and it shown that $\mathcal{M}_i$ is fully abstract w.r.t. a strong linear operational model for $\mathcal{L}_i$. (In this thesis, we mean by *strong models*, semantic models which do not abstract from *internal actions* usually denoted by $\tau$. Also, by *denotational models*, we mean semantic models constructed compositionally on the basis of some mathematical domain such as a *complete metric space* or a *complete partial order*. For more details, see Chapter 3.)

**Chapter 5. Fully Abstract Models for Communicating Processes with respect to Weak Linear Semantics with Divergence:** We treat the language $\mathcal{L}_{\bar{1}}$, which is a minor variant of the language $\mathcal{L}_1$ introduced in Chapter 4. (As described in § 1.3.1, the only difference between $\mathcal{L}_{\bar{1}}$ and $\mathcal{L}_1$ is in the way to treat recursion: $\mathcal{L}_{\bar{1}}$ (resp. $\mathcal{L}_1$) uses $\mu$-notation (resp. declarations) to treat recursion.) Two compositional models $\mathcal{C}_1$ and $\mathcal{C}_1^{\mathrm{i}}$ for $\mathcal{L}_1$ are defined operationally, and it is shown that $\mathcal{C}_1$ (resp. $\mathcal{C}_1^{\mathrm{i}}$) is fully abstract w.r.t. a weak linear operational model $\mathcal{O}_1$ (resp. $\mathcal{O}_1^{\mathrm{i}}$). (In this thesis, we mean by *weak models* semantic models which abstract from internal actions usually denoted by $\tau$.) The two operational models $\mathcal{O}_1$ and $\mathcal{O}_1^{\mathrm{i}}$ are different in that the latter involves only internal actions, whereas the former involves both external actions (communication with the environment) and internal actions.

**Chapter 6. A Fully Abstract Model for Communicating Processes Based on the Smyth Powerdomain of Failures:** A denotational model $\mathcal{D}_{\bar{1}}^{\mathrm{wf}}$ for $\mathcal{L}_1$ is constructed in an order-theoretic setting, and its full abstractness w.r.t. a weak linear operational model $\mathcal{O}_{\bar{1}}^{\mathrm{mf}}$ is established, where $\mathcal{O}_{\bar{1}}^{\mathrm{mf}}$ is a weak linear operational model similar to $\mathcal{O}_{\bar{1}}^{\mathrm{wm}}$ treated in Chapter 5, but the former is slightly different from the latter in the following sense: The model $\mathcal{O}_{\bar{1}}^{\mathrm{mf}}$ is based on the view that *divergence* is disastrous, whereas $\mathcal{O}_{\bar{1}}^{\mathrm{wm}}$ is based on another view that *divergence* is not particularly disastrous as deadlock is not.

**Part III. Models for Nonuniform Languages**

**Chapter 7. Fully Abstract Denotational Models for Nonuniform Concurrent Languages:** Two *nonuniform* languages $\mathcal{L}_3$ and $\mathcal{L}_4$ are treated. The first language $\mathcal{L}_3$ has *assignment* of values to individual variables, and recursion in addition to all constructs of $\mathcal{L}_0$. The second language $\mathcal{L}_4$ has *value passing* in addition. For each language $\mathcal{L}_i$ $(i = 3, 4)$ a denotational model $\mathcal{M}_i$ is constructed on the basis of a complete metric space, and it shown that $\mathcal{M}_i$ is fully abstract w.r.t. a strong linear operational model for $\mathcal{L}_i$.

**Chapter 8. A Fully Abstract Model for a Nonuniform Concurrent Languages with Parameterization and Locality:** The result for $\mathcal{L}_4$ presented in Chapter 7 is extended to a language $\mathcal{L}_5$ which is obtained from $\mathcal{L}_4$ by extending it so as to incorporate *parametrization* and *locality*.

Table 1.2: Interdependence of Chapters/Sections

| | | |
|---|---|---|
| Chap. 1 | | |
| Chap. 2 | § 2.1 | |
| | § 2.2 | § 2.1 |
| | § 2.3 | § 2.1 |
| | § 2.4 | §§ 2.1–2.3 |
| Chap. 3 | | § 2.1 |
| Chap. 4 | | §§ 2.1–2.2, Chap. 3 |
| Chap. 5 | | § 2.1, Chap. 3 |
| Chap. 6 | | §§ 2.1–2.4, Chap. 3, Chap. 5 |
| Chap. 7 | | §§ 2.1–2.2, Chap. 3 |
| Chap. 8 | | §§ 2.1–2.2, Chap. 3, (Chap. 4), Chap. 7 |

**Chapter 9. Conclusion:** A summary is given of the results obtained in the thesis, with several remarks about directions for future research.

### 1.3.3 Interdependence of the Chapters

Chapters of the thesis, especially Chapters 4–8, can be read almost independently, while there are certain logical dependences between the chapters/sections of the thesis. Logical dependence of chapters/sections is summarized in Table 1.2 (where the section/chapter in the left-hand column of each row depends on the sections/chapters in the right-hand column, with the sections/chapters in parentheses denoting non-essential dependence); it is also described in Figure 1.2 (where essential dependence (resp. non-essential dependence) of Chapter/Section $N'$ on Chapter/Section $N$ is denoted by $N$—$N'$ (resp. by $N - - - N'$)). Note that not all sections of Chapter 2 are necessary for each of Chapters 4–8: Each of Chapters 4–8 depends on § 2.1 (and Chapter 3), but only Chapter 6 depends on §§ 2.3 and 2.4.

## 1.4 Origins of the Chapters

Parts of the thesis appeared in several papers: Chapter 4 is based on [Hor 92c]. Chapter 5 is a minor revision of [Hor 92a]. Chapter 7 is an extension of joint work [HBR 90] with J. de Bakker and J. Rutten, and Chapter 8 is an extension of [Hor 93].

## 1.5 Related Work

The work reported in this thesis is closedly related to and based on the pioneering work of several people. In particular, Milner's *CCS* and *CSP* (which is due to

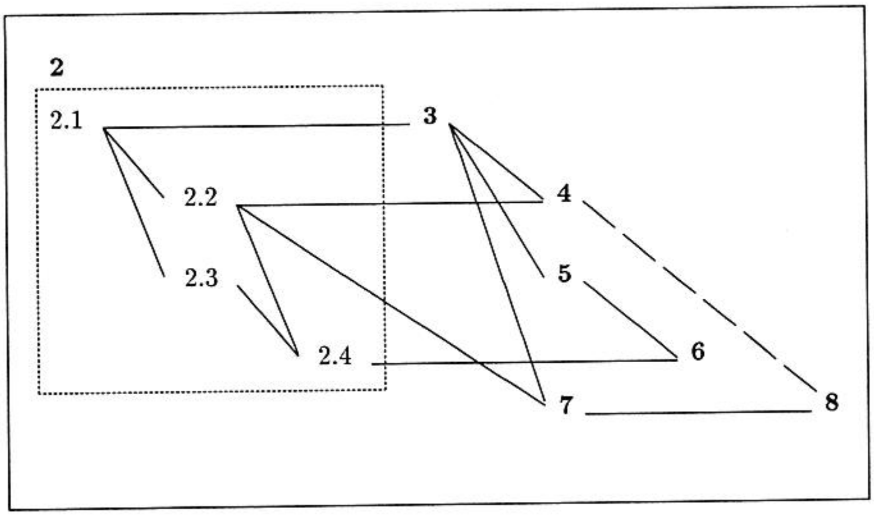


Figure 1.2: Interdependence of Chapters/Sections

Hoae and his colleagues) are closely related to the present work.

### 1.5.1 Languages

The languages $\mathcal{L}_i$ $(i = 0, 1, 2)$ treated in Parts I and II are very similar to CCS: The language $\mathcal{L}_1$ is the same as *pure CCS*, except that $\mathcal{L}_1$ has general *sequential composition* instread of more restricted *action-prefixing* of CCS, but does not have *relabeling*, which pure CCS has; likewise, $\mathcal{L}_2$ is the same as *full CCS*, except for certain minor differences.

The languages $\mathcal{L}_i$ $(i = 3, 4, 5)$ treated in Part III can be viewed as nonuniform versions of the languages $\mathcal{L}_i$ $(i = 0, 1, 2)$ treated in Parts I and II (see Figure 1.1). Also the nonuniform languages are similar to the original version of CSP ([Hoa 78]), in that both $\mathcal{L}_i$ $(i = 3, 4, 5)$ and the original CSP have variables to which values can be set, although there is an fundamental difference between variables of $\mathcal{L}_i$ $(i = 3, 4, 5)$ and those of the original CSP: Variables of $\mathcal{L}_i$ $(i = 3, 4, 5)$ can be shared by two or more processes, while variables in the original CSP cannot be shared.

### 1.5.2 Semantics

All the denotational models presented in this thesis are variants of the failures model first proposed for CSP by Brookes, Hoare, and Roscoe in [BHR 85]; thus as far as semantic models are concerned, this thesis is most closely related to the work on CSP semantics of Brookes, Hoare, and Roscoe ([BHR 85], [BHR 85], [BR 84], [Ros 84]). A detailed comparison between several fully abstract models presented in this thesis and the *improved failures semantics* due to Brookes and Roscoe ([BR 84]) is given in § 6.6.

On the other hand, all the operational semantics are provided in the style of Plotkin's Structured Operational Semantics ([Plo 81]).

We primarily exploit metric topology as mathematical framework for semantic construction, following the approach of De Bakker and Zucker ([BZ 82]) and their colleagues ([AR 88], [KR 90], [Rut 89]).

Hennessy and De Nicola emphasized the importance of full abstractness issues in the context of concurrency semantics ([DeN 85a], [DeN 85b], [DH 84], [Hen 88]), which encouraged the work reported in this thesis.

# Chapter 2

# Mathematical Preliminaries

As mathematical domains for our operational and denotational models we shall use *complete metric spaces* (*cms*'s) and *complete partial orders* (*cpo*'s) composed of (sets of) *sequences.* In this chapter, we first present basic notation for sets and functions in § 2.1. Then, in § 2.2 (resp. in § 2.3) we present standard notions on cms's (resp. on cpo's) with their basic properties, and some notions specific to domains of (sets of) sequences. Finally, in § 2.4, we will establish several connections between cms's and cpo's composed of sets of sequences.

There are two stylistic matters to be mentioned: Throughout the thesis, the end of a definition, theorem, and so on is marked by '▮', while the end of a proof is marked by '■'. The phrase "if$(B, E, E')$" is used as conditionals in mathematical formulas: If $B$ holds (resp. does not hold), then this phrase refers to $E$ (resp. to $E'$). (We prefer this notation because this clarifies the scope of conditionals, although some people write "if $B$ then $E$ else $E'$" instead.)

## 2.1 Basic Notation

In this thesis, we use the following notations on sets and functions.

**Notation 2.1** (Sets and Functions) Let $X$, $Y$ be sets.

**(1)** The usual $\lambda$-notation is used for denoting functions: For a variable $x$, and an expression $E(x)$, the expression $(\lambda x \in X.\ E(x))$ denotes the function which maps $x \in X$ to $E(x)$. We sometimes write $\langle E(x)\rangle_{x\in X}$ or $\langle E(x) : x \in X\rangle$ for $(\lambda x \in X.\ E(x))$.

**(2)** The *cardinality* of $X$ is denoted by $\sharp(X)$. The *powerset* of $X$ is denoted by $\wp(X)$. The set of *nonempty* subsets (resp. *finite* subsets) of $X$ is denoted by $\wp_+(X)$ (resp. by $\wp_{\mathrm{fin}}(X)$); the set of *nonempty finite* subsets of $X$ is denoted by $\wp_{+\mathrm{fin}}(X)$. The set of functions (resp. partial functions) from $X$ to $Y$ is denoted by $(X \to Y)$ or by $Y^X$ (resp. by $(X \nrightarrow Y)$). Let

$$(X \rightarrowtail Y) = \{f \in (X \to Y) : f \text{ is 1-1}\}.$$

The set of *natural numbers* is denoted by $\omega$, and each number $n \in \omega$ is identified with the set $\{i \in \omega : 0 \leq i < n\}$ as usual in set theory. For $n \in \omega$, let $\bar{n} = \{m \in \omega : 1 \leq m \leq n\}$. The *domain* and range of a function $f$ is denoted by $\mathrm{dom}(f)$ and $\mathrm{ran}(f)$, respectively. For $f \in (X \to Y)$ and a set $Z$, the *restriction* of $f$ to $Z$ is denoted by $f \ \ Z$. That is,

$$f \ \ Z = (\lambda x \in \mathrm{dom}(f) \cap Z.\ f(x)).$$

**(3)** The set of *integers* is denoted by **Int**. For $i, j \in \mathbf{Int}$, let

$$[i..j] = \{k \in \mathbf{Int} : i \leq k \leq j\}.$$

**(4)** Let $M$ be a metric space. We denote by $\wp_{\mathrm{cl}}(M)$ (resp. by $\wp_{+\mathrm{cl}}(M)$) the set of *closed* subsets (resp. *nonempty closed* subsets) of $M$. For $N \subseteq M$, the *closure* of $N$ is denoted by $N^{\mathrm{cls}}$. ∎

**Notation 2.2** (Sequences) Let $A$ be a set.

**(1)** The empty sequence is denoted by $\epsilon$. The sequence consisting of $a_0, \ldots, a_{n-1} \in A$ is denoted by $\langle a_0, \ldots, a_{n-1} \rangle$.

**(2)** The set of finite sequences of elements of $A$ is denoted by $A^{<\omega}$, and let $A^+ = A^{<\omega} \setminus \{\epsilon\}$. The set of finite or infinite (with length $\omega$) sequences of elements of $A$ is denoted by $A^{\leq\omega}$. For $a \in A$, we sometimes write simply $a$ instead of $\langle a \rangle$ to denote the sequence consisting only of $a$.

**(3)** Each sequence $w \in A^{\leq\omega}$ is regarded as a function whose domain is a member of $\omega \cup \{\omega\}$. Thus the *length* of $w$ is $\mathrm{dom}(w)$; referring to its length as $\mathrm{lgt}(w)$, one has

$$w = \langle w(i) \rangle_{i \in \mathrm{lgt}(w)} = (\lambda i \in \mathrm{lgt}(w).\ w(i)),$$

$$\mathrm{ran}(w) = \{w(i)\}_{i \in \mathrm{lgt}(w)}.$$

When an element $a$ of $A$ are considered a symbol, we write $a^\nu$ to denote the sequence $\langle a \rangle_{i \in \nu}$, where $\nu \in \omega \cup \{\omega\}$.

**(4)** For $w_1 \in A^{<\omega}$, $w_2 \in A^{\leq\omega}$, let $w_1 \cdot w_2$ denote the *concatenation* of $w_1$ and $w_2$. Also for $p_1 \subseteq A^{<\omega}$, $p_2 \subseteq A^{\leq\omega}$, let $p_1 \cdot p_2 = \{w_1 \cdot w_2 : w_1 \in p_1 \wedge w_2 \in p_2\}$.

**(5)** For $q \in A^+$, $\mathrm{rest}(q)$ denotes the unique sequence $\tilde{q} \in A^{\leq\omega}$ such that

$$\langle q(0) \rangle \cdot \tilde{q} = q,$$

and the last element of $q$ is denoted by $\mathrm{last}(q)$, i.e., let

$$\mathrm{last}(q) = q(\max(\mathrm{lgt}(q))) = q(\mathrm{lgt}(q) - 1).$$

**(6)** For $n \in \omega$ and a sequence $w \in A^{\leq\omega}$, the *truncation* of $w$ at level $n$, written as $w_{[n]}$, is the prefix of $w$ with length $n$ if $\mathrm{lgt}(w) \geq n$, or $q$ otherwise. Regarding $w$ as a function with domain $\mathrm{lgt}(w)$, $w_{[n]}$ is the restriction of $w$ to $n = \{0, \ldots, n-1\}$, i.e.,

$w_{[n]} = w \quad n.$

For a set of sequences $p \subseteq A^{\leq\omega}$, let $p_{[n]} = \{w_{[n]} : w \in p\}$.

**(7)** For $p \in \wp(A^{\leq\omega})$, and $w \in A^{<\omega}$, let $p[w] = \{w' \in A^{\leq\omega} : w \cdot w' \in p\}$.

**(8)** An *ordered pair* $(a_0, a_1)$ and a *ordered triple* $(a_0, a_1, a_2)$ $(= (a_0, (a_1, a_2)))$ are distinguished from, but treated as sequences $\langle a_i \rangle_{i \in n}$ with $n$ being 2 and 3, respectively; for $n = 2, 3$ and $i \in n$, we sometimes write $(a_i)_{i \in n}$ to denote $(a_0, \ldots, a_{n-1})$, and the $i$-th component of $t = (a_i)_{i \in n}$ is denoted by $\pi_i^n(t)$.

**(9)** Let $A^{\underline{2}} = A \times A$, and $A^{\underline{3}} = A \times A^{\underline{2}}$. We sometimes identify $A^{\underline{2}}$ (resp. $A^{\underline{3}}$) with $A^2 = (2 \to A)$ (resp. with $A^3 = (3 \to A)$), when no confusion can arise. ∎

**Notation 2.3** Let $X$, $Y$ be sets.

**(1)** For $n \in \omega$, $\vec{x} \in (n \rightarrowtail X)$, $\vec{y} \in (n \to Y)$, we denote by $\vec{y}/\vec{x}$ the mapping $\{(\vec{x}(i), \vec{y}(i)) : i \in n\}$. In particular, for $x \in X$ and $y \in Y$, $y/x = \{(x, y)\}$.

**(2)** For $f, g \in (X \nrightarrow Y)$, we denote by $f[g]$ the result of *overlaying* $f$ with $g$, i.e., $f[g] = (f \quad (\mathrm{dom}(f) \setminus \mathrm{dom}(g))) \cup g$. In particular, for $x_0 \in X$ and $y_0 \in Y$, $f[y_0/x_0]$ is the mapping which maps $x_0$ to $y_0$ and $x \neq x_0$ to $f(y)$. ∎

Throughout this thesis, the phrase "let $(x, y, \cdots \in)$ $X$ be $\cdots$" is used for introducing a set $X$ ranged over by typical elements $x$, $y$, $\cdots$.

## 2.2 Complete Metric Spaces

In this section, first we present standard notions on cms's and basic properties of them. Next, we present several notions specific to cms's consisting of sets of sequences, and establish basic properties of such cms's.

First, we assume the notions of *metric space*, *complete metric space*, *continuous function*, and *closed set* to be known (the reader might consult [Dugundji, 1966] or [Engelking, 1977]). A metric space $(M, d)$ is called a *ultra-metric space* (or an *non-Archimedian metric-space*), when it satisfies the following strong form of triangle inequality:

$$\forall x, y, z \in M[\ d(x, y) \leq \max\{d(x, y), d(y, z)\}\ ].$$

Let $(M_1, d_1)$ and $(M_2, d_2)$ be two metric spaces. The function $f : M_1 \to M_2$ is called *contracting* (or a *contraction*), when

$$\exists \varepsilon \in [0, 1)[\forall x, y \in M_1[d_2(f(x), f(y)) \leq \varepsilon \cdot d_1(x, y)]].$$

We call $M_1$ and $M_2$ *isometric* (which is denoted by $M_1 \cong M_2$, when there exists a bijective mapping $f : M_1 \to M_2$ such that

$$\forall x, y \in M_1[d_2(f(x), f(y)) = d_1(x, y)].$$

The following is known as *Banach's Fixed-Point Theorem*, and is conveniently used in the thesis:

*a contraction from a cms to itself has a unique fixed-point,* (2.1)

where a *fixed-point* of a mapping $f$ is an element $x \in \mathrm{dom}(f)$ such that $f(x) = x$. For a contraction $f$ from a cms to itself, the unique fixed-point of $f$ is denoted by $\mathrm{fix}(f)$.

We use the following operations on metric spaces. (In our definition the distance between two elements of a metric space is always bounded by 1.)

**Definition 2.1** (Operations on Metric Spaces) Let $(M, d)$, $(M_1, d_1)$, $\cdots$, $(M_n, d_n)$ be metric spaces.

**(1)** With $M_1 \uplus \cdots \uplus M_n$ we denote the *disjoint union* of $M_1, \cdots, M_n$, which can be defined as $\bigcup_{j\in\bar{n}}[\{j\} \times M_j]$. We define a metric $d_{\mathrm{U}}$ on $M_1 \uplus \cdots \uplus M_n$ as follows. For every $(i, x), (j, y) \in M_1 \uplus \cdots \uplus M_n$,

$$d_{\mathrm{U}}((i,x),(j,y)) = \begin{cases} d_i(x,y) & \text{if } i = j, \\ 1 & \text{otherwise.} \end{cases}$$

**(2)** We define a metric $d_{\mathrm{P}}$ on the Cartesian product $M_1 \times \cdots \times M_n$ as follows. For $(x_1, \cdots, x_n), (y_1, \cdots, y_n) \in M_1 \times \cdots \times M_n$

$$d_{\mathrm{P}}((x_1,\cdots,x_n),(y_1,\cdots,y_n)) = \max\{d_j(x_j,y_j) : j \in \bar{n}\}.$$

**(3)** Let $V$ be an arbitrary nonempty set. The function space $(V \to X)$ is a cms with a metric $d_{\mathrm{f}}$ defined as follows: For $f, g \in (V \to X)$,

$$d_{\mathrm{f}}(f,g) = \sup\{d(f(v),g(v)) : v \in V\}.$$

**(4)** For $X \subseteq M$, the *closure* of $X$ is denoted by $X^{\mathrm{cls}}$.

**(5)** Let $\wp_{\mathrm{cl}}(M) = \{X \in \wp(M) : X \text{ is closed}\}$. We define a metric $d_{\mathrm{H}}$ on $\wp_{\mathrm{cl}}(M)$, called the *Hausdorff distance*, as follows: For every $X, Y \in \wp_{\mathrm{cl}}(M)$,

$$d_{\mathrm{H}}(X,Y) = \max\{\sup\{\underline{d}(x,Y) : x \in X\}, \sup\{\underline{d}(y,X) : y \in Y\}\},$$

where $\underline{d}(x, Z) = \inf\{d(x, z) : z \in Z\}$ for every $Z \subseteq M$, $x \in X$. (Here we use the convention that $\sup(\emptyset) = 0$ and $\inf(\emptyset) = 1$.)
The space $\wp_{+\mathrm{cl}}(M) = \{X \in \wp(M) : X \text{ is closed and nonempty}\}$ is supplied with a metric by taking the restriction of $d_{\mathrm{H}}$ to it.

**(6)** For a real number $\varepsilon \in [0, 1)$, we define

$$id_\varepsilon((M,d)) = (M,d'),$$

where $d'(x, y) = \varepsilon \cdot d(x, y)$, for every $x, y \in M$.

**(7)** An arbitrary set $A$ can be supplied with a metric $d_A$, called the *discrete metric*, defined by

$$d_A(x,y) = \begin{cases} 0 & \text{if } x = y, \\ 1 & \text{if } x \neq y. \end{cases}$$

The space $(A, d_A)$ is an ultra-metric space.

**(8)** For two cms's $(X, d_X)$, $(Y, d_Y)$, and a positive number $\varepsilon$, let $(X \rightarrow^{\varepsilon} Y)$ be the set of functions satisfying the *Lipschitz condition* with coefficient $\varepsilon$, i.e., let

$$(X \rightarrow^{\varepsilon} Y) = \{\, f \in (X \rightarrow Y) : \forall x_0, x_1 \in X[\; d_Y(f(x_0), f(x_1)) \leq \varepsilon \cdot d_X(x_0, x_1)\;]\}. \blacksquare$$

Next, we state an important technical result due to Hahn which plays a central role in metric semantics of concurrency; we need a few preliminary definitions for this:

**Definition 2.2** Let $(M, d)$ be a cms.

**(1)** We denote by $\mathrm{CS}(M, d)$ the set of *Cauchy sequences* (CS's) in $M$.

**(2)** For $\langle p_n \rangle_{i \in \omega} \in (\omega \rightarrow \wp(M))$, let

$$\mathrm{Lim}(\langle p_n \rangle_{n \in \omega}) = \{\lim(\langle q_n \rangle_{n \in \omega}) : \langle q_n \rangle_{n \in \omega} \in \mathrm{CS}(M, d) \cap \prod_{n \in \omega} [p_n]\},$$

where $\lim(\langle q_n \rangle_{n \in \omega})$ is the (toplogical) limit of the CS $\langle q_n \rangle_{n \in \omega}$ in $M$. $\blacksquare$

In terms of the notions defined above, *Hahn's theorem* is as follows:

**Theorem 2.1** (Hahn's Theorem) *If $(M, d)$ is a cms, then so is $(\wp_{\mathrm{cl}}(M), d_{\mathrm{H}})$, with $d_{\mathrm{H}}$ is the Hausdorff distance defined in Definition 2.1 (5). Also, for $\langle p_n \rangle_{n \in \omega} \in \mathrm{CS}(\wp_{\mathrm{cl}}(M), d_{\mathrm{H}})$, we have*

$$\lim(\langle p_n \rangle_{n \in \omega}) = \mathrm{Lim}(\langle p_n \rangle_{n \in \omega}),$$

*where $\lim(\langle p_n \rangle_{n \in \omega})$ is the (topological) limit of $\langle p_n \rangle_{n \in \omega}$ in $\wp_{\mathrm{cl}}(M)$.* $\blacksquare$

**Proof.** See Theorem 2.9 of [BZ 82]. $\blacksquare$

Complete metric spaces consisting of sequences are introduced as solutions of appropriate domain equations as in [BZ 82] and [AR 88]. Namely, for arbitrary two sets $A$ and $B$, and for an arbitrary real number $\kappa$ with $0 < \kappa < 1$, there exists a complete metric space $(\mathbf{Q}, d_{\mathbf{Q}})$, which is unique up to isometry, satisfying the domain equation:

$$\mathbf{Q} \cong B \uplus (A \times \mathrm{id}_{\kappa}(\mathbf{Q})).$$

(The existence and uniqueness of such $\mathbf{Q}$ have been shown in [BZ 82] and [AR 88], respectively.) Note that $\mathrm{id}_{\kappa}$ is necessary for the associated functor with this domain equation to be contractive, which condition ensures the uniqueness of the solution (see [AR 88]). In the rest of the thesis, we fix a real number $\kappa$ with $0 < \kappa < 1$. The metric space $(\mathbf{Q}, d_{\mathbf{Q}})$ can be defined in terms *projection functions* introduced below, where the projection functions are very similar to the truncation functions of sequences but slightly different from them, as we will note below.

**Definition 2.3** (Projection Functions on Domain of Streams) Let $(Q, d_{\mathbf{Q}})$ be the unique cms which satisfies the following domain equation:

$$\mathbf{Q} \cong B \uplus (A \times \mathrm{id}_\kappa(\mathbf{Q})).$$

Actually the space $((A^{<\omega} \times B) \uplus A^\omega, d_Q)$ is the solution of this equation, where $d_Q$ is defined, in terms of truncation, by

$$d_Q(q_1, q_2) = \begin{cases} \kappa^{\min\{n:(q_1\;\, n) \neq (q_2\;\, n)\}-1} & \text{if } \exists n[\,(q_1\;\;\, n) \neq (q_2\;\;\, n)\,], \\ 0 & \text{otherwise.} \end{cases}$$

Here and in the sequel, we treat an element $(r, b)$ of $A^{<\omega} \times B$ as a sequence with length $\mathrm{lgt}(r) + 1$ instead of a pair of $r$ and $b$.

**(1)** For $n \in \omega$, the $n$-th *projection function* $\psi_n : \mathbf{Q} \to \mathbf{Q}$ is defined as follows. First, fix an arbitrary element $b_0$ of $B$. For $q \in \mathbf{Q}$,

  **(i)** $\psi_0(q) = b_0$,

  **(ii)** $\psi_{n+1}(q) = \begin{cases} b & \text{if } q \in B, \\ a \cdot \psi_n(q') & \text{if } q = a \cdot q'. \end{cases}$

**(2)** Let $\mathbf{P} = \wp_{+\mathrm{cl}}(\mathbf{Q})$. For $n \in \omega$ and $p \in \mathbf{P}$, let

$$\tilde{\psi}_n(p) = \{\psi_n(q) : q \in p\}. \blacksquare$$

**Notation 2.4** For $q \in \mathbf{Q}$ (resp. $p \in \wp(\mathbf{Q})$), we sometimes write $q^{[n]}$ (resp. $p^{[n]}$) for $\psi_n(q)$ (resp. $\tilde{\psi}_n(p)$). ∎

Note the difference between *truncation* and *projection*: The values of the projection functions are members of $\mathbf{Q}$, whereas the values of the truncation functions are members of $A^{<\omega}$ not of $\mathbf{Q}$.

The metric $d_Q$ is formulated also in terms of projection as follows:

**Lemma 2.1**

**(1)** *For* $q_1, q_2 \in \mathbf{Q}$,

$$d_Q(q_1, q_2) = \begin{cases} \kappa^{\min\{n:\psi_n(q_1) \neq \psi_n(q_2)\}-1} & \textit{if } \exists n[\,\psi_n(q_1) \neq \psi_n(q_2)\,], \\ 0 & \textit{otherwise.} \end{cases}$$

**(2)** *For* $p_1, p_2 \in \mathbf{P}$,

$$d_P(p_1, p_2) = \begin{cases} \kappa^{\min\{n:\tilde{\psi}_n(p_1) \neq \tilde{\psi}_n(p_2)\}-1} & \textit{if } \exists n[\,\tilde{\psi}_n(p_1) \neq \tilde{\psi}_n(p_2)\,], \\ 0 & \textit{otherwise.} \end{cases}$$

**(3)** $$\begin{array}{l} \forall n \in \omega, \exists \varepsilon > 0[ \\ \quad \forall p_1, p_2 \in \mathbf{P}[\; d_P(p_1, p_2) \leq \varepsilon \Rightarrow \tilde{\psi}_n(p_1) = \tilde{\psi}_n(p_2) \;]]. \blacksquare \end{array}$$

**Proof. (1)** Let $A$, $B$, $\mathbf{Q}$, $\psi_n$, $\mathbf{P}$, and $\tilde{\psi}_n$ be as in Definition 2.3 $(n \in \omega)$. For $n \in \omega$, let

$$\begin{array}{rl} \Phi(n) \Leftrightarrow & \forall q_1, q_2[(\psi_{n+1}(q_1) \neq \psi_{n+1}(q_2) \Rightarrow d_Q(q_1, q_2) \geq \kappa^n) \\ & \wedge(\psi_n(q_1) = \psi_n(q_2) \Rightarrow d_Q(q_1, q_2) \leq \kappa^n)\,]. \end{array}$$

It is sufficient to show $\forall n \in \omega[\Phi(n)]$, which will be established by induction.
**Induction Base:** Fix $q_1, q_2 \in \mathbf{Q}$.

First, let us show that

$$\psi_1(q_1) \neq \psi_1(q_2) \Rightarrow d_Q(q_1, q_2) = \kappa^0.$$

Suppose

$$\psi_1(q_1) \neq \psi_1(q_2). \tag{2.2}$$

**Case 1.** Suppose $q_1, q_2 \in B$. Then

$$\psi_1(q_1) = q_1 \wedge \psi_1(q_2) = q_2.$$

By this and (2.2), one has
$d_Q(q_1, q_2) = 1 = \kappa^0$.
**Case 2.** Suppose $q_1, q_2 \in A \times \mathbf{Q}$. Then, by (2.2), one has

$$\exists a_1, a_2 \in A, \exists q_1', q_2' \in \mathbf{Q}[\, q_1 = \langle a_1, q_1' \rangle \wedge q_2 = \langle a_2, q_2' \rangle \wedge a_1 \neq a_2 \,].$$

Hence $d_Q(q_1, q_2) = 1 = \kappa^0$.
**Case 3.** Otherwise, by the definition of $d_Q$, one has

$$d_Q(q_1, q_2) = 1 = \kappa^0$$

Thus one has

$$\psi_1(q_1) \neq \psi_1(q_2) \Rightarrow d_Q(q_1, q_2) = \kappa^0.$$

Moreover by the definition of $d_Q$, it holds that $d_Q(q_1, q_2) \leq \kappa^0$.
Thus one has $\Phi(0)$.
**Induction Step:** Fix $n \in \omega$ and assume $\Phi(n)$. Then fix $q_1, q_2 \in \mathbf{Q}$.

First, let us show that

$$\psi_{(n+1)+1}(q_1) \neq \psi_{(n+1)+1}(q_2) \Rightarrow d_Q(q_1, q_2') \geq \kappa^{n+1}.$$

Suppose

$$\psi_{(n+1)+1}(q_1) \neq \psi_{(n+1)+1}(q_2). \tag{2.3}$$

**Case 1.** Suppose $q_1, q_2 \in A \times \mathbf{Q}$. Then

$$\exists a_1, a_2 \in A, \exists q_1', q_2' \in \mathbf{Q}[\, q_1 = \langle a_1, q_1' \rangle \wedge q_2 = \langle a_2, q_2' \rangle \,].$$

If $a_1 \neq a_2$, then

$$d_Q(q_1, q_2) = 1 \geq \kappa^{n+1}.$$

Otherwise, (2.3) implies that

$$\psi_{n+1}(q_1') \neq \psi_{n+1}(q_2').$$

By this and the induction hypothesis, one has

$$d_Q(q'_1, q'_2) \geq \kappa^n,$$

and therefore

$$d_Q(q_1, q_2) = \kappa \cdot d_Q(q'_1, q'_2) \geq \kappa^{n+1}.$$

**Case 2.** Otherwise as in the induction base, one has

$$d_Q(q_1, q'_2) = 1 \geq \kappa^{n+1}.$$

Thus one has

$$\psi_{(n+1)+1}(q_1) \neq \psi_{(n+1)+1}(q_2) \Rightarrow d_Q(q_1, q_2) = \kappa^{n+1}.$$

Next let us show that

$$\psi_{n+1}(q_1) = \psi_{n+1}(q_2) \Rightarrow d_Q(q_1, q_2) \leq \kappa^{n+1}.$$

Suppose

$$\psi_{n+1}(q_1) = \psi_{n+1}(q_2). \tag{2.4}$$

Then, if $q_1, q_2 \in \mathbf{B}$, one has $q_1 = q_2$, and therefore, $d_Q(q_1, q_2) = 0 \leq \kappa^{n+1}$.
Otherwise, by the definition of $\psi_{n+1}$, one has

$$\exists a \in A, \exists q'_1, q'_2 \in \mathbf{Q}[\, q_1 = \langle a, q'_1 \rangle \wedge q_2 = \langle a, q'_2 \rangle \,].$$

By (2.4), one has

$$\psi_n(q'_1) = \psi_n(q'_2).$$

By this and the induction hypothesis, one has

$$d_Q(q'_1, q'_2) \leq \kappa^n,$$

and therefore

$$d_Q(q_1, q_2) = \kappa \cdot d_Q(q'_1, q'_2) \leq \kappa^{n+1}.$$

Thus one has

$$\psi_{n+1}(q_1) = \psi_{n+1}(q_2) \Rightarrow d_Q(q'_1, q'_2) \leq \kappa^{n+1}.$$

Summing up, one has $\Phi(n+1)$.
**(2)** By **(1)** and the definitions of $\tilde{\psi}_n$ $(n \in \omega)$ and $d_P$, one has

$$\begin{aligned}
&\forall n \in \omega, \forall p_1, p_2 \in \mathbf{P}[\\
&\quad (\tilde{\psi}_n(p_1) = \tilde{\psi}_n(p_2) \Rightarrow d(p_1, p_2) \leq \kappa^n)\\
&\quad \wedge (\tilde{\psi}_{n+1}(p_1) \neq \tilde{\psi}_{n+1}(p_2) \Rightarrow d(p_1, p_2) \geq \kappa^n)\,],
\end{aligned}$$

which implies the claim of this part.
**(3)** This part follows immediately from **(2)**. ■

The notion of *finitely characterized subset* is introduced for establishing that some subsets of a cms are also cms's.

**Definition 2.4** (Finitely Characterized Subsets) A subset $\mathbf{P}'$ of $\mathbf{P}$ is *finitely characterized* iff there exist $n \in \omega$ and $\mathbf{P}'' \subseteq \mathbf{P}$ such that

$$\forall p \in \mathbf{P}[\ p \in \mathbf{P}' \Leftrightarrow \tilde{\psi}_n(p) \in \mathbf{P}''\ ]. \tag{2.5}$$

Also, a subset $\mathbf{P}'$ of $\mathbf{P}$ is *finitely characterized in the wider sense*, when $\mathbf{P}'$ is the intersection of finitely characterized subsets of $\mathbf{P}$, i.e., when there exists $\mathcal{P} \subseteq \wp(\mathbf{P})$ such that $\mathbf{P}' = \bigcap(\mathcal{P})$ and

$$\forall \mathbf{P}'' \in \mathcal{P}[\ \mathbf{P}'' \text{ is finitely characterized subset of } \mathbf{P}\ ]. \blacksquare$$

A property defined for elements of $\mathbf{P}$ is called *finitely characterized*, if the set consisting of those elements of $\mathbf{P}$ which have the property is finitely characterized. The next example presents such a property.

**Example 2.1** Fix $n \in \omega$. An element $p \in \mathbf{P}$ said to be *nonempty at level $n$*, if $(p\quad n) \cap A^n \neq \emptyset$. Let $\mathbf{P}' = \{p \in \mathbf{P} : p \text{ is nonempty at level } n\}$. Then it immediately follows that

$$\forall p \in \mathbf{P}[p \in \mathbf{P}' \Leftrightarrow \psi_{n+1}(p) \in \mathbf{P}'].$$

Thus $\mathbf{P}'$ is finitely characterized, and therefore, the property "being *nonempty at level $n$*" is finitely characterized. Remark that $\mathbf{P}''$ in (2.5) is taken to be $\mathbf{P}'$ here. $\blacksquare$

The next lemma states that finitely characterized subsets and intersections of finitely characterized subsets are complete metric spaces with the original metric restricted to them. This lemma will be used in the proofs of full abstractness to show that the domains of denotational semantics to be presented in the following chapters are closed.

**Lemma 2.2**

**(1)** *If $\mathbf{P}' \subseteq \mathbf{P}$ is finitely characterized, then $\mathbf{P}'$ is closed in $\mathbf{P}$.*

**(2)** *If $\mathbf{P}' \subseteq \mathbf{P}$ is finitely characterized in the wider sense, then $\mathbf{P}'$ is closed in $\mathbf{P}$.* $\blacksquare$

**Proof. (1)** Suppose $\langle p_i \rangle_{i \in \omega}$ converges and $\forall i \in \omega[p_i \in \mathbf{P}']$.
Let $p = \lim_{i \in \omega}[p_i]$. Since $\mathbf{P}'$ is finitely characterized, there exist $n \in \omega$ and $\mathbf{P}'' \subseteq \mathbf{P}$ such that

$$\forall p \in \mathbf{P}[\ p \in \mathbf{P}' \Leftrightarrow \tilde{\psi}_n(p) \in \mathbf{P}''\ ]. \tag{2.6}$$

Fix such $n$ and $\mathbf{P}''$. By Lemma 2.1 (3), there exists $\varepsilon > 0$ such that

$$\forall p_1, p_2 \in \mathbf{P}[\ d_P(p_1, p_2) \leq \varepsilon \Rightarrow \tilde{\psi}_n(p_1) = \tilde{\psi}_n(p_2)\ ]. \tag{2.7}$$

Fix such $\varepsilon$.

$$\exists N \in \omega[\ d_P(p_N, p) \leq \varepsilon\ ].$$

Fix such $N$. By (2.7),

$$\tilde{\psi}_n(p_N) = \tilde{\psi}_n(p). \tag{2.8}$$

By (2.6) and the fact that $p_N \in \mathbf{P}'$, one has $\tilde{\psi}_n(p_N) \in \mathbf{P}'$. By this and (2.8), one has $\tilde{\psi}_n(p) \in \mathbf{P}'$. It follows from this and (2.6) that $p \in \mathbf{P}'$.
**(2)** By the definition of finitely characterized subsets in the wider sense, it suffices to show that

$$\mathcal{P} \subseteq \wp(\mathbf{P}) \wedge \forall \mathbf{P}'' \in \mathcal{P}[\ \mathbf{P}'' \text{ is finitely characterized }]$$
$$\Rightarrow \bigcap(\mathcal{P}) \text{ is closed in } \mathbf{P}.$$

This follows immediately from **(1)** and the fact that the intersection of closed sets is closed. ■

## 2.3 Complete Partial Orders

In this section, first we present standard notions on cpo's and basic properties of them; all the properties are standard and their proofs are omitted (see [DP 90] for detailed exposition; a concise description is found in § 1.2 of [Bar 84] and § 3.1 of [Hen 88].) Next, we present several notions specific to cpo's consisting of sets of sequences, and establish basic properties of such cpo's.

**Definition 2.5 (1)** For a set$X$ and a binary relation $\mathcal{R}$ on $X$, we define the set $\mathrm{Chain}(X, \mathcal{R})$ of *$\mathcal{R}$-chains* of elements of $X$ by

$$\mathrm{Chain}(X, \mathcal{R}) = \{f \in (\omega \to X) : \forall n \in \omega[\ f(n)\mathcal{R}f(n+1)\ ]\}.$$

Thus, for a *partially ordered set* (*pos*) $(X, \sqsubseteq)$, the set of *increasing sequences* of elements of $X$ is defined to be $\mathrm{Chain}(X, \sqsubseteq)$.

**(2)** A *complete partial order* (*cpo*) is a triple $(X, \sqsubseteq, \bot_S)$ such that $(X, \sqsubseteq)$ is a pos with $\bot$ being the least element (or *bottom*) of $(S, \sqsubseteq)$, and every chain $\langle x \rangle_{n\in\omega} \in \mathrm{Chain}(X, \sqsubseteq)$ has its *least upper bound* (*lub*) in $X$.

**(3)** For a cpo $(X, \sqsubseteq, \bot)$ and $\langle x_n \rangle_{n\in\omega} \in \mathrm{Chain}(X, \sqsubseteq, \bot)$, let us denote the lub of $\{x_n\}_{n\in\omega}$ in $X$ by $\bigsqcup(\langle x_n \rangle_{n\in\omega}, X)$ or more simply by $\bigsqcup(\langle x_n \rangle_{n\in\omega})$ or $\bigsqcup_{n\in\omega}[\langle x_n \rangle]$.

**(4)** Let $(X, \sqsubseteq, \bot)$ a cpo. A function $f : X \to X$ is called *continuous w.r.t.* $\sqsubseteq$, when $f$ is monotonic w.r.t. $\sqsubseteq$, and satisfies the following:

$$\forall \langle x_n \rangle_{n\in\omega} \in \mathrm{Chain}(X, \sqsubseteq)[\ f(\bigsqcup_{n\in\omega}[x_n]) = \bigsqcup_{n\in\omega}[f(x_n)]\ ]. \blacksquare$$

The following fact is due to Tarski (cf. Theorem 4.5 of [DP 90]).

**Lemma 2.3** *Let $(X, \sqsubseteq, \bot)$ a cpo, and let $f \in (X \to X)$ be continuous w.r.t. $\sqsubseteq$. Then, $\bigsqcup_{n\in\omega}[f^n(\bot)]$ is the least fixed-point of $f$, where $f^n$ is the $n$-th iteration of $f$ for $n \geq 1$ with $f^0$ being the identity function on $X$.* ■

**Notation 2.5** For a cpo $(X, \sqsubseteq, \bot)$ and $f \in (X \to X)$ which is continuous w.r.t. $\sqsubseteq$, we denote by $\mathrm{fix}_l(f)$ the least fixed-point of $f$. ■

By Lemma 2.3, we have

$$\mathrm{fix_l}(f) = \bigsqcup_{n\in\omega} [f^n(\bot)]. \tag{2.9}$$

for every cpo $(X, \sqsubseteq, \bot)$ and continuous $f \in (X \to X)$.

**Proposition 2.1** *Let $Z$ be an arbitrary set and $(X, \sqsubseteq, \bot)$ a cpo. Let $\tilde{\sqsubseteq}$ be a binary relation on $(Z \to X)$ defined by:*

$$\tilde{\sqsubseteq} = \{(f, g) \in (Z \to X)^2 : \forall z \in Z[\ f(z) \sqsubseteq g(z)\ ]\},$$

*and let $\tilde{\bot} = (\lambda z \in Z.\ \bot)$.*

**(1)** *$((Z \to X), \tilde{\sqsubseteq})$ is a pos.*

**(2)** *For every $\langle f_n\rangle_{n\in\omega} \in \mathrm{Chain}((Z \to X), \tilde{\sqsubseteq})$, the function*

$$(\lambda z \in Z.\ \bigsqcup_{n\in\omega} [f(z)])$$

*is the lub of $\{f_n\}_{n\in\omega}$ in $((Z \to X), \tilde{\sqsubseteq})$.*

**(3)** *$((Z \to X), \tilde{\sqsubseteq}, \tilde{\bot})$ is also a cpo.* ∎

**Definition 2.6** For two cpo's $(X, \sqsubseteq, \bot)$, $(Y, \sqsubseteq', \bot')$, let $[Y \to X]$ be the set of continuous functions from $(X, \sqsubseteq, \bot)$ to $(Y, \sqsubseteq', \bot')$. ∎

**Proposition 2.2** *Let $(X, \sqsubseteq, \bot)$ and $(Y, \sqsubseteq', \bot')$ be cpo's. Then $([Z \to X], \tilde{\sqsubseteq}, \tilde{\bot})$ is also a cpo, where $\tilde{\sqsubseteq}$ and $\tilde{\bot}$ are defined as in Proposition 2.1.* ∎

In the sequel of this section, we present several notions specific to cpo's consisting of sets of sequences, and establish basic properties of these cpo's.

**Definition 2.7** Let $A$ and $B$ be nonempty sets with $A \cap B = \emptyset$, and let $\bot$ be a symbol with $\bot \notin A \cup B$. Further, let $B_\bot = B \cup \{\bot\}$, and let $Q = A^{<\omega} \cdot B_\bot \cup A^\omega$.

**(1)** A function $\mathrm{strip}(\cdot) : Q \to A^{\leq\omega}$ is defined as follows: For $q \in Q$, let

$$\mathrm{strip}(q) = \begin{cases} q\ \ (\mathrm{lgt}(q) - 1) & \text{if } \mathrm{lgt}(q) < \omega, \\ q & \text{otherwise.} \end{cases}$$

**(2)** The *stream order* $\sqsubseteq \subseteq Q \times Q$ is defined as follows: For $q_1, q_2 \in Q$, let $q_1 \sqsubseteq q_2$, if one of the following two conditions is satisfied:

**(i)** $q_1 = q_2$,

**(ii)** $q_1 \in A^{<\omega} \cdot \{\langle\bot\rangle\}$ and $\mathrm{strip}(q_1) \leq_\mathrm{p} \mathrm{strip}(q_2)$.

**(3)** For each $n \in \omega$, the *projection* function $(\cdot)^{[n]} : Q \to A^{<\omega} \cdot B_\bot$ is defined as follows: For $q \in Q$, let

$$q^{[n]} = \begin{cases} q & \text{if } \mathrm{lgt}(q) \leq n, \\ (q\ \ n) \cdot \langle\bot\rangle & \text{otherwise.} \end{cases}$$

Also, for $p \in \wp(Q)$, let $p^{[n]} = \{q^{[n]} : q \in p\}$. ∎

Let us take $A$, $B$, and $Q$ as in the above definition and fix these sets in the sequel of this section.

**Lemma 2.4** *$(Q, \sqsubseteq, \langle\perp\rangle)$ is a cpo.* ∎

The ordered set $(Q, \sqsubseteq, \langle\perp\rangle)$ has the following special property:

**Proposition 2.3** *Let $q_1, q_2 \in Q$. If there exists $q \in Q$ such that $q_1 \sqsubseteq q$ and $q_2 \sqsubseteq q$, then $q_1$ and $q_2$ are* comparable *w.r.t. $\sqsubseteq$, i.e., either $q_1 \sqsubseteq q_2$ or $q_1 \sqsubseteq q_2$.* ∎

**Proof.** See Proposition 2.8 of [MV 88] for details. ∎

**Definition 2.8** **(1)** A binary relation $\sqsubseteq_s$, the *Smyth order* on $\wp(Q)$, is defined as follows: For $p_1, p_2 \in \wp(Q)$, let $p_1 \sqsubseteq_s p_2$, if

$$\forall q_2 \in p_2, \exists q_1 \in p_1 [\ q_1 \sqsubseteq q_2\ ].$$

**(2)** Let $p \in \wp(Q)$. We call $p$ *flat*, if $\forall q, q' \in p[\ q \sqsubseteq q' \Rightarrow q = q'\ ]$. Let

$$\wp_{\mathrm{f}}(Q) = \{q \in \wp(Q) : q \text{ is flat}\}.$$

**(3)** $\wp_{\mathrm{fcl}}(Q) = \{p \in \wp_{\mathrm{cl}}(Q) : p \text{ is flat}\}$,
where $\wp_{\mathrm{cl}}(Q)$ is the set of closed subsets of $Q$ (see § 2.2).

**(4)** $\wp_{+\mathrm{fcl}}(Q) = \wp_{\mathrm{fcl}}(Q) \setminus \{\emptyset\}$.

**(5)** $\wp_{\mathrm{fco}}(Q) = \{p \in \wp_{\mathrm{co}}(Q) : p \text{ is flat}\}$,
where $\wp_{\mathrm{co}}(Q)$ is the set of compact subsets of $Q$ (see § 2.2).

**(6)** $\wp_{+\mathrm{fco}}(Q) = \wp_{\mathrm{fco}}(Q) \setminus \{\emptyset\}$. ∎

As is shown in § 2.2, we can define a metric $d$ on $\wp_{\mathrm{cl}}(Q)$ in terms of the projection functions $(\cdot)^{[n]}$ as follows: For every $p_1, p_2 \in \wp_{\mathrm{cl}}(Q)$, putting

$$I = \{n : (p_1)^{[n]} \neq (p_2)^{[n]}\},$$

let

$$\tilde{d}(p_1, p_2) = \begin{cases} \kappa^{\min(I)} & \text{if } I \neq \emptyset, \\ 0 & \text{otherwise.} \end{cases} \tag{2.10}$$

The notion of *compactness* for subsets of $Q$ is introduced as usual; for the sace $(Q, d)$, we have a convenient characterization of compactness in terms of the notion of *boundedness* defined by:

**Definition 2.9** We say $p \in \wp(Q)$ is *bounded* iff $\forall n \in \omega[\ p^{[n]}$ is finite $]$. ∎

Then compactness is characterized as follows:

**Lemma 2.5** *Let $p \in \wp(Q)$. Then $p$ is compact in $(Q, d)$ iff $p$ is closed and* bounded. ∎

**Proof.** This proposition can be established using (2.10). See Theorem 3.18 of [MV 88] for details. ∎

**Lifting Method**

In [MV 88], Meyer and de Vink proved the following useful theorem which enables us to lift continuous operations on the domain $Q$ of sequences to continuous operations on the domain $\wp_{\text{co}}(Q)$ of compact sets of sequences:

**Theorem 2.2** (Lifting Method) *Let $f$ be a continuous function from $(Q, \sqsubseteq)$ to $(\wp_{\text{fco}}(Q), \sqsubseteq_{\text{s}})$, and let $F = (\lambda p \in \wp_{\text{fco}}(Q).\ \text{mini}(\bigcup(f[p])))$. Then,*

**(1)** $F : \wp_{\text{fco}}(Q) \to \wp_{\text{fco}}(Q)$.

**(2)** *The function $F$ is continuous w.r.t. $\sqsubseteq_{\text{s}}$.* ∎

Although the domain of sequences in [MV 88] is slightly more restricted than the one treated in the present paper, the above theorem can be proved in the same way as the corresponding result, Theorem 4.6, in [MV 88].

From the above lemma, we immediately obtain the following:

**Corollary 2.1** *Let $f \in (Q \to Q)$ such that $f$ is continuous w.r.t. $\sqsubseteq$, and let $F = (\lambda p \in \wp_{\text{fco}}(Q).\ \text{mini}(f[p]))$. Then,*

**(1)** $F : \wp_{\text{fco}}(Q) \to \wp_{\text{fco}}(Q)$.

**(2)** *The function $F$ is continuous w.r.t. $\sqsubseteq_{\text{s}}$.* ∎

## 2.4 Relating Cms's and Cpo's of Sets of Sequences

In this section, we will establish several connections between cms's and cpo's composed of sets of sequences. As is shown in § 2.2, $(\wp_{\text{cl}}(Q), d)$ is a complete ultrametric space with the metric bounded by 1.

**Definition 2.10** **(1)** Increasing sequences in $(Q, \sqsubseteq)$, i.e., elements of $\text{Chain}(Q, \sqsubseteq)$ are called (ascending) *chains* in $Q$.
Also for $P \subseteq \wp(Q)$, elements of $\text{Chain}(P, \sqsubseteq_{\text{s}})$ are called *chains* in $P$.

**(2)** A chain $\langle q_n \rangle_{n\in\omega} \in (\omega \to Q)$ is called *infinitely often increasing* (i.o.i.), if for infinitely many $n \in \omega$, $q_n \sqsubset q_{n+1}$; otherwise we say $\langle q_n \rangle_{n\in\omega} \in (\omega \to Q)$ is *stabilizing*.

**(3)** For $\langle q_n \rangle_{n\in\omega} \in \text{Chain}(Q, \sqsubseteq)$, we define $\text{lub}(\langle q_n \rangle_{n\in\omega})$ as follows: First, if $\langle q_n \rangle_{n\in\omega} \in \text{Chain}(Q, \sqsubseteq)$ is i.o.i., then for every $n$, $q_n \in A^{<\omega} \times \{\langle\bot\rangle\}$, and let

$$\text{lub}(\langle q_n \rangle_{n\in\omega}) = \bigcup_{n\in\omega} [\text{strip}(q_n)].$$

Otherwise, $\langle q_n \rangle_{n\in\omega} \in \text{Chain}(Q, \sqsubseteq)$ is stabilizing, and let

$$\text{lub}(\langle q_n \rangle_{n\in\omega}) = q_m,$$

where $m$ is the least number $k$ such that $\forall n \geq k[\ q_n = q_k\ ]$.

**(4)** For $\langle p_n \rangle_{n\in\omega} \in (\omega \to \wp(Q))$, let

$$\begin{aligned}&\mathrm{Lub}(\langle p_n \rangle_{n\in\omega})\\ &= \{\mathrm{lub}(\langle q_n \rangle_{n\in\omega}) : \langle q_n \rangle_{n\in\omega} \in \textstyle\prod_{n\in\omega}[p_n] \cap \mathrm{Chain}(Q, \sqsubseteq)\}. \blacksquare\end{aligned}$$

**Definition 2.11** **(1)** Let $(X, \sqsubseteq_X)$ be a cpo. For $\langle x_n \rangle_{n\in\omega} \in \mathrm{Chain}(X, \sqsubseteq_X)$, we denote by $\bigsqcup(\langle x_n \rangle_{n\in\omega}, X)$ the *least upper bound* (lub) of $\langle x_n \rangle_{n\in\omega}$ in $X$. We sometimes write $\bigsqcup(\langle x_n \rangle_{n\in\omega})$ or $\bigsqcup_{n\in\omega}[x_n]$ for $\bigsqcup(\langle x_n \rangle_{n\in\omega}, X)$, when it is clear what cpo is under consideration.

**(2)** Let $(X, d_X)$ be a complete metric space. For $\langle x_n \rangle_{n\in\omega} \in \mathrm{CS}(X, d_X)$, we denote by $\lim(\langle x_n \rangle_{n\in\omega}, X)$ the (topological) *limit* of $\langle x_n \rangle_{n\in\omega}$ in $X$. We sometimes write $\lim(\langle x_n \rangle_{n\in\omega})$ or $\lim_{n\in\omega}[x_n]$ for $\lim(\langle x_n \rangle_{n\in\omega}, X)$, when it is clear what cms is under consideration. $\blacksquare$

Chains of flat sets have the following *interpolation property*:

**Proposition 2.4** (Interpolation Lemma)

**(1)** *Let $k \geq 1$, $p_0 \in \wp_{\mathrm{f}}(Q)$ , and $p_1, \cdots, p_k \in \wp(Q)$. If $p_0 \sqsubseteq_{\mathrm{s}} p_1 \sqsubseteq_{\mathrm{s}} \cdots \sqsubseteq_{\mathrm{s}} p_k$, then for every $q \in p_0$ and $q' \in p_k$ such that $q \sqsubseteq q'$, there exists $\langle q_n \rangle_{n\in(k+1)} \in \prod_{n\in(k+1)}[p_n]$ such that*

$$q = q_0 \sqsubseteq q_1 \sqsubseteq \cdots \sqsubseteq q_k = q'.$$

**(2)** *Let $\langle p_n \rangle_{n\in\omega} \in \mathrm{Chain}(\wp_{\mathrm{f}}(Q), \sqsubseteq_{\mathrm{s}})$, and $\langle \nu(i) \rangle_{i\in\omega} \in (\omega \to \omega)$ be strict increasing. Then, for every $\langle q_i \rangle_{i\in\omega} \in \mathrm{Chain}(Q, \sqsubseteq) \cap \prod_{i\in\omega}[p_{\nu(i)}]$, there exists $\langle \tilde{q}_n \rangle_{n\in\omega} \in \mathrm{Chain}(q, \sqsubseteq) \cap \prod_{n\in\omega}[p_n]$ such that $\forall i \in \omega[\ \tilde{p}_{\nu(i)} = p_i\ ]$.* $\blacksquare$

**Proof.** **(1)** This part follows immediately from Proposition 2.3. See Proposition 2.19 of [MV 88] for details.
**(2)** This part can be established by successively applying part (1). $\blacksquare$

Closedness is preserved by $\mathrm{Lub}(\cdot)$ as is stated by part (1) of the following lemma, which is due to Back ([Bac 83]):

**Lemma 2.6** *Let $\langle p_n \rangle_{n\in\omega} \in \mathrm{Chain}(\wp_{\mathrm{fcl}}(Q), \sqsubseteq_{\mathrm{s}})$.*

**(1)** $$\mathrm{Lub}(\langle p_n \rangle_{n\in\omega}) \in \wp_{\mathrm{fcl}}(Q). \tag{2.11}$$

**(2)** *Also $\mathrm{Lub}(\langle p_n \rangle_{n\in\omega})$ is the lub of $\langle p_n \rangle_{n\in\omega}$ in $\wp_{\mathrm{fcl}}(Q)$, i.e.,*

$$\mathrm{Lub}(\langle p_n \rangle_{n\in\omega}) = \bigsqcup(\langle p_n \rangle_{n\in\omega}, \wp_{\mathrm{fcl}}(Q)). \tag{2.12}$$

**(3)** *$(\wp_{\mathrm{fcl}}(Q), \sqsubseteq_{\mathrm{s}}, \{\langle \bot \rangle\})$ is a cpo.* $\blacksquare$

**Proof.** **(1)** The flatness of $\mathrm{Lub}(\langle p_n \rangle_{n\in\omega})$ immediately follows from its definition. Its closedness can be shown using its definition and the definition of the characterization of the metric $d$ in terms of projections (Lemma 2.1 (1)). See [Bac 83] for details.
**(2)** This part follows from part (1) and Proposition 2.18 of [MV 88] stating that $\mathrm{Lub}(\langle p_n \rangle_{n\in\omega})$ is the lub of $\langle p_n \rangle_{n\in\omega}$ in $\wp_{\mathrm{fco}}(Q)$.
**(3)** This part immediately follows from part (2). $\blacksquare$

**Lemma 2.7** *Let* $\langle p_n\rangle_{n\in\omega} \in \mathrm{Chain}(\wp_{\mathrm{fcl}}(Q), \sqsubseteq_s) \cap \mathrm{CS}(\wp_{\mathrm{fcl}}(Q), d)$. *Then*

$$\begin{aligned}&\bigsqcup(\langle p_n\rangle_{n\in\omega}, \wp_{\mathrm{fcl}}(Q)) = \mathrm{Lub}(\langle p_n\rangle_{n\in\omega})\\ &= \mathrm{Lim}(\langle p_n\rangle_{n\in\omega}) = \lim(\langle p_n\rangle_{n\in\omega}, \wp_{\mathrm{fcl}}(Q)). \blacksquare\end{aligned} \tag{2.13}$$

**Proof.** Let $\langle p_n\rangle_{n\in\omega} \in \mathrm{Chain}(\wp_{\mathrm{fcl}}(Q), \sqsubseteq_s) \cap \mathrm{CS}(\wp_{\mathrm{fcl}}(Q), d)$. We will prove (2.13). Since we have shown that

$$\bigsqcup(\langle p_n\rangle_{n\in\omega}, \wp_{\mathrm{fcl}}(Q)) = \mathrm{Lub}(\langle p_n\rangle_{n\in\omega}) \quad \text{(the claim of Lemma 2.6 (2))}$$

and that

$$\mathrm{Lim}(\langle p_n\rangle_{n\in\omega}) = \lim(\langle p_n\rangle_{n\in\omega}, \wp_{\mathrm{fcl}}(Q)) \quad \text{(the claim of Lemma 2.1)},$$

it suffices to show the following:

$$\mathrm{Lub}(\langle p_n\rangle_{n\in\omega}) = \mathrm{Lim}(\langle p_n\rangle_{n\in\omega}). \tag{2.14}$$

From the definition of $\mathrm{Lub}(\langle p_n\rangle_{n\in\omega})$, it immediately follows that

$$\mathrm{Lub}(\langle p_n\rangle_{n\in\omega}, \wp_{\mathrm{fcl}}(Q)) \subseteq \mathrm{Lim}(\langle p_n\rangle_{n\in\omega}). \tag{2.15}$$

We will show the other part, i.e., that

$$\mathrm{Lim}(\langle p_n\rangle_{n\in\omega}) \subseteq \mathrm{Lub}(\langle p_n\rangle_{n\in\omega}). \tag{2.16}$$

Let $q \in \mathrm{Lim}(\langle p_n\rangle_{n\in\omega})$. We will show that $q \in \mathrm{Lub}(\langle p_n\rangle_{n\in\omega})$. By the definition of $\mathrm{Lim}(\langle p_n\rangle_{n\in\omega})$, there is $\langle q_n\rangle_{n\in\omega} \in \prod_{n\in\omega}[p_n] \cap \mathrm{CS}(Q, d)$ such that $q = \lim(\langle q_n\rangle_{n\in\omega})$. Fix such a sequence $\langle q_n\rangle_{n\in\omega}$. We distinguish two cases according to whether $q \in A^\omega$ or not.

**Case 1.** Suppose $q \notin A^\omega$. Then, $q$ must stabilize, i.e., $\exists N, \forall n \geq N[\ q_n = q\ ]$. Fix such a number $N$. Since $\langle p_n\rangle_{n\in\omega}$ is a chain, there exists an ascending sequence $\langle q'_n\rangle_{n\leq N} \in \prod_{n\leq N}[p_n]$ with $q'_N = q$. Let us define $\langle q''_n\rangle_{n\in\omega} \in \prod_{n\in\omega}[p_n]$ by:

$$q''_n = \begin{cases} q'_n & \text{if } n \leq N,\\ q & \text{if } n > N.\end{cases}$$

Obviously, $\langle q''_n\rangle_{n\in\omega} \in \prod_{n\in\omega}[p_n] \cap \mathrm{CS}(Q, \sqsubseteq)$ with $\mathrm{lub}(\langle q''_n\rangle_{n\in\omega}) = q$. Thus, $q \in \mathrm{Lub}(\langle p_n\rangle_{n\in\omega})$.

**Case 2.** Suppose $q \in A^\omega$. We distinguish two cases according to whether $\exists N, \forall n \geq N[\ q \in p_n\ ]$ or not.

**Subcase 2.1.** Suppose $\exists N, \forall n \geq N[\ q \in p_n\ ]$. Then, as in Case 1, we can define a sequence $\langle q''_n\rangle_{n\in\omega} \in \prod_{n\in\omega}[p_n] \cap \mathrm{CS}(Q, \sqsubseteq)$ such that $\forall n \geq N[\ q''_n = q\ ]$. Thus $q = \lim(\langle q''_n\rangle_{n\in\omega}) =\in \mathrm{Lub}(\langle p_n\rangle_{n\in\omega})$.

**Subcase 2.2.** Suppose $\forall N, \exists n \geq N[\ q \notin p_n\ ]$. Then, there exists a strictly increasing sequence $\langle \nu(i)\rangle_{i\in\omega} \in \mathrm{Chain}(\omega, <)$ such that $\forall i \in \omega[\ q \notin p_{\nu(i)}\ ]$. For each $i$, one has

$$d_*(q, p_{\nu(i)}) = \inf\{d(q, q') : q' \in p_{\nu(i)}\} > 0,$$

since $q \notin p_{\nu(i)}$ and since $p_{\nu(i)}$ is closed. By this and the fact $\lim_{n\in\omega}[d_*(q,p_n)] = 0$, we can assume, without loss of generality, that

$$\langle d_*(q, p_{\nu(i)})\rangle_{i\in\omega} \text{ is strictly decreasing.} \tag{2.17}$$

Let $i \in \omega$. Since $d_*(q, p_{\nu(i)}) > 0$, there exists $\tilde{q}_i \in p_{\nu(i)}$ such that

$$d(q, \tilde{q}_i) = d_*(q, p_{\nu(i)}).$$

Fix such an element $\tilde{q}_i$. Let $\ell(i) = \max\{k : \tilde{q}_i \;\; k = q \;\; k\}$. Then by the definition of $d$, one has $d_*(q, p_{\nu(i)}) = \kappa^{\ell(i)}$. Since $\langle d_*(q, p_{\nu(i)})\rangle_{i\in\omega}$ is strictly decreasing, one has

$$\langle \ell(i)\rangle_{i\in\omega} \text{ is strictly increasing.} \tag{2.18}$$

Since $p_{\nu(i)} \sqsubseteq_{\mathrm{s}} p_{\nu(i)+1} \sqsubseteq_{\mathrm{s}} \cdots \sqsubseteq_{\mathrm{s}} p_{\nu(i+1)}$, there exists an ascending sequence $\langle q_j^{(i)}\rangle_{j\in[\nu(i)..\nu(i+1)]} \in \prod_{j\in[\nu(i)..\nu(i+1)]}[p_j]$ with $q_{\nu(i+1)}^{(i)} = \tilde{q}_{i+1}$. By the definition of $\sqsubseteq$, either

$$q_{\nu(i)}^{(i)} = \tilde{q}_{i+1}, \tag{2.19}$$

or

$$q_{\nu(i)}^{(i)} \in A^{<\omega} \cdot \{\langle \perp \rangle\} \wedge \mathrm{strip}(q_{\nu(i)}^{(i)}) \leq_{\mathrm{p}} \mathrm{strip}(\tilde{q}_{i+1}). \tag{2.20}$$

Having (2.19) is impossible by (2.17). Thus one has (2.20).

Let us prove by contradiction that $\mathrm{lgt}(q_{\nu(i)}^{(i)}) \leq \mathrm{lgt}(\tilde{q}_i)$. Assume to the contrary that

$$\mathrm{lgt}(\tilde{q}_i) < \mathrm{lgt}(q_{\nu(i)}^{(i)}). \tag{2.21}$$

We distinguish two cases according to whether $\mathrm{lgt}(\tilde{q}_i) \leq \ell(i+1)$ or not.

When $\mathrm{lgt}(\tilde{q}_i) \leq \ell(i+1)$, one has

$$q_{\nu(i)}^{(i)} \;\; \mathrm{lgt}(\tilde{q}_i) = \tilde{q}_{\nu(i+1)} \;\; \mathrm{lgt}(\tilde{q}_i) = q \;\; \mathrm{lgt}(\tilde{q}_i),$$

and therefore,

$$\max\{k \in \omega : q_{\nu(i)}^{(i)} \;\; k = q \;\; k\} \geq \mathrm{lgt}(\tilde{q}) > \ell(i).$$

When $\ell(i+1) < \mathrm{lgt}(\tilde{q}_i)$, one has

$$q_{\nu(i)}^{(i)} \;\; \ell(i+1) = \tilde{q}_{\nu(i+1)} \;\; \ell(i+1) = q \;\; \ell(i+1),$$

and therefore,

$$\max\{k \in \omega : q_{\nu(i)}^{(i)} \;\; k = q \;\; k\} \geq \ell(i+1) > \ell(i).$$

Thus, in both cases, one has

$$\max\{k \in \omega : q_{\nu(i)}^{(i)} \;\; k = q \;\; k\} > \ell(i),$$

and therefore,

$$\begin{aligned} &d_*(q, p_{\nu(i)}) \le d(q, q^{(i)}_{\nu(i)}) = \kappa^{\max\{k \in \omega : q^{(i)}_{\nu(i)}\ \ k =_q k\}} \\ &< \kappa^{\ell(i)} = d(q, \tilde{q}_i). \end{aligned}$$

But this contradicts the fact that $d_*(q, p_{\nu(i)}) = d(q, \tilde{q}_i)$. Thus, (2.21) is false, i.e., $\mathrm{lgt}(q^{(i)}_{\nu(i)}) \le \mathrm{lgt}(\tilde{q}_i)$.

Moreover,

$$\begin{aligned} &\mathrm{strip}(q^{(i)}_{\nu(i)}) = q^{(i)}_{\nu(i)} \quad (\mathrm{lgt}(q^{(i)}_{\nu(i)}) - 1) \\ &= \tilde{q}_{i+1} \quad (\mathrm{lgt}(q^{(i)}_{\nu(i)}) - 1) = q \quad (\mathrm{lgt}(q^{(i)}_{\nu(i)}) - 1) \\ &\le_{\mathrm{p}} q \quad (\mathrm{lgt}(\tilde{q}_i) - 1) = \tilde{q}_i \quad (\mathrm{lgt}(\tilde{q}_i) - 1) = \mathrm{strip}(\tilde{q}_i). \end{aligned}$$

By this and (2.20), one has $q^{(i)}_{\nu(i)} \sqsubseteq \tilde{q}_i$. From this and the fact that $p_{\nu(i)}$ is flat, it follows that $q^{(i)}_{\nu(i)} = \tilde{q}_i$. Thus,

$$\tilde{q}_i = q^{(i)}_{\nu(i)} \sqsubseteq q^{(i)}_{\nu(i)+1} \sqsubseteq \cdots \sqsubseteq q^{(i)}_{\nu(i)} = \tilde{q}_{i+1}.$$

Fix such a sequence $\langle q^{(i)}_j \rangle_{j \in [\nu(i)..\nu(i+1)]}$ for each $i \in \omega$. Further, there is an ascending chain $\langle q'_j \rangle_{j \in [0..\nu(0)]} \in \prod_{j \in [0..\nu(0)]} [p_j]$ with $q'_{\nu(0)} = \tilde{q}_0$. Let $\langle q''_j \rangle_{j \in \omega} \in \mathrm{Chain}(Q, \sqsubseteq) \cap \prod_{j \in \omega} [p_j]$ be defined by:

$$\langle q''_j \rangle_{j \in \omega} = \langle q'_j \rangle_{j \in [0..\nu(0)]} \cup \bigcup_{i \in \omega} [\langle q^{(i)}_j \rangle_{j \in [\nu(i)..\nu(i+1)]}].$$

From this definition, it immediately follows that $\langle q''_j \rangle_{j \in \omega} \in \mathrm{Chain}(Q, \sqsubseteq)$, and that $\forall i \in \omega[\ q''_{\nu(i)} = \tilde{q}_i\ ]$. Thus

$$q = \mathrm{lub}(\langle \tilde{q}_i \rangle_{i \in \omega}) = \mathrm{lub}(\langle q''_j \rangle_{j \in \omega}) \in \mathrm{Lub}(\langle p_j \rangle_{j \in \omega}). \ \blacksquare$$

For $\wp_{\mathrm{fco}}(Q)$ instead of $\wp_{\mathrm{fcl}}(Q)$, we have the following stronger result:

**Lemma 2.8** *Let* $\langle p_n \rangle_{n \in \omega} \in \mathrm{Chain}(\wp_{\mathrm{fco}}(Q), \sqsubseteq_{\mathrm{s}})$.

**(1)** $\langle p_n \rangle_{n \in \omega} \in \mathrm{CS}(\wp_{\mathrm{fco}}(Q), d)$.

**(2)** $\mathrm{Lub}(\langle p_n \rangle_{n \in \omega}) = \bigsqcup(\langle p_n \rangle_{n \in \omega}, \wp_{\mathrm{fco}}(Q)) = \lim(\langle p_n \rangle_{n \in \omega}, \wp_{\mathrm{fco}}(Q))$. ▮

**Proof.** **(1)** This part is essentially the same as Corollary 3.30 of [MV 88], although the notion of sequences of [MV 88] is slightly more restricted than ours; here we give a proof for the sake of self-containedness with certain modifications in accordance with the present setting.

Let $\langle p_n \rangle_{n \in \omega} \in \mathrm{Chain}(\wp_{\mathrm{fco}}(Q))$.

First, let us show that $\langle p_n \rangle_{n \in \omega}$ is a CS, i.e. that $\langle p_n \rangle_{n \in \omega} \in \mathrm{CS}(\wp_{\mathrm{fco}}(Q))$. For this purpose, it suffices to show that

$$\forall m \in \omega, \exists N, \forall n \ge N[\ (p_n)^{[m]} = (p_N)^{[m]}\ ]. \qquad (2.22)$$

Fix $m \in \omega$. For each $n$, let $\hat{p}_n = (p_n)^{[m]}$. Note that $(*)$: *for each $n$, $\hat{p}_n$ is finite.* Let $\tilde{p} = \mathrm{Lub}(\langle \hat{p}_n \rangle_{n \in \omega})$. We will prove that

$$\exists N, \forall n \geq N[\ \hat{p}_n = \tilde{p}\ ]. \tag{2.23}$$

First, let us prove that

$$\exists N, \forall n \geq N[\ \tilde{p} \subseteq \hat{p}_n\ ]. \tag{2.24}$$

For this purpose, it suffices to show that

$$\begin{aligned} &\exists N \in \omega, \forall \langle q_n \rangle_{n \in \omega} \in \textstyle\prod_{n \in \omega}[\hat{p}_n] \cap \mathrm{Chain}(\wp_{\mathrm{fco}}(Q))[ \\ &\quad \forall n \geq N[\ q_n = q_N\ ]]. \end{aligned} \tag{2.25}$$

Assume, for the sake of contradiction, that (2.25) does not hold, i.e.,

$$\forall N \in \omega, \exists \langle q_n \rangle_{n \in \omega} \in \prod_{n \in \omega}[\hat{p}_n] \cap \mathrm{Chain}(\wp_{\mathrm{fco}}(Q))[\ \exists n > N[\ q_N \sqsubset q_n\ ]].$$

Then, we can define, by induction, a strictly increasing sequence $\langle \nu(i) \rangle_{i \in \omega} \in (\omega \to \omega)$ and a chain $\langle q'_i \rangle_{i \in \omega} \in \prod_{i \in \omega}[\hat{p}_{\nu(i)}]$ such that

$$\forall i \in \omega[\ q'_i \sqsubset q'_{i+1}\ ], \tag{2.26}$$

as follows.

First, let $\nu(0) = 0$ and choose $q_0$ so that

$$\begin{aligned} &\forall N \in \omega, \exists \langle q_n \rangle_{n \in \omega} \in \textstyle\prod_{n \in \omega}[\hat{p}_n] \cap \mathrm{Chain}(\wp_{\mathrm{fco}}(Q))[ \\ &\quad q_{\nu(0)} = q'_0\ \wedge\ \exists n \geq N[\ q_N \sqsubset q_n\ ]]. \end{aligned}$$

Let $k > 0$, and suppose $\langle \nu(i) \rangle_{i \in k}$ and $\langle q'_i \rangle_{i \in k}$ have been defined so that $\langle \nu(i) \rangle_{i \in k}$ is strictly increasing, $\langle q'_i \rangle_{i \in k} \in \prod_{i \in k}[\hat{p}_{\nu(i)}]$ with

$$\forall i \in k[\ i + 1 < k\ \Rightarrow\ q'_i \sqsubseteq q'_{i+1}\ ],$$

and

$$\begin{aligned} &\forall N \in \omega, \exists \langle q_n \rangle_{n \in \omega} \in \textstyle\prod_{n \in \omega}[\hat{p}_n] \cap \mathrm{Chain}(\wp_{\mathrm{fco}}(Q))[ \\ &\quad \forall i \in k[\ q_{\nu(i)} = q'_i\ ]\ \wedge\ \exists n > N[\ q_N \sqsubset q_n\ ]]. \end{aligned} \tag{2.27}$$

Then, taking $\nu(k-1)$ as $N$ in (2.27), we know there exists

$$\langle q_n \rangle_{n \in \omega} \in \prod_{n \in \omega}[\hat{p}_n] \cap \mathrm{Chain}(\wp_{\mathrm{fco}}(Q))$$

such that

$$\forall i \in k[\ q_{\nu(i)} = q'_i\ ]\ \wedge\ \exists n > \nu(k-1)[\ q_{\nu(k-1)} \sqsubset q_n\ ].$$

Fix such a chain $\langle q_n \rangle_{n \in \omega}$, and let $\nu(k) = \min\{n > \nu(k-1) : q_{\nu(k-1)} \sqsubset q_n\}$. We will show that

$$\begin{array}{l}\forall \langle \tilde{q}_n \rangle_{n\in\omega} \in \prod_{n\in\omega}[\hat{p}_n] \cap \mathrm{Chain}(\wp_{\mathrm{fco}}(Q))[\\ \quad \forall i \in k[\ \tilde{q}_{\nu(i)} = q'_i\ ] \Rightarrow \\ \quad \forall n \in [\nu(k-1)..\nu(k)-1][\ q'_{k-1} = \tilde{q}_{\nu(k-1)} = \tilde{q}_n\ ]].\end{array} \tag{2.28}$$

Assume, for the sake of contradiction, that (2.28) does not hold, i.e., there exists

$$\langle \tilde{q}_n \rangle_{n\in\omega} \in \prod_{n\in\omega}[\hat{p}_n] \cap \mathrm{Chain}(\wp_{\mathrm{fco}}(Q))$$

with $\forall i \in k[\ \tilde{q}_{\nu(i)} = q'_i\ ]$ such that

$$\exists n \in [\nu(k-1)..\nu(k)-1][\ q'_{k-1} = \tilde{q}_{\nu(k-1)} \mathbin{@} \tilde{q}_n\ ].$$

Fix such a number $n$. Then, $q_n = q'_{k-1} \mathbin{@} \tilde{q}_n$ with $q_n, \tilde{q}_n \in \hat{p}_n$, but this is impossible since $\hat{p}_n$ is flat. Thus (2.28) must hold. In a similar fashion, it can be shown that

$$\begin{array}{l}\forall \langle \tilde{q}_n \rangle_{n\in\omega} \in \prod_{n\in\omega}[\hat{p}_n] \cap \mathrm{Chain}(\wp_{\mathrm{fco}}(Q))[\\ \quad \forall i \in k[\ \tilde{q}_{\nu(i)} = q'_i\ ] \Rightarrow q'_{k-1} = \tilde{q}_{\nu(k-1)} \sqsubseteq \tilde{q}_{\nu(k)}\ ].\end{array}$$

Since $\hat{p}_{\nu(k)}$ is finite, there is $q' \in \hat{p}_{\nu(k)}$ such that $q'_{\nu(k-1)} \mathbin{@} q'$ and

$$\begin{array}{l}\forall N \in \omega, \exists \langle q_n \rangle_{n\in\omega} \in \prod_{n\in\omega}[\hat{p}_n] \cap \mathrm{Chain}(\wp_{\mathrm{fco}}(Q))[\\ \quad \forall i \in k[\ q_{\nu(i)} = q'_i\ ] \wedge q_{\nu(k)} = q' \wedge \exists n > N[\ q_N \mathbin{@} q_n\ ]].\end{array}$$

Define $q'_k$ to be such $q'$.

In this way, we define $\langle \nu(i) \rangle_{i\in\omega} \in (\omega \to \omega)$ and $\langle q'_i \rangle_{i\in\omega} \in \prod_{i\in\omega}[\hat{p}_{\nu(i)}]$ with the desired property (2.26). From (2.26), it follows that

$$\forall i \in \omega[\ q'_i \in A^{<\omega} \cdot \{\langle \bot \rangle\}\ ] \wedge \forall i \in \omega[\ \mathrm{strip}(q'_i) <_{\mathrm{p}} \mathrm{strip}(q'_{i+1})\ ].$$

This is, however, impossible, since $\forall i[\ \mathrm{lgt}(q'_i) \le m+1\ ]$. Hence, (2.25) must hold, and therefore, one has (2.24), i.e., $\exists N, \forall n \ge N[\ \tilde{p} \subseteq \hat{p}_n\ ]$. Fix such $N$ in the sequel of this proof.

Next, we will show, by contradiction, that

$$\exists N', \forall n \ge N'[\ \hat{p}_n \subseteq \tilde{p}\ ]. \tag{2.29}$$

Assume, to the contrary, that

$$\forall N', \exists n \ge N'[\ \hat{p}_n \setminus \tilde{p} \ne \emptyset\ ].$$

Then, we can define a strictly increasing sequence $\langle \tilde{\nu}(i) \rangle_{i\in\omega} \in (\omega \to \omega)$ such that

$$N \le \tilde{\nu}(0) \wedge \forall i \in \omega[\ \hat{p}_{\tilde{\nu}(i)} \setminus \tilde{p} \ne \emptyset\ ]. \tag{2.30}$$

Fix such a sequence $\langle \tilde{\nu}(i) \rangle_{i\in\omega}$.

Then, we will define, by induction, a chain $\langle \hat{q}_i \rangle_{i\in\omega} \in \prod_{i\in\omega}[\hat{p}_{\tilde{\nu}(i)} \setminus \tilde{p}]$.

From (2.30) and the fact that $\forall i > 0[\ \hat{p}_{\tilde{\nu}(0)} \sqsubseteq_{\mathrm{s}} \hat{p}_{\tilde{\nu}(i)}\ ]$, it follows that for $i > 0$ and $\hat{q} \in (\hat{p}_{\tilde{\nu}(i)} \setminus \tilde{p})$, there exisits $\hat{q}' \in \hat{p}_{\tilde{\nu}(0)}$ such that $\hat{q}' \sqsubseteq \hat{q}$. It is impossible that $\hat{q}' \in \tilde{p}$, since if $\hat{q}' \in \tilde{p}$ then

$$\hat{q}' \mathbin{@} \hat{q} \wedge \hat{q}' \in \hat{p}_{\tilde{\nu}(i)} \wedge \hat{q} \in \hat{p}_{\tilde{\nu}(i)},$$

which contradicts the flatness of $\hat{p}_{\tilde{\nu}(i)}$. Thus, one has

$$\forall i > 0, \forall \hat{q} \in (\hat{p}_{\tilde{\nu}(i)} \setminus \tilde{p}), \exists \hat{q}' \in (\hat{p}_{\tilde{\nu}(0)} \setminus \tilde{p})[\ \hat{q}' \sqsubseteq \hat{q}\ ].$$

From this and the fact that $(\hat{p}_{\tilde{\nu}(0)} \setminus \tilde{p})$ is finite, it follows that there exists $\hat{q}' \in (\hat{p}_{\tilde{\nu}(0)} \setminus \tilde{p})$ such that

$$\forall i > 0, \exists \hat{q} \in (\hat{p}_{\tilde{\nu}(i)} \setminus \tilde{p})[\ \hat{q}' \sqsubseteq \hat{q}\ ].$$

Let us define $\hat{q}_0$ to be such $\hat{q}'$.

Fix $k \in \omega$, and suppose we have defined a chain

$$\langle \hat{q}_0, \ldots, \hat{q}_k \rangle \in \prod_{i \in (k+1)} [\hat{p}_{\tilde{\nu}(i)} \setminus \tilde{p}]$$

with

$$\forall i > k, \exists q \in (\hat{p}_{\tilde{\nu}(i)} \setminus \tilde{p})[\ \hat{q}_k \sqsubseteq q\ ].$$

For each $i > (k+1)$ and $q \in (\hat{p}_{\tilde{\nu}(i)} \setminus \tilde{p})$ with $\hat{q}_k \sqsubseteq q$, it follows from the interpolation property (Lemma 2.4 (1)), that there is $\hat{q} \in \hat{p}_{\nu(k+1)}$ such that $\hat{q}_k \sqsubseteq \hat{q} \sqsubseteq q$. One has $\hat{q} \notin \tilde{p}$, since if $\hat{q} \in \tilde{p}$ then

$$\hat{q} \in \tilde{p} \subseteq \hat{p}_{\nu(i)} \ \wedge\ q \in (\hat{p}_{\nu(i)} \setminus \tilde{p}) \ \wedge\ \hat{q} \sqsubset q,$$

which contradicts the flatness of $\hat{p}_{\tilde{\nu}(i)}$. Thus $q$ must be an element of $\hat{p}_{\tilde{\nu}(k+1)} \setminus \tilde{p}$.

Since $\hat{p}_{\tilde{\nu}(k+1)} \setminus \tilde{p}$ is finite, there exists $\hat{q} \in (\hat{p}_{\tilde{\nu}(k+1)} \setminus \tilde{p})$ such that

$$\hat{q}_k \sqsubseteq \hat{q} \wedge \forall i > (k+1), \exists q \in (\hat{p}_{\tilde{\nu}(i)} \setminus \tilde{p})[\ \hat{q} \sqsubseteq q\ ].$$

Define $\hat{q}_{k+1}$ to be such $\hat{q}$.

In this way, a chain $\langle \hat{q}_i \rangle_{i \in \omega} \in \prod_{i \in \omega} [\hat{p}_{\tilde{\nu}(i)} \setminus \tilde{p}]$ is defined.

Since $\forall i \in \omega[\ \mathrm{lgt}(\hat{q}_i) \leq m + 1\ ]$, the sequence $\langle \hat{q}_i \rangle_{i \in \omega}$ must stabilize, i.e., there exists $\ell$ such that $\forall i \geq \ell[\ \hat{q}_i = \hat{q}_\ell\ ]$. Fix such $\ell$. One has,

$$\hat{q}_\ell = \mathrm{lub}(\langle \hat{q}_i \rangle_{i \in \omega}) \in \tilde{p}.$$

This, however, contradicts the fact $\hat{q}_\ell \in (\hat{p}_{\tilde{\nu}(\ell)} \setminus \tilde{p})$. Hence, (2.29) must hold.

From (2.24) and (2.29), it follows that

$$\exists N', \forall n \geq N[\ \tilde{p} = \hat{p}_n\ ].$$

Thus, one has (2.23), and therefore, (2.22).

**(2)** This part follows from part (1) and Lemma 2.7 as follows.

Let $\langle p_n \rangle_{n \in \omega} \in \mathrm{Chain}(\wp_{\mathrm{fco}}(Q), \sqsubseteq_{\mathrm{s}})$. From part (1), it follows that

$$\langle p_n \rangle_{n \in \omega} \in \mathrm{CS}(\wp_{\mathrm{fco}}(Q), d).$$

By this and Lemma 2.7, one has

$$\mathrm{Lub}(\langle p_n \rangle_{n \in \omega}) = \bigsqcup(\langle p_n \rangle_{n \in \omega}, \wp_{\mathrm{fcl}}(Q)) = \lim(\langle p_n \rangle_{n \in \omega}, \wp_{\mathrm{fcl}}(Q)). \tag{2.31}$$

From Lemma 2.5 and the fact that the property being *bounded* is finitely characterized, it follows that $\text{Lub}(\langle p_n\rangle_{n\in\omega}) \in \wp_{\text{fco}}(Q)$. By this and (2.31), one has

$$\text{Lub}(\langle p_n\rangle_{n\in\omega}) = \bigsqcup(\langle p_n\rangle_{n\in\omega}, \wp_{\text{fco}}(Q)) = \lim(\langle p_n\rangle_{n\in\omega}, \wp_{\text{fco}}(Q)). \blacksquare$$

We do not have a similar property for $\wp_{+\text{fcl}}(Q)$, i.e., not every $\langle p_n\rangle_{n\in\omega} \in \text{Chain}(\wp_{+\text{fcl}}(Q), \sqsubseteq_s)$, is a CS; this is demonstrated by the following example:

**Example 2.2** Suppose $B$ be infinite, and let $B = \{b_n : n \in \omega\}$ with $b_i \neq b_j$ for $i \neq j$. For each $n \in \omega$, let $p_n = \{\langle b_i\rangle : i \geq n\}$. Then, obviously, $\langle p_n\rangle_{n\in\omega} \in \text{Chain}(\wp_{+\text{fcl}}(Q), \sqsubseteq_s)$, but $\langle p_n\rangle_{n\in\omega}$ is not a CS. ∎

From Lemma 2.6 and Lemma 2.8, we have the following two corollaries.

**Corollary 2.2** **(1)** *Let* $\langle p_n\rangle_{n\in\omega} \in \text{Chain}(\wp_{\text{fco}}(Q), \sqsubseteq_s)$. *Then,* $\text{Lub}(\langle p_n\rangle_{n\in\omega}) \in \wp_{\text{fco}}(Q)$. *Also* $\text{Lub}(\langle p_n\rangle_{n\in\omega})$ *is the lub of* $\langle p_n\rangle_{n\in\omega}$ *in* $\wp_{\text{fco}}(Q)$, *i.e.,*

$$\text{Lub}(\langle p_n\rangle_{n\in\omega}) = \bigsqcup(\langle p_n\rangle_{n\in\omega}).$$

**(2)** $(\wp_{\text{fco}}(Q), \sqsubseteq_s, \{\langle\bot\rangle\})$ *is a cpo.* ∎

**Proof.** **(1)** Let $\langle p_n\rangle_{n\in\omega} \in \text{Chain}(\wp_{+\text{fco}}(Q), \sqsubseteq_s)$. Then, by Lemma 2.8, $\langle p_n\rangle_{n\in\omega} \in \text{CS}(\wp_{+\text{fco}}(Q), d)$ and

$$\text{Lub}(\langle p_n\rangle_{n\in\omega}) = \bigsqcup(\langle p_n\rangle_{n\in\omega}, \wp_{\text{fco}}(Q)) = \lim(\langle p_n\rangle_{n\in\omega}, \wp_{\text{fco}}(Q)).$$

Obviously, the property of *being flat and compact* is finitely characterized, and thus, $\lim_{n\in\omega}[p_n] \in \wp_{\text{fco}}(Q)$ (see § 2.2). Hence, $\text{Lub}(\langle p_n\rangle_{n\in\omega}) \in \wp_{\text{fco}}(Q)$, and therefore, $\text{Lub}(\langle p_n\rangle_{n\in\omega})$ is the lub of $\langle p_n\rangle_{n\in\omega}$ in $\wp_{\text{fco}}(Q)$.
**(2)** This part immediately follows from part (1). ∎

**Corollary 2.3** **(1)** *Let* $\langle p_n\rangle_{n\in\omega} \in \text{Chain}(\wp_{+\text{fco}}(Q), \sqsubseteq_s)$. *Then,* $\text{Lub}(\langle p_n\rangle_{n\in\omega}) \in \wp_{+\text{fco}}(Q)$. *Also* $\text{Lub}(\langle p_n\rangle_{n\in\omega})$ *is the lub of* $\langle p_n\rangle_{n\in\omega}$ *in* $\wp_{+\text{fco}}(Q)$, *i.e.,* $\text{Lub}(\langle p_n\rangle_{n\in\omega}) = \bigsqcup(\langle p_n\rangle_{n\in\omega})$.

**(2)** $(\wp_{+\text{fco}}(Q), \sqsubseteq_s, \{\langle\bot\rangle\})$ *is a cpo.* ∎

**Proof.** Similar to the proof of Corollary 2.2. In the proof, we exploit the fact that the property *being nonempty, flat, and compact* is finitely characterized. ∎

We give a few more propositions for later use.

**Proposition 2.5** **(1)** *For every* $\langle q_n\rangle_{n\in\omega}, \langle q'_n\rangle_{n\in\omega} \in \text{Chain}(Q, \sqsubseteq)$,

$$\forall n \in \omega[\ q_n \sqsubseteq q'_n\ ] \Rightarrow \text{lub}_{n\in\omega}[q_n] \sqsubseteq \text{lub}_{n\in\omega}[q'_n].$$

**(2)** *For every* $\langle p_n\rangle_{n\in\omega}, \langle p'_n\rangle_{n\in\omega} \in \text{Chain}(\wp_{\text{f}}(Q), \sqsubseteq_s)$,

$$\forall n \in \omega[\ p_n \sqsubseteq_s p'_n\ ] \Rightarrow \text{Lub}_{n\in\omega}[p_n] \sqsubseteq_s \text{Lub}_{n\in\omega}[p'_n].$$

**(3)** *For every* $\langle p_n\rangle_{n\in\omega}, \langle p'_n\rangle_{n\in\omega} \in \text{CS}(\wp_{\text{f}}(Q), \tilde{d})$,

$$\forall n \in \omega[\ p_n \sqsubseteq_{\mathrm{s}} p'_n\ ], \Rightarrow \lim_{n\in\omega}[p_n] \sqsubseteq_{\mathrm{s}} \lim_{n\in\omega}[p'_n]. \blacksquare$$

**Proof.** **(1)** This part follows immediately from the definition of $\mathrm{lub}_{n\in\omega}[q_n]$.
**(2)** Let $\langle p_n\rangle_{n\in\omega}, \langle p'_n\rangle_{n\in\omega} \in \mathrm{Chain}(\wp_{\mathrm{f}}(Q), \sqsubseteq_{\mathrm{s}})$, and suppose $\forall n \in \omega[\ p_n \sqsubseteq_{\mathrm{s}} p'_n\ ]$. Let $q' \in \mathrm{Lub}_{n\in\omega}[p'_n]$. We will show that

$$\exists q \in \mathrm{Lub}_{n\in\omega}[p_n][\ q \sqsubseteq q'\ ]. \tag{2.32}$$

By the definition of $\mathrm{Lub}_{n\in\omega}[p'_n]$, there exists $\langle q'_n\rangle_{n\in\omega} \in \mathrm{Chain}(Q, \sqsubseteq) \cap \prod_{n\in\omega}[p'_n]$ such that $q' = \mathrm{lub}_{n\in\omega}[q'_n]$. Fix such a sequence $\langle q'_n\rangle_{n\in\omega}$.

For every $n \in \omega$, there exists $q_n \in p_n$ such that $q_n \sqsubseteq q'_n$, because $p_n \sqsubseteq_{\mathrm{s}} p'_n$. Fix such $q_n$ for each $n$.

Fix $n \in \omega$. Since $p_n \sqsubseteq_{\mathrm{s}} p_{n+1}$, there exists $q''_n \in p_n$ such that $q''_n \sqsubseteq q_n$. Fix such $q''_n$. Then, $q''_n \sqsubseteq q_{n+1} \sqsubseteq q'_{n+1}$ and $q_n \sqsubseteq q'_n \sqsubseteq q'_{n+1}$. Thus, by Proposition 2.3, the two elements $q''_n$ and $q_n$ of $p_n$ are comparable. By this and the fact that $p_n$ is flat, one has $q''_n = q_n$, and therefore, $q_n \sqsubseteq q_{n+1}$.

Thus, $\langle p_n\rangle_{n\in\omega} \in \mathrm{Chain}(Q, \sqsubseteq)$. Further, by part (1),

$$\mathrm{lub}_{n\in\omega}[q_n] \sqsubseteq \mathrm{lub}_{n\in\omega}[q_n] = q'.$$

Thus, one has (2.32). Summarizing,

$$\forall q' \in \mathrm{Lub}_{n\in\omega}[p'_n], \exists q \in \mathrm{Lub}_{n\in\omega}[p'_n][\ q \sqsubseteq q'\ ],$$

i.e.,

$$\mathrm{Lub}_{n\in\omega}[p_n] \sqsubseteq_{\mathrm{s}} \mathrm{Lub}_{n\in\omega}[p'_n].$$

**(3)** Let $\langle p_n\rangle_{n\in\omega}, \langle p'_n\rangle_{n\in\omega} \in \mathrm{CS}(\wp_{\mathrm{f}}(Q), \tilde{d})$, and suppose

$$\forall n \in \omega[\ p_n \sqsubseteq_{\mathrm{s}} p'_n\ ]. \tag{2.33}$$

Let $p = \lim_{n\in\omega}[p_n]$ and $p' = \lim_{n\in\omega}[p'_n]$. We will prove

$$p \sqsubseteq_{\mathrm{s}} p'. \tag{2.34}$$

Fix $n \in \omega$. Since $\langle p_n\rangle_{n\in\omega}$ is a CS, there exists $N$ such that

$$\forall k \geq N[\ (p_k)^{[n]} = p^{[k]}\ ].$$

Likewise, there exists $N'$ such that

$$\forall k \geq N'[\ (p'_k)^{[n]} = (p')^{[k]}\ ].$$

Fix such $N$ and $N'$, and let $\ell = \max\{N, N'\}$. Then, $(p_\ell)^{[n]} = p^{[n]}$ and $(p'_\ell)^{[n]} = (p')^{[n]}$. By (2.33), $p_\ell \sqsubseteq_{\mathrm{s}} p'_\ell$, and therefore, $(p_\ell)^{[n]} \sqsubseteq_{\mathrm{s}} (p'_\ell)^{[n]}$. Thus, $p^{[n]} \sqsubseteq_{\mathrm{s}} (p')^{[n]}$. Summarizing,

$$\forall n \in \omega[\ p^{[n]} \sqsubseteq_{\mathrm{s}} (p')^{[n]}\ ].$$

From this and part (2), it follows that

$$p = \mathrm{Lub}_{n\in\omega}[p^{[n]}] \sqsubseteq_{\mathrm{s}} \mathrm{Lub}_{n\in\omega}[(p')^{[n]}] = p'.$$

Thus, one has (2.34). ■

**Definition 2.12** **(1)** Let $p \in \wp(Q)$. We say $q \in p$ is *minimal* in $p$ w.r.t. $\sqsubseteq$, when $\neg\exists q' \in p[\ q' \sqsubset q\ ]$.

**(2)** For $p \in \wp(Q)$, let

$$\mathrm{mini}(p) = \{q \in p : q \text{ is minimal in } p \text{ w.r.t. } \sqsubseteq\}.$$

**(3)** For $p_1, p_2 \in \wp(Q)$, let

$$p_1 \,\hat{\cup}\, p_2 = \mathrm{mini}(p_1 \cup p_2).$$

For $\mathbf{P}' \in \wp(Q)$, let

$$\hat{\bigcup}(\mathbf{P}') = \mathrm{mini}(\bigcup(\mathbf{P}')).$$

For $\langle p_i \rangle_{i\in I} \in (\wp(Q))^I$ with $I$ being some index set, let

$$\hat{\bigcup}_{i\in I}[p_i] = \hat{\bigcup}(\{p_i\}_{i\in I}).$$ ■

The following proposition can easily verified:

**Proposition 2.6** *Let $p \in \wp(Q)$ and $q \in p$. Then there exists unique $q' \in \mathrm{mini}(p)$ such that $q' \sqsubseteq q$.* ■

We introduce a notation for denoting such $q'$:

**Definition 2.13** For $p \in \wp(Q)$ and $q \in p$, let $\mathrm{mini}(p, q)$ be the unique $q' \in \mathrm{mini}(p)$ such that $q' \sqsubseteq q$. ■

Projection $(\cdot)^{[n]}$ and $\mathrm{mini}(\cdot)$ are interchangeable:

**Proposition 2.7** **(1)** $\forall p \in \wp_{\mathrm{f}}(Q), \forall n \in \omega[\ p^{[n]} \in \wp_{\mathrm{f}}(Q)\ ]$.

**(2)** $\forall p_1, p_2 \in \wp_{\mathrm{f}}(Q)[\ p_1 \sqsubseteq_{\mathrm{s}} p_2 \wedge p_2 \sqsubseteq_{\mathrm{s}} p_1 \Rightarrow p_1 = p_2\ ]$.

**(3)** $\forall p \in \wp(Q), \forall n \in \omega[\ (\mathrm{mini}(p))^{[n]} = \mathrm{mini}(p^{[n]})\ ]$. ■

**Proof.** **(1)** The claim is the same as Proposition 2.13 (a) of of [MV 88], where the proof is omitted. A proof is given here. Let $p \in \wp_{\mathrm{f}}(Q)$, $n \in \omega$, and $q_1, q_2 \in p$ such that

$$(q_1)^{[n]} \sqsubseteq (q_2)^{[n]}. \tag{2.35}$$

We will show $(q_1)^{[n]} = (q_2)^{[n]}$ by distinguishing two cases.
**Case 1.** Suppose $\mathrm{lgt}((q_1)^{[n]}) \leq n$. Then, by the definition of $(\cdot)^{[n]}$, one has

$$q_1 = (q_1)^{[n]} \sqsubseteq (q_2)^{[n]} \sqsubseteq q_2.$$

Thus, $q_1 = q_2$, because $q_1, q_2 \in p$ and $p$ is flat.
**Case 2.** Suppose $\mathrm{lgt}((q_1)^{[n]}) \leq n+1$. Then, by the definition of $(\cdot)^{[n]}$ one has $(q_1)^{[n]} = \mathrm{strip}((q_1)^{[n]}) \cdot \langle\bot\rangle$. Further, by the definition of $\sqsubseteq$, it follows from (2.35) that $\mathrm{strip}((q_1)^{[n]}) \leq_{\mathrm{p}} \mathrm{strip}((q_2)^{[n]})$, and thus, $\mathrm{lgt}(\mathrm{strip}((q_2)^{[n]})) = n$. Therefore, $\mathrm{lgt}((q_2)^{[n]}) = n+1$. Thus, by the definition of $(\cdot)^{[n]}$, it must hold that

$$(q_2)^{[n]} = \mathrm{strip}((q_2)^{[n]}) \cdot \langle\bot\rangle = \mathrm{strip}((q_1)^{[n]}) \cdot \langle\bot\rangle = (q_1)^{[n]}.$$

Summarizing, $\forall \hat{q}_1, \hat{q}_2 \in p^{[n]}[\ \hat{q}_1 \sqsubseteq \hat{q}_2 \Rightarrow \hat{q}_1 = \hat{q}_2\ ]$. That is, $p^{[n]}$ is flat.
**(2)** The claim is the same as Proposition 2.13 (b) of [MV 88], where the proof is omitted. A proof is given here. Let $p_1, p_2 \in \wp_{\mathrm{f}}(Q)$ such that $p_1 \sqsubseteq_{\mathrm{s}} p_2$ and $p_2 \sqsubseteq_{\mathrm{s}} p_1$. We will prove $p_2 \subseteq p_1$. (The fact that $p_1 \subseteq p_2$ can be shown in a similar fashion.)

Let $q \in p_2$. Then, since $p_1 \sqsubseteq_{\mathrm{s}} p_2$, there exists $q' \in p_1$ such that $q' \sqsubseteq q$. Further, since $p_2 \sqsubseteq_{\mathrm{s}} p_1$, there exists $q'' \in p_2$ such that $q'' \sqsubseteq q$. Thus, $q$ and $q''$ are elements of $p_2$ such that $q'' \sqsubseteq q$. Hence, by the fact $p_2$ is false, one has $q'' = q$, and therefore, $q' = q$. Summarizing, one has $p_2 \subseteq p_1$.
**(3)** The claim is the same as Proposition 2.23 (e) of [MV 88], except that the domain $Q$ is slightly more general than the one treated in [MV 88]. This is shown as in [MV 88], but with certain modifications.

By part (1), $(\mathrm{mini}(p))^{[n]}$ is flat. Hence, by part (2), it suffices to show that

$$(\mathrm{mini}(p))^{[n]} \sqsubseteq_{\mathrm{s}} \mathrm{mini}(p^{[n]}) \tag{2.36}$$

and that

$$\mathrm{mini}(p^{[n]}) \sqsubseteq_{\mathrm{s}} (\mathrm{mini}(p))^{[n]}. \tag{2.37}$$

First, let us show (2.36). Let $q \in p$ such that $q^{[n]} \in \mathrm{mini}(p^{[n]})$. Putting $q' = \mathrm{mini}(p, q)$, one has $q' \in \mathrm{mini}(p)$ and $q' \sqsubseteq q$. Thus, $(q')^{[n]} \in (\mathrm{mini}(p))^{[n]}$ and $(q')^{[n]} \sqsubseteq q^{[n]}$. Summarizing, one has

$$\forall \hat{q} \in \mathrm{mini}(p^{[n]}), \exists \hat{q}' \in (\mathrm{mini}(p))^{[n]}[\ \hat{q}' \sqsubseteq \hat{q}\ ],$$

i.e., (2.36) holds.

Next, let us show (2.37). Let $q \in \mathrm{mini}(p)$. Then, $q \in p$, and therefore, $q^{[n]} \in p^{[n]}$. Putting $\hat{p}'' = \mathrm{mini}(p^{[n]}, q^{[n]})$, one has $\hat{p}'' \in \mathrm{mini}(p^{[n]})$ and $\hat{p}'' \sqsubseteq q^{[n]}$. Summarizing, one has

$$\forall \hat{q} \in (\mathrm{mini}(p))^{[n]}, \exists \hat{q}'' \in \mathrm{mini}(p^{[n]})[\ \hat{q}'' \sqsubseteq \hat{q}\ ],$$

i.e., (2.37) holds. ■

From the above lemma, the next corollary immediately follows:

**Corollary 2.4** *For every* $p_1, p_2 \in \wp(Q)$ *and* $n \in \omega$,

$$(p_1)^{[n]} = (p_2)^{[n]} \Rightarrow (\mathrm{mini}(p_1))^{[n]} = (\mathrm{mini}(p_2))^{[n]}. ■$$

**Proof.** Let $p_1, p_2 \in \wp(Q)$, $n \in \omega$, and suppose $(*)$: $(p_1)^{[n]} = (p_2)^{[n]}$. Then, by Proposition 2.7 (2), one has

$$(\mathrm{mini}(p_1))^{[n]} = \mathrm{mini}((p_1)^{[n]}) = \mathrm{mini}((p_2)^{[n]}) = (\mathrm{mini}(p_2))^{[n]}. \blacksquare$$

The mapping mini : $\wp(Q) \to \wp(Q)$ is nonexpansive in the following sense:

**Lemma 2.9** *For every $p_1, p_2 \in \wp(Q)$, one has*

$$\tilde{d}(\mathrm{mini}(p_1), \mathrm{mini}(p_2)) \leq \tilde{d}(p_1, p_2). \blacksquare$$

**Proof.** First, we will prove that if one has

$$\begin{array}{l} \forall p_1, p_2 \in \wp(Q), \forall n \in \omega[ \\ \quad (p_1)^{[n]} = (p_2)^{[n]} \Rightarrow (\mathrm{mini}(p_1))^{[n]} = (\mathrm{mini}(p_2))^{[n]} \ ], \end{array} \tag{2.38}$$

then the desired result is obtained.

Let $p_1, p_2 \in \wp(Q)$. We will show

$$\tilde{d}(\mathrm{mini}(p_1), \mathrm{mini}(p_2)) \leq \tilde{d}(p_1, p_2). \tag{2.39}$$

We distinguish two cases according to whether or not

$$\forall n \in \omega[ \ (\mathrm{mini}(p_1))^{[n]} = (\mathrm{mini}(p_2))^{[n]} \ ]. \tag{2.40}$$

**Case 1.** Suppose that (2.40) holds. Then, $\tilde{d}(\mathrm{mini}(p_1), \mathrm{mini}(p_2)) = 0$, and therefore, one has (2.39).

**Case 2.** Suppose that (2.40) does not hold, i.e., that

$$\exists n \in \omega[ \ (\mathrm{mini}(p_1))^{[n]} \neq (\mathrm{mini}(p_2))^{[n]} \ ].$$

Let

$$m = \min\{n \in \omega : (\mathrm{mini}(p_1))^{[n]} \neq (\mathrm{mini}(p_2))^{[n]}\}.$$

By this and the characterization (2.10) of $\tilde{d}$, one has

$$\tilde{d}(\mathrm{mini}(p_1), \mathrm{mini}(p_2)) = \kappa^{m-1}.$$

By Corollary 2.4, one has $(p_1)^m \neq (p_2)^m$. Thus, putting

$$\ell = \min\{n \in \omega : (p_1)^{[n]} \neq (p_2)^{[n]}\},$$

one has $1 \leq \ell \leq m$. Hence

$$\tilde{d}(p_1, p_2) = \kappa^{\ell-1} \geq \kappa^{m-1} = \tilde{d}(\mathrm{mini}(p_1), \mathrm{mini}(p_2)).$$

Thus, one has (2.39). $\blacksquare$

# Chapter 3

# Introduction to Fully Abstract Models

In this chapter, we first give a definition of the notion of *full abstractness* in terms of the notion of *abstract contexts*, in § 3.1. The definitions are so abstract that they will apply to the various languages treated in the following chapters.

Next, in § 3.2, we give a fully abstract model for a very simple language involving a parallel construct but no communication; the model construction and the full abstractness proof are very easy, but they will still serve as an illustrative example of construction of fully abstract models.

## 3.1 Fully Abstract Models for Languages

We will give concrete definition of several languages in later parts of this thesis (in § 3.2 and in Chapters 4–8), where each of the languages is defined as the set of terms generated by a signature, sometimes with certain other constructs, such as *recursion* and *function abstraction.* However, no information about the construction of a language is necessary for the abstract definitions in this section; we fix a nonempty set $\mathcal{L}$, called a *language*, in this section.

### 3.1.1 Languages with the Notion of Contexts

In this thesis, we will define several languages as a set of *terms*, where we view a term as a finite ordered *tree* with its leaves and nodes labeled with symbols, rather than as strings. In other words, we are concerned with *abstract syntax* rather than *concrete syntax* (cf. § 1.2 of [Hen 90]; see also Remark 3.1 below for a related discussion).

Given a language as a set of terms, a *context* is usually defined as a term with a *hole* (or a *place-holder*) in it. (A hole can be represented in several ways, e.g., by a special symbol such as '[·]', by a variable, or by a sequence of integers specifying a *position* within a term (cf. [DJ 84]).)

Given a context $C[\cdot]$ in this sense, substitution of statements $s$ for the hole in $C[\cdot]$ determines an operation $(\lambda s \in \mathcal{L}.\ C[s])$ on $\mathcal{L}$, with $C[s]$ being the result of substituting $s$ for the hole in $C[\cdot]$. Further, two distinct contexts determine two distinct operations. Thus, we can identify a context $C[\cdot]$ with its associated operation on $\mathcal{L}$. Observing this, we give the following abstract definition of a *set of contexts*, where we take, as contexts, operations on $\mathcal{L}$.

**Definition 3.1** (Abstract Contexts)
Let $(\mathcal{S} \in)$ $\mathbf{Cont} \in \wp((\mathcal{L} \to \mathcal{L}))$. We call $\mathbf{Cont}$ a set of *abstract contexts* of $\mathcal{L}$, when the following two conditions are satisfied:

**(i)** $\mathrm{id}_{\mathcal{L}} \in \mathbf{Cont}$, with $\mathrm{id}_{\mathcal{L}}$ identity function on $\mathcal{L}$.

**(ii)** $\mathbf{Cont}$ is closed under composition, i.e.,

$$\forall \mathcal{S}_0, \mathcal{S}_1 \in \mathbf{Cont}[\ \mathcal{S}_0 \circ \mathcal{S}_1 \in \mathbf{Cont}\ ].\ \blacksquare$$

(In the sequel of this thesis, we will give a concrete definition of the set of contexts for each of the languages $\mathcal{L}_i$ $(i = 0, \ldots, 5)$; for each of the concretely defined sets of contexts, it will be easy to check that the two conditions in Definition 3.1 are satisfied.)

Whenever we consider the full abstractness of a model for a language in this thesis, it is always assumed that a set of contexts of the language is fixed. In the rest of this section, we fix a language $\mathcal{L}$ and a set $\mathbf{Cont}$ of contexts of $\mathcal{L}$.

## 3.1.2 Fully Abstract Models for Languages

Suppose a model $\mathcal{O}$ for $\mathcal{L}$ is given.

The notions of *compositionality* is defined in terms of contexts by:

**Definition 3.2** (Compositionality) Let $\mathcal{M} : \mathcal{L} \to \mathbf{D}$ be a model for $\mathcal{L}$. We say $\mathcal{M}$ is *compositional*, when

$$\begin{aligned} \forall s_0, s_1 \in \mathcal{L}[\ & \mathcal{M}[\![s_0]\!] = \mathcal{M}[\![s_1]\!] \Rightarrow \\ & \forall \mathcal{S} \in \mathbf{Cont}[\ \mathcal{M}[\![\mathcal{S}(s_0)]\!] = \mathcal{M}[\![\mathcal{S}(s_1)]\!]\ ]].\ \blacksquare \end{aligned} \tag{3.1}$$

(In the following, it will be helpful, for getting some engineering feeling, to assume that $\mathcal{O}$ represents some view of behavior of statements of $\mathcal{L}$, although this assumption is not necessary for the mathematical understanding of the following definitions.) The notions of *correctness*, *completeness*, *full abstractness* of another model $\mathcal{M}$ w.r.t. $\mathcal{O}$ are defined in terms of contexts as follows:

**Definition 3.3** Let $\mathcal{M}$ a model for $\mathcal{L}$.

**(1)** We say $\mathcal{M}$ is *correct* w.r.t. $\mathcal{O}$, when

$$\begin{aligned} \forall s_0, s_1 \in \mathcal{L}[\ & \mathcal{M}[\![s_0]\!] = \mathcal{M}[\![s_1]\!] \\ & \Rightarrow \forall \mathcal{S} \in \mathbf{Cont}[\ \mathcal{O}[\![\mathcal{S}(s_0)]\!] = \mathcal{O}[\![\mathcal{S}(s_1)]\!]\ ]]. \end{aligned}$$

**(2)** We say $\mathcal{M}$ is *complete* w.r.t. $\mathcal{O}$, when

$$\forall s_0, s_1 \in \mathcal{L}[\,\forall \mathcal{S} \in \mathbf{Cont}[\, \mathcal{O}[\![\mathcal{S}(s_0)]\!] = \mathcal{O}[\![\mathcal{S}(s_1)]\!]\,] \\ \Rightarrow \mathcal{M}[\![s_0]\!] = \mathcal{M}[\![s_1]\!]\,].$$

**(3)** We say $\mathcal{M}$ is *fully abstract* w.r.t. $\mathcal{O}$, when $\mathcal{M}$ is correct and complete w.r.t. $\mathcal{O}$, i.e., when

$$\forall s_0, s_1 \in \mathcal{L}[\,\mathcal{M}[\![s_0]\!] = \mathcal{M}[\![s_1]\!] \\ \Leftrightarrow \forall \mathcal{S} \in \mathbf{Cont}[\, \mathcal{O}[\![\mathcal{S}(s_0)]\!] = \mathcal{O}[\![\mathcal{S}(s_1)]\!]\,]\,].$$ ∎

**Remark 3.1** By the above definition, the property of full abstractness is sensitive to the choice of the set **Cont** of contexts, but there is usually only one *natural* definition of the set of contexts for a given language $\mathcal{L}$ defined by an abstract syntax. This is not the case, however, for a language defined by a *concrete syntax*, i.e., for a language regarded as a set of strings.

For example, let us consider a *concrete language* $(s \in)$ $\mathcal{L}_{\mathrm{bn}}$ defined by:

$$\mathcal{L}_{\mathrm{bn}} = \{0, 1\}^{+},$$

The language $\mathcal{L}_{\mathrm{bn}}$ is the set of *binary numerals*. Let $\mathcal{O}_{\mathrm{bn}}$ be the function which maps each numeral $s \in \mathcal{L}_{\mathrm{bn}}$ to its numeral value (in $\omega$). Let us consider two sets

$$\mathbf{Cont}_{\mathrm{bn}} = \{\text{“}s\cdot\text{”} : s \in \{0,1\}^{+}\}, \quad \mathbf{Cont}'_{\mathrm{bn}} = \{\text{“}\cdot s\text{”} : s \in \{0,1\}^{+}\}.$$

where "$s\cdot$" (resp. "$\cdot s$") is the function which maps each $\tilde{s} \in \mathcal{L}_{\mathrm{bn}}$ to $s\tilde{s}$ (resp. to $\tilde{s}s$). (Note that both $\mathbf{Cont}_{\mathrm{bn}}$ and $\mathbf{Cont}'_{\mathrm{bn}}$ satisfy conditions (i), (ii) in Definition 3.1.) The two sets $\mathbf{Cont}_{\mathrm{bn}}$ and $\mathbf{Cont}'_{\mathrm{bn}}$ correspond to the following two grammars, respectively:

$$\textbf{(i)}\ s ::= 0 \mid 1 \mid 0s \mid 1s, \quad \textbf{(ii)}\ s ::= 0 \mid 1 \mid s0 \mid s1. \tag{3.2}$$

(Note that both the grammars generate $\mathcal{L}_{\mathrm{bn}}$.)

Although the two sets $\mathbf{Cont}_{\mathrm{bn}}$ and $\mathbf{Cont}'_{\mathrm{bn}}$ are equally natural as a set of contexts, they have very different effects when we try to provide a fully abstract model. When we adopt $\mathbf{Cont}'_{\mathrm{bn}}$ as the set of contexts, $\mathcal{O}_{\mathrm{bn}}$ is compositional, and so, is fully abstract w.r.t. itself. But, this does not hold when we adopt $\mathbf{Cont}_{\mathrm{bn}}$ as the set of contexts, since $\mathcal{O}_{\mathrm{bn}}[\![01]\!] = \mathcal{O}_{\mathrm{bn}}[\![1]\!] = 1$, but

$$\mathcal{O}_{\mathrm{bn}}[\![s01]\!] = \mathcal{O}_{\mathrm{bn}}[\![101]\!] = (101)_2 = 5 \neq 3 = (11)_2 = \mathcal{O}_{\mathrm{bn}}[\![11]\!] = \mathcal{O}_{\mathrm{bn}}[\![s1]\!]$$

with $s \equiv 1$.

From the above example, we know that more than one notion of contexts is possible for a given *concrete syntax*, and different notions of contexts may derive different notions of full abstractness. On the basis of this fact, Mosses cast doubt on the appropriateness of full abstractness as an absolute criterion of the quality of semantics for programming languages. (Cf. § 3.1 of [Mos 90], in response to which this remark is made; cf. also § 1.2 of [Ten 91] for a related discussion.)

Our standpoint is different from this: An abstract syntax, which determines the notion of contexts, is more fundamental than the corresponding concrete syntax. ∎

**Definition 3.4** Let $\mathcal{M}_0$ and $\mathcal{M}_1$ be models for $\mathcal{L}$. We say $\mathcal{M}_0$ is *less abstract* than $\mathcal{M}_1$ (or $\mathcal{M}_1$ is *more abstract* than $\mathcal{M}_0$), when

$$\forall s_0, s_1 \in \mathcal{L}[\ \mathcal{M}_0[\![s_0]\!] = \mathcal{M}_0[\![s_1]\!] \Rightarrow \mathcal{M}_1[\![s_0]\!] = \mathcal{M}_1[\![s_1]\!]\ ]. \blacksquare$$

**Notation 3.1** Let $\mathcal{M}_0$ and $\mathcal{M}_1$ be models for $\mathcal{L}$. We write $\mathcal{M}_0 \leq_{\text{abs}} \mathcal{M}_1$ to denote that $\mathcal{M}_0$ is less abstract than $\mathcal{M}_1$. Also, we write $\mathcal{M}_0 \equiv_{\text{abs}} \mathcal{M}_1$ to denote that $\mathcal{M}_0 \leq_{\text{abs}} \mathcal{M}_1$ and $\mathcal{M}_1 \leq_{\text{abs}} \mathcal{M}_0$. Finally, we write $\mathcal{M}_0 <_{\text{abs}} \mathcal{M}_1$ to denote that

$$\mathcal{M}_0 \leq_{\text{abs}} \mathcal{M}_1 \wedge \neg(\mathcal{M}_0 \equiv_{\text{abs}} \mathcal{M}_1). \blacksquare$$

Given a model $\mathcal{O}$ for $\mathcal{L}$, we can always construct a model for $\mathcal{L}$ which is fully abstract w.r.t. $\mathcal{O}$, as a quotient of $\mathcal{L}$ by the congruence induced by $\mathcal{O}$:

**Definition 3.5** Let $\mathcal{O}$ be a model for $\mathcal{L}$.

**(1)** A binary relation $\cong^c_{\mathcal{O}}$ on $\mathcal{L}$, called the *congruence induced by* $\mathcal{O}$, is defined as follows: For $s_0, s_1 \in \mathcal{L}$,

$$s_0 \cong^c_{\mathcal{O}} s_1 \Leftrightarrow \forall \mathcal{S} \in \mathbf{Cont}[\ \mathcal{O}[\![\mathcal{S}(s_0)]\!] = \mathcal{O}[\![\mathcal{S}(s_1)]\!]\ ]. \tag{3.3}$$

Obviously, $\cong^c_{\mathcal{O}}$ is an equivalence relation on $\mathcal{L}$.

**(2)** For $s \in \mathcal{L}$, let $[s|\cong^c_{\mathcal{O}}]$ be the equivalence class of $s$ w.r.t. $\cong^c_{\mathcal{O}}$, i.e., let

$$[s|\cong^c_{\mathcal{O}}] = \{s' \in \mathcal{L} : s \cong^c_{\mathcal{O}} s'\}.$$

Let $\mathcal{L}/\cong^c_{\mathcal{O}}$ be the quotient of $\mathcal{L}$ by $\cong^c_{\mathcal{O}}$, i.e., let

$$\mathcal{L}/\cong^c_{\mathcal{O}} = \{[s|\cong^c_{\mathcal{O}}] : s \in \mathcal{L}\}.$$

**(3)** Let $\mathcal{C}^c_{\mathcal{O}} : \mathcal{L} \to \mathcal{L}/\cong^c_{\mathcal{O}}$ be defined by:

$$\mathcal{C}^c_{\mathcal{O}} = (\lambda s \in \mathcal{L}.\ [s|\cong^c_{\mathcal{O}}]).$$

We call $\mathcal{C}^c_{\mathcal{O}}$ the *canonical term model induced by* $\mathcal{O}$. $\blacksquare$

From Definition 3.3, the following immediately follows:

**Proposition 3.1** **(1)** *The canonical term model* $\mathcal{C}^c_{\mathcal{O}}$ *is fully abstract w.r.t.* $\mathcal{O}$.

**(2)** *A model* $\mathcal{M}$ *for* $\mathcal{L}$ *is fully abstract w.r.t.* $\mathcal{O}$ *iff* $\mathcal{M} \equiv_{\text{abs}} \mathcal{C}^c_{\mathcal{O}}$. $\blacksquare$

Fully abstract models are compositional:

**Proposition 3.2** *Let* $\mathcal{O}$ *and* $\mathcal{M}$ *be models for* $\mathcal{L}$. *If* $\mathcal{M}$ *is fully abstract w.r.t.* $\mathcal{O}$, *then* $\mathcal{M}$ *is compositional.* $\blacksquare$

**Proof.** Suppose $\mathcal{M}$ is fully abstract w.r.t. $\mathcal{O}$. Let $s_0, s_1 \in \mathcal{L}$ with $\mathcal{M}[\![s_0]\!] = \mathcal{M}[\![s_1]\!]$. We will show that

$$\forall \mathcal{S} \in \mathbf{Cont}[\ \mathcal{M}[\![\mathcal{S}(s_0)]\!] = \mathcal{M}[\![\mathcal{S}(s_1)]\!]\ ]. \tag{3.4}$$

Let $\mathcal{S} \in \mathbf{Cont}$. We want to show that $\mathcal{M}[\![\mathcal{S}(s_0)]\!] = \mathcal{M}[\![\mathcal{S}(s_1)]\!]$. For this purpose, it suffices, by Definition 3.3, to show that

$$\forall \mathcal{S}' \in \mathbf{Cont}[\ \mathcal{O}[\![\mathcal{S}'(\mathcal{S}(s_0))]\!] = \mathcal{O}[\![\mathcal{S}'(\mathcal{S}(s_1))]\!]\ ]. \tag{3.5}$$

This is shown as follows: Let $\mathcal{S}' \in \mathbf{Cont}$. Then, $\mathcal{S}' \circ \mathcal{S} \in \mathbf{Cont}$. Thus,

$$\mathcal{O}[\![\mathcal{S}'(\mathcal{S}(s_0))]\!] = \mathcal{O}[\![(\mathcal{S}' \circ \mathcal{S})(s_0)]\!] = \mathcal{O}[\![(\mathcal{S}' \circ \mathcal{S})(s_1)]\!] = \mathcal{O}[\![\mathcal{S}'(\mathcal{S}(s_1))]\!],$$

since $\mathcal{M}$ is fully abstract w.r.t. $\mathcal{O}$. Thus, one has (3.5) and therefore $\mathcal{M}[\![\mathcal{S}(s_0)]\!] = \mathcal{M}[\![\mathcal{S}(s_1)]\!]$. Since $\mathcal{S}$ was arbitrary, one has (3.4), and therefore, $\mathcal{M}$ is compositional. ■

When we identify equally abstract models, the models for $\mathcal{L}$ constitute a complete lattice $\mathbf{L}_{\text{model}}$ with $\leq_{\text{abs}}$ being the order of the lattice; Figure 3.1 illustrates the hierarchy of models for $\mathcal{L}$. The *trivial term model*, which maps each $s \in \mathcal{L}$ to itself, is the *bottom* of the lattice, and *the one-point model*, whose semantic domain consists of one point to which all $s \in \mathcal{L}$ are mapped, is the *top* of the lattice. The lattice is depicted by the dashed box in the figure. Within $\mathbf{L}_{\text{model}}$ compositional models constitute a sublattice $\mathbf{L}_{\text{compo}}$, which is also compete. The sublattice $\mathbf{L}_{\text{compo}}$ is depicted by the large rhombuse, in the figure.

Given an operational model $\mathcal{O}$ for $\mathcal{L}$, let $\mathcal{M}$ be the unique fully abstract model w.r.t. $\mathcal{O}$. We have two complementary characterizations of $\mathcal{D}$ in terms of the sublattice $\mathbf{L}_{\text{compo}}$ of compositional models. First, the compositional models which are correct w.r.t. $\mathcal{L}$ constitute a sublattice $\mathbf{L}_{\text{correct}}$, where $\mathcal{D}$ is the top; second, the compositional models which are complete w.r.t. $\mathcal{O}$ also constitute a sublattice $\mathbf{L}_{\text{complete}}$, where $\mathcal{D}$ is the bottom. In the figure, $\mathbf{L}_{\text{correct}}$ and $\mathbf{L}_{\text{complete}}$ are depicted by lower and upper small rhombuses, respectively.

In the sequel of this thesis, we look for various fully abstract models within $\mathbf{L}_{\text{correct}}$, i.e., within the compositional models which are correct w.r.t. $\mathcal{O}$. The other approach of looking for it in $\mathbf{L}_{\text{complete}}$ is also possible. For example, if the congruence $\equiv_{\mathcal{M}}$ induced by $\mathcal{M}$ is characterized by a set $E$ of (in)equalities, then the compositional models which are complete w.r.t. $\mathcal{D}$ can be characterized as the compositional models satisfying $E$, and $\mathcal{D}$ can be characterized as the *least* abstract model of these models. In this case, we may look for $\mathcal{D}$ within the compositional models satisfying $E$, i.e., within $\mathbf{L}_{\text{complete}}$ (see Proposition 1.4.13 of [Hen 88]).

In terms of the notion of *relative abstractness*, we have the following sufficient conditions for correctness and completeness:

**Proposition 3.3** *Let $\mathcal{M}$ be a compositional model for $\mathcal{L}$. If $\mathcal{M}$ is less abstract than $\mathcal{O}$, then $\mathcal{M}$ is correct w.r.t. $\mathcal{O}$.* ■

**Proof.** Let $\mathcal{M}$ be a compositional model for $\mathcal{L}$ and suppose that $\mathcal{M}$ is less abstract than $\mathcal{O}$. We will show that $\mathcal{M}$ is correct w.r.t. $\mathcal{O}$, i.e., that

$$\begin{aligned}&\forall s_0, s_1 \in \mathcal{L}[\ \mathcal{M}[\![s_0]\!] = \mathcal{M}[\![s_1]\!] \Rightarrow \\ &\quad \forall \mathcal{S} \in \mathbf{Cont}[\ \mathcal{O}[\![\mathcal{S}(s_0)]\!] = \mathcal{O}[\![\mathcal{S}(s_1)]\!]\ ]].\end{aligned} \tag{3.6}$$

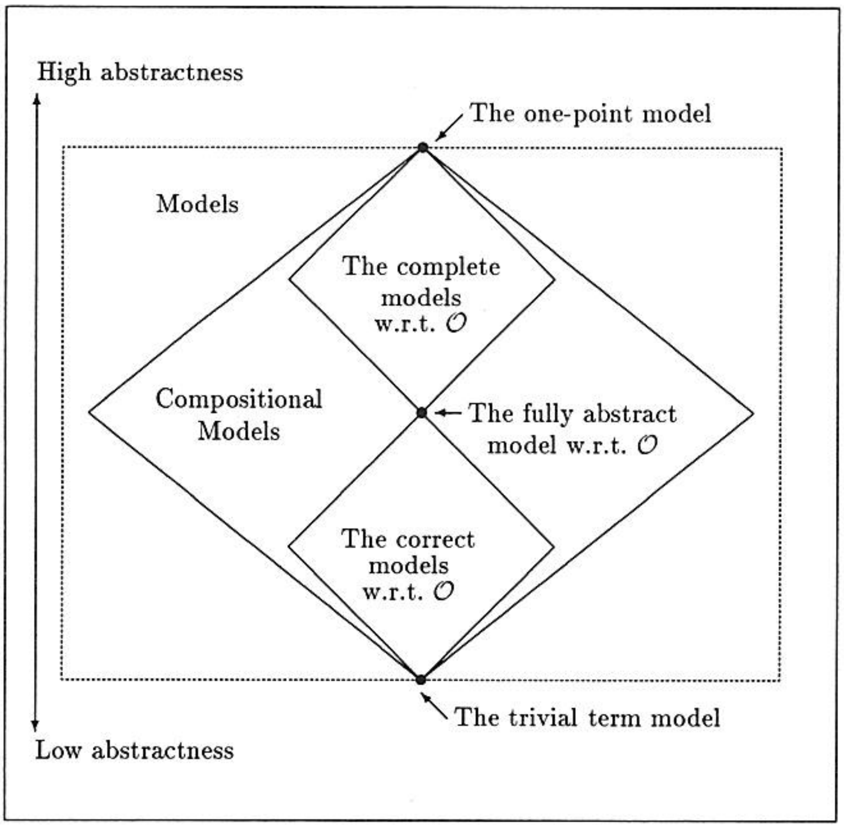


Figure 3.1: Hierarchy of Models

Let $s_0, s_1 \in \mathcal{L}$ with $\mathcal{M}[\![s_0]\!] = \mathcal{M}[\![s_1]\!]$, and let $\mathcal{S} \in \mathbf{Cont}$. Then, since $\mathcal{M}$ is compositional, one has $\mathcal{M}[\![\mathcal{S}(s_0)]\!] = \mathcal{M}[\![\mathcal{S}(s_1)]\!]$. From this and the fact $\mathcal{M}$ is less abstract than $\mathcal{O}$, it follows that $\mathcal{O}[\![\mathcal{S}(s_0)]\!] = \mathcal{O}[\![\mathcal{S}(s_1)]\!]$. Thus, one has (3.6). ∎

Using the above proposition, we obtain the following characterization of full abstractness in terms of *relative abstractness*:

**Proposition 3.4** *Let $\mathcal{M}$ be a model for $\mathcal{L}$. Then, $\mathcal{M}$* is fully abstract w.r.t. $\mathcal{O}$ *iff $\mathcal{M}$ is the most abstract model of those models which are compositional and less abstract than $\mathcal{O}$.* ∎

**Proof.** ($\Rightarrow$) Suppose $\mathcal{M}$ is a fully abstract model for $\mathcal{L}$ w.r.t. $\mathcal{O}$. Then, by Proposition 3.2, $\mathcal{M}$ is compositional. Further, $\mathcal{M}$ is less abstract than $\mathcal{O}$, since for every $s_0, s_1 \in \mathcal{L}$ with $\mathcal{M}[\![s_0]\!] = \mathcal{M}[\![s_1]\!]$, one has

$$\forall \mathcal{S} \in \mathbf{Cont}[\ \mathcal{O}[\![\mathcal{S}(s_0)]\!] = \mathcal{O}[\![\mathcal{S}(s_1)]\!]\ ]$$

by Definition 3.3, and therefore,

$$\mathcal{O}[\![s_0]\!] = \mathcal{O}[\![\mathrm{id}_{\mathcal{L}}(s_0)]\!] = \mathcal{O}[\![\mathrm{id}_{\mathcal{L}}(s_1)]\!] = \mathcal{O}[\![s_1]\!],$$

by taking $\mathrm{id}_{\mathcal{L}}$ as $\mathcal{S}$.

Let $\mathcal{M}'$ be a compositional model such that $\mathcal{M}'$ is less abstract than $\mathcal{O}$. We will show that $\mathcal{M}$ is more abstract than $\mathcal{M}'$. By Proposition 3.3, $\mathcal{M}'$ is correct w.r.t. $\mathcal{O}$. Let $s_0, s_1 \in \mathcal{L}$ with $\mathcal{M}'[\![s_0]\!] = \mathcal{M}'[\![s_1]\!]$. Then, since $\mathcal{M}'$ is correct w.r.t. $\mathcal{O}$, one has

$$\forall \mathcal{S} \in \mathbf{Cont}[\ \mathcal{O}[\![\mathcal{S}(s_0)]\!] = \mathcal{O}[\![\mathcal{S}(s_1)]\!]\ ].$$

From this and the full abstractness of $\mathcal{M}$ w.r.t. $\mathcal{O}$, it follows that $\mathcal{M}[\![s_0]\!] = \mathcal{M}[\![s_1]\!]$. Thus, one has

$$\forall s_0, s_1 \in \mathcal{L}[\ \mathcal{M}'[\![s_0]\!] = \mathcal{M}'[\![s_1]\!] \Rightarrow \mathcal{M}[\![s_0]\!] = \mathcal{M}[\![s_1]\!]\ ],$$

i.e., $\mathcal{M}$ is more abstract than $\mathcal{M}'$.

($\Leftarrow$) Suppose $\mathcal{M}$ is the most abstract model of those models which are compositional and less abstract than $\mathcal{O}$. We will show that $\mathcal{M}$ is fully abstract w.r.t. $\mathcal{O}$. First, it follows from Proposition 3.3, that $\mathcal{M}$ is correct w.r.t. $\mathcal{O}$.

The completeness of $\mathcal{M}$ is shown as follows. The canonical term model $\mathcal{C}^{\mathrm{c}}_{\mathcal{O}}$ is compositional and less abstract than $\mathcal{O}$. Therefore, $\mathcal{M}$ is more abstract than $\mathcal{C}^{\mathrm{c}}_{\mathcal{O}}$, i.e.,

$$\forall s_0, s_1 \in \mathcal{L}[\ \mathcal{C}^{\mathrm{c}}_{\mathcal{O}}[\![s_0]\!] = \mathcal{C}^{\mathrm{c}}_{\mathcal{O}}[\![s_1]\!] \Rightarrow \mathcal{M}[\![s_0]\!] = \mathcal{M}[\![s_1]\!]\ ]. \tag{3.7}$$

By the definition of $\mathcal{C}^{\mathrm{c}}_{\mathcal{O}}$, one has

$$\begin{aligned}\forall s_0, s_1 \in \mathcal{L}[\ & \mathcal{C}^{\mathrm{c}}_{\mathcal{O}}[\![s_0]\!] = \mathcal{C}^{\mathrm{c}}_{\mathcal{O}}[\![s_1]\!] \\ & \Leftrightarrow \forall \mathcal{S} \in \mathbf{Cont}[\ \mathcal{O}[\![\mathcal{S}(s_0)]\!] = \mathcal{O}[\![\mathcal{S}(s_1)]\!]\ ]].\end{aligned}$$

From this and (3.7), it follows that

$$\begin{aligned}\forall s_0, s_1 \in \mathcal{L}[\ & \forall \mathcal{S} \in \mathbf{Cont}[\ \mathcal{O}[\![\mathcal{S}(s_0)]\!] = \mathcal{O}[\![\mathcal{S}(s_1)]\!]\ ] \\ & \Rightarrow \mathcal{M}[\![s_0]\!] = \mathcal{M}[\![s_1]\!]\ ],\end{aligned}$$

i.e., $\mathcal{M}$ is complete w.r.t. $\mathcal{O}$.

Thus, $\mathcal{M}$ is fully abstract w.r.t. $\mathcal{O}$. ■

**Notation 3.2** Let $\mathcal{M}_0$ and $\mathcal{M}_1$ be models for $\mathcal{L}$. We write $\mathcal{M}_0 \leq_{\mathrm{correct}} \mathcal{M}_1$ to denote that $\mathcal{M}_0$ is correct w.r.t. $\mathcal{M}_1$. Also, we write $\mathcal{M}_0 \leq_{\mathrm{complete}} \mathcal{M}_1$ to denote that $\mathcal{M}_0$ is complete w.r.t. $\mathcal{M}_1$. ■

From the definition of contexts, it follows that the relations $\leq_{\mathrm{correct}}$ and $\leq_{\mathrm{complete}}$ are transitive:

**Proposition 3.5** *Let $\mathcal{M}_0, \mathcal{M}_1, \mathcal{M}_2$ models for $\mathcal{L}$.*

**(1)** *If $\mathcal{M}_0 \leq_{\mathrm{correct}} \mathcal{M}_1$ and $\mathcal{M}_1 \leq_{\mathrm{correct}} \mathcal{M}_2$, then $\mathcal{M}_0 \leq_{\mathrm{correct}} \mathcal{M}_2$.*

**(2)** *If* $\mathcal{M}_0 \leq_{\text{complete}} \mathcal{M}_1$ *and* $\mathcal{M}_1 \leq_{\text{complete}} \mathcal{M}_2$*, then* $\mathcal{M}_0 \leq_{\text{complete}} \mathcal{M}_2$*.* ∎

**Proof.** **(1)** Suppose $\mathcal{M}_0 \leq_{\text{correct}} \mathcal{M}_1$ and $\mathcal{M}_1 \leq_{\text{correct}} \mathcal{M}_2$. We will show that $\mathcal{M}_0 \leq_{\text{correct}} \mathcal{M}_2$, i.e., that

$$\begin{aligned}&\forall s_0, s_1 \in \mathcal{L}[\ \mathcal{M}_0[\![s_0]\!] = \mathcal{M}_0[\![s_1]\!] \\ &\quad \Rightarrow \forall \mathcal{S} \in \textbf{Cont}[\ \mathcal{M}_2[\![\mathcal{S}(s_0)]\!] = \mathcal{M}_2[\![\mathcal{S}(s_1)]\!]\ ]].\end{aligned} \tag{3.8}$$

Let $s_0, s_1 \in \mathcal{L}$ and suppose that $\mathcal{M}_0[\![s_0]\!] = \mathcal{M}_0[\![s_1]\!]$. It suffices to show that

$$\forall \mathcal{S} \in \textbf{Cont}[\ \mathcal{M}_2[\![\mathcal{S}(s_0)]\!] = \mathcal{M}_2[\![\mathcal{S}(s_1)]\!]\ ]. \tag{3.9}$$

Let $\mathcal{S} \in \textbf{Cont}$. Then, $\mathcal{M}_1[\![\mathcal{S}(s_0)]\!] = \mathcal{M}_1[\![\mathcal{S}(s_1)]\!]$, since $\mathcal{M}_0 \leq_{\text{correct}} \mathcal{M}_1$. From this and the fact $\mathcal{M}_1 \leq_{\text{correct}} \mathcal{M}_2$, it follows that $\mathcal{M}_2[\![\text{id}_{\mathcal{L}}(\mathcal{S}(s_0))]\!] = \mathcal{M}_2[\![\text{id}_{\mathcal{L}}(\mathcal{S}(s_1))]\!]$, since $\text{id}_{\mathcal{L}} \in \textbf{Cont}$. Thus, one has $\mathcal{M}_2[\![\mathcal{S}(s_0)]\!] = \mathcal{M}_2[\![\mathcal{S}(s_1)]\!]$. Since $\mathcal{S}$ was arbitrary, one has (3.9).

**(2)** Suppose $\mathcal{M}_0 \leq_{\text{complete}} \mathcal{M}_1$ and $\mathcal{M}_1 \leq_{\text{complete}} \mathcal{M}_2$. We will show that $\mathcal{M}_0 \leq_{\text{complete}} \mathcal{M}_2$, i.e., that

$$\begin{aligned}&\forall s_0, s_1 \in \mathcal{L}[\ \forall \mathcal{S} \in \textbf{Cont}[\ \mathcal{M}_2[\![\mathcal{S}(s_0)]\!] = \mathcal{M}_2[\![\mathcal{S}(s_1)]\!]\ ] \\ &\quad \Rightarrow \mathcal{M}_0[\![s_0]\!] = \mathcal{M}_0[\![s_1]\!]\ ].\end{aligned} \tag{3.10}$$

Let $s_0, s_1 \in \mathcal{L}$ and suppose that

$$\forall \mathcal{S} \in \textbf{Cont}[\ \mathcal{M}_2[\![\mathcal{S}(s_0)]\!] = \mathcal{M}_2[\![\mathcal{S}(s_1)]\!]\ ]. \tag{3.11}$$

We will show that $\mathcal{M}_0[\![s_0]\!] = \mathcal{M}_0[\![s_1]\!]$. For this purpose, it suffices to show

$$\forall \mathcal{S} \in \textbf{Cont}[\ \mathcal{M}_1[\![\mathcal{S}(s_0)]\!] = \mathcal{M}_1[\![\mathcal{S}(s_1)]\!]\ ], \tag{3.12}$$

since $\mathcal{M}_0 \leq_{\text{complete}} \mathcal{M}_1$. Let $\mathcal{S} \in \textbf{Cont}$. We want to show that $\mathcal{M}_1[\![\mathcal{S}(s_0)]\!] = \mathcal{M}_1[\![\mathcal{S}(s_1)]\!]$. For this purpose, it suffices to show

$$\forall \mathcal{S}' \in \textbf{Cont}[\ \mathcal{M}_2[\![\mathcal{S}'(\mathcal{S}(s_0))]\!] = \mathcal{M}_2[\![\mathcal{S}'(\mathcal{S}(s_1))]\!]\ ], \tag{3.13}$$

since $\mathcal{M}_1 \leq_{\text{complete}} \mathcal{M}_2$. Let $\mathcal{S}' \in \textbf{Cont}$. Then, $\mathcal{S}' \circ \mathcal{S} \in \textbf{Cont}$, and therefore,

$$\mathcal{M}_2[\![(\mathcal{S}' \circ \mathcal{S})(s_0)]\!] = \mathcal{M}_2[\![(\mathcal{S}' \circ \mathcal{S})(s_1)]\!],$$

by (3.11). Thus,

$$\mathcal{M}_2[\![\mathcal{S}'(\mathcal{S}(s_0))]\!] = \mathcal{M}_2[\![\mathcal{S}'(\mathcal{S}(s_1))]\!].$$

Since $\mathcal{S}'$ was arbitrary, one has (3.13), and therefore, $\mathcal{M}_1[\![\mathcal{S}(s_0)]\!] = \mathcal{M}_1[\![\mathcal{S}(s_1)]\!]$. Since $\mathcal{S}$ was arbitrary, one has (3.12). ∎

### 3.1.3 Models for Languages without Recursion

A (single-sorted) *signature* is a par of set **Fun** of combinators and a function arity : **Fun** $\to \omega$ which maps each combinator to its *arity*. (In this thesis, we regard constant symbols as nullary combinators.) Let **Sig** = (**Fun**, arity) be a signature, and $\mathcal{X}$ a set of variables. For every $r \in \omega$, let

$$\mathbf{Fun}^{(r)} = \{F \in \mathbf{Fun} : \text{arity}(F) = r\}.$$

A language $\mathcal{L}$ is defined as the set of *terms* generated by **Sig** and $\mathcal{X}$:

**Definition 3.6** Let

$$\mathcal{L} = \mathbf{Term}(\mathbf{Sig}, \mathcal{X})$$

where $\mathbf{Term}(\mathbf{Sig}, \mathcal{X})$ is the set of terms generated by **Sig** and $\mathcal{X}$. For $S \in \mathcal{L}$, let $\text{FV}(S)$ be the set of variables contained in $S$. For $\mathcal{Y} \subseteq \mathcal{X}$, let

$$\mathcal{L}[\mathcal{Y}] = \{S \in \mathcal{L} : \text{FV}(S) \subseteq \mathcal{Y}\}.$$

For $X \in \mathcal{X}$, we simply write $\mathcal{L}[X]$ instead of $\mathcal{L}[\{X\}]$. ∎

Further, let the set **Cont** of contexts be defined in terms of substitution:

**Definition 3.7** Let

$$\mathbf{Cont} = \{S[\cdot/X] : X \in \mathcal{X} \wedge S \in \mathcal{L}[X]\},$$

where

$$S[\cdot/X] = (\lambda s \in \mathcal{L}.\ S[s/X]),$$

with $S[s/X]$ the result of substituting $s$ for $X$ is $S$. ∎

**Definition 3.8** Let $\mathcal{M}$ be a model for $\mathcal{L}$. We say $\mathcal{M}$ is *compositional* w.r.t. the combinators in **Fun**, when the following holds for every $r \in \omega$ and $F \in \mathbf{Fun}^{(r)}$:

$$\begin{array}{l} \forall s_0, s'_0, \ldots, s_{r-1}, s'_{r-1} \in \mathcal{L}[\ \forall i \in r[\ \mathcal{D}[\![s_i]\!] = \mathcal{D}[\![s'_i]\!]\ ] \\ \quad \Rightarrow \mathcal{M}[\![F(s_0, \ldots, s_{r-1})]\!] = \mathcal{M}[\![F(s'_0, \ldots, s'_{r-1})]\!]\ ]. \end{array} \tag{3.14}$$ ∎

For a model $\mathcal{M}$ for $\mathcal{L}$, compositionality of $\mathcal{M}$ in the sense of Definition 3.2 is equivalent to its compositionality w.r.t. the combinators in **Fun**:

**Proposition 3.6** *A model $\mathcal{M}$ for $\mathcal{L}$ is compositional in the sense of Definition 3.2 iff $\mathcal{M}$ is compositional w.r.t. the combinators in* **Fun**. ∎

**Proof.** ($\Rightarrow$) Suppose $\mathcal{M}$ for $\mathcal{L}$ is compositional in the sense of Definition 3.2. Let $r \in \omega$ and $F \in \mathbf{Fun}^{(r)}$. We will show (3.14). For this purpose, it suffices to show that the following holds for every $i \in r$:

$$\begin{array}{l} \forall s_0, \ldots, s_{i-1}, s_{i+1}, \ldots, s_{r-1}, s'_i, s''_i \in \mathcal{L}[\emptyset][\ \mathcal{M}[\![s'_i]\!] = \mathcal{M}[\![s''_i]\!] \Rightarrow \\ \quad \mathcal{M}[\![F(s_0, \ldots, s_{i-1}, s'_i, s_{i+1}, \ldots, s_{r-1})]\!] \\ \quad = \mathcal{M}[\![F(s_0, \ldots, s_{i-1}, s''_i, s_{i+1}, \ldots, s_{r-1})]\!]\ ]. \end{array} \tag{3.15}$$

Let $s_0, \ldots, s_{i-1}, s_{i+1}, \ldots, s_{r-1}, s'_i, s''_i \in \mathcal{L}[\emptyset]$ with $\mathcal{M}[\![s'_i]\!] = \mathcal{M}[\![s''_i]\!]$, and set

$$S \equiv F(s_0, \ldots, s_{i-1}, X, s_{i+1}, \ldots, s_{r-1}).$$

Then, since $\mathcal{M}$ is compositional in the sense of Definition 3.2, one has

$$\mathcal{M}[\![S[s'_i/X]]\!] = \mathcal{M}[\![S[s''_i/X]]\!],$$

i.e.,

$$\begin{aligned}&\mathcal{M}[\![F(s_0, \ldots, s_{i-1}, s'_i, s_{i+1}, \ldots, s_{r-1})]\!]\\ &= \mathcal{M}[\![F(s_0, \ldots, s_{i-1}, s''_i, s_{i+1}, \ldots, s_{r-1})]\!].\end{aligned}$$

Thus, one has (3.15).
($\Leftarrow$) Suppose $\mathcal{M}$ is compositional w.r.t. the combinators in **Fun**. Let $s_0, s_1 \in \mathcal{L}[\emptyset]$ with $\mathcal{M}[\![s_0]\!] = \mathcal{M}[\![s_1]\!]$. Then, we can show the following by induction on the structure of $S \in \mathcal{L}$:

$$\forall X \in \mathcal{X}, \forall S \in \mathcal{L}[X][\ \mathcal{M}[\![S[s_0/X]]\!] = \mathcal{M}[\![S[s_1/X]]\!]\ ].$$

Thus, $\mathcal{M}$ is compositional in the sense of Definition 3.2. ∎

Given a model $\mathcal{M}$ which is compositional in the sense of Definition 3.2, we can define an interpretation of each combinator of $\mathcal{L}$ as a function on the codomain of $\mathcal{M}$:

**Definition 3.9** Let $\mathcal{M}$ be a compositional model for $\mathcal{L}$. For $r \in \omega$ and $F \in \mathbf{Fun}^{(r)}$, let $F^{\mathcal{M}} : (\mathcal{M}[\mathcal{L}])^{\underline{r}} \to \mathcal{M}[\mathcal{L}]$, the interpretation of $F$ induced by $\mathcal{M}$, be defined as follows: For $p_0, \ldots, p_{r-1} \in \mathcal{M}[\mathcal{L}[\emptyset]]$,

$$F^{\mathcal{M}}(p_0, \ldots, p_{r-1}) = \mathcal{M}[\![F(s_0, \ldots, s_{r-1})]\!],$$

where for $i \in r$, $s_i$ is chosen so that $\mathcal{M}[\![s_i]\!] = p_i$. Note that the value $\mathcal{M}[\![F(s_1, \ldots, s_r)]\!]$ is not dependent of the choice of $s_i$, since $\mathcal{M}$ is compositional w.r.t. the combinators in **Fun**. ∎

Given the above definition of contexts, Definition 3.3 is rephrased as follows:

**Definition 3.10** (Full Abstractness) Let $\mathcal{O}$ and $\mathcal{D}$ be models for $\mathcal{L}$. We say $\mathcal{D}$ is *fully abstract* w.r.t. $\mathcal{O}$ iff the following holds for every $s_1, s_2 \in \mathcal{L}$:

$$\begin{aligned}&\mathcal{D}[\![s_1]\!] = \mathcal{D}[\![s_2]\!] \Leftrightarrow\\ &\forall X \in \mathcal{X}_{\mathcal{P}}, \forall S \in \mathcal{L}[X][\ \mathcal{O}[\![S_{\langle X\rangle}[s_1]]\!] = \mathcal{O}[\![S_{\langle X\rangle}[s_2]]\!]\ ].\ \blacksquare\end{aligned}$$

For such a language $\mathcal{L}$ and an operational model $\mathcal{O}$ for it, there always exists a model $\mathcal{D}$ which is fully abstract w.r.t. $\mathcal{O}$. Actually, we can construct such a model $\mathcal{D}$ as a term model, i.e., a model whose carrier is a quotient set of $\mathcal{L}$. Moreover such a model is unique in the following sense:

**Lemma 3.1** (Uniqueness and Existence of Fully Abstract Models) *If two models $\mathcal{M}_0$ and $\mathcal{M}_1$ are fully abstract w.r.t. $\mathcal{O}$, then there is an isomorphism from $\mathcal{M}_0[\mathcal{L}]$ to $\mathcal{M}_1[\mathcal{L}]$, i.e., there is a bijection $\varphi : \mathcal{M}_0[\mathcal{L}] \to \mathcal{M}_1[\mathcal{L}]$ such that for every combinator $F$ in $\mathcal{L}$ with arity $r$, and $p_1, \ldots, p_r \in \mathcal{M}_0[\mathcal{L}]$, the following holds:*

$$\varphi(F^{\mathcal{M}_0}(p_1,\ldots,p_r)) = F^{\mathcal{M}_1}(\varphi(p_1),\ldots,\varphi(p_r)),$$

*where for $i \in 2$, $F^{\mathcal{M}_i}$ is defined as in Definition 3.9.*

*In other words, the fully abstract compositional model is unique except for isomorphism.* ∎

**Proof.** See Proposition 7.1.1 of [BKO 88]. ∎

Thus, for such languages, the full abstractness problem is reduced to the problem of how to *characterize* the congruence relation $\cong^c_{\mathcal{O}}$ induced by $\mathcal{O}$ (cf. Definition 3.5). In § 3.2, we present an example of such a characterization for a very simple language, which has parallel composition but does not allow communication.

**Remark 3.2** Such a characterization may have several practical significances: For example, by an appropriate characterization of this kind, it may be shown to be decidable whether $s_1 \cong^c_{\mathcal{O}} s_2$ or not for arbitrarily given two statements $s_1$, $s_2$, where $\cong^c_{\mathcal{O}}$ is the congruence induced by $\mathcal{O}$ (cf. Definition 3.5 (1)). (Indeed, it is easily checked that the fully abstract model $\mathcal{D}_0$ to be given in § 3.2 serves as a basis of such a decision procedure.) Note that even when the function $\mathcal{O}[\![\cdot]\!]$ is computable, this is not necessarily decidable, because of the infinitary nature of the definition of $\cong^c_{\mathcal{O}}$ (note that in the right-hand side of (3.3), we use universal quantification over **Cont** which is usually infinite). Moreover, such a characterization may serve as a basis for *complete axiomatization* of the congruence $\cong^c_{\mathcal{O}}$ (cf., e.g., Proposition 1.3.8 of [Bro 83]). ∎

### 3.1.4 Models for Languages with Recursion

Let $\mathcal{L}$ be a language which involves some form of recursive construct. Although there always exists a (compositional) model for $\mathcal{L}$ which is fully abstract w.r.t. a given model $\mathcal{O}$, such a model $\mathcal{D}$ is not always constructed as a *denotational model*, as was shown in [AP 86] (cf. also Remark 3.4 at the end of this subsection). Thus, for languages with recursion, there are two issues concerning the full abstractness problem: One is the problem of how to characterize the fully abstract model, as for languages with no recursion; the other is to investigate whether it is possible or not to construct a fully abstract model *denotationally*.

In this thesis, we consider, as *denotational models*, only *domain-based continuous models*, i.e., *least fixed-point models* based on *cpo*'s and *fixed-point models* based on *cms*'s. That is, we consider as *denotational models* only those models satisfying the following conditions (1) and (2):

**(1)** First, $\mathcal{M}$ is *compositional* in the sense that the meaning of a composite program (say $F(s_1, s_2)$) under $\mathcal{M}$ is specified as a function of the meanings of its syntactic subcomponents (say $\mathcal{M}[\![s_1]\!]$ and $\mathcal{M}[\![s_2]\!]$).

**(2)** The semantic domain, the codomain of $\mathcal{M}$, is a *cms* or *cpo*, and the meanings of recursive programs under $\mathcal{M}$ are obtained as fixed-points in the domain in such a way that either of the following conditions (i) and (ii) holds:

**(i)** The semantic domain is a cms, and the meaning of a recursive program $s$ (of the form say $(\mu X.\ S(X))$) is the *limit* of the iteration sequence

$$\langle p_0, f(p_0), f(f(p_0)), \ldots \rangle$$

with $p_0$ an arbitrary point in the cms, and $f$ a *contractive* mapping (from the cms to itself) obtained as the interpretation of the body $S(X)$ of $s$. (In this case, the meaning of $s$ is a fixed-point of $f$, i.e., $f(\lim_n[f^n(p_0)]) = \lim_n[f^n(p_0)]$, which is shown by:

$$\begin{aligned}
& f(\lim_n[f^n(p_0)]) \\
& = \lim_n[f(f^n(p_0))] \\
& \qquad \text{(since } f \text{ is contractive, and so, continuous)} \\
& = \lim_n[f^n(p_0)].\ )
\end{aligned}$$

**(ii)** The semantic domain is a cpo, and the meaning of a recursive program is the *least upper bound* (*lub*) of the iteration sequence

$$\langle \bot, f(\bot), f(f(\bot)), \ldots \rangle,$$

with $\bot$ the bottom of the cpo, and $f$ a *continuous* mapping (from the cpo to itself) obtained as above. (In this case as well, the meaning of $s$ is a fixed-point of $f$, i.e., $f(\mathrm{lub}_n[f^n(\bot)]) = \mathrm{lub}_n[f^n(\bot)]$, which can be shown in a similar fashion to case (i) by:

$$\begin{aligned}
& f(\mathrm{lub}_n[f^n(p_0)]) \\
& = \mathrm{lub}_n[f(f^n(p_0))] \quad \text{(since } f \text{ is continuous)} \\
& = \mathrm{lub}_n[f^n(p_0)].\ )
\end{aligned}$$

**Remark 3.3** There are other kinds of models which might be called *denotational*, such as fixed-point models using *category-theoretic (co)limits*, and models based on *non-well-founded sets*; such models are useful for certain purposes (cf. [Rut 92] for the latter). The mathematical background of these models are, however, rather different from that of domain-based continuous models described above; we cannot afford to discuss these models in this thesis. ∎

**Remark 3.4** Let $\mathcal{O}$ be a given operational model. As De Nicola and Hennessy have shown in their work (such as [DeN 85a], [DeN 85b], [DH 84], and [Hen 88]), when there is an (in)equational proof system which is sound and complete w.r.t. the induced congruence $\cong^{c}_{\mathcal{O}}$, we can construct a cpo-based fully abstract model w.r.t. $\mathcal{O}$ (as the ideal completion of a certain term model induced from the proof system). However, finding such a proof system (with the proof of its soundness and completeness) seems to be almost as difficult as that denotational construction of a fully abstract model which is independent of such a proof system, although it is very pleasant to have the equivalence of three complementary views of the semantics of concurrency —an operational, denotational, and proof-theoretic views. ∎

## 3.2 A Simple Example of Fully Abstract Models

In this section, we give a fully abstract model for a very simple language involving a parallel construct but no communication; the model construction and the full abstractness proof are very easy, but they will still serve as an illustrative example of construction of fully abstract models.

### 3.2.1 A Simple Language $\mathcal{L}_0$

**Definition 3.11** Let $(a \in)$ $\mathbf{A}_0$ be the set of *atomic actions*. No assumption is made about $\mathbf{A}_0$ except that $\mathbf{A}_0$ is nonempty.

**(1)** A (single-sorted) *signature* $\mathbf{Sig}_0 = (\mathbf{Fun}_0, \text{arity}(\cdot))$ is defined as follows.

$$\mathbf{Fun}_0 = \{\mathbf{0}, \mathbf{e}\} \cup \{a : a \in \mathbf{A}_0\} \cup \{+, \|, ;\},$$

$$\text{arity}(\text{op}) = \begin{cases} 0 & \text{if op} \in \{\mathbf{0}, \mathbf{e}\} \cup \{a : a \in \mathbf{A}_0\}, \\ 2 & \text{if op} \in \{+, \|, ;\}, \end{cases}$$

where the constant symbols: "$a$" ($a \in \mathbf{A}_0$), "$\mathbf{0}$", and "$\mathbf{e}$" represent an *atomic action*, the *inaction*, and the *termination*, respectively; the combinators: ";", "+", and "$\|$" represent *sequential composition*, *alternative composition*, and *parallel composition*, respectively. For $r \in 3$, we put

$$\mathbf{Fun}_0^{(r)} = \{\text{op} \in \mathbf{Fun}_0 : \text{arity}(\text{op}) = r\}.$$

**(2)** Let $(X \in)$ $\mathcal{X}_{\mathcal{P}}$ be the set of *statement variables*. The language $(S \in)$ $\mathcal{L}_0$ is defined by the following grammar:

$$S ::= \ a \mid \mathbf{0} \mid \mathbf{e} \mid (S_1; S_2) \mid (S_1 + S_2) \mid (S_1 \parallel S_2) \mid X. \tag{3.16}$$

For $S \in \mathcal{L}_0$, let FV($S$) be the set of statement variables contained in $S$. For $\mathcal{Y} \subseteq \mathcal{X}_{\mathcal{P}}$ and $X \in \mathcal{X}_{\mathcal{P}}$, $\mathcal{L}_0[\mathcal{Y}]$ and $\mathcal{L}_0[X]$ are defined as in Definition 3.6. ∎

### 3.2.2 Operational Models for $\mathcal{L}_0$

**A Linear Operational Model $\mathcal{O}_0$**

**Definition 3.12** (Transition Relations $\xrightarrow{\alpha}_0$) A family of binary transition relations $\xrightarrow{a}_0 \subseteq \mathcal{L}_0[\emptyset] \times \mathcal{L}_0[\emptyset]$ $(a \in \mathbf{A}_0)$, and a unary relation $\xrightarrow{\surd}_0 \subseteq \mathcal{L}_0[\emptyset]$ are defined as the smallest sets satisfying the following rules (1) to (5.2). (Let us set $(\alpha \in)$ $\mathbf{A}_0^{\surd} = \mathbf{A}_0 \cup \{\surd\}$ for short.) Intuitively, the expression "$s_1 \xrightarrow{a}_0 s_2$" means that *the process $s_1$ may perform the action $a$ as a first step, resulting in the process $s_2$*, and "$s_1 \xrightarrow{\surd}_0$" means that $s_1$ *may terminate successfully* or that $s_1$ *is in a final state* (following the terminology of classical automata theory). In the sequel, we use the notational convention that *for $\alpha \in \mathbf{A}_0^{\surd}$, the phrase "$s_1 \xrightarrow{\alpha}_0 \left(s_2\right)$" means $s_1 \xrightarrow{\surd}_0$, if $\alpha = \surd$; otherwise, it means $s_1 \xrightarrow{\alpha}_0 s_2$.*

**(1)** $a \xrightarrow{a}_0 \mathbf{e} \quad (a \in \mathbf{A}_0)$.

**(2)** $\mathbf{e} \xrightarrow{\surd}_0$.

**(3.1)**

$$\frac{s_1 \xrightarrow{a}_0 s_1'}{(s_1; s_2) \xrightarrow{a}_0 (s_1'; s_2).} \quad (a \in \mathbf{A}_0).$$

**(3.2)**

$$\frac{s_1 \xrightarrow{\surd}_0,\ s_2 \xrightarrow{\alpha}_0 \left(s_2'\right)}{(s_1; s_2) \xrightarrow{\alpha}_0 \left(s_2'\right)} \quad (\alpha \in \mathbf{A}_0^{\surd}).$$

**(4)**

$$\frac{s_1 \xrightarrow{\alpha}_0 \left(s_1'\right)}{\begin{array}{c}(s_1 + s_2) \xrightarrow{\alpha}_0 \left(s_1'\right) \\ (s_2 + s_1) \xrightarrow{\alpha}_0 \left(s_1'\right)\end{array}} \quad (\alpha \in \mathbf{A}_0^{\surd}).$$

**(5.1)**

$$\frac{s_1 \xrightarrow{a}_0 s_1'}{\begin{array}{c}(s_1 \parallel s_2) \xrightarrow{a}_0 (s_1' \parallel s_2) \\ (s_2 \parallel s_1) \xrightarrow{a}_0 (s_2 \parallel s_1')\end{array}} \quad (a \in \mathbf{A}_0).$$

**(5.2)**

$$\frac{s_1 \xrightarrow{\surd}_0, s_2 \xrightarrow{\surd}_0}{(s_1 \parallel s_2) \xrightarrow{\surd}_0}. \blacksquare$$

We call $(\mathcal{L}_0[\emptyset], \langle \xrightarrow{\alpha}_0 : \alpha \in \mathbf{A}_0^{\surd} \rangle)$ the *labeled transition system* (*LTS*) of $\mathcal{L}_0$.

**Notation 3.3** Let $(\mathbf{S}, \langle \xrightarrow{\alpha} : \alpha \in \mathbf{A}^{\surd} \rangle$ be a LTS such that $\mathbf{A}^{\surd} = \mathbf{A} \cup \{\surd\}$, $\xrightarrow{\surd} \subseteq \mathbf{S}$, and $\forall \alpha \in \mathbf{A}[\ \xrightarrow{\alpha} \subseteq \mathbf{S} \times \mathbf{S}\ ]$. As in [Mil 80], we can assign each state $s \in \mathbf{S}$ a *synchronization tree* (*ST*) which is a rooted unordered tree labeled with elements of $\mathbf{A}^{\surd}$. In the sequel of the thesis, we exploit diagrams as the one below, for depicting a synchronization tree:

$$s_0 = \bullet \begin{cases} \xrightarrow{a_0} \circ \\ \xrightarrow{a_1} \bullet \xrightarrow{a_2} \circ \\ \xrightarrow{a_3} \bullet \xrightarrow{a_4} \bullet \xrightarrow{a_5} \bullet \end{cases}$$

This diagram represents a synchronization tree $T$ defined as follows: First, $T$ has $s_0$ as its root, six other nodes $s_1$ to $s_6$ (placed from the upper left to the lower right), and six arcs $\{(s_0, s_1), (s_0, s_2), (s_2, s_3), (s_0, s_4), (s_4, s_5), (s_5, s_6)\}$. The arcs are labeled by a labeling function $\ell$ defined by:

$$\ell((s_0, s_1)) = a_0,\ \ \ell((s_0, s_2)) = a_1,\ \ \ell((s_2, s_3)) = a_2,$$
$$\ell((s_0, s_4)) = a_3,\ \ \ell((s_4, s_5)) = a_4,\ \ \ell((s_5, s_6)) = a_5.$$

Finally, the states $s_1$, $s_3$ represented by $\circ$ are *final states* (states labeled with $\surd$), and the others represented by $\bullet$ are nonfinal states (states not labeled with $\surd$). ∎

**Definition 3.13** For $w = \langle a_0, \cdots, a_{n-1} \rangle \in (\mathbf{A}_0)^{<\omega}$, a binary relation $\xrightarrow{w}_{0*}$ is defined by:

$$\xrightarrow{w}_{0*} = \xrightarrow{a_0}_0 \xrightarrow{a_1}_0 \cdots \xrightarrow{a_{n-1}}_0 .$$

(Here and in the sequel, juxtaposition is used for denoting *relational composition*, e.g., $\xrightarrow{a}_0 \xrightarrow{a'}_0 = \{(s, s') : \exists s''[\, s \xrightarrow{a}_0 s'' \xrightarrow{a'}_0 s' \,]\}$.) For $a \in \mathbf{A}_0$, $w \in (\mathbf{A}_0)^{<\omega}$, we write $s \xrightarrow{a}_0$ and $s \xrightarrow{w}_{0*}$, to denote that $\exists s'[s \xrightarrow{a}_0 s']$ and $\exists s'[s \xrightarrow{w}_{0*} s']$, respectively. ∎

**Definition 3.14** (A Linear Model $\mathcal{O}_0$ for $\mathcal{L}_0$)

**(1)** Let $\mathrm{act}_0 : \mathcal{L}_0[\emptyset] \to \wp(\mathbf{A}_0^{\surd})$ be defined as follows: For $s \in \mathcal{L}_0[\emptyset]$,

$$\mathrm{act}_0(s) = \{\alpha \in \mathbf{A}_0^{\surd} : s \xrightarrow{\alpha}_0\}.$$

**(2)** Let

$$(\rho \in)\ \mathbf{Q}_0^{\mathrm{l}} = (\mathbf{A}_0)^{<\omega} \cdot \{\langle \surd \rangle, \langle \delta \rangle\},$$

where $\surd$ and $\delta$ are symbols representing (successful) *termination* and *deadlock*, respectively.

**(3)** The function $\mathcal{O}_0 : \mathcal{L}_0[\emptyset] \to \wp(\mathbf{Q}_0^{\mathrm{l}})$ is defined as follows: Let $s \in \mathcal{L}_0[\emptyset]$. For $\rho \in \mathbf{Q}_0^{\mathrm{l}}$, we put $\rho \in \mathcal{O}_0[\![s]\!]$ iff either of (3.17) or (3.18) below holds:

$$\exists w \in (\mathbf{A}_0)^{<\omega}[\, \rho = w \cdot \langle \surd \rangle \wedge \exists s'[\, s \xrightarrow{w}_{0*} s' \wedge \surd \in \mathrm{act}_0(s') \,]]. \tag{3.17}$$

$$\exists w \in (\mathbf{A}_0)^{<\omega}[\, \rho = w \cdot \langle \delta \rangle \wedge \exists s'[\, s \xrightarrow{w}_{0*} s' \wedge \mathrm{act}_0(s') = \emptyset \,]]. \tag{3.18}$$

The model $\mathcal{O}_0$ is not compositional as the following example shows:

**Example 3.1** Let

$$s_0 = a + (a; a), \quad s_1 \equiv a; (\mathrm{e} + a).$$

The two statements $s_0$ and $s_1$ are depicted by:

$$s_0 = \bullet \begin{cases} \xrightarrow{a} \circ \\ \xrightarrow{a} \bullet \xrightarrow{a} \circ \end{cases}$$

$$s_1 = \bullet \xrightarrow{a} \circ \xrightarrow{a} \circ$$

Then,

$$\mathcal{O}_0[\![s_0]\!] = \mathcal{O}_0[\![s_1]\!] = \{\langle a, \surd \rangle, \langle a, a, \surd \rangle\}.$$

However, $\mathcal{O}_0[\![s_0 \parallel \mathbf{0}]\!] \neq \mathcal{O}_0[\![s_1 \parallel \mathbf{0}]\!]$, since

$$\langle a, \delta \rangle \in \mathcal{O}_0[\![s_0 \parallel \mathbf{0}]\!] \setminus \mathcal{O}_0[\![s_1 \parallel \mathbf{0}]\!].$$ ∎

**A Compositional Model $\mathcal{C}_0$ for $\mathcal{L}_0$**

**Definition 3.15** (An Intermediate Model $\mathcal{C}_0$ for $\mathcal{L}_0$)

**(1)** Let $\mathbf{Q}_0^{\mathrm{f}}$, the semantic domain of $\mathcal{C}_0$, be defined by:

$$\mathbf{Q}_0^{\mathrm{f}} = (\mathbf{A}_0)^{<\omega} \cdot \{\langle\surd\rangle, \langle\delta(\{\surd\})\rangle, \langle\delta(\emptyset)\rangle\},$$

where $\delta(\Upsilon) = (\,\delta\,, \Upsilon)$ for $\Upsilon \in \wp(\{\surd\})$.
Let us use a variable $\Upsilon$ ranging over $\wp(\{\surd\})$.

**(2)** We define $\mathcal{C}_0 : \mathcal{L}_0[\emptyset] \to \wp(\mathbf{Q}_0^{\mathrm{f}})$ as follows: Let $s \in \mathcal{L}_0[\emptyset]$. For $\rho \in \mathbf{Q}_0^{\mathrm{l}}$, we put $\rho \in \mathcal{C}_0[\![s]\!]$ iff either of (3.17) above or (3.19) below holds:

$$\begin{aligned}&\exists w \in (\mathbf{A}_0)^{<\omega}, \exists \Upsilon \in \wp(\{\surd\})[\rho = w \cdot \langle\delta(\Upsilon)\rangle \wedge \\ &\quad \exists s'[\; s \xrightarrow{w}_{0*} s' \wedge \mathrm{act}_0(s') \subseteq \{\surd\} \wedge \mathrm{act}_0(s') \cap \Upsilon = \emptyset \;]]. \blacksquare\end{aligned} \tag{3.19}$$

We define an abstraction function $\mathcal{A}_0 : \wp(\mathbf{Q}_0^{\mathrm{f}}) \to \wp(\mathbf{Q}_0^{\mathrm{l}})$, in terms of which we will establish a connection between $\mathcal{O}_0$ and $\mathcal{C}_0$:

**Definition 3.16** For $p \in \wp(\mathbf{Q}_0^{\mathrm{l}})$, let

$$\mathcal{A}_0(p) = (p \cap ((\mathbf{A}_0)^{<\omega} \cdot \{\langle\surd\rangle\})) \cup \{w \cdot \langle\,\delta\,\rangle : w \cdot \langle\delta(\{\surd\})\rangle \in p\}. \blacksquare$$

From the definitions of $\mathcal{C}_0$, $\mathcal{O}_0$, and $\mathcal{A}_0$, the following lemma immediately follows:

**Lemma 3.2** $\mathcal{O}_0 = \mathcal{A}_0 \circ \mathcal{C}_0$. $\blacksquare$

Also the following lemma immediately follows from the definition of $\mathcal{C}_0$:

**Lemma 3.3** *For every $s \in \mathcal{L}_0[\emptyset]$ and $w \in (\mathbf{A}_0)^{<\omega}$, one has*

$$w \cdot \langle\delta(\emptyset)\rangle \in \mathcal{C}_0[\![s]\!] \Leftrightarrow w \cdot \langle\delta(\{\surd\})\rangle \in \mathcal{C}_0[\![s \,\|\, \mathbf{0}]\!]. \blacksquare$$

### 3.2.3 A Denotational Model $\mathcal{D}_0$ for $\mathcal{L}_0$

In this subsection, we define the denotational model $\mathcal{D}_0$. As a preliminary to the definition, we define semantic operations $\widetilde{\mathsf{op}}$ corresponding to the combinators $\mathsf{op} \in \mathbf{Fun}$. First, we need an auxiliary function.

**Definition 3.17** For $w_0, w_1 \in (\mathbf{A}_0)^{<\omega}$, we define $\mathrm{mrg}(w_0, w_1) \in \wp((\mathbf{A}_0)^{<\omega})$ by induction on $\mathrm{lgt}(w_0) + \mathrm{lgt}(w_0)$ as follows:

$$\mathrm{mrg}(\epsilon, \epsilon) = \{\epsilon\};$$

for $(w_0, w_1) \neq (\epsilon, \epsilon)$,

$$\begin{aligned}&\mathrm{mrg}(w_0, w_1) \\ &= \bigcup\{w_i(0) \cdot \mathrm{mrg}(\mathrm{rest}(w_i), w_j) : (i,j) \in \{(0,1),(1,0)\} \wedge w_i \neq \epsilon\}. \blacksquare\end{aligned}$$

First, the interpretations of constant symbols $\mathsf{op} \in \mathbf{Fun}_0^{(0)}$ are defined by:

**Definition 3.18** For $\mathsf{op} \in (\mathbf{Fun}_0)^{(0)}$, let $\widetilde{\mathsf{op}} = \mathcal{C}_0[\![\mathsf{op}]\!]$. ∎

Then, the interpretations of binary operations $\mathsf{op} \in \mathbf{Fun}_0^{(2)}$ are defined by:

**Definition 3.19** For $\mathsf{op} \in \mathbf{Fun}_0^{(2)}$, the semantic operations $\widetilde{\mathsf{op}} : (\wp(\mathbf{Q}_0^{\mathrm{f}}))^{*2} \to \wp(\mathbf{Q}_0^{\mathrm{f}})$ are defined as follows:

**(1)** For $p_0, p_1 \in \wp(\mathbf{Q}_0^{\mathrm{f}})$, let

$$\tilde{;}(p_0, p_1) = \tilde{;}_{\surd}(p_0, p_1) \cup \tilde{;}_{\delta}(p_0, p_1). \tag{3.20}$$

where

$$\begin{aligned}\tilde{;}_{\surd}(p_0, p_1) = \{\, & w_0 \cdot w_1 \cdot \langle\surd\rangle : \\ & w_0 \cdot \langle\surd\rangle \in p_0 \ \wedge\ w_1 \cdot \langle\surd\rangle \in p1\},\end{aligned}$$

$$\begin{aligned}&\tilde{;}_{\delta}(p_0, p_1) \\ = \ & \{w \cdot \langle\delta(\Upsilon)\rangle : w \cdot \delta(\Upsilon \cup \{\surd\}) \in p_0\} \\ & \cup\{w \cdot \langle\delta(\Upsilon)\rangle : w \cdot \langle\delta(\Upsilon \setminus \{\surd\})\rangle \in p_0 \ \wedge\ \langle\delta(\Upsilon)\rangle \in p_1\} \\ & \cup\{\, w_0 \cdot w_1 \cdot \delta(\Upsilon) : w_0 \cdot \langle\surd\rangle \in p_0 \\ & \qquad \wedge w_1 \neq \epsilon \ \wedge\ w_1 \cdot \delta(\Upsilon) \in p_1\}.\end{aligned}$$

**(2)** For $p_0, p_1 \in \wp(\mathbf{Q}_0^{\mathrm{f}})$, let

$$\begin{aligned}\tilde{+}(p_0, p_1) = \ & (\{\langle\delta(\Upsilon)\rangle : \Upsilon \in \wp(\{\surd\})\} \cap p_0 \cap p_1) \\ & \cup (p_1 \setminus \{\langle\delta(\Upsilon)\rangle : \Upsilon \in \wp(\{\surd\})\}) \\ & \cup (p_2 \setminus \{\langle\delta(\Upsilon)\rangle : \Upsilon \in \wp(\{\surd\})\}).\end{aligned} \tag{3.21}$$

**(3)** For $p_0, p_1 \in \wp(\mathbf{Q}_0^{\mathrm{f}})$, let

$$\tilde{\|}(p_0, p_1) = \tilde{\|}_{\surd}(p_0, p_1) \cup \tilde{\|}_{\delta}(p_0, p_1), \tag{3.22}$$

where

$$\begin{aligned}\tilde{\|}_{\surd}(p_0, p_1) = \ \{w \cdot \langle\surd\rangle : \ & \exists w_0, w_1[\ w \in \mathrm{mrg}(w_0, w_1) \wedge \\ & w_0 \cdot \langle\surd\rangle \in p_0 \ \wedge\ w_1 \cdot \langle\surd\rangle \in p_1 \ ]\},\end{aligned}$$

$$\begin{aligned}&\tilde{\|}_{\delta}(p_0, p_1) \\ &= \{\, w \cdot \langle\delta(\Upsilon)\rangle : \Upsilon \in \wp(\{\surd\}) \wedge \\ &\qquad \exists w_0, w_1[\ w \in \mathrm{mrg}(w_0, w_1) \wedge \\ &\qquad ((w_0 \cdot \langle\delta(\Upsilon)\rangle \in p_0 \ \wedge\ w_1 \cdot \langle\delta(\Upsilon \setminus \{\surd\})\rangle \in p_1) \vee \\ &\qquad (w_0 \cdot \langle\delta(\Upsilon \setminus \{\surd\})\rangle \in p_0 \ \wedge\ w_1 \cdot \langle\delta(\Upsilon)\rangle \in p_1))\ ]\}.\end{aligned}$$ ∎

**Definition 3.20** For $s \in \mathcal{L}_0[\emptyset]$, $\mathcal{D}_0[\![s]\!] \in \wp(\mathbf{Q}_0^{\mathrm{f}})$ is defined by induction on the structure of $s$ as follows: For every $r \in 3$ and $\mathsf{op} \in \mathbf{Fun}_0^{(r)}$,

$$\mathcal{D}_0[\![\mathsf{op}(s_0, \ldots, s_{r-1})]\!] = \widetilde{\mathsf{op}}(\mathcal{D}_0[\![s_0]\!], \ldots, \mathcal{D}_0[\![s_{r-1}]\!]).$$ ∎

Obviously $\mathcal{D}_0$ is compositional by its definition:

**Lemma 3.4** *For every* $s_0, s_1 \in \mathcal{L}_0[\emptyset]$, *one has*

$$\mathcal{D}_0[\![s_0]\!] = \mathcal{D}_0[\![s_1]\!] \Rightarrow \\ \forall X \in \mathcal{X}_\mathcal{P}, \forall S \in \mathcal{L}_0[X][\ \mathcal{D}_0[\![S[s_0/X]]\!] = \mathcal{D}_0[\![S[s_1/X]]\!]\ ]. \blacksquare$$

The model $\mathcal{D}_0$ is more distinctive (less abstract) than $\mathcal{O}_0$:

**Example 3.2** Let two statements $s_0$ and $s_1$ be defined as in Example 3.1. Then,

$$\mathcal{O}_0[\![s_0]\!] = \mathcal{O}_0[\![s_1]\!] = \{\langle a, \surd\rangle, \langle a, a, \surd\rangle\}.$$

However, $\mathcal{D}_0[\![s_0]\!] \neq \mathcal{D}_0[\![s_1]\!]$, since

$$\langle a, \delta(\emptyset)\rangle \in \mathcal{D}_0[\![s_0]\!] \setminus \mathcal{D}_0[\![s_1]\!]. \blacksquare$$

**Equivalence between $\mathcal{D}_0$ and $\mathcal{C}_0$**

The two models $\mathcal{D}_0$ and $\mathcal{C}_0$ turn out to be identical, as is shown below.

First, it turns out that $\mathcal{C}_0$ is compositional w.r.t. all the combinators in $\mathbf{Fun}_0$:

**Lemma 3.5** *For every* $r \in 3$, $\mathrm{op} \in \mathbf{Fun}_0^{(r)}$, *one has*

$$\forall s_0, \ldots, s_{r-1} \in \mathcal{L}_0[\emptyset][ \\ \mathcal{C}_0[\![\mathrm{op}(s_0, \ldots, s_{r-1})]\!] = \widetilde{\mathrm{op}}(\mathcal{C}_0[\![s_0]\!], \ldots, \mathcal{C}_0[\![s_{r-1}]\!])\ ]. \blacksquare$$

From this lemma, the following lemma immediately follows:

**Lemma 3.6** (Equivalence between $\mathcal{D}_0$ and $\mathcal{C}_0$)

$$\forall s \in \mathcal{L}_0[\emptyset][\ \mathcal{D}_0[\![s]\!] = \mathcal{C}_0[\![s]\!]\ ]. \blacksquare$$

**Proof.** Recalling the inductive definition of $\mathcal{D}_0$, we can prove the claim by induction on the structure of $s \in \mathcal{L}_0[\emptyset]$ using Lemma 3.5. ■

From the above lemma and Lemma 3.2, the following corollary immediately follows:

**Corollary 3.1** $\mathcal{O}_0 = \mathcal{A}_0 \circ \mathcal{D}_0$. ■

### 3.2.4 Full Abstractness of $\mathcal{D}_0$ w.r.t. $\mathcal{O}_0$

The model $\mathcal{D}_0$ is fully abstract w.r.t. $\mathcal{O}_0$:

**Theorem 3.1** *For every* $s_0, s_1 \in \mathcal{L}_0[\emptyset]$, *one has the following:*

$$\mathcal{D}_0[\![s_0]\!] = \mathcal{D}_0[\![s_1]\!] \Leftrightarrow \\ \forall X \in \mathcal{X}_\mathcal{P}, \forall S \in \mathcal{L}_0[X][\ \mathcal{O}_0[\![S[s_0/X]]\!] = \mathcal{O}_0[\![S[s_1/X]]\!]\ ]. \tag{3.23}$$

**Proof.** The $\Rightarrow$-part of (3.23) immediately follows from Lemma 3.4 and Corollary 3.1.

Let us show the $\Leftarrow$-part of (3.23). For this purpose, it suffices to show the following:

$$\forall s_0, s_1 \in \mathcal{L}_0[\emptyset][\ \mathcal{D}_0[\![s_0]\!] \neq \mathcal{D}_0[\![s_1]\!] \Rightarrow \\ \exists X \in \mathcal{X}_{\mathcal{P}}, \exists S \in \mathcal{L}_0[X][\ \mathcal{O}_0[\![S[s_0/X]]\!] \neq \mathcal{O}_0[\![S[s_1/X]]\!]\ ]].$$

Let $s_0, s_1 \in \mathcal{L}_0[\emptyset]$ with $\mathcal{D}_0[\![s_0]\!] \neq \mathcal{D}_0[\![s_1]\!]$. We will show

$$\exists X \in \mathcal{X}_{\mathcal{P}}, \exists S \in \mathcal{L}_0[X][\ \mathcal{O}_0[\![S[s_0/X]]\!] \neq \mathcal{O}_0[\![S[s_1/X]]\!]\ ]. \tag{3.24}$$

When $\mathcal{O}_0[\![s_0]\!] \neq \mathcal{O}_0[\![s_1]\!]$, we immediately have (3.24).

Suppose $\mathcal{O}_0[\![s_0]\!] = \mathcal{O}_0[\![s_1]\!]$. Then, we can assume, without loss of generality, that

$$\exists w \in (\mathbf{A}_0)^{<\omega}[\ w \cdot \langle \delta(\emptyset) \rangle \in \mathcal{D}_0[\![s_0]\!] \setminus \mathcal{D}_0[\![s_1]\!]\ ].$$

Fix such a sequence $w$. Then, by Lemma 3.3, one has

$$w \cdot \langle \delta(\{\surd\}) \rangle \in \mathcal{D}_0[\![s_0 \parallel \mathbf{0}]\!] \setminus \mathcal{D}_0[\![s_1 \parallel \mathbf{0}]\!]\ ],$$

and thus,

$$w \cdot \langle\ \delta\ \rangle \in \mathcal{A}_0(\mathcal{D}_0[\![s_0 \parallel \mathbf{0}]\!]) \setminus \mathcal{A}_0(\mathcal{D}_0[\![s_1 \parallel \mathbf{0}]\!]) \\ = \mathcal{O}_0[\![s_0 \parallel \mathbf{0}]\!] \setminus \mathcal{O}_0[\![s_1 \parallel \mathbf{0}]\!].$$

Thus, taking $S \equiv (X \parallel \mathbf{0})$, one has (3.24). ■

# Part II

# Models for Uniform Languages

# Chapter 4

# Full Abstraction of Metric Semantics for Communicating Processes with Value-Passing

## 4.1 Introduction

This chapter investigates *full abstractness* of denotational models w.r.t. operational ones for two concurrent languages $\mathcal{L}_i$ $(i = 1, 2)$. The first language, $\mathcal{L}_1$, contains *atomic actions*, *termination*, *inaction*, *sequential* and *parallel composition*, *nondeterministic choice*, *action restriction*, and a form of *guarded* recursion. The second one, $\mathcal{L}_2$, is an *applicative* language based on $\mathcal{L}_1$; in addition to the constructs of $\mathcal{L}_1$, $\mathcal{L}_2$ has *value-passing*. For each of $\mathcal{L}_i$ $(i = 1, 2)$, three operational models $\mathcal{O}_i$, $\mathcal{O}_i^{\mathrm{m}}$, $\mathcal{C}_i$ are defined in terms of a Plotkin-style transition system. The first two models are *linear* in that the meaning of each program in both $\mathcal{O}_i$ and $\mathcal{O}_i^{\mathrm{m}}$ is a set of action sequences the program may perform; they differ from each other in that $\mathcal{O}_i^{\mathrm{m}}$ is a so-called *maximal trace semantics*, whereas $\mathcal{O}_i$ is not. The third one $\mathcal{C}_i$ is a *failures* model. For each of $\mathcal{L}_i$ $(i = 1, 2)$, a denotational model $\mathcal{M}_i$, which is equivalent to $\mathcal{C}_i$, is defined compositionally using interpreted operations of the language, with meanings of recursive programs as fixed-points in an appropriate complete metric space. It is shown that $\mathcal{M}_i$ is *fully abstract* w.r.t. $\mathcal{O}_i$, and also, under an additional condition, w.r.t. $\mathcal{O}_i^{\mathrm{m}}$ $(i = 1, 2)$.

The main body of this chapter consists of §§ 4.2–4.3 treating $\mathcal{L}_1$ and $\mathcal{L}_2$, respectively. Although the first language, $\mathcal{L}_1$, is of some interest of its own, the treatment of $\mathcal{L}_1$ will serve as a preliminary to that of $\mathcal{L}_2$.

First, in § 4.2, for a simple language $\mathcal{L}_1$ of "uniform and pure processes", the *full abstractness* of a *failures model* w.r.t. two *linear semantics* is established. (By "uniform and pure processes" we mean processes whose actions are pre-defined and uninterpreted and whose only form of communication is synchronization of complementary actions.)

The full abstractness of a failures model w.r.t. a linear model was first established in [BKO 88] for a language without recursion. In [Rut 89], the problem was investigated in the framework of a complete metric space for a language with recursion, which language is very similar to our first language $\mathcal{L}_1$. There, the problem whether the failures model is fully abstract w.r.t. a linear model, which we denote by $\mathcal{O}_1^{\mathrm{i}}$, was investigated. The model $\mathcal{O}_1^{\mathrm{i}}$ differs from $\mathcal{O}_1$ and $\mathcal{O}_1^{\mathrm{m}}$ in that it is a so-called *internal action model* taking into account only internal actions, whereas $\mathcal{O}_1$ and $\mathcal{O}_1^{\mathrm{m}}$ take into account communication actions in addition to internal actions; an error was found in the full abstractness proof in [Rut 89], but a way to overcome it was found by Franck van Breugel under a certain additional assumption (see § 4.2.5, for a related argument). This chapter investigates full abstractness issues similar to the one investigated in [Rut 89]; we treat two operational models $\mathcal{O}_1$ and $\mathcal{O}_1^{\mathrm{m}}$ which are slightly different from $\mathcal{O}_1^{\mathrm{i}}$, and it is shown that the failures model $\mathcal{M}_1$, which is the same as the one in [Rut 89], is fully abstract w.r.t. $\mathcal{O}_1$, and also, under a certain additional condition, w.r.t. $\mathcal{O}_1^{\mathrm{m}}$.

Second, in § 4.3, an extension $\mathcal{L}_2$ of $\mathcal{L}_1$ so as to include value-passing is presented; for $\mathcal{L}_2$ also, it is shown that a failures model is fully abstract w.r.t. two linear semantics. As described in [HI 90], there are two ways to extend $\mathcal{L}_1$ so as to include *value-passing*: One is to extend to an *applicative* language based on $\mathcal{L}_1$; the other is to extend to a *nonuniform* language also based on $\mathcal{L}_1$. This chapter presents an applicative language $\mathcal{L}_2$ based on $\mathcal{L}_1$; it is shown that a failures model $\mathcal{M}_2$ for $\mathcal{L}_2$ is also fully abstract w.r.t. two linear models $\mathcal{O}_2$ and $\mathcal{O}_2^{\mathrm{m}}$ for $\mathcal{L}_2$ (under an additional assumption w.r.t. $\mathcal{O}_2^{\mathrm{m}}$). The full abstractness result for $\mathcal{L}_2$ is established in essentially the same way as the proof of the corresponding result for $\mathcal{L}_1$, by defining $\mathcal{M}_2$ so that the following holds: For a term $(\lambda x.\ E_{(x)}) \in \mathcal{L}_2$,

$$\mathcal{M}_2[\![(\lambda x.\ E_{(x)})]\!] = (\lambda v \in \mathbf{V}.\ [\![E_{(x)}]\!]^{\mathcal{I}_2}(v)),$$

where $\mathcal{I}_2$ is the interpretation (a type-respecting mapping which maps each syntactical construct to a corresponding semantic operation) from which the model $\mathcal{M}_2$ is derived, and $[\![E_{(x)}]\!]^{\mathcal{I}_2}(v)$ denotes the interpretation of $E_{(x)}$ under $\mathcal{I}_2$ with the valuation associating $v$ with $x$.

As far as we are aware, a similar full abstractness result for an applicative language with value-passing first appeared in [HI 90], where a fully abstract denotational model w.r.t. an operational model induced by a notion of *testing* was proposed. This work has much in common with our research, although their operational model based on *testing* is different from our operational models $\mathcal{O}_2$ and $\mathcal{O}_2^{\mathrm{m}}$.

A similar full abstractness result for a *nonuniform language* which is based on $\mathcal{L}_1$ and supports value-passing appeared in [HBR 90]; this result was refined in Chapter 7 so as to treat another operational model which is more abstract than that of [HBR 90] in that it ignores *states* while the model of [HBR 90] does not. Note that a combinatorial technique, called the *testing method*, had to be employed for establishing the result for the nonuniform language in [HBR 90], whereas the corresponding result for $\mathcal{L}_2$ is established in essentially the same way as the proof of the corresponding result for $\mathcal{L}_1$.

Finally, in § 4.4, some remarks on related works and future work are given.

Several mathematical proofs are deferred to the appendixes.

## 4.2 A Language $\mathcal{L}_1$ for Pure Processes

In this section, a simple language $\mathcal{L}_1$ of "uniform and pure processes" is presented. By "uniform and pure processes" we mean processes whose actions are pre-defined and uninterpreted and whose only form of communication is synchronization of complementary actions. The language $\mathcal{L}_1$ is a minor extension of the one investigated in [Rut 89], containing *atomic actions*, *termination*, *inaction*, *sequential* and *parallel composition*, *nondeterministic choice*, *action restriction*, and a form of *guarded* recursion. This is the language investigated in [Hor 92a] in the context of weak semantics.

Three operational models $\mathcal{O}_1$, $\mathcal{O}_1^{\mathrm{m}}$, and $\mathcal{C}_1$ for $\mathcal{L}_1$ are presented; they are defined operationally in terms of a transition relation $\rightarrow_1$ for $\mathcal{L}_1$. The first two models $\mathcal{O}_1$, $\mathcal{O}_1^{\mathrm{m}}$ are *linear models* in that the meaning of each program in both $\mathcal{O}_1$ and $\mathcal{O}_1^{\mathrm{m}}$ is a set of action sequences the program may perform; they differ from each other in that $\mathcal{O}_1^{\mathrm{m}}$ is a so-called *maximal* trace semantics, whereas $\mathcal{O}_1$ is not. The third one $\mathcal{C}_1$ is a *failures model* which was first introduced in [BHR 85]. This model is also defined denotationally: A denotational model $\mathcal{M}_1$ is defined using explicit semantic operations with meanings of recursive programs as fixed-points in an appropriate complete metric space, and the semantic equivalence between $\mathcal{C}_1$ and $\mathcal{M}_1$ is established.

The importance of the failures model $\mathcal{M}_1$ lies in the fact that it is *fully abstract* w.r.t. $\mathcal{O}_1$ and $\mathcal{O}_1^{\mathrm{m}}$, that is, for $\mathcal{O} = \mathcal{O}_1, \mathcal{O}_1^{\mathrm{m}}$, it makes *just enough* distinctions in order to be compositional and correct w.r.t. $\mathcal{O}$.

The full abstractness of a failures model w.r.t. a linear model was first established in [BKO 88] for a language without recursion. In [Rut 89], the problem was investigated for a language with recursion in the framework of a complete metric space. There a linear model, which we denote by $\mathcal{O}_1^{\mathrm{i}}$, is treated; the model $\mathcal{O}_1^{\mathrm{i}}$ is linear in the above sense and an *internal action model* in that the meaning of each program in $\mathcal{O}_1^{\mathrm{i}}$ is a set of sequences of *internal actions* the program may perform. In [Rut 89], the full abstractness of the failures model w.r.t. $\mathcal{O}_1^{\mathrm{i}}$ was investigated; an error was found in the full abstractness proof in [Rut 89], but a way to overcome it was found by Franck van Breugel under a certain additional assumption (see § 4.2.5, for an related argument). In this section, we establish similar full abstractness results for $\mathcal{C}_1$ w.r.t. two linear models $\mathcal{O}_1$ and $\mathcal{O}_1^{\mathrm{m}}$, rather than w.r.t. $\mathcal{O}_1^{\mathrm{i}}$; the model $\mathcal{O}_1$ is defined in the same way as the model $\mathcal{O}_1^{\mathrm{i}}$ treated in [Rut 89] except that $\mathcal{O}_1$ involves *communication actions* besides *internal actions*.

### 4.2.1 The Language $\mathcal{L}_1$

For the definition of $\mathcal{L}_1$, a (possibly infinite) set $(c \in)\ \mathbf{C}_1$ of *communication actions* is assumed to be given. As in CCS ([Mil 80]), a *complementation function* $\bar{\cdot} : \mathbf{C}_1 \to \mathbf{C}_1$ is assumed to be given such that it is a bijection, and $\bar{\cdot} \circ \bar{\cdot} = \mathrm{id}_{\mathbf{C}_1}$. For $C \subseteq \mathbf{C}_1$, let $\overline{C} = \{\bar{c} : c \in C\}$. The *internal action*, which may be the synchronization of two communication actions which are complementary to each other, is denoted by $\tau$. The set of *actions*, denoted by $\mathbf{A}_1$, consists of *communication actions* and $\tau$, i.e.,

$$(a \in)\ \mathbf{A}_1 = \mathbf{C}_1 \cup \{\tau\}.$$

Further, let $\surd$ be a special action standing for *successful termination* such that $\surd \notin \mathbf{A}_1$, and let

$$(\alpha \in)\ \mathbf{A}_1^{\surd} = \mathbf{A}_1 \cup \{\surd\}.$$

We extend the function $\bar{\cdot}$ on $\mathbf{C}_1$ to $\mathbf{A}_1$ so that $\bar{\tau} = \tau$. Thus $\bar{\cdot} \circ \bar{\cdot} = \mathrm{id}_{\mathbf{A}_1}$.

Let $P$ range over $\mathcal{RV}$, the set of *recursion variables*, and let $X$ range over $\mathcal{X_P}$, the set of *statement variables*. Note that recursion variables are used as *names* of recursively defined statements, while statement variables are used as *place holders* for defining *contexts* of the language.

**Definition 4.1** (Language $\mathcal{L}_1$) The set of statements of the concurrent language $(S \in)\ \mathcal{L}_1$ is defined by the following BNF grammar:

$$S ::= \mathbf{0} \mid \mathbf{e} \mid a \mid (S_1; S_2) \mid (S_1 + S_2) \mid (S_1 \parallel S_2) \mid \partial_C(S) \mid P \mid X,$$

where $P$ ranges over the set $\mathcal{RV}$ of recursion variables; $X$ range over the set $\mathcal{X_P}$ of place-holders used for defining contexts; $a$ ranges over the set $\mathbf{A}_1$ of actions; $C$ rages over $\wp(\mathbf{C}_1)$; the constants $\mathbf{0}$ and $\mathbf{e}$ represent *inaction* and (successful) *termination*, respectively; the combinator ;, +, $\parallel$, and $\partial_C(\cdot)$ represent *sequential composition*, *nondeterministic choice*, *parallel composition*, and *action restriction*, respectively. We regard elements of $\mathcal{RV}$ as *constant symbols* rather than as *variables*; for $S \in \mathcal{L}_1$, let $\mathrm{FV}(S)$ be the set of elements of $\mathcal{X_P}$ occurring in $S$. For $\mathcal{L} \subseteq \mathcal{L}_1$ and $\mathcal{X} \subseteq \mathcal{X_P}$, let

$$\mathcal{L}[\mathcal{X}] = \{S \in \mathcal{L} : \mathrm{FV}(S) \subseteq \mathcal{X}\}.$$

For $X \in \mathcal{X_P}$, we sometimes write $\mathcal{L}[X]$ for $\mathcal{L}[\{X\}]$. We use the variable $s$ ranging over $\mathcal{L}_1[\emptyset]$.

Then the set $(g \in)\ \mathcal{G}_1$ of *guarded statements* is defined by the following BNF grammar:

$$g ::= \mathbf{0} \mid a \mid (g; s) \mid (g_1 + g_2) \mid (g_1 \parallel g_2) \mid \partial_C(g).$$

We assume that each recursion variable $P$ is associated with an element $g_P$ of $\mathcal{G}_1$ by a set of *declarations*:

$$D = \{\langle P, g_P \rangle\}_{P \in \mathcal{RV}}.$$

Actually a declaration set $D$ is a function from $\mathcal{RV}$ into $\mathcal{G}_1$. ∎

In the sequel of this section, we fix a declaration set $D = \{(P, g_P)\}_{P\in\mathcal{RV}}$.

We can characterize $\mathcal{L}_1$ as the set of terms generated by a *signature*

$$\mathbf{Sig}_1^* = (\mathbf{Fun}_1^*, \mathrm{arity}_1),$$

a pair of a set of syntactical combinators and an arity function:

**Definition 4.2** The arity function $\mathrm{arity}_1$ assigns each $F \in \mathbf{Fun}_1^*$ a natural number called the *arity* of $F$. For $k \in \omega$, let $(\mathbf{Fun}_1^*)^{(k)} = (\mathrm{arity}_1)^{-1}[\{k\}]$. We define $\mathbf{Sig}_1^*$ as follows:

$$(\mathbf{Fun}_1^*)^{(k)} = \begin{cases} \{\mathbf{0}, \mathbf{e}\} \cup \{a : a \in \mathbf{A}_1\} \cup \mathcal{RV} & \text{if } k = 0, \\ \{\partial_C(\cdot) : C \subseteq \mathbf{C}_1\} & \text{if } k = 1, \\ \{;, +, \|\} & \text{if } k = 2, \\ \emptyset & \text{otherwise.} \end{cases}$$

Let $\mathbf{Sig}_1$ be the result of removing $\mathcal{RV}$ from $\mathbf{Fun}_1^*$, i.e., let

$$\mathbf{Fun}_1 = (\mathbf{Fun}_1^* \setminus \mathcal{RV}), \quad \mathbf{Sig}_1 = (\mathbf{Fun}_1, \mathrm{arity}_1).$$ ∎

Having defined $\mathbf{Sig}_1^*$, it is easy to check that the set $\mathcal{L}_1$ can be characterized as the set of terms generated by $\mathbf{Sig}_1^*$.

We use the following notation in the sequel:

**Notation 4.1** For $S \in \mathcal{L}_1$, $k \in \omega$, $\vec{X} \in (\mathcal{X}_\mathcal{P})^n$, and $\vec{S} \in (\mathcal{L}_1)^n$, the expression $S_{\vec{X}}[\vec{S}]$ denotes the result of simultaneously replacing all the free occurrences of $\vec{X}(i)$ by $\vec{S}(i)$ $(i \in n)$. We sometimes write $S[\vec{S}/\vec{X}]$ for $S_{\vec{X}}[\vec{S}]$. For $n = 1$, we write $S_{\langle X\rangle}[S_0]$ or $S[S_0/X]$. ∎

**Remark 4.1** The language $\mathcal{L}_1$ is very similar to the language treated in [Rut 89], except that $\mathcal{L}_1$ has *action restriction*, the inaction $\mathbf{0}$, and the termination $\mathbf{e}$ in addition. The termination $\mathbf{e}$ corresponds to **SKIP** in [BHR 85]; the inaction $\mathbf{0}$ is a basic construct of CCS (cf. [Mil 80], [Mil 85], [Mil 89]). ∎

**Example 4.1** (Description of Dining Philosophers in $\mathcal{L}_1$) The problem of $n$ dining philosophers with $n \geq 2$ can be solved without any centralized control mechanism (such as a monitor or a semaphore), only by appropriately specifying which of the two forks $f_i$ and $f_{i+1}$ must be got first by the philosopher $P_i$, in the configuration where $f_i$ is placed next to $P_i$ on the left $(0 \leq i < n)$. For example, we can construct a deadlock-free and starvation-free system under the assumption of fairness, by specifying that $P_i$ with $i$ even (resp. with $i$ odd) must take $f_i$ (resp. $f_{i-1}$) first $(0 \leq i < n)$. An instance of this specification for $n = 5$ is shown in Figure 4.1, where "$P_i \rightarrow f_j$" denotes that $P_i$ must take $f_j$ first.

Figure 4.2 gives a description of the system of dining philosophers in $\mathcal{L}_1$. (This is a $\mathcal{L}_1$ version of the LOTOS description given in § 3.1 of [OH 89].) We name the description `Philosophers`; it is the parallel composition of five processes named $P_{\langle n,1,1\rangle}$ $(n \in 5)$ with the restriction of actions denoted by $C$.

The process $P_{(n,i,j)}$ represents the $n$-th philosopher in the *thinking mode* ($(n,i,j) \in 5\times2\times2$), where the fact that $i = 1$ (resp. $j = 1$) signifies that the fork to be got first (resp. second) is available. For each $n$, the action $\overline{\mathtt{think}_n}$ (resp. $\overline{\mathtt{eat}_n}$) signals that the $n$-th philosopher is thinking (resp. that the $n$-th philosopher is eating). The actions $\overline{\mathtt{get}_{\imath(n)}}$ (resp. $\overline{\mathtt{get}_{\jmath(n)}}$) signals that the $n$-th philosopher gets the fork to be got first (resp. second); the actions $\overline{\mathtt{put}_{\imath(n)}}$ (resp. $\overline{\mathtt{put}_{\jmath(n)}}$) signals that the $n$-th philosopher puts the fork to be got first (resp. second). The actions $\overline{\mathtt{get}_{\imath(n)}}$, $\overline{\mathtt{get}_{\jmath(n)}}$, $\overline{\mathtt{put}_{\imath(n)}}$, $\overline{\mathtt{put}_{\jmath(n)}}$ executed by the $n$-th philosopher must synchronize with their complementary actions $\mathtt{get}_{\imath(n)}$, $\mathtt{get}_{\jmath(n)}$, $\mathtt{put}_{\imath(n)}$, $\mathtt{put}_{\jmath(n)}$ executed by an adjacent philosopher.

The process $P'_{(n,i,j,k)}$ represents the $n$-th philosopher in the *eating mode* ($(n,i,j) \in 5 \times 2 \times 2$), where the values of $i$ and $j$ have the same meanings as in $P_{(n,i,j)}$, and the fact $k = 1$ signifies that the $n$-th philosopher has already got the first fork. Also, $P''_{(n,i,j)}$ represents the $n$-th philosopher in the *post-eating mode* ($(n,i,j) \in 5 \times 2 \times 2$).

For each $n \in 5$, $\imath(n)$ (resp. $\jmath(n)$) is the index of the fork to be got first (resp. second); it is assumed that

$$\forall n \in 5[\ \imath(n), \jmath(n) \in 5\ ] \wedge \{\mathtt{get}_n : n \in 5\} \cap \{\mathtt{put}_n : n \in 5\} = \emptyset.$$

As illustrated in Figure 4.1, we define $\imath(n)$ and $\jmath(n)$ by:

$$\imath(n) = \begin{cases} n & \text{if } i \text{ is even} \\ (n-1) \bmod 5 & \text{otherwise,} \end{cases}$$

$$\jmath(n) = \begin{cases} (n-1) \bmod 5 & \text{if } i \text{ is even} \\ n & \text{otherwise.} \end{cases} \blacksquare$$

### 4.2.2 Three Operational Models $\mathcal{O}_1$, $\mathcal{O}_1^{\mathrm{m}}$, $\mathcal{C}_1$ for $\mathcal{L}_1$

In this subsection, three operational models $\mathcal{O}_1$, $\mathcal{O}_1^{\mathrm{m}}$, and $\mathcal{C}_1$ for $\mathcal{L}_1$ are defined in terms of a transition system $(\mathcal{L}_1[\emptyset], \langle \xrightarrow{\alpha}_1 : \alpha \in \mathbf{A}_1^{\surd} \rangle)$ in the style of [Plo 81]. The transition relations $\xrightarrow{a}_1 \subseteq \mathcal{L}_1[\emptyset] \times \mathcal{L}_1[\emptyset]$ ($a \in \mathbf{A}_1$) and $\xrightarrow{\tau}_1 \subseteq \mathcal{L}_1[\emptyset]$ are defined as follows. (For $s_1, s_2 \in \mathcal{L}_1[\emptyset]$, and $a \in \mathbf{A}_1$, we write $s_1 \xrightarrow{a}_1 s_2$ (resp. $s_1 \xrightarrow{\tau}$) for $(s_1, s_2) \in \xrightarrow{a}_1$ (resp. $s_1 \in \xrightarrow{\tau}$), as usual.

**Definition 4.3** (Transition Relations $\xrightarrow{\alpha}_1$) A family of binary transition relations $\xrightarrow{a}_1 \subseteq \mathcal{L}_1 \times \mathcal{L}_1$ ($a \in \mathbf{A}_1$), and a unary relation $\xrightarrow{\surd}_1 \subseteq \mathcal{L}_1$ are defined as the smallest sets satisfying the following rules (1)–(7). Intuitively, the expression "$s_1 \xrightarrow{a}_1 s_2$" means that *the process $s_1$ may perform the action $a$ as a first step, resulting in the process $s_2$*, and "$s_1 \xrightarrow{\surd}_1$" means that $s_1$ *may terminate successfully* or that $s_1$ *is in a final state* (following the terminology of classical automata theory). In the sequel, we use the notational convention that *for $\alpha \in \mathbf{A}_1^{\surd}$, the phrase "$s_1 \xrightarrow{\alpha}_1 \left(s_2\right)$" means $s_1 \xrightarrow{\surd}_1$, if $\alpha = \surd$; otherwise, it means $s_1 \xrightarrow{\alpha}_1 s_2$.*

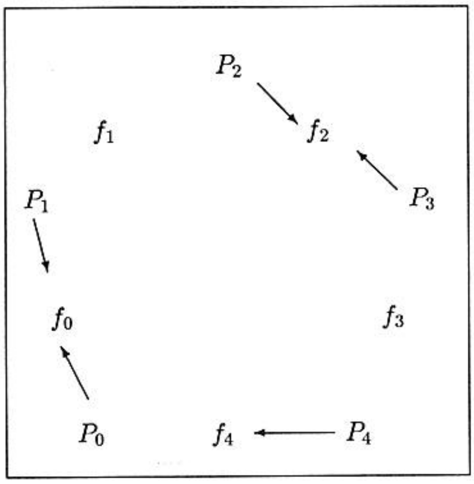


Figure 4.1: Arrangement of Dininging Philosophers

**(1)** $a \xrightarrow{a}_1 \mathbf{e} \quad (a \in \mathbf{A}_1)$.

**(2)** $\mathbf{e} \xrightarrow{\surd}_1$.

**(3.1)**

$$\frac{s_1 \xrightarrow{a}_1 s_1'}{(s_1; s_2) \xrightarrow{a}_1 (s_1'; s_2).} \quad (a \in \mathbf{A}_1).$$

**(3.2)**

$$\frac{s_1 \xrightarrow{\surd}_1,\ s_2 \xrightarrow{\alpha}_1 \left(s_2'\right)}{(s_1; s_2) \xrightarrow{\alpha}_1 \left(s_2'\right)} \quad (\alpha \in \mathbf{A}_1^{\surd}).$$

**(4)**

$$\frac{s_1 \xrightarrow{\alpha}_1 \left(s_1'\right)}{\begin{array}{l}(s_1 + s_2) \xrightarrow{\alpha}_1 \left(s_1'\right) \\ (s_2 + s_1) \xrightarrow{\alpha}_1 \left(s_1'\right)\end{array}} \quad (\alpha \in \mathbf{A}_1^{\surd}).$$

**(5.1)**

$$\frac{s_1 \xrightarrow{a}_1 s_1'}{\begin{array}{l}(s_1 \parallel s_2) \xrightarrow{a}_1 (s_1' \parallel s_2) \\ (s_2 \parallel s_1) \xrightarrow{a}_1 (s_2 \parallel s_1')\end{array}} \quad (a \in \mathbf{A}_1).$$

**(5.2)**

$$\frac{s_1 \xrightarrow{c}_1 s_1',\ s_2 \xrightarrow{\bar{c}}_1 s_2'}{(s_1 \,\|\, s_2) \xrightarrow{\tau}_1 (s_1' \,\|\, s_2')} \quad (c \in \mathbf{C}_1).$$

**(5.3)**

$$\frac{s_1 \xrightarrow{\surd}_1,\ s_2 \xrightarrow{\surd}_1}{(s_1 \,\|\, s_2) \xrightarrow{\surd}_1}.$$

**(6)**

$$\frac{s \xrightarrow{\alpha}_1 \left(s'\right)}{\partial_C(s) \xrightarrow{\alpha} \left(\partial_C(s')\right)} \quad (\alpha \in \mathbf{A}_1^{\surd} \setminus C).$$

See Remark 4.2 below for the motivation for this rule.

**(7)** For each $\langle P, g_P \rangle \in D$,

$$\frac{g_P \xrightarrow{\alpha}_1 \left(s'\right)}{P \xrightarrow{\alpha}_1 \left(s'\right)}.$$

The last rule is called the *recursion rule.* ∎

**Remark 4.2** In many of the references (e.g., [Mil 89], [Hen 88]), the following rule, which is sligtly different from rule (6) in Definition 4.3, is adopted for action restriction:

$$\frac{s \xrightarrow{\alpha}_1 \left(s'\right)}{\partial_C'(s) \xrightarrow{\alpha} \left(\partial_C(s')\right)} \quad (\alpha \in \mathbf{A}_1^{\surd} \setminus (C \cup \overline{C})), \tag{4.1}$$

where $\partial_C'(\cdot)$ stands for another restriction combinator specified by this rule. We adopt, however, rule (6) in Definition 4.3 for the following reasons:

**(i)** It is easy to check that $\partial_C'(\cdot)$ can be defined in terms of $\partial_C(\cdot)$ in the sense that the transitional behavior of $\partial_C'(s)$ is the same as that of $\partial_{(C \cup \bar{C})}(s)$ $(s \in \mathcal{L}_1[\emptyset])$. Thus $\partial_C(\cdot)$ is more expressive than $\partial_C'(\cdot)$.

**(i)** While the combinator $\partial_C'(\cdot)$ restricts the communications contained in $C$, *together with their complementary communications*, $\partial_C(\cdot)$ restricts only the the communications contained in $C$, which situation we may want for certain purposes. For example, we may want to forbid inputs through a channel without forbidding outputs through it.

**(ii)** The full abstraction result in Chapter 6 depends on rule (6) in Definition 4.3, although the results in this chapter and Chapter 5 can be established using rule (4.1) instead of rule (6) in Definition 4.3. ∎

**Definition 4.4**

**(1)** For $w = \langle a_0, \cdots, a_{n-1}\rangle \in (\mathbf{A}_1)^{<\omega}$, a binary relation $\xrightarrow{w}_{1*}$ is defined by:

$$\xrightarrow{w}_{1*} = \xrightarrow{a_0}_1 \xrightarrow{a_1}_1 \cdots \xrightarrow{a_{n-1}}_1 .$$

Here and in the sequel, juxtaposition is used for denoting relational composition, e.g.,

$$\xrightarrow{a}_1 \xrightarrow{a'}_1 = \{\langle s, s'\rangle : \exists s''[\ s \xrightarrow{a}_1 s'' \xrightarrow{a'}_1 s'\ ]\}.$$

For $a \in \mathbf{A}_1$, $w \in (\mathbf{A}_1)^{<\omega}$, we write $s \xrightarrow{a}_1$ and $s \xrightarrow{w}_{1*}$, to denote that $\exists s'[s \xrightarrow{a}_1 s']$ and $\exists s'[s \xrightarrow{w}_{1*} s']$, respectively.

**(2)** For $w \in (\mathbf{A}_1)^{\omega}$, a unary relation $\xrightarrow{w}_{1\omega}$ is defined as follows: For $s \in \mathcal{L}_\emptyset$, $s \xrightarrow{w}_{1\omega}$ iff there exists $\langle s_i\rangle_{i\in\omega} \in (\mathcal{L}_\emptyset)^{\omega}$ such that $s = s_0$ and $\forall i \in \omega[\ s_i \xrightarrow{w(i)} s_{i+1}\ ]$. ∎

The transition system $(\mathcal{L}_1[\emptyset], \langle \xrightarrow{\alpha}_1 : \alpha \in \mathbf{A}_1^{\surd}\rangle)$ is *finitely branching* in the following sense:

**Lemma 4.1** *For every $s \in \mathcal{L}_1[\emptyset]$, the set $\{(a, s') \in \mathbf{A}_1 \times \mathcal{L}_1 : s \xrightarrow{a}_1 s'\}$ of branches is finite.* ∎

Note that the property of being finitely branching relies on the guardedness restriction Definition 4.1.

The first two models $\mathcal{O}_1$, $\mathcal{O}_1^{\mathrm{m}}$ are defined in terms of the transition system $(\mathcal{L}_1[\emptyset], \langle \xrightarrow{\alpha}_1 : \alpha \in \mathbf{A}_1^{\surd}\rangle)$ These models are called *linear*, because they yield sets of non-branching streams as the meaning of a statement:

**Definition 4.5** (Two Linear Models $\mathcal{O}_1$ and $\mathcal{O}_1^{\mathrm{m}}$ for $\mathcal{L}_1$)

**(1)** Let $\mathrm{act}_1 : \mathcal{L}_1[\emptyset] \to \wp(\mathbf{A}_1^{\surd})$ be defined as follows: For $s \in \mathcal{L}_1[\emptyset]$,

$$\mathrm{act}_1(s) = \{\alpha \in \mathbf{A}_1^{\surd} : s \xrightarrow{\alpha}_1\}.$$

**(2)** Let $\mathbf{C}_1^{\surd} = \mathbf{C}_1 \cup \{\surd\}$, and let

$$(\rho \in)\ \mathbf{B}_1^{!} = (\mathbf{A}_1)^{\omega} \cup ((\mathbf{A}_1)^{<\omega} \cdot \{\langle\surd\rangle\}) \cup (\mathbf{A}_1)^{<\omega} \cdot \{\langle\,\delta\,\rangle\}.$$

**(3)** The function $\mathcal{O}_1 : \mathcal{L}_1[\emptyset] \to \wp(\mathbf{B}_1^{!})$ is defined as follows: Let $s \in \mathcal{L}_1[\emptyset]$. For $\rho \in \mathbf{B}_1^{!}$, we put $\rho \in \mathcal{O}_1[\![s]\!]$ iff one of the following conditions (4.2), (4.3), (4.4) holds:

$$\exists\langle s_i\rangle_{i\in\omega}, \exists\langle a_i\rangle_{i\in\omega}[\ \rho = \langle a_i\rangle_{i\in\omega} \wedge s_0 \equiv s \wedge \forall i \in \omega[s_i \xrightarrow{a_i}_1 s_{i+1}]\ ]. \tag{4.2}$$

$$\exists n \in \omega, \exists\langle s_i\rangle_{i\in(n+1)}, \exists\langle a_i\rangle_{i\in n}[\ \rho = \langle a_i\rangle_{i\in n} \cdot \langle\surd\rangle \wedge s_0 \equiv s \wedge \forall i \in n[\ s_i \xrightarrow{a_i}_1 s_{i+1}\ ] \wedge \surd \in \mathrm{act}_1(s_n)\ ]. \tag{4.3}$$

$$\exists n \in \omega, \exists \langle s_i \rangle_{i \in (n+1)}, \exists \langle a_i \rangle_{i \in n}[\ \rho = \langle a_i \rangle_{i \in n} \cdot \langle\ \delta\ \rangle \wedge s_0 \equiv s \\ \wedge \forall i \in n[\ s_i \xrightarrow{a_i}_1 s_{i+1}\ ] \wedge \mathrm{act}_1(s_n) \subseteq \mathbf{C}_1\ ]. \tag{4.4}$$

**(4)** The function $\mathcal{O}_1^{\mathrm{m}} : \mathcal{L}_1[\emptyset] \to \wp(\mathbf{B}_1^{\mathrm{l}})$ is defined as follows: Let $s \in \mathcal{L}_1[\emptyset]$. For $\rho \in \mathbf{B}_1^{\mathrm{l}}$, we put $\rho \in \mathcal{O}_1^{\mathrm{m}}[\![s]\!]$ iff one of the propositions (4.2), (4.3) above, and (4.5) below holds.

$$\exists n \in \omega, \exists \langle s_i \rangle_{i \in (n+1)}, \exists \langle a_i \rangle_{i \in n}[\ \rho = \langle a_i \rangle_{i \in n} \wedge s_0 \equiv s \\ \wedge \forall i \in n[\ s_i \xrightarrow{a_i}_1 s_{i+1}\ ] \wedge \mathrm{act}_1(s_n) = \emptyset\ ]. \blacksquare \tag{4.5}$$

**Example 4.2** Let $c_0, c_1, c_2 \in \mathbf{C}_1$, and put

$$s \equiv (c_0; c_1; ((\tau; \mathbf{0}) + c_2)) + P_0,$$

where it is assumed that $(P_0, (\tau; P_0)) \in D$. The process $s$ is depicted by:

$$s = \bullet \left\{ \begin{array}{l} \xrightarrow{c_0} \bullet \xrightarrow{c_1} \bullet \left\{ \begin{array}{l} \xrightarrow{\tau} \bullet \\ \xrightarrow{c_2} \circ \end{array} \right. \\ \xrightarrow{\tau} \bullet \xrightarrow{\tau} \bullet \xrightarrow{\tau} \bullet \xrightarrow{\tau} \bullet \cdots \end{array} \right.$$

By the definitions of $\mathcal{O}_1$ and $\mathcal{O}_1^{\mathrm{m}}$, one has

$$\mathcal{O}_1[\![s]\!] = \{\langle c_0,\ \delta\ \rangle, \langle c_0, c_1, \tau,\ \delta\ \rangle, \langle c_0, c_1, c_2, \surd \rangle, \tau^\omega\},$$

and

$$\mathcal{O}_1^{\mathrm{m}}[\![s]\!] = \{\langle c_0, c_1, \tau,\ \delta\ \rangle, \langle c_0, c_1, c_2, \surd \rangle, \tau^\omega\}. \blacksquare$$

The last model $\mathcal{C}_1$ is a *failures model* which was first introduced in [BHR 85]; for this model, two alternative definitions are given: an *operational* one and a *denotational* one. First, in this subsection, the operational definition is given by:

**Definition 4.6** (Failures Model $\mathcal{C}_1$ for $\mathcal{L}_1$) Let $\delta$ be a symbol standing for *deadlock*, as in Chapter 3.

**(1)** Let

$$\mathbf{Q}_1 = (\mathbf{A}_1)^\omega \cup ((\mathbf{A}_1)^{<\omega} \cdot \{\langle \surd \rangle\}) \cup ((\mathbf{A}_1)^{<\omega} \cdot \{\langle \delta(\Gamma) \rangle : \Gamma \subseteq \mathbf{C}_1^{\surd}\}),$$

where $\delta(\Gamma) = (\ \delta\ , \Gamma)$.

**(2)** The function $\mathcal{C}_1 : \mathcal{L}_1[\emptyset] \to \wp(\mathbf{Q}_1)$ is defined as follows: Let $s \in \mathcal{L}_1[\emptyset]$. For $\rho \in \mathbf{Q}_1$, we put $\rho \in \mathcal{C}_1[\![s]\!]$ iff one of the conditions (4.2), (4.3) above, and (4.6) below holds.

$$\exists n \in \omega, \exists \langle s_i \rangle_{i \in (n+1)}, \exists \langle a_i \rangle_{i \in n}, \exists \Gamma \in \wp(\mathbf{C}_1^{\surd})[ \\ \rho = \langle a_i \rangle_{i \in n} \cdot \langle \delta(\Gamma) \rangle \wedge s_0 \equiv s \\ \wedge \forall i \in n[s_i \xrightarrow{a_i}_1 s_{i+1}] \wedge \tau \notin \mathrm{act}_1(s_n) \\ \wedge \Gamma \cap \mathrm{act}_1(s_n) = \emptyset\ ]. \blacksquare \tag{4.6}$$

**Example 4.3** Let $s$ be defined as in Example 4.2. Then, by the definition of $\mathcal{C}_1$, one has

$$\begin{aligned}\mathcal{C}_1[\![s]\!] = &\{\langle c_0, \delta(\Gamma)\rangle : \Gamma \subseteq \mathbf{C}_1^{\checkmark} \setminus \{c_1\}\} \cup \{\langle c_0, c_1, \tau, \delta(\Gamma)\rangle : \Gamma \subseteq \mathbf{C}_1^{\checkmark}\}\\ &\cup\{\langle c_0, c_1, c_2, \delta(\Gamma)\rangle : \Gamma \subseteq \mathbf{C}_1\} \cup \{\langle c_0, c_1, c_2, \checkmark\rangle, \tau^{\omega}\}. \blacksquare\end{aligned}$$

The model $\mathcal{C}_1$ can also be characterized as the fixed-point of a higher-order function $\Phi_1^{\mathrm{f}}$. That is, we can define $\Phi_1^{\mathrm{f}}$ and show $\mathcal{C}_1 = \mathrm{fix}(\Phi_1^{\mathrm{f}})$. This characterization will be of use when establishing the semantic equivalence between $\mathcal{C}_1$ and a denotational model presented in § 4.2.3.

**Definition 4.7** (Fixed-Point Formulation of Failures Model for $\mathcal{L}_1$)

**(1)** By defining a metric $d_{\mathbf{Q}_1}$ on $\mathbf{Q}_1$ in terms of *truncations* as usual, it is easily shown that $(\mathbf{Q}_1, d_{\mathbf{Q}_1})$ is the unique solution to

$$\mathbf{Q}_1 \cong (\wp(\mathbf{C}_1^{\checkmark}) \cup \{\checkmark\}) \uplus (\mathbf{A}_1 \times \mathrm{id}_{\kappa}(\mathbf{Q}_1)),$$

in the category of complete metric spaces.
The set $\mathbf{P}_1$, the domain of the failures model for $\mathcal{L}_1$, is given by

$$\mathbf{P}_1 = \wp_{+\mathrm{cl}}(\mathbf{Q}_1).$$

The *Hausdorff distance* $d_{\mathbf{P}_1}$ (on $\mathbf{P}_1$) is induced from $d_{\mathbf{Q}_1}$ such that $(\mathbf{P}_1, d_{\mathbf{P}_1})$ is a complete metric space (see Chapter 1 for the definition of *Hausdorff distance*).

**(2)** Let $\mathbf{M}_1^{\mathcal{O}} = (\mathcal{L}_1[\emptyset] \to \mathbf{P}_1)$, and let $\Phi_1^{\mathrm{f}} : \mathbf{M}_1^{\mathcal{O}} \to \mathbf{M}_1^{\mathcal{O}}$ be defined as follows: For $F \in \mathbf{M}_1^{\mathcal{O}}$ and $s \in \mathcal{L}_1[\emptyset]$,

$$\begin{aligned}\Phi_1^{\mathrm{f}}(F)[\![s]\!] = &\,\mathrm{if}(\checkmark \in \mathrm{act}_1(s), \{\langle\checkmark\rangle\}, \emptyset)\\ &\cup\{\langle\delta(\Gamma)\rangle : \Gamma \subseteq \mathbf{C}_1^{\checkmark} \wedge \tau \notin \mathrm{act}_1(s) \wedge \Gamma \cap \mathrm{act}_1(s) = \emptyset\}\\ &\cup\bigcup\{\langle a\rangle \cdot F[\![s']\!] : a \in \mathbf{A}_1 \wedge s' \in \mathcal{L}_1[\emptyset] \wedge s \xrightarrow{a}_1 s'\}.\end{aligned}$$

By Lemma 4.1, it is straightforward to check that the right-hand side of the above equation is nonempty and closed.

**(3)** Let $\hat{\mathcal{C}}_1 = \mathrm{fix}(\Phi_1^{\mathrm{f}})$. By definition one has the following for every $s \in \mathcal{L}_1[\emptyset]$:

$$\begin{aligned}\hat{\mathcal{C}}_1[\![s]\!] = &\,\mathrm{if}(\checkmark \in \mathrm{act}_1(s), \{\langle\checkmark\rangle\}, \emptyset)\\ &\cup\{\langle\delta(\Gamma)\rangle : \Gamma \subseteq \mathbf{C}_1^{\checkmark} \wedge \tau \notin \mathrm{act}_1(s) \wedge \Gamma \cap \mathrm{act}_1(s) = \emptyset\}\\ &\cup\bigcup\{\langle a\rangle \cdot \hat{\mathcal{C}}_1[\![s']\!] : a \in \mathbf{A}_1 \wedge s' \in \mathcal{L}_1[\emptyset] \wedge s \xrightarrow{a}_1 s'\}. \blacksquare\end{aligned} \tag{4.7}$$

One has the following fixed-point characterization of $\mathcal{C}_1$, i.e. the equivalence of $\mathcal{C}_1$ and $\hat{\mathcal{C}}_1$:

**Lemma 4.2** (Fixed-Point Characterization of $\mathcal{C}_1$)

**(1)** $\forall s \in \mathcal{L}_1[\emptyset][\, \mathcal{C}_1[\![s]\!] \in \mathbf{P}_1 \,]$.

**(2)** $\mathcal{C}_1 = \hat{\mathcal{C}}_1$. $\blacksquare$

**Proof. (1)** Let $s \in \mathcal{L}_1[\emptyset]$. It suffices to show that $\mathcal{C}_1[\![s]\!]$ is *nonempty* and *closed* in $\mathbf{Q}_0$.

**Nonemptiness.** Let us show that $\mathcal{C}_1[\![s]\!]$ is nonempty, by distinguishing two cases according to whether

$$\exists s', \exists w \in (\mathbf{A}_1)^{<\omega}[\ s \xrightarrow{w}_{1*} s' \wedge \tau \notin \mathrm{act}_1(s')\ ]. \tag{4.8}$$

**Case 1.** Suppose (4.8) holds, and choose $s'$, $w$ so that (4.8) holds. Then

$$w \cdot \langle \delta(\emptyset) \rangle \in \mathcal{C}_1[\![s]\!]$$

by the definition of $\mathcal{C}_1[\![\cdot]\!]$.

**Case 2.** Suppose (4.8) does not hold. Then, we can define an infinite sequences $\langle s_n \rangle_{n\in\omega} \in (\mathcal{L}_1)^\omega$ such that $s_0 \equiv s$ and $\forall n \in \omega[\ s_n \xrightarrow{\tau} s_{n+1}\ ]$. Then, by the definition $\mathcal{C}_1[\![\cdot]\!]$, one has $\langle \tau \rangle_{n\in\omega} \in \mathcal{C}_1[\![s]\!]$.

**Closedness.** Let $w \in (\mathcal{C}_1[\![s]\!])^{\mathrm{cls}} \setminus \mathcal{C}_1[\![s]\!]$. It immediately follows that $w \in (\mathbf{A}_1)^\omega$; putting $w = \langle a_n \rangle_{n\in\omega}$, and let us show $\langle a_n \rangle_{n\in\omega} \in \mathcal{C}_1[\![s]\!]$.

First, let us define a sequence $\langle s_n \rangle_{n\in\omega}$ by induction such that

$$s_0 \equiv s \wedge \forall n \in \omega[\ s_n \xrightarrow{a_n} s_{n+1}\ ]. \tag{4.9}$$

Let $s_0 \equiv s$. For $k \in \omega$, suppose $s_k$ has been defined. When

$$\langle a_{k+n} \rangle_{n\in\omega} \in (\mathcal{C}_1[\![s_k]\!])^{\mathrm{cls}}, \tag{4.10}$$

there are only a finite number of statements $s'_0, \cdots, s'_{m-1}$, by Lemma 4.1, such that $s_k \xrightarrow{ak} s'_i$ $(i \in m)$. From the fact that $\langle a_{k+n} \rangle_{n\in\omega} \in (\mathcal{C}_1[\![s_k]\!])^{\mathrm{cls}}$, it follows that

$$\exists i \in m[\ \langle a_{k+1+n} \rangle_{n\in\omega} \in (\mathcal{C}_1[\![s'_i]\!])^{\mathrm{cls}}\ ];$$

choosing such $i$, let $s_{k+1} \equiv s'_i$. (Then obviously one has (4.10) with $k$ replaced by $k+1$.) When (4.10) does not hold, let $s_{k+1} \equiv \mathbf{0}$.

Then, it follows, by induction, that (4.10) holds for every $k \in \omega$. Thus one has (4.9). By this and the definition of $\mathcal{C}_1$, one has $\langle a_n \rangle_{n\in\omega} \in \mathcal{C}_1[\![s]\!]$.

**(2)** Having established part (1), it is easy to check that

$$\forall s \in \mathcal{L}_1[\emptyset][\ \Phi_1^{\mathrm{f}}(\mathcal{C}_1)[\![s]\!] = \mathcal{C}_1[\![s]\!]\ ].$$

Thus, one has

$$\mathcal{C}_1 = \mathrm{fix}(\Phi_1^{\mathrm{f}}) = \hat{\mathcal{C}}_1. \blacksquare$$

Two *abstraction functions* $\mathcal{A}_0^{\mathrm{l}}, \mathcal{A}_0^{\mathrm{c}} : \wp(\mathbf{Q}_1) \to \wp((\mathbf{A}_1)^{\leq\omega})$ are defined by:

**Definition 4.8** (Abstraction Functions $\mathcal{A}_0^{\mathrm{l}}$, $\mathcal{A}_0^{\mathrm{c}}$) Two functions

$$\mathcal{A}_0^{\mathrm{l}}, \mathcal{A}_0^{\mathrm{c}} : \mathbf{P}_1 \to \wp((\mathbf{A}_1)^{\leq\omega})$$

are defined as follows: For $p \in \mathbf{P}_1$,

**(1)** $$\mathcal{A}_0^{\mathrm{l}}(p) = ((\mathbf{A}_1)^\omega \cap p) \cup (((\mathbf{A}_1)^\omega \cdot \{\langle\surd\rangle\}) \cap p) \cup \{w \cdot \langle\,\delta\,\rangle : \exists \Gamma \in \wp(\mathbf{C}_1^{\surd})[\; w \cdot \langle\delta(\Gamma)\rangle \in p \wedge \surd \in \Gamma\;]\}.$$

**(2)** $$\mathcal{A}_0^{\mathrm{c}}(p) = ((\mathbf{A}_1)^\omega \cap p) \cup (((\mathbf{A}_1)^\omega \cdot \{\langle\surd\rangle\}) \cap p) \{w \cdot \langle\,\delta\,\rangle : w \cdot \langle\delta(\mathbf{C}_1^{\surd})\rangle \in p\}. \blacksquare$$

Having defined $\mathcal{A}_0^{\mathrm{l}}$ and $\mathcal{A}_0^{\mathrm{c}}$, we immediately obtain the following lemma from the definitions of $\mathcal{O}_1$, $\mathcal{O}_1^{\mathrm{m}}$, and $\mathcal{C}_1$:

**Lemma 4.3** (Relative Abstractness of $\mathcal{C}_1$ w.r.t. $\mathcal{O}_1$ and $\mathcal{O}_1^{\mathrm{m}}$)

**(1)** $\mathcal{O}_1 = \mathcal{A}_0^{\mathrm{l}} \circ \mathcal{C}_1$.

**(2)** $\mathcal{O}_1^{\mathrm{m}} = \mathcal{A}_0^{\mathrm{c}} \circ \mathcal{C}_1$. $\blacksquare$

### 4.2.3 A Denotational Model $\mathcal{M}_1$ for $\mathcal{L}_1$

In this subsection, a *denotational* model $\mathcal{M}_1$ is defined using explicit semantic operations (corresponding to the constructs of $\mathcal{L}_1$), with meanings of recursive programs as fixed-points in the complete metric space $(\mathbf{P}_1, d_{\mathbf{P}_1})$. Then, the semantic equivalence between $\mathcal{M}_1$ and $\mathcal{C}_1$ is established along the lines of [KR 90], [Rut 89], and [BR 91].

For the definition of $\mathcal{M}_1$, we use the following notation:

**Notation 4.2** For $p \in \mathbf{P}_1$, let

$$\tilde{\mathcal{T}}_1(p) = p \cap ((\mathbf{A}_1)^{<\omega} \cdot \{\langle\surd\rangle\}), \quad \tilde{\mathcal{T}}_1^{\omega}(p) = p \cap (\mathbf{A}_1)^\omega,$$

$$\tilde{\mathcal{F}}_1(p) = p \cap ((\mathbf{A}_1)^{<\omega} \cdot \{\langle\delta(\Gamma)\rangle : \Gamma \in \wp(\mathbf{C}_1^{\surd})\}). \blacksquare$$

An *interpretation* $\mathcal{I}_1$ which maps each syntactical construct of $\mathcal{L}_1$ to a corresponding semantic operation is defined on the basis of the domain $\mathbf{P}_1$ by:

**Definition 4.9** (Interpretation $\mathcal{I}_1$ for $\mathcal{L}_1$) We define an interpretation $\mathcal{I}_1$, a type-respecting mapping which maps each combinator of the subsignature $\mathbf{Sig}_1$, introduced in Definition 4.2, to a semantic operation as follows.

**(1)** The semantic interpretation $\tilde{\mathbf{0}}_1$ of $\mathbf{0}$ is defined by:

$$\tilde{\mathbf{0}}_1 = \{\langle\delta(\Gamma)\rangle : \Gamma \in \wp(\mathbf{C}_1^{\surd})\}.$$

**(2)** The semantic interpretation $\tilde{\mathbf{e}}_1$ of $\mathbf{e}$ is defined by:

$$\tilde{\mathbf{e}}_1 = \{\langle\delta(\Gamma)\rangle : \Gamma \in \wp(\mathbf{C}_1^{\surd}) \wedge \surd \notin \Gamma\} \cup \{\langle\surd\rangle\}.$$

**(3)** For $a \in \mathbf{A}_1$, the semantic interpretation $\tilde{a}_1$ is defined by distinguishing two cases according to $a = \tau$ or not:

**(i)** $\tilde{\tau}_1 = \{\langle\tau\rangle\} \cdot \tilde{\mathbf{e}}_1$.

**(ii)** For $a \neq \tau$,

$$\tilde{a}_1 = \{\langle\delta(\Gamma)\rangle : \Gamma \subseteq \mathbf{C}_1^{\surd} \wedge a \notin \Gamma\} \cup \{\langle a\rangle\} \cdot \tilde{\mathbf{e}}_1.$$

**(4)** With an auxiliary operation $\mathcal{F}_1^{;} : \mathbf{P}_1 \times \mathbf{P}_1 \to \mathbf{P}_1$, the semantic operation $\tilde{;}_1 : \mathbf{P}_1 \times \mathbf{P}_1 \to \mathbf{P}_1$ corresponding to the sequential composition ';' is defined as follows: For $p_1, p_2 \in \mathbf{P}_1$,

$$\begin{aligned} p_1 \tilde{;}_1 p_2 = &\{w \cdot q : w \cdot \langle \surd \rangle \in \tilde{\mathcal{T}}_1(p_1) \wedge q \in \tilde{\mathcal{T}}_1(p_2)\} \\ &\cup \mathcal{F}^{;}(p_1, p_2) \cup \tilde{\mathcal{T}}_1^{\omega}(p_1) \\ &\cup \{w \cdot q : w \cdot \langle \surd \rangle \in \tilde{\mathcal{T}}_1(p_1) \wedge q \in \tilde{\mathcal{T}}_1^{\omega}(p_2)\}. \end{aligned}$$

where

$$\begin{aligned} &\mathcal{F}^{;}(p_1, p_2) \\ &= \{w \cdot \langle \delta(\Gamma) \rangle : (w \in (\mathbf{A}_1)^{<\omega} \wedge \Gamma \subseteq \mathbf{C}_1^{\surd}) \wedge \\ &\quad ((w \cdot \langle \delta(\Gamma \cup \{\surd\}) \rangle \in \tilde{\mathcal{F}}_1(p_1)) \\ &\quad \vee (w \cdot \langle \delta(\Gamma \setminus \{\surd\}) \rangle \in \tilde{\mathcal{F}}_1(p_1) \wedge \langle \delta(\Gamma) \rangle \in \tilde{\mathcal{F}}_1(p_2)))\} \\ &\quad \cup \{w \cdot q : w \cdot \langle \surd \rangle \in \tilde{\mathcal{T}}_1(p_1) \\ &\quad \wedge q \in (\tilde{\mathcal{F}}_1(p_2) \setminus \{\langle \delta(\Gamma) \rangle : \Gamma \in \wp(\mathbf{C}_1^{\surd})\})\}. \end{aligned}$$

**(5)** For $p \in \mathbf{P}_1$, $p \setminus \tilde{\mathbf{0}}_1$ is called the *action part* of $p$ and denoted by $p^{\oplus}$. For $p_1, p_2 \in \mathbf{P}_1$, $p_1 \tilde{+}_1 p_2$ is defined by:

$$\begin{aligned} p_1 \tilde{+}_1 p_2 = &(p_1)^{\oplus} \cup (p_2)^{\oplus} \\ &\cup \{\langle \delta(\Gamma) \rangle : \Gamma \subseteq \mathbf{C}_1^{\surd} \wedge \langle \delta(\Gamma) \rangle \in p_1 \cap p_2\}. \end{aligned}$$

A process $p \in \mathbf{P}_1$ is said to be *downward closed at level* 0, if

$$\forall \Gamma \in \wp(\mathbf{C}_1^{\surd})[\ \langle \delta(\Gamma) \rangle \in p \Rightarrow \forall \Gamma' \in \wp(\mathbf{C}_1^{\surd})[\ \Gamma' \subseteq \Gamma \Rightarrow \langle \delta(\Gamma') \rangle \in p\ ]].$$

Obviously, if $p_1$ and $p_2$ are downward closed at level 0, then

$$\begin{aligned} p_1 \tilde{+}_1 p_2 = &(p_1)^{\oplus} \cup (p_2)^{\oplus} \\ &\cup \{\langle \delta(\Gamma) \rangle : \exists \langle \delta(\Gamma_1) \rangle \in p_1, \exists \langle \delta(\Gamma_2) \rangle \in p_2[\ \Gamma \subseteq \Gamma_1 \cap \Gamma_2]\}. \end{aligned}$$

**(6)** We have the unique operation $\tilde{\|}_1 : \mathbf{P}_1 \times \mathbf{P}_1 \to \mathbf{P}_1$ satisfying the following (for the existence and uniqueness of such an operation see [Rut 89]): For $p_1, p_2 \in \mathbf{P}_1$,

$$\begin{aligned} p_1 \tilde{\|}_1 p_2 = &(\{\langle \surd \rangle\} \cap (p_1 \cap p_2)) \cup (p_1 \tilde{\|}_1^{\delta} p_2) \\ &\cup (p_1 \tilde{\|\!\!\_}_1 p_2) \cup (p_2 \tilde{\|\!\!\_}_1 p_1) \cup (p_1 \tilde{|}_1 p_2) \end{aligned}$$

where

$$p_1 \tilde{\|\!\!\_}_1 p_2 = \bigcup \{\langle a \rangle \cdot (p_1[\langle a \rangle] \tilde{\|}_1 p_2) : a \in \mathbf{A}_1 \wedge p_1[\langle a \rangle] \neq \emptyset\},$$

$$\begin{aligned} &p_1 \tilde{|}_1 p_2 \\ &= (\bigcup \{\langle \tau \rangle \cdot (p_1[(c)] \tilde{\|}_1 p_2[(\bar{c})]) : c \in \mathbf{C}_1 \\ &\quad \wedge p_1[(c)] \neq \emptyset \wedge p_2[(\bar{c})] \neq \emptyset\ \})^{\text{cls}}, \end{aligned} \tag{4.11}$$

and

$$
\begin{aligned}
& p_1 \tilde{\|}_1^{\delta} p_2 \\
&= \{\langle\delta(\Gamma)\rangle : \exists\langle\delta(\Gamma_1)\rangle \in p_1, \exists\langle\delta(\Gamma_2)\rangle \in p_2[ \\
&\qquad (\mathbf{C}_1 \setminus \Gamma_1) \cap \overline{(\mathbf{C}_1 \setminus \Gamma_2)} = \emptyset \wedge \\
&\qquad (((\Gamma \setminus \{\surd\}) \subseteq \Gamma_1 \ \wedge \Gamma \subseteq \Gamma_2) \\
&\qquad\ \vee (\Gamma \subseteq \Gamma_1 \ \wedge \ (\Gamma \setminus \{\surd\}) \subseteq \Gamma_2)) \ ]\}.
\end{aligned}
$$

Note that taking closure in the right-hand side of (4.11) is necessary as was shown by Example 4 in [HBR 90].

**(7)** For every $C \in \wp(\mathbf{C}_1)$, the operation $\tilde{\partial}^1_C(\cdot) : \mathbf{P}_1 \to \mathbf{P}_1$, corresponding the combinator '$\partial_C(\cdot)$', is defined as follows: For every $p \in \mathbf{P}_1$,

$$
\begin{aligned}
\tilde{\partial}^1_C(p) = \ & \{w \in \tilde{T}_1(p) \cup \tilde{T}_1^{\omega}(p) : \mathrm{ran}(w) \cap C = \emptyset\} \\
& \cup\{w \cdot \langle\delta(\Gamma)\rangle : w \in (\mathbf{A}_1)^{<\omega} \ \wedge \ \Gamma \in \wp(\mathbf{C}_1^{\checkmark}) \\
& \qquad \wedge \mathrm{ran}(w) \cap C = \emptyset \\
& \qquad \wedge w \cdot \langle\delta(\Gamma \setminus C)\rangle \in \tilde{\mathcal{F}}_1(p)\}.
\end{aligned}
$$

**(8)** Representing a mapping as a set of pairs as usual, we define $\mathcal{I}_1$ by:

$$
\begin{aligned}
\mathcal{I}_1 = \ & \{(\mathbf{0}, \tilde{\mathbf{0}}_1), (\mathbf{e}, \tilde{\mathbf{e}}_2), (;, \tilde{;}_1), (+, \tilde{+}_1), (\|, \tilde{\|}_1)\} \\
& \cup\{(\partial_C, \tilde{\partial}^1_C) : C \in \wp(\mathbf{C}_1)\}. \ \blacksquare
\end{aligned}
$$

In terms of the interpretation $\mathcal{I}_1$, the denotational model $\mathcal{M}_1$ is defined. For the definition, we need the following definition and lemma:

**Definition 4.10** Let $\tilde{\mathcal{I}}$ be an interpretation for $\mathbf{Sig}_1^*$. For $n \in \omega$, $\vec{X} \in (n \rightarrowtail \mathcal{X}_{\mathcal{P}})$, $S \in \mathcal{L}_1$ with $\mathrm{FV}(S) \subseteq \mathrm{ran}(\vec{X})$, and $\vec{p} \in (\mathbf{P}_1)^n$, let $[\![S_{\vec{X}}]\!]^{\tilde{\mathcal{I}}}(\vec{p})$ denote the *interpretation* of $S$ in $\tilde{\mathcal{I}}$ with the valuation associating $\vec{p}(i)$ to $\vec{X}(i)$ $(i \in n)$. (For $n = 1$ and $\vec{p} = \langle p\rangle$, we write $[\![S_{\langle X\rangle}]\!]^{\tilde{\mathcal{I}}}(p)$.) Formally $[\![S_{\vec{X}}]\!]^{\tilde{\mathcal{I}}}(\vec{p})$ is defined by induction on the structure of $S$ using the following rules:

**(i)** For $i \in n$, $[\![(\vec{X}(i))_{\vec{X}}]\!]^{\tilde{\mathcal{I}}}(\vec{p}) = \vec{p}(i)$;

**(ii)** For $k \in \omega$, $F \in (\mathbf{Fun}_1^*)^{(k)}$, $\vec{S} \in (\mathcal{L}_1)^k$,

$$
[\![(F(\vec{S}))_{\vec{X}}]\!]^{\tilde{\mathcal{I}}}(\vec{p}) = \tilde{\mathcal{I}}(F)(\langle[\![(\vec{S}(i))_{\vec{X}}]\!]^{\tilde{\mathcal{I}}}(\vec{p})\rangle_{i \in k}). \ \blacksquare
$$

**Lemma 4.4** *Let $D = \{(P, g_P) : P \in \mathcal{RV}\}$ be a set of declarations, and let $\mathbf{M}_1^{\mathcal{M}} = (\mathcal{RV} \to \mathbf{P}_1)$. We define $\Pi_1 : \mathbf{M}_1^{\mathcal{M}} \to \mathbf{M}_1^{\mathcal{M}}$ as follows: For $\mathbf{p} \in \mathbf{M}_1^{\mathcal{M}}$, $\mathcal{I}_1 \cup \mathbf{p}$ is an interpretation for $\mathbf{Sig}_1^*$; for $P \in \mathcal{RV}$, let*

$$
\Pi_1(\mathbf{p})(P) = [\![g_P]\!]^{\mathcal{I}_1 \cup \mathbf{p}}.
$$

*Then, the mapping $\Pi_1$ is a contraction from $\mathbf{M}_1^{\mathcal{M}}$ to $\mathbf{M}_1^{\mathcal{M}}$.* $\blacksquare$

**Proof.** See § 4.A. $\blacksquare$

By means of the above lemma, the denotational model $\mathcal{M}_1$ is defined by:

**Definition 4.11** (Denotational Model $\mathcal{M}_1$ for $\mathcal{L}_1$) Let $D = \{(P, g_P)\}_{P\in\mathcal{RV}}$ be the set of declarations. Let $\mathbf{M}_1^{\mathcal{M}}$ and $\Pi_1$ be as in Lemma 4.4, and let $\tilde{\mathbf{p}}_1 = \text{fix}(\Pi_1)$, whose existence is guaranteed by Lemma 4.4. Let us set $\tilde{\mathcal{I}}_1 = \mathcal{I}_1 \cup \tilde{\mathbf{p}}_1$. Then, $\tilde{\mathcal{I}}_1$ is an interpretation for $\mathbf{Sig}_1^*$. By means of $\tilde{\mathcal{I}}_1$, the model $\mathcal{M}_1 : \mathcal{L}_1[\emptyset] \to \mathbf{P}_1$ is defined by $\mathcal{M}_1[\![s]\!] = [\![s]\!]^{\tilde{\mathcal{I}}_1}$ ($s \in \mathcal{L}_1[\emptyset]$). Equivalently, $\mathcal{M}_1[\![s]\!]$ is defined by induction on the structure of $s \in \mathcal{L}_1[\emptyset]$ as follows:

**(1)** For $P \in \mathcal{RV}$, $P^{\mathcal{M}_1}$, the denotational meaning of $P$, is defined by:

$$P^{\mathcal{M}_1} = \tilde{\mathbf{p}}_1(P).$$

**(2)** For $k \in \omega$, $F \in (\mathbf{Fun}_1^*)^{(k)}$, and $s_1, \cdots, s_k \in \mathcal{L}_2[\emptyset]$, let

$$\mathcal{M}_1[\![F(s_1, \cdots, s_k)]\!] = \mathcal{I}_1(F)(\mathcal{M}_1[\![s_1]\!], \cdots, \mathcal{M}_1[\![s_k]\!]).$$

where $\mathcal{I}_1(F)$ is the semantic operation in $\mathcal{I}_1$ corresponding to $F$. ∎

The denotational model $\mathcal{M}_1$ is compositional by definition:

**Lemma 4.5** (Compositionality of $\mathcal{M}_1$) *Let $X \in \mathcal{X}_\mathcal{P}$ and $S \in \mathcal{L}_1[X]$. Then,*

$$\forall s_1, s_2 \in \mathcal{L}_1[\emptyset][\ \mathcal{M}_1[\![s_1]\!] = \mathcal{M}_1[\![s_2]\!] \Rightarrow \mathcal{M}_1[\![S_{\langle X\rangle}[s_1]]\!] = \mathcal{M}_1[\![S_{\langle X\rangle}[s_2]]\!]\ ].$$ ∎

**Proof.** It can be shown, by induction on the structure of $S \in \mathcal{L}_1$, that

$$\forall s \in \mathcal{L}_1[\emptyset][\ \mathcal{M}_1[\![S_{\langle X\rangle}[s]]\!] = [\![S_{\langle X\rangle}]\!]^{\tilde{\mathcal{I}}_1}(\mathcal{M}_1[\![s]\!])\ ].$$

From this, the claim immediately follows. ∎

It will be shown that the operational model $\mathcal{C}_1$ is also compositional, as stated by Lemma 4.6 below. From this, the equivalence $\mathcal{C}_1$ and $\mathcal{M}_1$, the claim of Lemma 4.8 below, immediately follows.

**Lemma 4.6** (Compositionality of $\mathcal{C}_1$)

**(1)** *For each $k \in \omega$, $F \in ((\mathbf{Fun}_1^*)^{(k)} \setminus \mathcal{RV})$, and $s_1, \cdots, s_k \in \mathcal{L}_1[\emptyset]$, the following holds:*

$$\mathcal{C}_1[\![F(s_1, \cdots, s_k)]\!] = \mathcal{I}_1(F)(\mathcal{C}_1[\![s_1]\!], \cdots, \mathcal{C}_1[\![s_k]\!]).$$

**(2)** *Putting $\tilde{\mathbf{p}}_1' = (\lambda P \in \mathcal{RV}.\ \mathcal{C}_1[\![P]\!])$, one has the following for every $s \in \mathcal{L}_1[\emptyset]$:*

$$\mathcal{C}_1[\![s]\!] = [\![s]\!]^{\mathcal{I}_1 \cup \tilde{\mathbf{p}}_1'}.$$ ∎

As a preliminary to the Proof of Lemma 4.6, we give the following lemma stating two useful properties of the operation $\tilde{\|}_1$:

**Lemma 4.7 (1)** *For every $p_1, p_2 \in \mathbf{P}_1$, $p_1 \tilde{\|}_1 p_2 = p_2 \tilde{\|}_1 p_1$.*

**(2)** *For every $p_1, p_1', p_2 \in \mathbf{P}_1$, $(p_1 \cup p_1')\tilde{\|}_1 p_2 = (p_1 \tilde{\|}_1 p_2) \cup (p_1' \tilde{\|}_1 p_2)$.* ∎

**Proof.** This is established in a similar fashion to the proof of a corresponding result, Lemma 18 in [HBR 90]. ∎

**Proof of Lemma 4.6.** **(1)** This is established in a similar fashion to the proof of a corresponding result, Lemma 19 (1) in [HBR 90]. Here, we prove the claim for the combinator $F \equiv \|$. For the other combinators, this is proved (more straightforwardly) in a similar fashion.

Let $\mathbf{H}_1 = (\mathcal{L}_1[\emptyset] \times \mathcal{L}_1[\emptyset] \to \mathbf{P}_1)$, and let $F, G \in \mathbf{H}_1$ be defined as follows: For $s_1, s_2 \in \mathcal{L}_1[\emptyset]$,

$$F(s_1, s_2) = \mathcal{C}_1[\![s_1 \parallel s_2]\!],$$

$$G(s_1, s_2) = \mathcal{C}_1[\![s_1]\!] \tilde{\|}_1 \mathcal{C}_1[\![s_1]\!].$$

Let us define a mapping $\mathcal{H}_1 : \mathbf{H}_1 \to \mathbf{H}_1$ as follows: For $H \in \mathbf{H}_1$, and $s_1, s_2 \in \mathcal{L}_1[\emptyset]$,

$$\begin{aligned}\mathcal{H}_1(H)(s_1, s_2) = {} & \mathcal{H}_1^{\checkmark}(s_1, s_2) \cup \mathcal{H}_1^{\delta}(s_1, s_2) \\ & \cup \mathcal{H}_1^{\lfloor\!\lfloor}(H)(s_1, s_2) \cup \mathcal{H}_1^{\lfloor\!\lfloor}(H)(s_2, s_1) \cup \mathcal{H}_1^{|}(H)(s_1, s_2),\end{aligned}$$

where

$$\begin{aligned}&\mathcal{H}_1^{\checkmark}(s_1, s_2) \\ &= \{\langle\surd\rangle\} \cap \mathrm{act}_1(s_1) \cap \mathrm{act}_1(s_2),\end{aligned}$$

$$\begin{aligned}&\mathcal{H}_1^{\delta}(s_1, s_2) \\ &= \{\langle\delta(\Gamma)\rangle : \Gamma \subseteq \mathbf{C}_1^{\checkmark} \wedge \tau \notin \mathrm{act}_1(s_1) \wedge \tau \notin \mathrm{act}_1(s_2) \\ &\quad \wedge (\mathrm{act}_1(s_1) \cap \mathbf{C}_1) \cap \overline{(\mathrm{act}_1(s_2) \cap \mathbf{C}_1)} = \emptyset \\ &\quad \wedge \Gamma \cap ((\mathrm{act}_1(s_1) \cap \mathbf{C}_1) \cup (\mathrm{act}_1(s_2) \cap \mathbf{C}_1)) = \emptyset \\ &\quad \wedge \Gamma \cap (\{\surd\} \cap \mathrm{act}_1(s_1) \cap \mathrm{act}_1(s_2)) = \emptyset\ \},\end{aligned}$$

$$\begin{aligned}&\mathcal{H}_1^{\lfloor\!\lfloor}(H)(s_1, s_2) \\ &= \bigcup\{\langle a\rangle \cdot \mathcal{H}_1(s_1', s_2) : a \in \mathbf{A}_1 \wedge s_1' \in \mathcal{L}_1[\emptyset] \wedge s_1 \xrightarrow{a}_1 s_1'\},\end{aligned}$$

$$\begin{aligned}&\mathcal{H}_1^{|}(H)(s_1, s_2) \\ &= \bigcup\{\langle\tau\rangle \cdot \mathcal{H}_1(s_1', s_2') : \exists c \in \mathbf{C}_1[\ s_1', s_2' \in \mathcal{L}_1[\emptyset] \wedge s_1 \xrightarrow{c}_1 s_1' \wedge s_2 \xrightarrow{\bar{c}}_1 s_2'\ ]\}.\end{aligned}$$

The mapping $\mathcal{H}_1$ is a contraction. By the definition of $\mathcal{C}_1$ and $\xrightarrow{\alpha}_1$, it holds for $s_1, s_2 \in \mathcal{L}_1[\emptyset]$ that $F(s_1, s_2) = \mathcal{H}_1(F)(s_1, s_2)$. That is, $F = \mathrm{fix}(\mathcal{H}_1)$.

Let us show $G = \mathcal{H}_1(G)$. By the definition of $\tilde{\|}_1$, one has

$$\begin{aligned}G(s_1, s_2) = {} & (\{\langle\surd\rangle\} \cap (\mathcal{C}_1[\![s_1]\!] \cap \mathcal{C}_1[\![s_2]\!])) \cup (\mathcal{C}_1[\![s_1]\!] \tilde{\|}_1^{\delta} \mathcal{C}_1[\![s_2]\!]) \\ & \cup (\mathcal{C}_1[\![s_1]\!] \tilde{\lfloor\!\lfloor}_1 \mathcal{C}_1[\![s_2]\!]) \cup (\mathcal{C}_1[\![s_2]\!] \tilde{\lfloor\!\lfloor}_1 \mathcal{C}_1[\![s_1]\!]) \\ & \cup (\mathcal{C}_1[\![s_2]\!] \tilde{|}_1 \mathcal{C}_1[\![s_1]\!]).\end{aligned}$$

It is easy to check that

$$\{\langle\surd\rangle\} \cap (\mathcal{C}_1[\![s_1]\!] \cap \mathcal{C}_1[\![s_2]\!]) = \mathcal{H}_1^{\checkmark}(s_1, s_2). \tag{4.12}$$

It immediately follows from the definition of $\tilde{\|}_1^{\delta}$ that

$$\mathcal{C}_1[\![s_1]\!]\,\tilde{|}|_1^\delta\,\mathcal{C}_1[\![s_2]\!] = \mathcal{H}_1^\delta(s_1, s_2). \tag{4.13}$$

We can show that

$$\mathcal{C}_1[\![s_1]\!]\,\tilde{\lfloor\!\lfloor}_1\,\mathcal{C}_1[\![s_2]\!] = \mathcal{H}_1^{\lfloor\!\lfloor}(G)(s_1, s_2) \tag{4.14}$$

as follows:

$$\begin{aligned}
&\mathcal{C}_1[\![s_1]\!]\,\tilde{\lfloor\!\lfloor}_1\,\mathcal{C}_1[\![s_2]\!]\\
&= \bigcup\{\langle a\rangle \cdot (\bigcup\{\mathcal{C}_1[\![s_1']\!] : s_1 \xrightarrow{a}_1 s_1'\}\tilde{||}\mathcal{C}_1[\![s_2]\!]) : a \in \mathbf{A}_1 \wedge \exists s_1'[\ s_1 \xrightarrow{a}_1 s_1'\ ]\}\\
&= \bigcup\{\langle a\rangle \cdot (\bigcup\{\mathcal{C}_1[\![s_1']\!]\tilde{||}\mathcal{C}_1[\![s_2]\!] : s_1 \xrightarrow{a}_1 s_1'\}) : a \in \mathbf{A}_1 \wedge \exists s_1'[\ s_1 \xrightarrow{a}_1 s_1'\ ]\}\\
&\qquad \text{(by Lemma 4.7)}\\
&= \bigcup\{\langle a\rangle \cdot (\mathcal{C}_1[\![s_1']\!]\tilde{||}\mathcal{C}_1[\![s_2]\!]) : a \in \mathbf{A}_1 \wedge s_1 \xrightarrow{a}_1 s_1'\}\\
&= \bigcup\{\langle a\rangle \cdot G(s_1', s_2) : a \in \mathbf{A}_1 \wedge s_1 \xrightarrow{a}_1 s_1'\}\\
&\qquad \text{(by the definition of } G(\cdot,\cdot))\\
&= \mathcal{H}_1^{\lfloor\!\lfloor}(G)(s_1, s_2)\\
&\qquad \text{(by the definition of } \mathcal{H}_1^{\lfloor\!\lfloor}(G)(\cdot,\cdot)).
\end{aligned}$$

Thus, one has (4.14). Likewise, one has

$$\mathcal{C}_1[\![s_2]\!]\,\tilde{\lfloor\!\lfloor}_1\,\mathcal{C}_1[\![s_1]\!] = \mathcal{H}_1^{\lfloor\!\lfloor}(G)(s_2, s_1). \tag{4.15}$$

Also, the following can be shown in a similar fashion to (4.14):

$$\mathcal{C}_1[\![s_1]\!]\,\tilde{|}_1\,\mathcal{C}_1[\![s_2]\!] = \mathcal{H}_1^{|}(G)(s_1, s_2). \tag{4.16}$$

By (4.12), (4.13), (4.14), (4.15), and (4.16), one has $G(s_1, s_2) = \mathcal{H}_1(G)(s_1, s_2)$ for every $s_1, s_2 \in \mathcal{L}_1[\emptyset]$. Thus, $G = \text{fix}(\mathcal{H}_1) = F$. That is,

$$\forall s_1, s_2 \in \mathcal{L}_1[\emptyset][\ \mathcal{C}_1[\![s_1 \parallel s_2]\!] = \mathcal{C}_1[\![s_1]\!]\tilde{||}_1\mathcal{C}_1[\![s_1]\!]\ ].$$

**(2)** This part can be established using part (1) by induction on the structure $s \in \mathcal{L}_1[\emptyset]$. ■

The following semantic equivalence result immediately follows from the above lemma:

**Lemma 4.8** (Semantic Equivalence for $\mathcal{L}_1$) $\mathcal{C}_1 = \mathcal{M}_1$. ■

**Proof.** First, let us show

$$\forall P \in \mathcal{RV}[\ \mathcal{C}_1[\![P]\!] = \tilde{\mathbf{p}}_1(P)\ ], \tag{4.17}$$

where $\tilde{\mathbf{p}}_1 = \text{fix}(\Pi_1)$ is the mapping defined in Definition 4.11. For this purpose, it suffices to show that $\tilde{\mathbf{p}}_1' = (\lambda P \in \mathcal{RV}.\ \mathcal{C}_1[\![P]\!])$ is the fixed-point of $\Pi_1$, i.e., that

$$\tilde{\mathbf{p}}_1' = \Pi_1(\tilde{\mathbf{p}}_1'). \tag{4.18}$$

This is shown as follows: Let $P \in \mathcal{RV}$. Then,

$$\begin{aligned}\tilde{\mathbf{p}}_1'(P) &= \mathcal{C}_1[\![P]\!] \quad \text{(by the definition of } \tilde{\mathbf{p}}_1'\text{)}\\ &= \mathcal{C}_1[\![g_P]\!] \quad \text{(by the definition of } \mathcal{C}_1\text{)}\\ &= [\![g_P]\!]^{\mathcal{I}_1 \cup \tilde{\mathbf{p}}_1'} \quad \text{(by Lemma 4.6 (2))}\\ &= \Pi_1(\tilde{\mathbf{p}}_1')(P) \quad \text{(by the definition of } \Pi_1\text{)}.\end{aligned}$$

Thus one has (4.18), and therefore, (4.17).

Using (4.17) and Lemma 4.6 (1), one can establish the following by induction on the structure of $s$:

$$\forall s \in \mathcal{L}_1[\emptyset][\ \mathcal{C}_1[\![s]\!] = \mathcal{M}_1[\![s]\!]\ ].\ \blacksquare$$

From Lemmas 4.3 and 4.8, the following proposition immediately follows:

**Proposition 4.1** (Relative Abstractness of $\mathcal{M}_1$ w.r.t. $\mathcal{O}_1$ and $\mathcal{O}_1^{\mathrm{m}}$)

**(1)** $\mathcal{O}_1 = \mathcal{A}_0^{\mathrm{l}} \circ \mathcal{M}_1$.

**(2)** $\mathcal{O}_1^{\mathrm{m}} = \mathcal{A}_0^{\mathrm{c}} \circ \mathcal{M}_1$. ∎

### 4.2.4 The Full Abstraction of $\mathcal{M}_1$ w.r.t. $\mathcal{O}_1$ and $\mathcal{O}_1^{\mathrm{m}}$

The full abstractness of $\mathcal{M}_1$ w.r.t. $\mathcal{O}_1$ and $\mathcal{O}_1^{\mathrm{m}}$ is established along the lines of the proof of a similar statement in [BKO 88]. First, the full abstractness of $\mathcal{M}_1$ w.r.t. $\mathcal{O}_1$ is established. Then, under the assumption that $\mathbf{C}_1$ is infinite, its full abstractness w.r.t. $\mathcal{O}_1^{\mathrm{m}}$ is also established.

As preliminaries to the proof, Lemmas 4.9–4.11 below are presented.

First, we observe that each statement of $\mathcal{L}_1$ is *sort-finite* at any given depth, which is formally stated by Lemma 4.9. For making precise this observation, we define the *sort-set* of a statement at a fixed depth by:

**Definition 4.12** For $s \in \mathcal{L}_1[\emptyset]$, and $n \in \omega$, let

$$\mathcal{S}_1^{(n)}(s) = \{\alpha \in \mathbf{A}_1^{\checkmark} : \exists w \in (\mathbf{A}_1)^n[\ s \xrightarrow{w}_{1*} \xrightarrow{\alpha}_1\ ]\}.$$

The set $\mathcal{S}_1^{(n)}(s)$ is called the *sort-set* of $s$ at depth $n$. ∎

Then, one has the following lemma:

**Lemma 4.9** *For $s \in \mathcal{L}_1[\emptyset]$, and $n \in \omega$, the sort-set $\mathcal{S}_1^{(n)}(s)$ is finite.* ∎

**Lemma 4.10** *For $s \in \mathcal{L}_1[\emptyset]$, $w \in (\mathbf{A}_1)^{<\omega}$, and $\Gamma \subseteq \mathbf{C}_1$, the following holds:*

$$w \cdot \langle \delta(\Gamma) \rangle \in \mathcal{M}_1[\![s]\!] \Leftrightarrow w \cdot \langle \delta(\Gamma \cup \{\surd\}) \rangle \in \mathcal{M}_1[\![s \,\|\, \mathbf{0}]\!].\ \blacksquare$$

**Lemma 4.11** *Let $s \in \mathcal{L}_1[\emptyset]$, $w \in (\mathbf{A}_1)^{<\omega}$, $C \subseteq \mathbf{C}_1$, and $\Gamma \subseteq \mathbf{C}_1^{\checkmark}$.
If $C \cap \mathrm{ran}(w) = \emptyset$, then*

$$w \cdot \langle \delta(\Gamma) \rangle \in \mathcal{M}_1[\![s]\!] \Leftrightarrow w \cdot \langle \delta(\Gamma \cup C) \rangle \in \mathcal{M}_1[\![\partial_C(s)]\!].\ \blacksquare$$

**Theorem 4.1** (Full Abstraction of $\mathcal{M}_1$ w.r.t. $\mathcal{O}_1$ and $\mathcal{O}_1^{\mathrm{m}}$)

**(1)** *The following holds for every* $s_1, s_2 \in \mathcal{L}_1[\emptyset]$:

$$\begin{aligned}&\mathcal{M}_1[\![s_1]\!] = \mathcal{M}_1[\![s_2]\!]\\ &\Leftrightarrow \forall X \in \mathcal{X}_{\mathcal{P}}, \forall S \in \mathcal{L}_1[X][\ \mathcal{O}_1[\![S_{\langle X\rangle}[s_1]]\!] = \mathcal{O}_1[\![S_{\langle X\rangle}[s_2]]\!]\ ].\end{aligned} \tag{4.19}$$

**(2)** *If* $\mathbf{C}_1$ *is infinite, then the following holds for every* $s_1, s_2 \in \mathcal{L}_1[\emptyset]$:

$$\begin{aligned}&\mathcal{M}_1[\![s_1]\!] = \mathcal{M}_1[\![s_2]\!] \Leftrightarrow\\ &\forall X \in \mathcal{X}_{\mathcal{P}}, \forall S \in \mathcal{L}_1[X][\ \mathcal{O}_1^{\mathrm{m}}[\![S_{\langle X\rangle}[s_1]]\!] = \mathcal{O}_1^{\mathrm{m}}[\![S_{\langle X\rangle}[s_2]]\!]\ ].\ \blacksquare\end{aligned} \tag{4.20}$$

**Proof.** **(1)** Let $s_1, s_2 \in \mathcal{L}_1[\emptyset]$. The $\Rightarrow$-part of (4.19) immediately follows from the relative abstractness of $\mathcal{M}_1$ w.r.t. $\mathcal{O}_1$ (Proposition 4.1) and the compositionality of $\mathcal{M}_1$ (Lemma 4.5) as follows: Let $\mathcal{M}_1[\![s_1]\!] = \mathcal{M}_1[\![s_2]\!]$, $X \in \mathcal{X}_{\mathcal{P}}$, and $S \in \mathcal{L}_1[X]$. Then

$$\begin{aligned}&\mathcal{O}_1[\![S_{\langle X\rangle}[s_1]]\!]\\ &= \mathcal{A}_0^{\mathrm{l}}(\mathcal{M}_1[\![S_{\langle X\rangle}[s_1]]\!]) \quad \text{(by Proposition 4.1)}\\ &= \mathcal{A}_0^{\mathrm{l}}([\![S_{\langle X\rangle}]\!]^{\tilde{\mathcal{I}}_1}(\mathcal{M}_1[\![s_1]\!])) \quad \text{(by Lemma 4.5)}\\ &= \mathcal{A}_0^{\mathrm{l}}([\![S_{\langle X\rangle}]\!]^{\tilde{\mathcal{I}}_1}(\mathcal{M}_1[\![s_2]\!])) \quad \text{(since } \mathcal{M}_1[\![s_1]\!] = \mathcal{M}_1[\![s_2]\!])\\ &= \mathcal{A}_0^{\mathrm{l}}(\mathcal{M}_1[\![S_{\langle X\rangle}[s_2]]\!]) \quad \text{(by Lemma 4.5)}\\ &= \mathcal{O}_1[\![S_{\langle X\rangle}[s_2]]\!] \quad \text{(by Proposition 4.1).}\end{aligned}$$

For establishing the $\Leftarrow$-part of (4.19), it suffices to show that if $\mathcal{M}_1[\![s_1]\!] \neq \mathcal{M}_1[\![s_2]\!]$, then

$$\exists X \in \mathcal{X}_{\mathcal{P}}, \exists S \in \mathcal{L}_1[X][\ \mathcal{O}_1[\![S_{\langle X\rangle}[s_1]]\!] \neq \mathcal{O}_1[\![S_{\langle X\rangle}[s_1]]\!]\ ]. \tag{4.21}$$

Suppose

$$\mathcal{M}_1[\![s_1]\!] \neq \mathcal{M}_1[\![s_2]\!]. \tag{4.22}$$

Let us show (4.21). By (4.22), one of the following propositions holds:

$$\begin{aligned}&\textbf{(i)}\ \tilde{\mathcal{T}}_1(\mathcal{M}_1[\![s_1]\!]) \neq \tilde{\mathcal{T}}_1(\mathcal{M}_1[\![s_2]\!]),\\ &\textbf{(ii)}\ \tilde{\mathcal{T}}_1^{\omega}(\mathcal{M}_1[\![s_1]\!]) \neq \tilde{\mathcal{T}}_1^{\omega}(\mathcal{M}_1[\![s_2]\!]),\\ &\textbf{(iii)}\ \tilde{\mathcal{F}}_1(\mathcal{M}_1[\![s_1]\!]) \neq \tilde{\mathcal{F}}_1(\mathcal{M}_1[\![s_2]\!]).\end{aligned} \tag{4.23}$$

When (4.23) (i) or (ii) holds, the desired consequence (4.21) immediately follows by putting $S \equiv X$.

Let us consider the case that (4.23) (iii) holds. We can assume without loss of generality that there exist $w \in (\mathbf{A}_1)^{<\omega}$, $\Gamma \in \wp(\mathbf{C}_1^{\checkmark})$ such that

$$\textbf{(i)}\ w \cdot \langle \delta(\Gamma)\rangle \in \mathcal{M}_1[\![s_1]\!], \quad \textbf{(ii)}\ w \cdot \langle \delta(\Gamma)\rangle \notin \mathcal{M}_1[\![s_2]\!]. \tag{4.24}$$

We distinguish two cases according to whether $\surd \in \Gamma$ or not.

**Case 1.** Suppose $\surd \in \Gamma$, and put $m = \mathrm{lgt}(w)$. Then, by Lemma 4.9, the set $\mathcal{S}_1^{(m)}(s_2)$ is finite. Let $\Gamma' = (\Gamma \cap \mathcal{S}_1^{(m)}(s_2)) \cup \{\surd\}$. Then $\Gamma'$ is finite and contains $\surd$, thus we can put $\Gamma' = \{\surd, c_1, \cdots, c_k\}$. By (4.24), one has

$$\textbf{(i) } w \cdot \langle \delta(\Gamma') \rangle \in \mathcal{M}_1[\![s_1]\!], \quad \textbf{(ii) } w \cdot \langle \delta(\Gamma') \rangle \notin \mathcal{M}_1[\![s_2]\!]. \tag{4.25}$$

Let $\hat{s} \equiv \tau^{m+1}$, and

$$\hat{s}' \equiv \mathbf{e} + (\overline{c_1}; \hat{s}) + \cdots + (\overline{c_k}; \hat{s}).$$

Let us show

$$\textbf{(i) } w \cdot \langle \delta \rangle \in \mathcal{O}_1[\![s_1 \parallel \hat{s}']\!], \quad \textbf{(ii) } w \cdot \langle \delta \rangle \notin \mathcal{O}_1[\![s_2 \parallel \hat{s}']\!]. \tag{4.26}$$

First, (4.26) (i) is shown straightforwardly as follows: By (4.25) (i), there is $s_1'$ such that

$$s_1 \xrightarrow{w}_{1*} s_1' \wedge \tau \notin \mathrm{act}_1(s_1') \wedge \mathrm{act}_1(s_1') \cap \Gamma' = \emptyset. \tag{4.27}$$

With this $s_1'$, one has

$$\begin{aligned} &(s_1 \parallel \hat{s}') \xrightarrow{w}_{1*} (s_1' \parallel \hat{s}') \\ &\wedge \mathrm{act}_1(s_1' \parallel \hat{s}') = \mathrm{act}_1(s_1') \cup (\mathrm{act}_1(\hat{s}') \setminus \{\surd\}) \subseteq \mathbf{C}_1. \end{aligned}$$

Thus, by the definition of $\mathcal{O}_1$, one has (4.26) (i).

Next, let us show (4.26) (ii) by contradiction. Assume, to the contrary, that

$$w \cdot \langle \delta \rangle \in \mathcal{O}_1[\![s_2 \parallel \hat{s}']\!]. \tag{4.28}$$

Then, there is $s'$ such that

$$\textbf{(i) } (s_2 \parallel \hat{s}') \xrightarrow{w}_{1*} s', \quad \textbf{(ii) } \mathrm{act}_1(s') \subseteq \mathbf{C}_1. \tag{4.29}$$

We observe that each action in $w$ must stem from either $s_2$ or $\hat{s}'$, or it must be the synchronization of two actions stemming from $s_2$ and $\hat{s}'$. However, it can be shown that every action in $w$ must stem from $s_2$ as follows: Assume, for the sake of contradiction, that some action of $w$ stems from $\hat{s}'$, or it is the synchronization of two actions stemming from $s_2$ and $\hat{s}'$. Then, by the form of $\hat{s}'$, there must be at at least $(m+1)$ $\tau$-actions in $w$, which contradicts the fact that $m = \mathrm{lgt}(w)$. Thus, every action in $w$ must stem from $s_2$, and therefore, there is $s_2'$ such that

$$\textbf{(i) } s_2 \xrightarrow{w}_{1*} s_2', \quad \textbf{(ii) } s' \equiv (s_2' \parallel \hat{s}'). \tag{4.30}$$

By (4.29) (ii), and (4.30) (ii), one has $\mathrm{act}_1(s_2' \parallel \hat{s}') \subseteq \mathbf{C}_1$. Thus, $\tau \notin \mathrm{act}_1(s_2')$ and

$$\mathrm{act}_1(s_2') \cap \{\surd, c_1, \cdots, c_k\} = \mathrm{act}_1(s_2') \cap \Gamma' = \emptyset.$$

From this and (4.30) (i), it follows that

$$w \cdot \langle \delta(\Gamma') \rangle \in \mathcal{M}_1[\![s_2]\!],$$

which contradicts (4.25) (ii). Summarizing, (4.28) does not hold, i.e., one has (4.26) (ii).

**Case 2.** Suppose $\surd \in \Gamma$. Applying Lemma 4.10, one obtains the following from (4.24):

$$w \cdot \langle \delta(\Gamma \cup \{\surd\}) \rangle \in \mathcal{M}_1[\![s_1 \,\|\, \mathbf{0}]\!] \wedge w \cdot \langle \delta(\Gamma \cup \{\surd\}) \rangle \notin \mathcal{M}_1[\![s_2 \,\|\, \mathbf{0}]\!].$$

Thus, this case is reduced to Case 1.

**(2)** Suppose $\mathbf{C}_1$ is infinite, and let $s_1, s_2 \in \mathcal{L}_1[\emptyset]$. As in part (1), the $\Rightarrow$-part of (4.20) follows from the relative abstractness of $\mathcal{M}_1$ w.r.t. $\mathcal{O}_1^{\mathrm{m}}$ and the compositionality of $\mathcal{M}_1$. For establishing the $\Leftarrow$-part of (4.20), it suffices to show that if $\mathcal{M}_1[\![s_1]\!] \neq \mathcal{M}_1[\![s_2]\!]$, then

$$\exists X \in \mathcal{X}_{\mathcal{P}}, \exists S \in \mathcal{L}_1[X][\; \mathcal{O}_1^{\mathrm{m}}[\![S_{\langle X \rangle}[s_1]]\!] \neq \mathcal{O}_1^{\mathrm{m}}[\![S_{\langle X \rangle}[s_1]]\!] \;]. \tag{4.31}$$

Suppose

$$\mathcal{M}_1[\![s_1]\!] \neq \mathcal{M}_1[\![s_2]\!]. \tag{4.32}$$

Let us show (4.32). As in part (1), one of the propositions (4.23) (i)–(iii) holds; when (4.23) (i) or (ii) holds, the desired consequence (4.31) follows immediately.

Let us consider the case that (4.23) (iii) holds. As in part (1), we can assume without loss of generality that there exist $w \in (\mathbf{A}_1)^{<\omega}$, $\Gamma \in \wp(\mathbf{C}_1^{\surd})$ satisfying (4.24) above. As in part (1), we distinguish two cases according to whether $\surd \in \Gamma$ or not.

**Case 1.** Suppose $\surd \in \Gamma$. Then, as in Case 1 in part (1), there exist $w \in (\mathbf{A}_1)^{<\omega}$ and a finite set $\Gamma' \subseteq \mathbf{C}_1^{\surd}$ such that $\surd \in \Gamma'$ and (4.25) above holds. Setting $\Gamma' = \{\surd, c_1, \cdots, c_k\}$ and $m = \mathrm{lgt}(w)$, let

$$w = \tau^{\ell(0)} \cdot c_0' \cdot \tau^{\ell(1)} \cdot c_1' \cdot \;\cdots\; \cdot \tau^{\ell(\nu-1)} \cdot c_{\nu-1}' \cdot \tau^{\ell(\nu)}.$$

Further, let

$$\begin{aligned} C_0 = (&\mathcal{S}_1^{(m)}(s_1) \cup \bigcup\{\mathcal{S}_1^{(i)}(s_2) : i \in (m + 2\cdot\nu)\} \\ &\cup\{c_1, \cdots, c_k\} \cup \{c_0', \cdots, c_{\nu-1}'\}). \end{aligned} \tag{4.33}$$

Then, $C_0$ is finite, and therefore, the set $\mathbf{C}_1 \setminus (C_0 \cup \overline{C_0})$ is nonempty under the assumption that $\mathbf{C}_1$ is infinite. Choosing an element $c$ of this set, let us define $\hat{s}'$, and $\hat{s}''$ as follows:

$$\hat{s}' \equiv \mathbf{e} + \overline{c_1} + \cdots + \overline{c_k},$$

$$\hat{s}'' \equiv (c; \overline{c_0'}; c); \cdots; (c; \overline{c_{\nu-1}'}; c); c; \hat{s}'.$$

Finally, let $C = \mathbf{C}_1 \setminus \{c, \bar{c}\}$. Putting

$$w' = (\tau^{\ell(0)} \cdot c \cdot \tau \cdot c) \cdot \;\cdots\; \cdot (\tau^{\ell(\nu-1)} \cdot c \cdot \tau \cdot c) \cdot (\tau^{\ell(\nu)} \cdot c), \tag{4.34}$$

let us show

$$\text{(i) } w' \cdot \langle \delta \rangle \in \mathcal{O}_1^{\mathrm{m}}[\![\partial_C(s_1 \,\|\, \hat{s}'')]\!], \quad \text{(ii) } w' \cdot \langle \delta \rangle \notin \mathcal{O}_1^{\mathrm{m}}[\![\partial_C(s_2 \,\|\, \hat{s}'')]\!]. \tag{4.35}$$

First, it follows from (4.24) (i) that there exists $s_1'$ satisfying (4.27) above. With this $s_1'$, one has

$$(s_1 \parallel \hat{s}'') \xrightarrow{w'}_{1*} (s_1' \parallel \hat{s}'). \tag{4.36}$$

We observe that

$$\mathrm{act}_1(s_1' \parallel \hat{s}') = (\mathrm{act}_1(s_1') \setminus \{\surd\}) \cup (\mathrm{act}_1(\hat{s}') \setminus \{\surd\}) \not\ni c, \bar{c}, \tau, \surd,$$

and therefore,

$$\mathrm{act}_1(s_1' \parallel \hat{s}') \subseteq \mathbf{C}_1 \setminus \{c, \bar{c}\} = C. \tag{4.37}$$

By (4.36) and the fact that $c \notin C$, one has

$$\partial_C(s_1 \parallel \hat{s}'') \xrightarrow{w'}_{1*} \partial_C(s_1' \parallel \hat{s}').$$

Also, one has $\mathrm{act}_1(\partial_C(s_1' \parallel \hat{s}')) = \emptyset$, by (4.37). Thus,

$$w' \cdot \langle \delta \rangle \in \mathcal{O}_1^{\mathrm{m}}[\![\partial_C(s_1 \parallel \hat{s}'')]\!],$$

i.e., one has (4.35) (i).

Next, let us show (4.35) (ii) by contradiction. Assume, to the contrary, that

$$w' \cdot \langle \delta \rangle \in \mathcal{O}_1^{\mathrm{m}}[\![\partial_C(s_2 \parallel \hat{s}'')]\!]. \tag{4.38}$$

Then, there exists $s'$ such that

$$\text{(i) } (s_2 \parallel \hat{s}'') \xrightarrow{w'} s', \quad \text{(ii) } \mathrm{act}_1(s') \subseteq C. \tag{4.39}$$

Since $c \in \mathbf{C}_1 \setminus (C_0 \cup \overline{C_0})$ with $C_0$ defined in (4.33), all the $c$-actions in $w'$ must stem from $\hat{s}''$, and therefore, the $i$-th $\tau$-action (in $w'$) preceded and followed by a $c$-action must be the synchronization of $c_i'$- and $\overline{c_i'}$- actions stemming from $s_2$ and $\hat{s}''$, respectively ($i \in \nu$). Moreover, all the other actions in $w'$, which consist of $(\ell(0) + \cdots + \ell(\nu))$ $\tau$-actions, must stem from $s_2$, since $\hat{s}''$ must perform exactly $3\nu$ actions until it perform the $(2\nu + 1)$-th $c$-action. Thus, the actions:

$$(\tau^{\ell(0)} \cdot c_0) \cdot \cdots \cdot (\tau^{\ell(\nu-1)} \cdot c_{\nu-1}) \cdot \tau^{\ell(\nu)} = w$$

(in $w'$) must stem from $s_2$, and therefore, there is $s_2'$ such that

$$\text{(i) } s_2 \xrightarrow{w}_{2*} s_2', \quad \text{(ii) } s' \equiv (s_2' \parallel \hat{s}'). \tag{4.40}$$

By (4.39) (ii) and (4.40) (ii), one has $\mathrm{act}_1(s_2' \parallel \hat{s}') \subseteq C$. Thus, $\tau \notin \mathrm{act}_1(s_2')$ and $\mathrm{act}_1(s_2') \cap \Gamma' = \emptyset$. From this and (4.40) (i), it follows that $w \cdot \langle \delta(\Gamma') \rangle \in \mathcal{M}_1[\![s_2]\!]$, which contradicts (4.25) (ii). Summarizing, (4.38) does not hold, i.e., one has (4.35) (ii).

**Case 2.** Consider the case $\surd \in \Gamma$. This case is reduced to Case 1, as in part (1). ■

### 4.2.5 An Internal Action Model

In [Rut 89], the argument for the full abstractness of $\mathcal{C}_1$ (or $\mathcal{M}_1$) w.r.t. the internal linear model $\mathcal{O}_1^{\mathrm{i}}$, claimed by Theorem 6.13 in [Rut 89], relies on Proposition 4.2 below (with some adaptation to the present setting); this, however, does not hold as Example 4.4 below exhibits.

The setting in [Rut 89] is slightly more general than that of this chapter. That is, in [Rut 89] the set $\mathbf{I}$ of *internal actions* is assumed to be given such that $\tau \in \mathbf{I}$ but it may contain other elements than $\tau$, whereas in this chapter we assume that the set of internal actions consists only of $\tau$. However, this difference is irrelevant to whether Proposition 4.2 holds or not.

As mentioned in § 4.1, Franck van Breugel found a way to overcome this error under an additional assumption that the $\mathbf{I}$ is infinite; under this assumption, the full abstractness of $\mathcal{M}_1$ w.r.t. $\mathcal{O}_1^{\mathrm{i}}$ can be established in a similar manner to the proof of Theorem 4.1 (see page 423 of [BR 92]).

**Proposition 4.2** (False) *Let* $s \in \mathcal{L}_1[\emptyset]$, $w = (a_0, \cdots, a_{n-1}) \in (\mathbf{A}_1)^{<\omega}$, *and* $\{a_{k(0)}, \cdots, a_{k(m-1)}\} = \{a_0, \cdots, a_{n-1}\} \cap \mathbf{C}_1$ *with* $k(0), \cdots, k(m-1) \in n$ *and* $k(0) < \cdots < k(m-1)$. *If*

$$\underbrace{(\tau, \cdots, \tau)}_{n} \cdot q \in \mathcal{O}_1^{\mathrm{i}}[\![s \| (\overline{a_{k(0)}}; \cdots; \overline{a_{k(m-1)}})]\!]$$

*with some* $q$, *then there exists* $s'_1, \cdots, s'_n \in \mathcal{L}_1[\emptyset]$ *such that*

$$s \xrightarrow{a_0} s'_1 \xrightarrow{a_1} \cdots \xrightarrow{a_{n-1}} s'_n. \blacksquare$$

**Example 4.4** *Put* $w = \langle c \rangle$ *with some* $c \in \mathbf{C}_1$, *and* $s \equiv \tau$. *Then*

$$\langle \tau \rangle \cdot q \in \mathcal{O}_1^{\mathrm{i}}[\![s \| (\bar{c}; \mathbf{0})]\!]$$

*with some* $q$, *but there is no* $s' \in \mathcal{L}_1[\emptyset]$ *such that* $s \xrightarrow{c}_1 s'$. $\blacksquare$

## 4.3 An Applicative Language $\mathcal{L}_2$ with Value-Passing

As described in [HI 90], there are two ways to extend $\mathcal{L}_1$ so as to include *value-passing*: One is to extend to an *applicative* language based on $\mathcal{L}_1$; the other is to extend to a *nonuniform* language also based on $\mathcal{L}_1$. This chapter presents an applicative language $\mathcal{L}_2$ based on $\mathcal{L}_1$; it is shown that a failures model $\mathcal{M}_2$ for $\mathcal{L}_2$ is also fully abstract w.r.t. two linear models $\mathcal{O}_2$ and $\mathcal{O}_2^{\mathrm{m}}$ for $\mathcal{L}_2$ (under an additional assumption w.r.t. $\mathcal{O}_2^{\mathrm{m}}$). The full abstractness result for $\mathcal{L}_2$ is established in essentially the same way as the proof of the corresponding result for $\mathcal{L}_1$, by defining $\mathcal{M}_2$ so that the following holds: For a term $(\lambda x.\ E_{(x)}) \in \mathcal{L}_2$,

$$\mathcal{M}_2[\![(\lambda x.\ E_{(x)})]\!] = (\lambda v \in \mathbf{V}.\ [\![E_{(x)}]\!]^{\mathcal{I}_2}(v)),$$

where $\mathcal{I}_2$ is the interpretation (a set of semantic operations) from which the model $\mathcal{M}_2$ is derived, and $[\![E_{(x)}]\!]^{\mathcal{I}_2}(v)$ denotes the interpretation of $E_{(x)}$ under $\mathcal{I}_2$ with the *assignment* associating $v$ with $x$.

### 4.3.1 The Language $\mathcal{L}_2$

For the definition of $\mathcal{L}_2$, the following two sets **Chan** and **V** are assumed to be given:

**(i)** a set of (possibly infinite) *communication channels* $(\mathbf{c} \in)$ **Chan**,

**(ii)** the set of (possibly infinite) *values* $(v \in)$ **V**.

It is also assumed that **V** contains a distinguished value **nil** standing for the logical value **false**.[1] Let $(x \in)\ \mathcal{X}_\mathcal{V}$ be the set of *value variables*, $(E \in)\ \tilde{\mathcal{E}}$ the set of *value expressions* which is predefined, and let

$$(e \in)\ \mathcal{E} = \{E \in \tilde{\mathcal{E}} : \mathrm{FV}(E) = \emptyset\}.$$

Further, an *evaluation function* $[\![\cdot]\!] : \mathcal{E} \to \mathbf{V}$ is assumed to be given, and it is convenient, for later purposes, to postulate that $\mathbf{V} \subseteq \mathcal{E}$.

Note that elements of $\mathcal{X}_\mathcal{V}$ have a very different nature from that of so-called *individual variables*, which are used in *nonuniform languages* to introduce *assignment statements* such as "$x := e$". These variables in *nonuniform languages* are used to store values, whereas elements of $\mathcal{X}_\mathcal{V}$ serve as parameters, e.g., as $x$ in "$(\lambda x.\ E)$".

Moreover, let $(Y \in)\ \mathcal{RV}^{(1)}$ be the set of *recursion variables* of type $\mathcal{P}^{(1)}$, which is the type of *parameterized statements* (with one parameter). Also, let $(\eta \in)\ \mathcal{X}_\mathcal{P}^{(1)}$ be the set of variables of type $\mathcal{P}^{(1)}$. We put $(\zeta \in)\ \mathcal{X}_\mathcal{P}^* = \mathcal{X}_\mathcal{P} \cup \mathcal{X}_\mathcal{P}^{(1)}$, and $\mathcal{X}^* = \mathcal{X}_\mathcal{V} \cup \mathcal{X}_\mathcal{P}^*$.

**Definition 4.13** (Language $\mathcal{L}_2$)

**(1)** The set of statements of the applicative concurrent language $(S \in)\ \mathcal{L}_2$ is defined simultaneously with the set of *parameterized statements* $(U \in)\ \mathcal{L}_2^{(1)}$, by the following BNF grammar:

$$\begin{cases} S ::= \ \mathbf{0} \mid \mathbf{e} \mid \tau \mid \mathbf{c}!(E) \mid \mathbf{c}?(T) \mid (S_1; S_2) \mid (S_1 + S_2) \mid (S_1 \parallel S_2) \mid \\ \qquad\ \ \partial_C(S) \mid \mathbf{if}(E, S_1, S_2) \mid T(E) \mid P \mid X, \\ T ::= (\lambda x.\ S) \mid Y \mid \eta, \end{cases}$$

where $P$ and $X$ are as in Definition 4.1; $Y$ (resp. $\eta$) ranges over $\mathcal{RV}^{(1)}$ (resp. $\mathcal{X}_\mathcal{P}^{(1)}$), $\mathbf{c}$ ranges over **Chan**, and $C$ (resp. $E$) ranges over $\wp(\mathbf{Chan})$ (resp. $\tilde{\mathcal{E}}$). The constructs $\mathbf{c}!(E)$ and $\mathbf{c}?(T)$ represent *output* and *input* through the channel $\mathbf{c}$; the constants $\mathbf{0}$, $\mathbf{e}$ and the combinators ;, +, $\parallel$, $\partial_C(\cdot)$ are as in $\mathcal{L}_1$; $T(E)$ is the *application* of a *parameterized process* $T$ to an actual parameter $E$.

Let

[1] It is possible to introduce *Boolean* expressions, instead of introducing the value **nil** for defining conditional statements. The present approach of introducing **nil**, as in Lisp and the C language, is adopted for simplifying the semantic definitions in § 4.3.3.

$$(U \in)\ \mathcal{L}_2^* = \mathcal{L}_2 \cup \mathcal{L}_2^{(1)}.$$

**(2)** Let

$$(Z \in)\ \mathcal{RV}^* = \mathcal{RV} \cup \mathcal{RV}^{(1)}.$$

We regard elements of $\mathcal{RV}^*$ as *constant symbols* rather than as *variables.* The construct "$(\lambda x.\ \cdots)$" has the usual binding property. For $U \in \mathcal{L}_2^*$, let $\mathrm{FV}(U)$ be the set of elements of $\mathcal{X}^*$ which have a free occurrence in $U$.

Then the set $\mathcal{L}_2[\emptyset]$ of *statements* (resp. the set $\mathcal{L}_2^{(1)}[\emptyset]$ of *parameterized statements*) in $\mathcal{L}_2$ are defined as the *closed* elements of $\mathcal{L}_2$ (resp. $\mathcal{L}_2^{(1)}$) by:

$$(s \in)\ \mathcal{L}_2 = \{S \in \mathcal{L}_2 : \mathrm{FV}(S) = \emptyset\},$$

$$(t \in)\ \mathcal{L}_2^{(1)} = \{T \in \mathcal{L}_2^{(1)} : \mathrm{FV}(T) = \emptyset\}.$$

Let

$$(u \in)\ \mathcal{L}_2^* = \mathcal{L}_2 \cup \mathcal{L}_2^{(1)}.$$

**(3)** The set of *guarded statements* $(G \in)\ \mathcal{G}_2$ and the set of *guarded parameterized statements* $(H \in)\ \mathcal{G}_2^{(1)}$ are defined as follows:

$$\begin{cases} G ::= \mathbf{0} \mid \tau \mid \mathbf{c}!(E) \mid \mathbf{c}?(T) \\ \qquad \mid (G;S) \mid (G_1 + G_2) \mid (G_1 \parallel G_2) \\ \qquad \mid \partial_C(G) \mid \mathbf{if}(E, G_1, G_2) \mid H(E), \\ H ::= (\lambda x.\ G). \end{cases}$$

Then let

$$(g \in)\ \mathcal{G}_2 = \{G \in \mathcal{G}_2 : \mathrm{FV}(G) = \emptyset\},$$

$$(h \in)\ \mathcal{G}_2^{(1)} = \{H \in \mathcal{G}_2^{(1)} : \mathrm{FV}(H) = \emptyset\}.$$

**(4)** Let

$$(\tilde{g} \in)\ \mathcal{G}_2^* = \mathcal{G}_2 \cup \mathcal{G}_2^{(1)}.$$

We assume that each recursion variable $Z \in \mathcal{RV}^*$ are associated with an element of $\mathcal{G}_2^*$, by a set of declarations:

$$D = \{(Z, \tilde{g}_Z) : Z \in \mathcal{RV}^*\},$$

where we assume that $D$ *respects types* in that

$$\begin{aligned} &\forall Z \in \mathcal{RV}^*[\ (Z \in \mathcal{RV} \Rightarrow \tilde{g}_Z \in \mathcal{G}_2[\emptyset]) \\ &\quad \wedge (Z \in \mathcal{RV}^{(1)} \Rightarrow \tilde{g}_Z \in \mathcal{G}_2^{(1)}[\emptyset])\ ]. \blacksquare \end{aligned}$$

We give an example program in $\mathcal{L}_2$ in Example 4.5 below; the program is very simple but makes an essential use of the parameterization feature of $\mathcal{L}_2$:

**Example 4.5** A description of a process `Accumulating-Responder` in $\mathcal{L}_2$ is given in Figure 4.3. Let $N$ be a fixed number representing a limit of accumulation of `Accumulating-Responder`. In the $n$-th step ($n \in \omega$), the program inputs $i_n$ through the channel $\mathbf{c}_0$, and outputs the value $\sum_{j\in[0..n]}[i_j]$ through the channel $\mathbf{c}_1$ on the input $i_n$, so long as $\sum_{j\in[0..n]}[i_j] \leq N$; otherwise it outputs 0 though $\mathbf{c}_2$ and restarts. ∎

In the sequel of this section, we fix a declaration set $D = \{(Z, \tilde{g}_Z) : Z \in \mathcal{RV}^*\}$.

We can characterize $\mathcal{L}_2$ and $\mathcal{L}_2^{(1)}$ as the set of terms of a certain type, generated by a *signature* $\mathbf{Sig}_2^*$ in the sense of the standard typed $\lambda$-calculus (cf. [Mit 90]):

**Definition 4.14** The signature $\mathbf{Sig}_2^*$ is defined by:

$$\mathbf{Sig}_2^* = (\{\mathcal{V}, \mathcal{P}\}, \mathbf{Fun}_2^*, \mathrm{type}_2)$$

with $\mathcal{V}$ and $\mathcal{P}$ are the *base types* of *values* and *statements*, respectively, $\mathbf{Fun}_2^*$ being a set of function symbols, and 'type$_2$' a function assigning a type to each function symbol. The set of types of $\mathbf{Sig}_2^*$ is $\{\mathcal{V}, \mathcal{P}, \mathcal{P}^{(1)}\}$ with $\mathcal{P}^{(1)}$ standing for $(\mathcal{V} \to \mathbf{P})$, and for each $F \in \mathbf{Fun}_2^*$, $\mathrm{type}_2(F) = \langle \mathcal{V}^k \cdot \mathcal{P}^\ell \cdot (\mathcal{P}^{(1)})^m, \mathcal{S}\rangle$ with $k, \ell, m \in \omega$ and $\mathcal{S} \in \{\mathcal{P}, \mathcal{P}^{(1)}\}$. For $k, \ell, m \in \omega$ and $\mathcal{S} \in \{\mathcal{P}, \mathcal{P}^{(1)}\}$ let

$$(\mathbf{Fun}_2^*)^{((k,\ell,m),\mathcal{S})} = \mathrm{type}_2^{-1}[\{(\mathcal{V}^k \cdot \mathcal{P}^\ell \cdot (\mathcal{P}^{(1)})^m, \mathcal{S})\}].$$

Let $(\tilde{X} \in)\ \mathcal{X} = \mathcal{X}_\mathcal{V} \cup \mathcal{X}_\mathcal{P}^*$. We define $\mathbf{Sig}_2^*$ as follows:

$$\mathbf{Fig}_{2*}^{((0,0,0),\mathcal{P})} = \{\mathbf{e}, \mathbf{0}, \tau\} \cup \mathcal{RV};$$

$$\mathbf{Fig}_{2*}^{((1,0,0),\mathcal{P})} = \{\mathbf{c}!(\cdot) : \mathbf{c} \in \mathbf{Chan}\};$$

$$\mathbf{Fig}_{2*}^{((0,0,1),\mathcal{P})} = \{\mathbf{c}?(\cdot) : \mathbf{c} \in \mathbf{Chan}\};$$

$$\mathbf{Fig}_{2*}^{((0,1,0),\mathcal{P})} = \{\partial_C(\cdot) : C \subseteq \mathbf{Chan}\};$$

$$\mathbf{Fig}_{2*}^{((0,2,0),\mathcal{P})} = \{;, +, \|\};$$

$$\mathbf{Fig}_{2*}^{((1,2,0),\mathcal{P})} = \{\mathbf{if}(\cdot,\cdot,\cdot)\};$$

$$\mathbf{Fig}_{2*}^{((0,0,1),\mathcal{P}^{(1)})} = \mathcal{RV}^{(1)};$$

$$\mathbf{Fig}_{2*}^{((k,\ell,m),\mathcal{S})} = \emptyset \quad \text{for other indexes } ((k,\ell,m),\mathcal{S}).$$

Let $\mathbf{Sig}_2$ be the result of removing $\mathcal{RV}^*$ from $\mathbf{Fun}_2^*$, i.e., let

$$\mathbf{Fun}_2 = (\mathbf{Fun}_2^* \setminus \mathcal{RV}^*), \quad \mathbf{Sig}_2 = (\{\mathcal{V}, \mathcal{P}\}, \mathbf{Fun}_2, \mathrm{type}_2). \blacksquare$$

Having defined $\mathbf{Sig}_2^*$, it is easy to check that the sets $\mathcal{L}_2$ and $\mathcal{L}_2^{(1)}$ can be characterized as the sets of terms generated by $\mathbf{Sig}_2^*$, $\mathcal{X}^*$, and $\tilde{\mathcal{E}}$, of type $\mathcal{P}$ and $\mathcal{P}^{(1)}$, respectively.

### 4.3.2 Three Operational Models $\mathcal{O}_2$, $\mathcal{O}_2^{\mathrm{m}}$, $\mathcal{C}_2$ for $\mathcal{L}_2$

Three operational models $\mathcal{O}_2$, $\mathcal{O}_2^{\mathrm{m}}$, and $\mathcal{C}_2$ for $\mathcal{L}_2$ are defined in terms of a transition system $(\mathcal{L}_2[\emptyset], \langle \xrightarrow{\alpha}_2 : \alpha \in \mathbf{A}_2^{\surd} \rangle)$, where $\mathbf{A}_2^{\surd}$ is the set of actions of $\mathcal{L}_2$ defined below. The following definition is given as a preliminary to the definition of $\xrightarrow{\alpha}_2$ $(\alpha \in \mathbf{A}_2^{\surd})$.

**Definition 4.15** (Actions for $\mathcal{L}_2$)

**(1)** For $C \subseteq \mathbf{Chan}$, let

$$C! = \{\mathbf{c}! : \mathbf{c} \in C\}, \quad C? = \{\mathbf{c}? : \mathbf{c} \in C\},$$

and

$$C!? = C! \cup C?.$$

The set $(\gamma \in)$ $\mathbf{C}_2$ of *communication sorts* is given by:

$$\mathbf{C}_2 = \mathbf{Chan}!?.$$

For $\gamma \in \wp(\mathbf{C}_2)$, let $\bar{\gamma} = \mathbf{c}?$ if $\gamma = \mathbf{c}!$ for some $\mathbf{c} \in \mathbf{Chan}$; otherwise $\gamma = \mathbf{c}?$ for some $\mathbf{c} \in \mathbf{Chan}$, and let $\bar{\gamma} = \mathbf{c}!$. For $\Gamma \subseteq \mathbf{C}_2$, let $\bar{C} = \{\bar{\gamma} : \gamma \in \Gamma\}$. Let $\mathbf{C}_2^{\surd} = \mathbf{C}_2 \cup \{\surd\}$.

**(2)** The set of *actions*, $(a \in)$ $\mathbf{A}_2$, is given by

$$\mathbf{A}_2 = (\mathbf{C}_2 \times \mathbf{V}) \cup \{\tau\}.$$

Let

$$\mathbf{A}_2^{\surd} = \mathbf{A}_2 \cup \{\surd\}.$$

In this section, let the variable $c$ range over $\mathbf{C}_2 \times \mathbf{V}$, the set of *communication actions*. For $c = \langle \gamma, v \rangle \in \mathbf{C}_2 \times \mathbf{V}$, let $\bar{c} = \langle \bar{\gamma}, v \rangle$.

**(3)** The set of *action sorts*, $(A \in)$ $\mathbf{ASort}$, is given by:

$$\mathbf{ASort} = \mathbf{C}_2^{\surd} \cup \{\tau\}.$$

**(4)** The function sort : $\mathbf{A}_2^{\surd} \to \mathbf{ASort}$ is defined as follows: For $\alpha \in \mathbf{A}_2^{\surd}$,

$$\mathrm{sort}(\alpha) = \begin{cases} \gamma & \text{if } \exists \gamma, \exists v[\ \alpha = \langle \gamma, v \rangle\ ], \\ \alpha & \text{otherwise.} \end{cases}$$

**(5)** For $A \in \mathbf{ASort}$, let chan$(A)$ be the unique $\mathbf{c} \in \mathbf{Chan}$ such that $\gamma = \mathbf{c}!$ or $\gamma = \mathbf{c}?$ if $A \in \mathbf{C}_2$; otherwise $A \in \{\tau, \surd\}$ and let chan$(A) = A$. ∎

Next, let us define the transition relations $\xrightarrow{a}_2 \subseteq \mathcal{L}_2[\emptyset] \times \mathcal{L}_2[\emptyset]$ $(a \in \mathbf{A}_2)$ and $\xrightarrow{\surd}_2 \subseteq \mathcal{L}_2[\emptyset]$. For $s_1, s_2 \in \mathcal{L}_2[\emptyset]$ and $a \in \mathbf{A}_2$, we write $s_1 \xrightarrow{a}_2 s_2$ (resp. $s_1 \xrightarrow{\surd}_2$) for $\langle s_1, s_2 \rangle \in \xrightarrow{a}_2$ (resp. for $s_1 \in \xrightarrow{\surd}_2$), as usual. For $\mathbf{c}!, \mathbf{c}? \in \mathbf{C}_2$ and $v \in \mathbf{V}$, we sometimes write $\mathbf{c}!v$ and $\mathbf{c}?v$ for $(\mathbf{c}!, v)$ and $(\mathbf{c}?, v)$, respectively.

**Definition 4.16** (Transition Relations $\xrightarrow{\alpha}_2$) The transition relations $\xrightarrow{\alpha}_2$ ($\alpha \in \mathbf{A}_2^{\checkmark}$) are defined as the smallest relations satisfying the following rules (1)–(10.2):

**(1.1)**

$$\tau \xrightarrow{\tau}_2 \mathbf{e}.$$

**(1.2)**

$$\mathbf{c}!(e) \xrightarrow{\mathbf{c}![\![e]\!]}_2 \mathbf{e}.$$

**(1.3)**

$$\mathbf{c}?(t) \xrightarrow{\mathbf{c}?v}_2 t(v) \quad (v \in \mathbf{V}).$$

**(2)**

$$\mathbf{e} \xrightarrow{\checkmark}_2 .$$

**(3.1)**

$$\frac{s_1 \xrightarrow{a}_2 s_1'}{(s_1; s_2) \xrightarrow{a}_2 (s_1'; s_2).} \quad (a \in \mathbf{A}_2).$$

**(3.2)**

$$\frac{s_1 \xrightarrow{\checkmark}_2,\ s_2 \xrightarrow{\alpha}_2 \left(s_2'\right)}{(s_1; s_2) \xrightarrow{\alpha}_2 \left(s_2'\right)} \quad (\alpha \in \mathbf{A}_2^{\checkmark}).$$

**(4)**

$$\frac{s_1 \xrightarrow{\alpha}_2 \left(s_1'\right)}{\begin{array}{l}(s_1 + s_2) \xrightarrow{\alpha}_2 \left(s_1'\right)\\ (s_2 + s_1) \xrightarrow{\alpha}_2 \left(s_1'\right)\end{array}} \quad (\alpha \in \mathbf{A}_2^{\checkmark}).$$

**(5.1)**

$$\frac{s_1 \xrightarrow{a}_2 s_1'}{\begin{array}{l}(s_1 \parallel s_2) \xrightarrow{a}_2 (s_1' \parallel s_2)\\ (s_2 \parallel s_1) \xrightarrow{a}_2 (s_2 \parallel s_1')\end{array}} \quad (a \in \mathbf{A}_2).$$

**(5.2)**

$$\frac{s_1 \xrightarrow{c}_2 s_1',\ s_2 \xrightarrow{\bar{c}}_2 s_2'}{(s_1 \parallel s_2) \xrightarrow{\tau}_2 (s_1' \parallel s_2')} \quad (c \in \mathbf{C}_2).$$

**(5.3)**

$$\frac{s_1 \xrightarrow{\checkmark}_2, s_2 \xrightarrow{\checkmark}_2}{(s_1 \parallel s_2) \xrightarrow{\checkmark}_2}.$$

**(6)**

$$\frac{s \xrightarrow{\alpha}_2 \left(s'\right)}{\partial_C(s) \xrightarrow{\alpha}_2 \left(\partial_C(s')\right)} \quad (\alpha \in (\mathbf{A}_2^{\checkmark}) \setminus (C!? \times \mathbf{V}))$$

**(7.1)**

$$\frac{s_1 \xrightarrow{\alpha}_2 \left(s\right)}{\mathbf{if}(e, s_1, s_2) \xrightarrow{\alpha}_2 \left(s\right)} \quad \text{if } [\![e]\!] \neq \mathbf{nil}.$$

**(7.2)**

$$\frac{s_2 \xrightarrow{\alpha}_2 \left(s\right)}{\mathbf{if}(e, s_1, s_2) \xrightarrow{\alpha}_2 \left(s\right)} \quad \text{if } [\![e]\!](\sigma) = \mathbf{nil}.$$

**(8)** The following rule is called the *pre-evaluation rule*:

$$\frac{u([\![e]\!]) \xrightarrow{\alpha}_2 \left(s'\right)}{u(e) \xrightarrow{\alpha}_2 \left(s'\right)} \quad (e \in \mathcal{E}).$$

**(9)** The following rule is called the *λ-rule*:

$$\frac{S[v/x] \xrightarrow{\alpha}_2 \left(s'\right)}{(\lambda x.\ S)(v) \xrightarrow{\alpha}_2 \left(s'\right)} \quad (v \in \mathbf{V}).$$

**(10.1)** For each $(Z, \tilde{g}_Z) \in D$ with $Z \in \mathcal{RV}$:

$$\frac{\tilde{g}_Z \xrightarrow{\alpha}_2 \left(s'\right)}{Z \xrightarrow{\alpha}_2 \left(s\right)'}.$$

**(10.2)** For each $(Z, \tilde{g}_Z) \in D$ with $Z \in \mathcal{RV}^{(1)}$:

$$\frac{\tilde{g}_Z(v) \xrightarrow{\alpha}_2 \left(s'\right)}{Z(v) \xrightarrow{\alpha}_2 \left(s'\right)} \quad (v \in \mathbf{V}).$$

The last two rules (10.1) and (10.2) are called the first and second *recursion rules.* ∎

**Definition 4.17** **(1)** For each $w \in (\mathbf{A}_2)^{<\omega}$, the binary relation $\xrightarrow{w}_{2*}$ is defined as in Definition 4.4.

**(2)** Also, for $a \in \mathbf{A}_2$, $w \in (\mathbf{A}_2)^{<\omega}$, $w' \in (\mathbf{A}_2)^{\omega}$, unary relations $\xrightarrow{a}_2$, $\xrightarrow{w}_{2*}$, $\xrightarrow{w'}_{2\omega}$ are defined as in Definition 4.4.

**(3)** Let $\mathrm{act}_2 : \mathcal{L}_2[\emptyset] \to \wp(\mathbf{A}_2^{\checkmark})$ be defined as follows: For $s \in \mathcal{L}_2[\emptyset]$,

$$\mathrm{act}_2(s) = \{\alpha \in \mathbf{A}_2^{\checkmark} : s \xrightarrow{\alpha}_2\}. \blacksquare$$

The transition system $(\mathcal{L}_2[\emptyset], \langle \xrightarrow{\alpha}_2 : \alpha \in \mathbf{A}_2^{\surd} \rangle)$ is *image-finite* in the sense of part (1) of the following lemma:

**Lemma 4.12** *For every $s \in \mathcal{L}_2[\emptyset]$, the following hold:*

**(1)** *For every $a \in \mathbf{A}_2$, the* image set $\{s' : s \xrightarrow{a}_2 s'\}$ *is finite.*

**(2)** sort[act$_2(s)$] *is finite.*

**(3)** *For every $\mathbf{c} \in \mathbf{Chan}$, the set $\{v \in \mathbf{V} : (\mathbf{c}!, v) \in \mathrm{act}_2(s)\}$ is finite.* ∎

**Definition 4.18** (Two Linear Models $\mathcal{O}_2$ and $\mathcal{O}_2^{\mathrm{m}}$ for $\mathcal{L}_2$)

**(1)** Let

$$(\rho \in)\ \mathbf{B}_2^{\mathrm{l}} = (\mathbf{A}_2)^{\omega} \cup (\mathbf{A}_2)^{<\omega} \cdot \{\langle \surd \rangle\} \cup (\mathbf{A}_2)^{<\omega} \cdot \{\langle \delta \rangle\}.$$

**(2)** The function $\mathcal{O}_2 : \mathcal{L}_2[\emptyset] \to \wp(\mathbf{B}_2^{\mathrm{l}})$ is defined as follows: Let $s \in \mathcal{L}_2[\emptyset]$. For $\rho \in \mathbf{B}_2^{\mathrm{l}}$, we put $\rho \in \mathcal{O}_2[\![s]\!]$ iff one of the following conditions (4.41), (4.42), (4.43) holds:

$$\begin{array}{l} \exists \langle s_i \rangle_{i\in\omega} \in (\mathcal{L}_2)^{\omega}, \exists \langle a_i \rangle_{i\in\omega} \in (\mathbf{A}_2)^{\omega}[ \\ \quad \rho = \langle a_i \rangle_{i\in\omega} \wedge s_0 \equiv s \wedge \forall i \in \omega[\ s_i \xrightarrow{a_i}_2 s_{i+1}\ ]]. \end{array} \tag{4.41}$$

$$\begin{array}{l} \exists n \in \omega, \exists \langle s_i \rangle_{i\in(n+1)} \in (\mathcal{L}_2)^{n+1}, \exists \langle a_i \rangle_{i\in n} \in (\mathbf{A}_2)^{n}[ \\ \quad \rho = \langle a_i \rangle_{i\in n} \cdot \langle \surd \rangle \wedge s_0 \equiv s \wedge \forall i \in n[\ s_i \xrightarrow{a_i}_2 s_{i+1}\ ] \\ \quad \wedge \surd \in \mathrm{act}_2(s_n)\ ]. \end{array} \tag{4.42}$$

$$\begin{array}{l} \exists n \in \omega, \exists \langle s_i \rangle_{i\in(n+1)} \in (\mathcal{L}_2)^{n+1}, \exists \langle a_i \rangle_{i\in n} \in (\mathbf{A}_2)^{n}[ \\ \quad \rho = \langle a_i \rangle_{i\in n} \cdot \langle \delta \rangle \wedge s_0 \equiv s \wedge \forall i \in n[\ s_i \xrightarrow{a_i}_2 s_{i+1}\ ] \\ \quad \wedge \mathrm{sort}[\mathrm{act}_2(s_n)] \subseteq \mathbf{C}_2\ ]. \end{array} \tag{4.43}$$

**(3)** The function $\mathcal{O}_2^{\mathrm{m}} : \mathcal{L}_2[\emptyset] \to \wp(\mathbf{B}_2^{\mathrm{l}})$ is defined as follows: Let $s \in \mathcal{L}_2[\emptyset]$. For $\rho \in \mathbf{B}_2^{\mathrm{l}}$, we put $\rho \in \mathcal{O}_2^{\mathrm{m}}[\![s]\!]$ iff one of the propositions (4.41), (4.42) above, and (4.44) below holds.

$$\begin{array}{l} \exists n \in \omega, \exists \langle s_i \rangle_{i\in(n+1)} \in (\mathcal{L}_2)^{(n+1)}, \exists \langle a_i \rangle_{i\in n} \in (\mathbf{A}_2)^{n}[ \\ \quad \rho = \langle a_i \rangle_{i\in n} \cdot \langle \delta \rangle \wedge s_0 \equiv s \wedge \forall i \in n[\ s_i \xrightarrow{a_i}_2 s_{i+1}\ ] \\ \quad \wedge \mathrm{act}_2(s_n) = \emptyset\ ]. \end{array} \tag{4.44}$$
∎

**Example 4.6** Let $\mathbf{c}_0, \mathbf{c}_1, \mathbf{c}_2 \in \mathbf{Chan}$, and put

$$s \equiv P + \mathbf{c}_0?(\lambda x_0.\ \mathbf{c}_1!(x_0 + 1, (\tau; \mathbf{0}) + \mathbf{c}_2!(x_0 + 2, \mathrm{e}))),$$

where it is assumed that $(P, (\tau; P)) \in D$. Let $\mathbf{V} = \{v_i : i \in \omega\}$ with $v_i \neq v_j$ for distinct $i$ and $j$. Then, $s$ is depicted by:

$$s = \bullet \begin{cases} \xrightarrow{\tau} \bullet \xrightarrow{\tau} \bullet \xrightarrow{\tau} \bullet \xrightarrow{\tau} \bullet \xrightarrow{\tau} \bullet \cdots \\ \xrightarrow{\mathbf{c}_0?v_0} \bullet \xrightarrow{\mathbf{c}_1!(v_0+1)} \bullet \begin{cases} \xrightarrow{\tau} \bullet \\ \xrightarrow{\mathbf{c}_2!(v_0+2)} \circ \end{cases} \\ \xrightarrow{\mathbf{c}_0?v_1} \bullet \xrightarrow{\mathbf{c}_1!(v_1+1)} \bullet \begin{cases} \xrightarrow{\tau} \bullet \\ \xrightarrow{\mathbf{c}_2!(v_1+2)} \circ \end{cases} \\ \vdots \end{cases}$$

By the definitions of $\mathcal{O}_2$ and $\mathcal{O}_2^{\mathrm{m}}$, one has

$$\begin{aligned} \mathcal{O}_2[\![s]\!] = \{\tau^\omega\} \cup \{\langle \mathbf{c}_0?v,\ \delta\ \rangle : v \in \mathbf{V}\} \cup \{\langle \mathbf{c}_0?v, \mathbf{c}_1!(v+1), \tau,\ \delta\ \rangle : v \in \mathbf{V}\} \\ \cup\{\langle \mathbf{c}_0?v, \mathbf{c}_1!(v+1), \mathbf{c}_2!(v+2), \surd\rangle : v \in \mathbf{V}\}, \end{aligned}$$

$$\begin{aligned} \mathcal{O}_2^{\mathrm{m}}[\![s]\!] = \{\tau^\omega\} \cup \{\langle \mathbf{c}_0?v, \mathbf{c}_1!(v+1), \tau,\ \delta\ \rangle : v \in \mathbf{V}\} \\ \cup\{\langle \mathbf{c}_0?v, \mathbf{c}_1!(v+1), \mathbf{c}_2!(v+2), \surd\rangle : v \in \mathbf{V}\}. \blacksquare \end{aligned}$$

Next, as for $\mathcal{L}_1$, a *failures model* $\mathcal{C}_2$ is defined in two ways (operationally and denotationally). The major difference between $\mathcal{C}_2$ and $\mathcal{C}_1$ is that the the meaning of a statement in $\mathcal{C}_2$ consists of elements like $w_1 \cdot \langle\delta(\Gamma_1)\rangle$ with $w_1$ a sequence of actions and $\Gamma_1$ a set of *action sorts*, while the one in $\mathcal{C}_1$ consists of elements like $w_0 \cdot \langle\delta(\Gamma_0)\rangle$ with $w_0$ like $w_1$ and $\Gamma_0$ a set of *actions*. First, an operational definition of the failures model is given by:

**Definition 4.19** (Failures Model $\mathcal{C}_2$ for $\mathcal{L}_2$)

**(1)** Let

$$\mathbf{Q}_2 = (\mathbf{A}_2)^\omega \cup ((\mathbf{A}_2)^{<\omega} \cdot \{\langle\surd\rangle\}) \cup ((\mathbf{A}_2)^{<\omega} \cdot \{\langle\delta(\Gamma)\rangle : \Gamma \subseteq \mathbf{C}_2^{\surd}\}).$$

**(2)** The function $\mathcal{C}_2 : \mathcal{L}_2^*[\emptyset] \to (\wp(\mathbf{Q}_2) \cup (\mathbf{V} \to \wp(\mathbf{Q}_2)))$ is defined in the following two stages:

**(i)** First, for $s \in \mathcal{L}_2[\emptyset]$, $\mathcal{C}_2[\![s]\!] \in \wp(\mathbf{Q}_2)$ is defined as follows: For $\rho \in \mathbf{Q}_2$, we put $\rho \in \mathcal{C}_2[\![s]\!]$ iff one of the conditions (4.41), (4.42) above, and (4.45) below holds.

$$\begin{aligned} &\exists n \in \omega, \exists\langle s_i\rangle_{i\in(n+1)}, \exists\langle a_i\rangle_{i\in n}, \exists\Gamma \in \wp(\mathbf{C}_2^{\surd})[ \\ &\quad \rho = \langle a_i\rangle_{i\in n} \cdot \langle\delta(\Gamma)\rangle \wedge s_0 \equiv s \wedge \forall i \in n[\ s_i \xrightarrow{a_i}_2 s_{i+1}\ ] \\ &\quad \wedge \tau \notin \mathrm{act}_2(s_n) \wedge \Gamma \cap \mathrm{sort}[\mathrm{act}_2(s_n)] = \emptyset\ ]. \end{aligned} \tag{4.45}$$

**(ii)** Having defined $\mathcal{C}_2[\![s]\!]$ for $s \in \mathcal{L}_2[\emptyset]$, $\mathcal{C}_2[\![t]\!] \in (\mathbf{V} \to \wp(\mathbf{Q}_2))$ for $t \in \mathcal{L}_2^{(1)}[\emptyset]$ is defined by:

$$\mathcal{C}_2[\![t]\!] = (\lambda v \in \mathbf{V}.\ \mathcal{C}_2[\![t(v)]\!]). \blacksquare$$

**Example 4.7** Let $s$ be put as in Example 4.6. Then, by the definition of $\mathcal{C}_2$, one has

$$
\begin{aligned}
\mathcal{C}_2[\![s]\!] = \{\tau^\omega\} \\
&\cup\{\langle \mathbf{c}_0?v, \delta(\Gamma)\rangle : v \in \mathbf{V} \wedge \Gamma \in \wp(\mathbf{C}_2^{\surd}) \wedge \mathbf{c}_1! \notin \Gamma\} \\
&\cup\{\langle \mathbf{c}_0?v, \mathbf{c}_1!(v+1), \tau, \delta(\Gamma)\rangle : v \in \mathbf{V} \wedge \Gamma \in \wp(\mathbf{C}_2^{\surd})\} \\
&\cup\{\langle \mathbf{c}_0?v, \mathbf{c}_1!(v+1), \mathbf{c}_2!(v+2), \surd\rangle : v \in \mathbf{V} \wedge \Gamma \in \wp(\mathbf{C}_2^{\surd})\} \\
&\cup\{\langle \mathbf{c}_0?v, \mathbf{c}_1!(v+1), \mathbf{c}_2!(v+2), \delta(\Gamma)\rangle : v \in \mathbf{V} \wedge \Gamma \in \wp(\mathbf{C}_2)\}. \blacksquare
\end{aligned}
$$

As for $\mathcal{C}_1$, the model $\mathcal{C}_2$ can also be characterized as the fixed-point of a higher-order mapping $\Phi_2^{\mathrm{f}}$:

**Definition 4.20** (Fixed-Point Formulation of Failures Model for $\mathcal{L}_2$)

**(1)** By defining a metric $d_{\mathbf{Q}_2}$ on $\mathbf{Q}_2$ in terms of *truncations* as usual, it is easily shown that $(\mathbf{Q}_2, d_{\mathbf{Q}_2})$ is the unique solution to

$$\mathbf{Q}_2 \cong (\delta[\wp(\mathbf{C}_2^{\surd})] \cup \{\surd\}) \uplus (\mathbf{A}_2 \times \mathrm{id}_\kappa(\mathbf{Q}_2)).$$

in the category of complete metric spaces.
The set $\mathbf{P}_2$, the domain of the failures model for $\mathcal{L}_2$, is given by:

$$(p \in)\ \mathbf{P}_2 = \wp_{+\mathrm{cl}}(\mathbf{Q}_2).$$

The *Hausdorff distance* $d_{\mathbf{P}_2}$ on $\mathbf{P}_2$ induced from $d_{\mathbf{Q}_2}$ is defined such that $(\mathbf{P}_2, d_{\mathbf{P}_2})$ is a complete metric space. Let

$$(r \in)\ \mathbf{P}_2^{(1)} = (\mathbf{V} \to \mathbf{P}_2),$$

$$(\mathbf{p} \in)\ \mathbf{P}_2^* = \mathbf{P}_2 \cup \mathbf{P}_2^{(1)}.$$

**(2)** Let

$$
\begin{aligned}
\mathbf{M}_2^{\wp} = \{F \in (\mathcal{L}_2^*[\emptyset] \to \mathbf{P}_2^*) : \\
&\forall u \in \mathcal{L}_2^*[\emptyset][\ u \in \mathcal{L}_2[\emptyset] \Leftrightarrow F(u) \in \mathbf{P}_2\ ]\},
\end{aligned}
$$

and let $\Phi_2^{\mathrm{f}} : \mathbf{M}_2^{\wp} \to \mathbf{M}_2^{\wp}$ be defined as follows: For $F \in \mathbf{M}_2^{\wp}$ and $u \in \mathcal{L}_2^*[\emptyset]$, $\Phi_2^{\mathrm{f}}(F)[\![u]\!]$ is defined in the following two stages:

**(i)** First, for $s \in \mathcal{L}_2[\emptyset]$,

$$
\begin{aligned}
&\Phi_2^{\mathrm{f}}(F)[\![s]\!] \\
&= \{\langle\surd\rangle : \surd \in \mathrm{act}_2(s)\} \\
&\quad \cup\{\langle\delta(\Gamma)\rangle : \Gamma \subseteq \mathbf{C}_2^{\surd} \wedge \tau \notin \mathrm{act}_2(s) \wedge \\
&\qquad \Gamma \cap \mathrm{sort}[\mathrm{act}_2(s)] = \emptyset\} \\
&\quad \cup \bigcup\{\langle a\rangle \cdot F[\![s']\!] : a \in \mathbf{A}_2 \wedge s' \in \mathcal{L}_2[\emptyset] \wedge s \xrightarrow{a}_2 s'\}.
\end{aligned}
$$

By Lemma 4.12, it is straightforward to check that the right-hand side of the above equation is nonempty and closed.

**(ii)** Having defined $\Phi_2^{\mathrm{f}}(F)[\![s]\!]$ for $s \in \mathcal{L}_2[\emptyset]$, $\Phi_2^{\mathrm{f}}(F)[\![t]\!]$ for $t \in \mathcal{L}_2^{(1)}[\emptyset]$ is defined by:

$$\Phi_2^{\mathrm{f}}(F)[\![t]\!] = (\lambda v \in \mathbf{V}.\ \Phi_2^{\mathrm{f}}(F)[\![t(v)]\!]).$$

**(3)** Let $\hat{\mathcal{C}}_2 = \mathrm{fix}(\Phi_2^{\mathrm{f}})$. By definition one has the following:

**(i)** For $s \in \mathcal{L}_2[\emptyset]$,

$$\begin{aligned}&\hat{\mathcal{C}}_2[\![s]\!]\\ &= \ \{\langle\surd\rangle : \surd \in \mathrm{act}_2(s)\}\\ &\quad \cup\{\langle\delta(\Gamma)\rangle : \Gamma \subseteq \mathbf{C}_2^{\surd} \ \wedge\ \tau \notin \mathrm{act}_2(s) \wedge\\ &\qquad \Gamma \cap \mathrm{sort}[\mathrm{act}_2(s)] = \emptyset\}\\ &\quad \cup \bigcup\{\langle a\rangle \cdot \hat{\mathcal{C}}_2[\![s']\!] : a \in \mathbf{A}_2 \ \wedge\ s' \in \mathcal{L}_2[\emptyset] \ \wedge\ s \xrightarrow{a}_2 s'\}.\end{aligned} \tag{4.46}$$

**(ii)** For $t \in \mathcal{L}_2^{(1)}[\emptyset]$,

$$\hat{\mathcal{C}}_2[\![t]\!] = (\lambda v \in \mathbf{V}.\ \hat{\mathcal{C}}_2[\![t(v)]\!]).\ \blacksquare \tag{4.47}$$

As for $\mathcal{L}_1$, one obtains the fixed-point characterization of $\mathcal{C}_2$, i.e. the equivalence of $\mathcal{C}_2$ and $\hat{\mathcal{C}}_2$:

**Lemma 4.13** (Fixed-Point Characterization of $\mathcal{C}_2$)

**(1)** *The function $\mathcal{C}_2$ is type-respecting in the following sense:*

$$\text{(i) } \forall s \in \mathcal{L}_2[\emptyset][\ \mathcal{C}_2[\![s]\!] \in \mathbf{P}_2\ ], \quad \text{(ii) } \forall t \in \mathcal{L}_2^{(1)}[\emptyset][\ \mathcal{C}_2[\![t]\!] \in \mathbf{P}_2^{(1)}\ ].$$

**(2)** $\mathcal{C}_2 = \hat{\mathcal{C}}_2$. ∎

**Proof.** Similar to the proof of Lemma 4.2. ∎

As for $\mathcal{L}_1$, two *abstraction functions* $\mathcal{A}_2^{\mathrm{l}}, \mathcal{A}_2^{\mathrm{c}} : \mathbf{P}_2 \to \wp((\mathbf{A}_2)^{\leq\omega})$ are defined by:

**Definition 4.21** (Abstraction Functions $\mathcal{A}_2^{\mathrm{l}}$, $\mathcal{A}_2^{\mathrm{c}}$) Two functions $\mathcal{A}_2^{\mathrm{l}}, \mathcal{A}_2^{\mathrm{c}} : \mathbf{P}_2 \to \wp((\mathbf{A}_2)^{\leq\omega})$ are defined as follows: For $p \in \mathbf{P}_2$,

**(1)** $$\begin{aligned}\mathcal{A}_2^{\mathrm{l}}(p) = {}& ((\mathbf{A}_2)^{\omega} \cap p) \cup ((\mathbf{A}_2)^{<\omega} \cdot \{\langle\surd\rangle\} \cap p)\\ & \{w \cdot \langle\delta\rangle : \exists\Gamma \in \wp(\mathbf{C}_2^{\surd})[\ w \cdot \langle\delta(\Gamma)\rangle \in p \ \wedge\ \surd \in \Gamma\ ]\}.\end{aligned}$$

**(2)** $$\begin{aligned}\mathcal{A}_2^{\mathrm{c}}(p) = {}& ((\mathbf{A}_2)^{\omega} \cap p) \cup ((\mathbf{A}_2)^{<\omega} \cdot \{\langle\surd\rangle\} \cap p)\\ & \cup\{w \cdot \langle\,\delta\,\rangle : w \cdot \langle\delta(\mathbf{C}_2^{\surd}\rangle \in p\}.\ \blacksquare\end{aligned}$$

As for $\mathcal{L}_1$, one has the *correctness* of $\mathcal{C}_2$ w.r.t. $\mathcal{O}_2$ and $\mathcal{O}_2^{\mathrm{m}}$ by means of these abstraction functions:

**Lemma 4.14** (Relative Abstractness of $\mathcal{C}_2$ w.r.t. $\mathcal{O}_2$ and $\mathcal{O}_2^{\mathrm{m}}$)

**(1)** $\forall s \in \mathcal{L}_2[\emptyset][\ \mathcal{O}_2[\![s]\!] = \mathcal{A}_2^{\mathrm{l}}(\mathcal{C}_2[\![s]\!])\ ].$

**(2)** $\forall s \in \mathcal{L}_2[\emptyset][\ \mathcal{O}_2^{\mathrm{m}}[\![s]\!] = \mathcal{A}_2^{\mathrm{c}}(\mathcal{C}_2[\![s]\!])\ ].$ ∎

### 4.3.3 A Denotational Model $\mathcal{M}_2$ for $\mathcal{L}_2$

In this subsection, a *denotational* model $\mathcal{M}_2$ is defined, as for $\mathcal{L}_1$, using explicit semantic operations (corresponding to constructs of $\mathcal{L}_2$), with meanings of recursive programs as fixed-points in the complete metric space $(\mathbf{P}_2, d_{\mathbf{P}_2})$. Then, the semantic equivalence between $\mathcal{M}_2$ and $\mathcal{C}_2$ is established as in § 4.2.3.

For the definition of $\mathcal{M}_2$, we use the following notation:

**Notation 4.3** For $p \in \mathbf{P}_2$, let

$$\tilde{\mathcal{T}}_2(p) = p \cap ((\mathbf{A}_2)^{<\omega} \cdot \{\langle\surd\rangle\}), \quad \tilde{\mathcal{T}}_2^{\omega}(p) = p \cap (\mathbf{A}_2)^{\omega},$$

$$\tilde{\mathcal{F}}_2(p) = p \cap ((\mathbf{A}_2)^{<\omega} \cdot \{\langle\delta(\Gamma)\rangle : \Gamma \in \wp(\mathbf{C}_2^{\surd})\}). \blacksquare$$

We define semantic operations corresponding the syntactical constructs introduced in Definition 4.14 as follows:

**Definition 4.22** (Interpretation $\mathcal{I}_2$ for $\mathcal{L}_2$) We define an interpretation $\mathcal{I}_2$, a type-respecting mapping which maps each combinator in the subsignature $\mathbf{Sig}_2$, introduced in Definition 4.13, to a semantic operation as follows:

**(1)** The semantic interpretation $\tilde{\mathbf{0}}_2$ of $\mathbf{0}$ is defined as in Definition 4.9 (1) using $\mathbf{C}_2^{\surd}$ instead of $\mathbf{C}_1^{\surd}$ by:

$$\tilde{\mathbf{0}}_2 = \{\langle\delta(\Gamma)\rangle : \Gamma \in \wp(\mathbf{C}_2^{\surd})\}.$$

**(2)** The semantic interpretation $\tilde{\mathbf{e}}_2$ of $\mathbf{e}$ is defined as in Definition 4.9 (1) using $\mathbf{C}_2^{\surd}$ instead of $\mathbf{C}_1^{\surd}$ by:

$$\tilde{\mathbf{e}}_2 = \{\langle\delta(\Gamma)\rangle : \Gamma \in \wp(\mathbf{C}_2^{\surd}) \wedge \surd \notin \Gamma\} \cup \{\langle\surd\rangle\}.$$

**(3)** The interpretation $\tilde{\tau}_2$ of $\tau$ is defined as in Definition 4.9 (3) (i) using $\mathbf{C}_2^{\surd}$ instead of $\mathbf{C}_1^{\surd}$ by:

$$\tilde{\tau}_2 = \{\langle\tau\rangle\} \cdot \tilde{\mathbf{e}}_2.$$

**(4)** For every $\mathbf{c} \in \mathbf{Chan}$, the interpretion $\tilde{\mathbf{c}}! : \mathbf{V} \to \mathbf{P}_2$ of $\mathbf{c}!(\cdot)$ is defined as follows: For $v \in \mathbf{V}$,

$$\tilde{\mathbf{c}}!(v) = \{\langle\delta(\Gamma)\rangle : \Gamma \subseteq \mathbf{C}_2^{\surd} \wedge \mathbf{c}! \notin \Gamma\} \cup \{\langle\mathbf{c}!v\rangle\} \cdot \tilde{\mathbf{e}}_2.$$

**(5)** For every $\mathbf{c} \in \mathbf{Chan}$, the interpretation $\tilde{\mathbf{c}}? : \mathbf{P}_2^{(1)} \to \mathbf{P}_2$ of $\mathbf{c}?(\cdot)$ is defined as follows: For $r \in \mathbf{P}_2^{(1)}$,

$$\tilde{\mathbf{c}}?(r) = \{\langle\delta(\Gamma)\rangle : \Gamma \subseteq \mathbf{C}_2^{\surd} \wedge \mathbf{c}? \notin \Gamma\} \cup \bigcup_{v \in \mathbf{V}} [\{\langle\mathbf{c}?v\rangle\} \cdot r(v)].$$

**(6)** The semantic operation $\tilde{;}_2 : \mathbf{P}_2 \times \mathbf{P}_2 \to \mathbf{P}_2$ is defined in the same way as $\tilde{;}_1$ in Definition 4.9 (4) except that $\mathbf{A}_2$ and $\mathbf{C}_2^{\surd}$ are used instead of $\mathbf{A}_1$ and $\mathbf{C}_1^{\surd}$, respectively.

**(7)** The semantic operation $\tilde{+}_2 : \mathbf{P}_2 \times \mathbf{P}_2 \to \mathbf{P}_2$ is defined in the same way as $\tilde{+}_1$ in Definition 4.9 (5) except that $\mathbf{A}_2$ and $\mathbf{C}_2^{\surd}$ are used instead of $\mathbf{A}_1$ and $\mathbf{C}_1^{\surd}$, respectively.

**(8)** As we have $\tilde{\|}_1$ for $\mathcal{L}_1$, we have the unique operation $\tilde{\|}_2 : \mathbf{P}_2 \times \mathbf{P}_2 \to \mathbf{P}_2$ satisfying the following: For $p_1, p_2 \in \mathbf{P}_2$,

$$\begin{aligned} p_1 \tilde{\|}_2 p_2 = {} & (\{\langle\surd\rangle\} \cap (p_1 \cap p_2)) \cup (p_1 \tilde{\|}_2^{\delta} p_2) \\ & \cup (p_1 \tilde{\lfloor\!\lfloor}_2 p_2) \cup (p_2 \tilde{\lfloor\!\lfloor}_2 p_1) \cup (p_1 \tilde{\lfloor}_2 p_2) \cup (p_2 \tilde{\lfloor}_2 p_1). \end{aligned}$$

where the operations $\tilde{\lfloor\!\lfloor}_2$ and $\tilde{\|}_2^{\delta}$ are defined in the same way as $\tilde{\lfloor\!\lfloor}_1$ and $\tilde{\|}_1^{\delta}$ in Definition 4.9 (6) except that $\mathbf{A}_2$ and $\mathbf{C}_2$ are used instead of $\mathbf{A}_1$ and $\mathbf{C}_1$, and the operation $\tilde{\lfloor}_2$ is defined as $\tilde{|}_1$ in Definition 4.9 (6) by:

$$\begin{aligned} & p_1 \tilde{\lfloor}_2 p_2 \\ & = (\bigcup\{ \langle\tau\rangle \cdot (p_1[\langle \mathbf{c}!v\rangle] \tilde{\|}_2 p_2[\langle \mathbf{c}?v\rangle]) : \mathbf{c} \in \mathbf{Chan} \wedge v \in \mathbf{V} \\ & \qquad \wedge p_1[\langle \mathbf{c}!v\rangle] \neq \emptyset \wedge p_2[\langle \mathbf{c}?v\rangle] \neq \emptyset \})^{\mathrm{cls}}. \end{aligned} \tag{4.48}$$

**(9)** For every $C \in \wp(\mathbf{Chan})$, the operation $\tilde{\partial}_C^2(\cdot) : \mathbf{P}_2 \to \mathbf{P}_2$, corresponding to the combinator '$\partial_C(\cdot)$', is defined as follows: For every $p \in \mathbf{P}_2$,

$$\begin{aligned} & \tilde{\partial}_C^2(p) \\ = {} & \{q \in \tilde{\mathcal{T}}_2(p) \cup \tilde{\mathcal{T}}_2^{\omega}(p) : \mathrm{chan}[\mathrm{sort}[\mathrm{ran}(w)]] \cap C = \emptyset\} \\ & \cup \{w \cdot \langle\delta(\Gamma)\rangle : w \in (\mathbf{A}_2)^{<\omega} \wedge \Gamma \in \wp(\mathbf{C}_2^{\surd}) \\ & \qquad \wedge \mathrm{chan}[\mathrm{sort}[\mathrm{ran}(w)]] \cap C = \emptyset \wedge w \cdot \langle\delta(\Gamma \setminus C!?)\rangle \in \tilde{\mathcal{F}}_2(p)\}. \end{aligned}$$

**(10)** A semantic operation $\mathrm{if}_2(\cdot,\cdot,\cdot) : \mathbf{V} \times \mathbf{P}_2 \times \mathbf{P}_2 \to \mathbf{P}_2$ is defined as follows: For $v \in \mathbf{V}$, $p_1, p_2 \in \mathbf{P}_2$, let

$$\mathrm{if}_2(v, p_1, p_2) = \begin{cases} p_1 & \text{if } v \neq \mathbf{nil}, \\ p_2 & \text{if } v = \mathbf{nil}. \end{cases}$$

**(11)** Let

$$\begin{aligned} \mathcal{I}_2 = {} & \{(\mathbf{0}, \tilde{\mathbf{0}}_2), (\mathrm{e}, \tilde{\mathrm{e}}_2), (\tau, \tilde{\tau}_2)\} \\ & \cup \{(\mathbf{c}!(\cdot), \tilde{\mathbf{c}}!(\cdot)) : \mathbf{c} \in \mathbf{Chan}\} \\ & \cup \{(\mathbf{c}?(\cdot), \tilde{\mathbf{c}}?(\cdot)) : \mathbf{c} \in \mathbf{Chan}\} \\ & \cup \{(;, \tilde{;}_2), (+, \tilde{+}_2), (\|, \tilde{\|}_2), (\mathbf{if}(\cdot,\cdot,\cdot), \mathrm{if}_2(\cdot,\cdot,\cdot))\} \\ & \cup \{\langle \partial_C, \tilde{\partial}_C^2 \rangle : C \in \wp(\mathbf{Chan})\}. \blacksquare \end{aligned}$$

In terms of the interpretation $\mathcal{I}_2$, the denotational model $\mathcal{M}_2$ is defined. For the definition, we need the following definition and lemma:

**Definition 4.23** (Compositional Interpretation of Statements in $\mathcal{L}_2$) Let $U \in \tilde{\mathcal{E}} \cup \mathcal{L}_2^*$, $n \in \omega$, and let $\vec{\zeta} \in (n \rightarrowtail \mathcal{X}^*)$ with $\mathrm{FV}(U) \subseteq \mathrm{ran}(\vec{\zeta})$; let $\vec{\mathbf{p}} \in (\mathbf{V} \cup \mathbf{P}_2^*)^n$ such that the *types* of $\vec{\zeta}$ and $\vec{\mathbf{p}}$ agree in the sense that

$$\begin{aligned} \forall i \in n[ \ & (\vec{\zeta}(i) \in \mathcal{X}_{\mathcal{V}} \Rightarrow \vec{\mathbf{p}}(i) \in \mathbf{V}) \\ & \wedge (\vec{\zeta}(i) \in \mathcal{X}_{\mathcal{P}} \Rightarrow \vec{\mathbf{p}}(i) \in \mathbf{P}_2) \\ & \wedge (\vec{\zeta}(i) \in \mathcal{X}_{\mathcal{P}}^{(1)} \Rightarrow \vec{\mathbf{p}}(i) \in \mathbf{P}_2^{(1)}) \ ]. \end{aligned}$$

Let $\tilde{\mathcal{I}}$ be an interpretation for the signature $\mathbf{Sig}_2^*$. Let us define the *interpretation* of $U$ in $\tilde{\mathcal{I}}$ with the assignment $\vec{\mathbf{p}}(i)$ to $\vec{\zeta}(i)$ $(i \in n)$, written $[\![U_{\vec{\zeta}}]\!]^{\tilde{\mathcal{I}}}(\vec{\mathbf{p}})$, by induction on the structure of $U$ as follows:

**(i)** If $U \in \mathcal{X}^*$, then there is unique $i \in n$ such that $U \equiv \vec{\zeta}(i)$, and let

$$[\![U_{ze\vec{t}a}]\!]^{\tilde{\mathcal{I}}}(\vec{\mathbf{p}}) = \vec{\mathbf{p}}(i).$$

**(ii)** For $U \in \mathcal{RV}^*$, let $[\![U_{\vec{\zeta}}]\!]^{\tilde{\mathcal{I}}}(\vec{\mathbf{p}}) = \tilde{\mathcal{I}}(U)$.

**(iii)** If $U \in \tilde{\mathcal{E}}$, then let $[\![U_{\vec{\zeta}}]\!]^{\tilde{\mathcal{I}}}(\vec{\mathbf{p}}) = [\![U_{\vec{\zeta}}[\vec{\mathbf{p}}]]\!]$.

**(iv)** If $U \equiv F(\langle U_i \rangle_{i \in r})$ with $F \in \mathbf{Fun}_2^* \setminus \mathcal{RV}^*$, then let

$$[\![U_{\vec{\zeta}}]\!]^{\tilde{\mathcal{I}}}(\vec{\mathbf{p}}) = \tilde{\mathcal{I}}(F)(\langle [\![(U_i)_{\vec{\zeta}}]\!]^{\tilde{\mathcal{I}}}(\vec{\mathbf{p}}) \rangle_{i \in r}),$$

where $\tilde{\mathcal{I}}(F)$ is the semantic operation in $\tilde{\mathcal{I}}$ corresponding to $F$.

**(v)** If $U \equiv (\lambda x.\ S)$ with $S \in \mathcal{L}_2$, then let

$$[\![U_{\vec{\zeta}}]\!]^{\tilde{\mathcal{I}}}(\vec{\mathbf{p}}) = (\lambda v \in \mathbf{V}.\ [\![(S[v/x])_{\vec{\zeta}}]\!]^{\tilde{\mathcal{I}}}(\vec{\mathbf{p}})).$$

**(vi)** If $U \equiv T(e)$ with $T \in \mathcal{L}_2^{(1)}$ and $e \in \mathcal{E}$, then let

$$[\![U_{\vec{\zeta}}]\!]^{\tilde{\mathcal{I}}}(\vec{\mathbf{p}}) = [\![T_{\vec{\zeta}}]\!]^{\tilde{\mathcal{I}}}(\vec{\mathbf{p}})([\![e]\!]). \blacksquare$$

**Lemma 4.15** *Let $D = \{(Z, \tilde{g}_Z) : Z \in \mathcal{RV}^*\}$ be the set of declarations. We say a function $f \in (\mathcal{RV}^* \to \mathbf{P}_2^*)$ respects types iff*

$$\forall Z \in \mathcal{RV}^*[\ (Z \in \mathcal{RV} \Rightarrow f(Z) \in \mathbf{P}_2)$$
$$\wedge (Z \in \mathcal{RV}^{(1)} \Rightarrow f(Z) \in \mathbf{P}_2^{(1)})\ ].$$

*Let $\mathbf{M}_2^{\mathcal{M}} = \{f \in (\mathcal{RV}^* \to \mathbf{P}_2^*) : f$ respects types$\}$. (Obviously, $\mathbf{M}_2^{\mathcal{M}}$ is closed in the complete metric space $(\mathcal{RV}^* \to \mathbf{P}_2^*)$, and therefore, $\mathbf{M}_2^{\mathcal{M}}$ itself is a complete metric space.) Finally, let $\Pi_2 : \mathbf{M}_2^{\mathcal{M}} \to \mathbf{M}_2^{\mathcal{M}}$ be defined as follows: For $\mathbf{p} \in \mathbf{M}_2^{\mathcal{M}}$ and for $Z \in \mathcal{RV}^*$, let*

$$\Pi_2(\mathbf{p})(Z) = [\![\tilde{g}_Z]\!]^{\mathcal{I}_2 \cup \mathbf{p}}.$$

*(Here note that $\mathcal{I}_2 \cup \mathbf{p}$ is an interpretation for $\mathbf{Sig}_2^*$, and so, writing $[\![\tilde{g}_Z]\!]^{\mathcal{I}_2 \cup \mathbf{p}}$ makes sense.) Then, the mapping $\Pi_2$ is a contraction from $\mathbf{M}_2^{\mathcal{M}}$ to $\mathbf{M}_2^{\mathcal{M}}$.* $\blacksquare$

**Proof.** This is shown in a similar way to the proof of Lemma 4.4. See § 4.B. $\blacksquare$

By means of the above lemma, the denotational model $\mathcal{M}_2$ is defined by:

**Definition 4.24** (Denotational Model $\mathcal{M}_2$ for $\mathcal{L}_2$) Let $D = \{(Z, \tilde{g}_Z) : Z \in \mathcal{RV}^*\}$ be the set of declarations. Let $\mathbf{M}_2^{\mathcal{M}}$ and $\Pi_2$ be as in Lemma 4.15, and let $\tilde{\mathbf{p}}_2 = \text{fix}(\Pi_2)$. Let us set $\tilde{\mathcal{I}}_2 = \mathcal{I}_2 \cup \tilde{\mathbf{p}}_2$. Then, $\tilde{\mathcal{I}}_2$ is an interpretation for $\mathbf{Sig}_2^*$. By means of $\tilde{\mathcal{I}}_2$, the model $\mathcal{M}_2 : \mathcal{L}_2^* \to \mathbf{P}_2^*$ is defined by $\mathcal{M}_2[\![u]\!] = [\![u]\!]^{\tilde{\mathcal{I}}_2}$ $(u \in \mathcal{L}_2^*[\emptyset])$. Equivalently, $\mathcal{M}_2[\![u]\!]$ is defined by induction on the structure of $u \in \mathcal{L}_2^*[\emptyset]$ using the following rules (1) to (4) (in the sequel, we use the convention that for $k \in \omega$, $\vec{u} \in (\mathcal{L}_2^*[\emptyset])^k$ (resp. $\vec{e} \in \mathcal{E}^k$) $\mathcal{M}_2[\![\vec{u}]\!]$ (resp. $[\![\vec{e}]\!]$) denotes the sequence $\langle \mathcal{M}_2[\![\vec{u}(i)]\!] \rangle_{i \in k}$ (resp. $\langle [\![\vec{e}(i)]\!] \rangle_{i \in k}$) ):

**(1)** For $Z \in \mathcal{RV}^*$, let us define $Z^{\mathcal{M}_2}$, the denotational meaning of $Z$, by:

$$Z^{\mathcal{M}_2} = \tilde{\mathbf{p}}_2(Z).$$

**(2)** For each $(k, l, m) \in \omega^3$, and $F \in (\mathbf{Fun}_2^*)^{((k,l,m),\mathcal{P})}$, $\vec{e} \in (\mathcal{E})^k$, $\vec{s} \in (\mathcal{L}_2[\emptyset])^\ell$, $\vec{t} \in (\mathcal{L}_2^{(1)}[\emptyset])^m$, let

$$\mathcal{M}_2[\![F(\vec{e} \cdot \vec{s} \cdot \vec{t})]\!] = \mathcal{I}_2(F)([\![\vec{e}]\!] \cdot \mathcal{M}_2[\![\vec{s} \cdot \vec{t}]\!]),$$

where $\mathcal{I}_2(F)$ is the semantic operation in $\mathcal{I}_2$ corresponding to $F$.

**(3)** Let $\mathcal{M}_2[\![(\lambda x.\ S)]\!] = (\lambda v \in \mathbf{V}.\ \mathcal{M}_2[\![S[v/x]]\!])$.

**(4)** Let $\mathcal{M}_2[\![t(e)]\!] = \mathcal{M}_2[\![t]\!]([\![e]\!])$. ∎

**Lemma 4.16** (Compositionality of $\mathcal{M}_2$) *Let $X \in \mathcal{X}_{\mathcal{P}}$, $s \in \mathcal{L}_2^*[\emptyset]$, and $U \in \mathcal{L}_2^*[X]$. Then,*

$$\mathcal{M}_2[\![U_{\langle X\rangle}[s]]\!] = [\![U_{\langle X\rangle}]\!]^{\tilde{\mathcal{I}}_2}(\mathcal{M}_2[\![s]\!]). \blacksquare \tag{4.49}$$

**Proof.** This is established by induction on the structure of $U \in \mathcal{L}_2^*$, more specifically, by induction on $\deg(U)$, the number of combinators, $\lambda$-constructs, and *application* constructs contained in $U$. Assume that (4.49) holds for every $U$ with $\deg(U) \leq n$. Let us show that (4.49) also holds for $U$ with $\deg(U) = n + 1$. Here we consider the case that $U \equiv (\lambda x.\ S)$. (In the other cases, the consequence is obtained more straightforwardly.) One has

$$\begin{aligned}
&\mathcal{M}_2[\![U_{\langle X\rangle}[s]]\!] = \mathcal{M}_2[\![(\lambda x.\ S)_{\langle X\rangle}[s]]\!] = \mathcal{M}_2[\![(\lambda x.\ S_{\langle X\rangle}[s])]\!] \\
&= (\lambda v \in \mathbf{V}.\ \mathcal{M}_2[\![(S_{\langle X\rangle}[s])[v/x]]\!]) \quad \text{(by the definition of } \mathcal{M}_2) \\
&= (\lambda v \in \mathbf{V}.\ \mathcal{M}_2[\![(S[v/x])_{\langle X\rangle}[s]]\!]) \\
&= (\lambda v \in \mathbf{V}.\ [\![(S[v/x])_{\langle X\rangle}]\!]^{\tilde{\mathcal{I}}_2}(\mathcal{M}_2[\![s]\!])) \quad \text{(by the induction hypothesis)} \\
&= [\![(\lambda x.\ S)_{\langle X\rangle}]\!]^{\tilde{\mathcal{I}}_2}(\mathcal{M}_2[\![s]\!]) \quad \text{(by the definition of } [\![\cdot]\!]^{\tilde{\mathcal{I}}_2}) \\
&= [\![U_{\langle X\rangle}]\!]^{\tilde{\mathcal{I}}_2}(\mathcal{M}_2[\![s]\!]). \blacksquare
\end{aligned}$$

The model $\mathcal{C}_2$ is also compositional in the following sense:

**Lemma 4.17** (Compositionality of $\mathcal{C}_2$)

**(1)** *For $u \in \mathcal{L}_2^*[\emptyset]$, $\mathcal{C}_2[\![u]\!]$ is defined in terms of the meanings of subterms of $u$ as follows:*

**(i)** *Let $(k, \ell, m) \in \omega^3$ and $\mathcal{S} \in \{\mathcal{P}, \mathcal{P}^{(1)}\}$. If $u \equiv F(\vec{e} \cdot \vec{s} \cdot \vec{t})$ with $F \in ((\mathbf{Fun}_2^*)^{((k,\ell,m),\mathcal{S})} \setminus \mathcal{RV}^*)$, $\vec{e} \in \mathcal{E}^k$, $\vec{s} \in (\mathcal{L}_2[\emptyset])^\ell$, $\vec{t} \in (\mathcal{L}_2^{(1)}[\emptyset])^m$, then*

$$\mathcal{C}_2[\![u]\!] = \mathcal{I}_2(F)([\![\vec{e}]\!] \cdot \mathcal{C}_2[\![\vec{s} \cdot \vec{t}]\!]).$$

**(ii)** *If $u \equiv (\lambda x.\ S)$ with $S \in \mathcal{L}_2$, then*

$$\mathcal{C}_2[\![u]\!] = (\lambda v \in \mathbf{V}.\ \mathcal{C}_2[\![S[v/x]]\!]).$$

**(iii)** *If $u \equiv t(e)$ with $t \in \mathcal{L}_2^{(1)}[\emptyset]$ and $e \in \mathcal{E}$, then*

$$\mathcal{C}_2[\![u]\!] = \mathcal{C}_2[\![t]\!]([\![e]\!]).$$

**(2)** *Putting* $\tilde{\mathbf{p}}_2' = (\lambda Z \in \mathcal{RV}^*.\ \mathcal{C}_2[\![P]\!])$, *one has the following for every* $u \in \mathcal{L}_2^*[\emptyset]$:

$$\mathcal{C}_2[\![u]\!] = [\![u]\!]^{\mathcal{I}_2 \cup \tilde{\mathbf{p}}_2'}. \blacksquare$$

**Proof.** **(1)** Part (i) can be established in a similar fashion to the proof of Lemma 4.6 (1).

Part (ii) follows from the definition of $\mathcal{C}_2$, since the only rule which derives actions of $(\lambda x.\ S)(v)$ is the $\lambda$*-rule* (the rule (9) in Definition 4.16), and therefore, the actions of $(\lambda x.\ S)(v)$ are the same as the actions of $S[v/x]$.

Likewise, part (iii) follows from the definition of $\mathcal{C}_2$, since the only rule which derives actions of $u(e)$ is the *pre-evaluation rule* (the rule (8) in Definition 4.16), and therefore, the actions of $t(e)$ are the same as the actions of $t([\![e]\!])$.

**(2)** This part follows from part (1) by induction on the structure of $u \in \mathcal{L}_2^*[\emptyset]$. ■

The following semantic equivalence result for $\mathcal{L}_2$ immediately follows from the above lemma, as Lemma 4.8 follows from Lemma 4.6.

**Lemma 4.18** (Semantic Equivalence for $\mathcal{L}_2$)

$$\forall u \in \mathcal{L}_2^*[\emptyset][\ \mathcal{C}_2[\![u]\!] = \mathcal{M}_2[\![u]\!]\ ]. \blacksquare$$

**Proof.** First, let us show

$$\forall Z \in \mathcal{RV}^*[\ \mathcal{C}_2[\![Z]\!] = \tilde{\mathbf{p}}_2(Z)\ ], \tag{4.50}$$

where $\tilde{\mathbf{p}}_2 = \mathrm{fix}(\Pi_2)$ is the mapping defined in Definition 4.24. For this purpose, it suffices to show that $\tilde{\mathbf{p}}_2' = (\lambda Z \in \mathcal{RV}^*.\ \mathcal{C}_2[\![Z]\!])$ is the fixed-point of $\Pi_2$, i.e., that $(*)$: $\tilde{\mathbf{p}}_2' = \Pi_2(\tilde{\mathbf{p}}_2')$. This is shown as follows: Let $Z \in \mathcal{RV}^*$. Then,

$$\begin{aligned}
\tilde{\mathbf{p}}_2'(Z) &= \mathcal{C}_2[\![Z]\!] \quad \text{(by the definition of } \tilde{\mathbf{p}}_2')\\
&= \mathcal{C}_2[\![\tilde{g}_Z]\!] \quad \text{(by the definition of } \mathcal{C}_2)\\
&= [\![\tilde{g}_Z]\!]^{\mathcal{I}_2 \cup \tilde{\mathbf{p}}_2'} \quad \text{(by Lemma 4.17 (2))}\\
&= \Pi_2(\tilde{\mathbf{p}}_2')(Z) \quad \text{(by the definition of } \Pi_2).
\end{aligned}$$

Thus one has $(*)$, and therefore, (4.50).

Using (4.50) and Lemma 4.17 (1), one can establish the following by induction on the structure of $u$:

$$\forall u \in \mathcal{L}_2^*[\emptyset][\ \mathcal{C}_2[\![u]\!] = \mathcal{M}_2[\![u]\!]\ ]. \blacksquare$$

### 4.3.4 The Full Abstraction of $\mathcal{M}_2$ w.r.t. $\mathcal{O}_2$ and $\mathcal{O}_2^{\mathrm{m}}$

The full abstractness of $\mathcal{M}_2$ w.r.t. $\mathcal{O}_2$ and $\mathcal{O}_2^{\mathrm{m}}$ can be shown in a similar way to the proof of Theorem 4.1, by adjusting the *contexts* used there so that they fit into the setting of $\mathcal{L}_2$; we give the proof, in this subsection, to demonstrate that this is indeed the case.

As in § 4.2.4, the following three lemmas are presented as preliminaries to the full abstractness proof. First, we define the *sort-set* of a statement at a fixed depth by:

**Definition 4.25** For $s \in \mathcal{L}_2[\emptyset]$, and $n \in \omega$, let

$$\mathcal{S}_2^{(n)}(s) = \{\mathrm{sort}(\alpha) : \alpha \in \mathbf{A}_2^{\surd} \wedge \exists w \in (\mathbf{A}_1)^n[\ s \xrightarrow{w}_{2*} \xrightarrow{a}_2\ ]\}.$$

The set $\mathcal{S}_2^{(n)}(s)$ is called the *sort-set* of $s$ at depth $n$. ∎

Then, one has the following lemma:

**Lemma 4.19** *For $s \in \mathcal{L}_2[\emptyset]$, and $n \in \omega$, the sort-set $\mathcal{S}_2^{(n)}(s)$ is finite.* ∎

**Lemma 4.20** *For $s \in \mathcal{L}_2[\emptyset]$, $w \in (\mathbf{A}_2)^{<\omega}$, and $\Gamma \subseteq \mathbf{C}_2$, the following holds:*

$$w \cdot \langle \delta(\Gamma) \rangle \in \mathcal{M}_2[\![s]\!] \Leftrightarrow w \cdot \langle \delta(\Gamma \cup \{\surd\}) \rangle \in \mathcal{M}_2[\![s \,\|\, \mathbf{0}]\!].$$ ∎

**Lemma 4.21** *Let $s \in \mathcal{L}_2[\emptyset]$, $w \in (\mathbf{A}_1)^{<\omega}$, $C \subseteq$ **Chan**, and $\Gamma \subseteq \mathbf{C}_2^{\surd}$. If $C \cap \mathrm{chan}[\mathrm{sort}[\mathrm{ran}(w)]] = \emptyset$, then*

$$w \cdot \langle \delta(\Gamma) \rangle \in \mathcal{M}_2[\![s]\!] \Leftrightarrow w \cdot \langle \delta(\Gamma \cup C!?) \rangle \in \mathcal{M}_2[\![\partial_C(s)]\!].$$ ∎

**Theorem 4.2** (Full Abstraction of $\mathcal{M}_2$ w.r.t. $\mathcal{O}_2$ and $\mathcal{O}_2^{\mathrm{m}}$)

**(1)** *The following holds for every $s_1, s_2 \in \mathcal{L}_2[\emptyset]$:*

$$\begin{aligned} &\mathcal{M}_2[\![s_1]\!] = \mathcal{M}_2[\![s_2]\!] \\ &\Leftrightarrow \forall X \in \mathcal{X}_{\mathcal{P}}, \forall S \in \mathcal{L}_2[X][\ \mathcal{O}_2[\![S_{\langle X \rangle}[s_1]]\!] = \mathcal{O}_2[\![S_{\langle X \rangle}[s_2]]\!]\ ]. \end{aligned} \tag{4.51}$$

**(2)** *If* **Chan** *is infinite, then the following holds for every $s_1, s_2 \in \mathcal{L}_2[\emptyset]$:*

$$\begin{aligned} &\mathcal{M}_2[\![s_1]\!] = \mathcal{M}_2[\![s_2]\!] \Leftrightarrow \\ &\forall X \in \mathcal{X}_{\mathcal{P}}, \forall S \in \mathcal{L}_2[X][\ \mathcal{O}_2^{\mathrm{m}}[\![S_{\langle X \rangle}[s_1]]\!] = \mathcal{O}_2^{\mathrm{m}}[\![S_{\langle X \rangle}[s_2]]\!]\ ]. \end{aligned} \tag{4.52}$$ ∎

**Proof.** **(1)** Let $s_1, s_2 \in \mathcal{L}_2[\emptyset]$. The $\Rightarrow$-part of (4.51) follows from the relative abstractness of $\mathcal{M}_2$ w.r.t. $\mathcal{L}_2^{\mathrm{l}}$ and the compositionality of $\mathcal{M}_2$. For establishing the $\Leftarrow$-part of (4.51), it suffices to show that if $\mathcal{M}_2[\![s_1]\!] \neq \mathcal{M}_2[\![s_2]\!]$, then

$$\exists X \in \mathcal{X}_{\mathcal{P}}, \exists S \in \mathcal{L}_2[X][\ \mathcal{O}_2[\![S_{\langle X \rangle}[s_1]]\!] \neq \mathcal{O}_2[\![S_{\langle X \rangle}[s_1]]\!]\ ]. \tag{4.53}$$

Suppose

$$\mathcal{M}_2[\![s_1]\!] \neq \mathcal{M}_2[\![s_2]\!]. \tag{4.54}$$

Let us show (4.53). By (4.54), one of the following propositions holds:

$$\begin{aligned} &\textbf{(i) } \tilde{\mathcal{T}}_2(\mathcal{M}_2[\![s_1]\!]) \neq \tilde{\mathcal{T}}_2(\mathcal{M}_2[\![s_2]\!]), \\ &\textbf{(ii) } \tilde{\mathcal{T}}_2^{\omega}(\mathcal{M}_2[\![s_1]\!]) \neq \tilde{\mathcal{T}}_2^{\omega}(\mathcal{M}_2[\![s_2]\!]), \\ &\textbf{(iii) } \tilde{\mathcal{F}}_2(\mathcal{M}_2[\![s_1]\!]) \neq \tilde{\mathcal{F}}_2(\mathcal{M}_2[\![s_2]\!]). \end{aligned} \tag{4.55}$$

When (4.55) (i) or (ii) holds, the desired consequence (4.53) immediately follows by putting $S \equiv X$.

Let us consider the case that (4.55) (iii) holds. We can assume without loss of generality that there exist $w \in (\mathbf{A}_2)^{<\omega}$, $\Gamma \in \wp(\mathbf{C}_2^{\surd})$ such that

$$\text{(i) } w \cdot \langle \delta(\Gamma) \rangle \in \mathcal{M}_2[\![s_1]\!], \quad \text{(ii) } w \cdot \langle \delta(\Gamma) \rangle \notin \mathcal{M}_2[\![s_2]\!]. \tag{4.56}$$

We distinguish two cases according to whether $\surd \in \Gamma$ or not.

**Case 1.** Suppose $\surd \in \Gamma$, and put $m = \mathrm{lgt}(w)$. Then, by Lemma 4.19, the set $\mathcal{S}_2^{(m)}(s_2)$ is finite. Let $\Gamma' = (\Gamma \cap \mathcal{S}_2^{(m)}(s_2)) \cup \{\surd\}$. Then $\Gamma'$ is finite and contains $\surd$, thus we can put $\Gamma' = \{\surd, \gamma_1, \cdots, \gamma_k\}$. By (4.56), one has

$$\text{(i) } w \cdot \langle \delta(\Gamma') \rangle \in \mathcal{M}_2[\![s_1]\!], \quad \text{(ii) } w \cdot \langle \delta(\Gamma') \rangle \notin \mathcal{M}_2[\![s_2]\!]. \tag{4.57}$$

We will construct an appropriate statement $\hat{s}'$ such that

$$\text{(i) } w \cdot \langle \delta \rangle \in \mathcal{O}_2[\![s_1 \parallel \hat{s}']\!], \quad \text{(ii) } w \cdot \langle \delta \rangle \notin \mathcal{O}_2[\![s_2 \parallel \hat{s}']\!]. \tag{4.58}$$

A few preliminaries are needed for constructing such $\hat{s}'$. First, fix $\hat{v} \in \mathbf{V}$ and $\hat{x} \in \mathcal{X}_{\mathcal{V}}$. Then, for $\gamma \in \wp(\mathbf{C}_2)$, let us define $\chi(\gamma) \in \mathcal{L}_2$ as follows:

$$\chi(\gamma) = \begin{cases} \mathbf{c}!(\hat{v}) & \text{if } \gamma = \mathbf{c}!, \\ \mathbf{c}?((\lambda \hat{x}.\, \mathbf{e})) & \text{if } \gamma = \mathbf{c}?. \end{cases}$$

Let $\hat{s} \equiv \tau^{m+1}$. Finally, let

$$\hat{s}' \equiv \mathbf{e} + (\chi(\overline{\gamma_1}); \hat{s}) + \cdots + (\chi(\overline{\gamma_k}); \hat{s}).$$

First, (4.58) (i) is shown straightforwardly as follows: By (4.56) (i), there is $s_1'$ such that

$$s_1 \xrightarrow{w}_{2*} s_1' \wedge \tau \notin \mathrm{act}_2(s_1') \wedge \mathrm{sort}[\mathrm{act}_2(s_1')] \cap \Gamma' = \emptyset. \tag{4.59}$$

With this $s_1'$, one has

$$\begin{aligned} &(s_1 \parallel \hat{s}') \xrightarrow{w}_{2*} (s_1' \parallel \hat{s}') \\ &\wedge \mathrm{sort}[\mathrm{act}_2(s_1' \parallel \hat{s}')] = \mathrm{sort}[\mathrm{act}_2(s_1') \cup (\mathrm{act}_2(\hat{s}') \setminus \{\surd\})] \subseteq \mathbf{C}_2. \end{aligned}$$

Thus, by the definition of $\mathcal{O}_2$, one has (4.58) (i).

Next, let us show (4.58) (ii) by contradiction. Assume, to the contrary, that

$$w \cdot \langle \delta \rangle \in \mathcal{O}_2[\![s_2 \parallel \hat{s}']\!]. \tag{4.60}$$

Then, there is $s'$ such that

$$\text{(i) } (s_2 \parallel \hat{s}') \xrightarrow{w}_{2*} s', \quad \text{(ii) } \mathrm{sort}[\mathrm{act}_2(s')] \subseteq \mathbf{C}_2. \tag{4.61}$$

We observe that each action in $w$ must stem from either $s_2$ or $\hat{s}'$, or it must be the synchronization of two actions stemming from $s_2$ and $\hat{s}'$. However, it can be shown that every action in $w$ must stem from $s_2$ as follows: Assume, for the sake of contradiction, that some action of $w$ stems from $\hat{s}'$, or it is the synchronization of two actions stemming from $s_2$ and $\hat{s}'$. Then, by the form of $\hat{s}'$, there must be at at least $(m+1)$ $\tau$-actions in $w$, which contradicts the fact that $m = \mathrm{lgt}(w)$. Thus, every action in $w$ must stem from $s_2$, and therefore, there is $s_2'$ such that

$$\text{(i) } s_2 \xrightarrow{w}_{2*} s_2', \quad \text{(ii) } s' \equiv (s_2' \parallel \hat{s}'). \tag{4.62}$$

By (4.61) (ii), and (4.62) (ii), one has $\mathrm{sort}[\mathrm{act}_2(s_2' \parallel \hat{s}')] \subseteq \mathbf{C}_2$. Thus, $\tau \notin \mathrm{act}_2(s_2')$ and

$$\mathrm{sort}[\mathrm{act}_2(s_2')] \cap \{\surd, \gamma_1, \cdots, \gamma_k\} = \mathrm{sort}[\mathrm{act}_2(s_2')] \cap \Gamma' = \emptyset.$$

From this and (4.62) (i), it follows that $w \cdot \langle \delta(\Gamma') \rangle \in \mathcal{M}_2[\![s_2]\!]$, which contradicts (4.57) (ii). Summarizing, (4.60) does not hold, i.e., one has (4.58) (ii).

**Case 2.** Suppose $\surd \in \Gamma$. Applying Lemma 4.20, one obtains the following from (4.56):

$$w \cdot \langle \delta(\Gamma \cup \{\surd\}) \rangle \in \mathcal{M}_2[\![s_1 \parallel \mathbf{0}]\!] \wedge w \cdot \langle \delta(\Gamma \cup \{\surd\}) \rangle \notin \mathcal{M}_2[\![s_2 \parallel \mathbf{0}]\!].$$

Thus, this case is reduced to Case 1.

**(2)** Suppose **Chan** is infinite, and let $s_1, s_2 \in \mathcal{L}_2[\emptyset]$. As in part (1), the $\Rightarrow$-part of (4.20) follows from the relative abstractness of $\mathcal{M}_2$ w.r.t. $\mathcal{O}_2^{\mathrm{m}}$ and the compositionality of $\mathcal{M}_2$. For establishing the $\Leftarrow$-part of (4.52), it suffices to show that if $\mathcal{M}_2[\![s_1]\!] \neq \mathcal{M}_2[\![s_2]\!]$, then

$$\exists X \in \mathcal{X}_{\mathcal{P}}, \exists S \in \mathcal{L}_2[X][\ \mathcal{O}_2^{\mathrm{m}}[\![S_{\langle X \rangle}[s_1]]\!] \neq \mathcal{O}_2^{\mathrm{m}}[\![S_{\langle X \rangle}[s_1]]\!]\ ]. \tag{4.63}$$

Suppose

$$\mathcal{M}_2[\![s_1]\!] \neq \mathcal{M}_2[\![s_2]\!].$$

Let us show (4.63). As in part (1), one of the propositions (4.55) holds; when (4.55) (i) or (ii) holds, the desired consequence (4.63) follows immediately.

Let us consider the case that (4.55) (iii) holds. As in part (1), we can assume without loss of generality that there exist $w \in (\mathbf{A}_2)^{<\omega}$, $\Gamma \in \wp(\mathbf{C}_2^{\surd})$ satisfying (4.56) above. As in part (1), we distinguish two cases according to whether $\surd \in \Gamma$ or not.

**Case 1.** Suppose $\surd \in \Gamma$. Then, as in Case 1 in part (1), there exist $w \in (\mathbf{A}_2)^{<\omega}$ and a finite set $\Gamma' \subseteq \mathbf{C}_2^{\surd}$ such that $\surd \in \Gamma'$ and (4.57) above holds. Let $\Gamma' = \{\surd, \gamma_1, \cdots, \gamma_k\}$, $m = \mathrm{lgt}(w)$, and let

$$w = \tau^{\ell(0)} \cdot c_1' \cdot \tau^{\ell(1)} \cdot c_1' \cdot \ \cdots \ \cdot \tau^{\ell(\nu-1)} \cdot c_{\nu-1}' \cdot \tau^{\ell(\nu)}.$$

Let

$$\begin{aligned} C_0 = \mathrm{chan}[&\mathcal{S}_2^{(m)}(s_1) \cup \bigcup \{\mathcal{S}_2^{(i)}(s_1) : i \in (m + 2 \cdot \nu)\} \\ &\cup \{\gamma_1, \ldots, \gamma_k\} \cup \mathrm{sort}[\{c_0', \cdots, c_{\nu-1}'\}]\ ]. \end{aligned} \tag{4.64}$$

Then, $C_0$ is finite, and therefore, the set $(\mathbf{Chan} \setminus C_0)$ is nonempty under the assumption that **Chan** is infinite. Choosing an element $\mathbf{c}$ of this set, let us define $\hat{s}'$ and $\hat{s}''$ as follows:

$$\hat{s}' \equiv \mathbf{e} + \chi(\overline{\gamma_1}) + \cdots + \chi(\overline{\gamma_k}),$$

$$\hat{s}'' \equiv (\mathbf{c}!(\hat{v}); \overline{c_0'}; \mathbf{c}!(\hat{v})); \cdots; (\mathbf{c}!(\hat{v}); \overline{c_{\nu-1}'}; \mathbf{c}!(\hat{v})); \mathbf{c}!(\hat{v})); \hat{s}'.$$

Finally, let $C = \mathbf{Chan} \setminus \{\mathbf{c}\}$, and put

$$w' = (\tau^{\ell(0)} \cdot \mathbf{c}!\hat{v} \cdot \tau \cdot \mathbf{c}!\hat{v}) \cdot \cdots \cdot (\tau^{\ell(\nu-1)} \cdot \mathbf{c}!\hat{v} \cdot \tau \cdot \mathbf{c}!\hat{v}) \cdot \tau^{\ell(\nu)} \cdot \mathbf{c}!\hat{v}. \quad (4.65)$$

Let us show

$$\textbf{(i)}\ w' \cdot \langle\delta\rangle \in \mathcal{O}_2^{\mathrm{m}}[\![\partial_C(s_1 \parallel \hat{s}'')]\!], \quad \textbf{(ii)}\ w' \cdot \langle\delta\rangle \notin \mathcal{O}_2^{\mathrm{m}}[\![\partial_C(s_2 \parallel \hat{s}'')]\!]. \quad (4.66)$$

First, it follows from (4.56) (i) that there exists $s_1'$ satisfying (4.59) above. With this $s_1'$, one has

$$(s_1 \parallel \hat{s}'') \xrightarrow{w'}_{2*} (s_1' \parallel \hat{s}'). \quad (4.67)$$

We observe that

$$\mathrm{act}_2(s_1' \parallel \hat{s}') = (\mathrm{act}_2(s_1') \setminus \{\surd\}) \cup (\mathrm{act}_2(\hat{s}') \setminus \{\surd\}) \not\ni \tau, \surd,$$

and

$$\mathbf{c} \notin \mathrm{chan}[\mathrm{sort}[\mathrm{act}_2(s_1' \parallel \hat{s}')]].$$

Thus,

$$\mathrm{chan}[\mathrm{sort}[\mathrm{act}_2(s_1' \parallel \hat{s}')]] \subseteq \mathbf{Chan} \setminus \{\mathbf{c}\} = C. \quad (4.68)$$

By (4.67) and the fact that $\mathbf{c} \notin C_0$, one has

$$\partial_C(s_1 \parallel \hat{s}'') \xrightarrow{w'}_{2*} \partial_C(s_1' \parallel \hat{s}').$$

Also, by (4.68), one has $\mathrm{act}_2(\partial_C(s_1' \parallel \hat{s}')) = \emptyset$. Thus, one has

$$w' \cdot \langle\delta\rangle \in \mathcal{O}_2^{\mathrm{m}}[\![\partial_C(s_1 \parallel \hat{s}'')]\!],$$

i.e., one has (4.66) (i).

Next, let us show (4.66) (ii) by contradiction. Assume, to the contrary,

$$w' \cdot \langle\delta\rangle \in \mathcal{O}_2^{\mathrm{m}}[\![\partial_C(s_2 \parallel \hat{s}'')]\!]. \quad (4.69)$$

Then, there exists $s'$ such that

$$\textbf{(i)}\ (s_2 \parallel \hat{s}'') \xrightarrow{w'} s', \quad \textbf{(ii)}\ \mathrm{chan}[\mathrm{sort}[\mathrm{act}_2(s')]] \subseteq C. \quad (4.70)$$

Since $\mathbf{c} \in (\mathbf{Chan} \setminus C_0)$ with $C_0$ defined in (4.64), all the $\mathbf{c}!\hat{v}$-actions in $w'$ must stem from $\hat{s}''$, and therefore, the $i$-th $\tau$-action in $w'$ preceded and followed by a $\mathbf{c}!\hat{v}$-action must be the synchronization of two actions $c_i'$- and $\overline{c_i'}$ stemming from $s_2$ and $\hat{s}''$, respectively ($i \in \nu$). Moreover, all the other actions in $w'$, which consist of $\ell(0) + \cdots + \ell(\nu)$ $\tau$-actions, must stem from $s_2$, since $\hat{s}''$ must perform exactly $(3\nu + 1)$ actions until it perform the $(2\nu + 1)$-th $\mathbf{c}!\hat{v}$-action. Thus, the actions:

$$(\tau^{\ell(0)} \cdot c_0) \cdot \cdots \cdot (\tau^{\ell(\nu-1)} \cdot c_{\nu-1}) \cdot \tau^{\ell(\nu)} = w$$

(in $w'$) must stem from $s_2$, and therefore, there is $s_2'$ such that

$$\textbf{(i)}\ s_2 \xrightarrow{w} s_2', \quad \textbf{(ii)}\ s' \equiv (s_2' \parallel \hat{s}'). \quad (4.71)$$

By (4.70) (ii) and (4.71) (ii), one has $\mathrm{chan}[\mathrm{sort}[\mathrm{act}_2(s_2' \parallel \hat{s}')]] \subseteq C$. Thus,

$$\tau \notin \mathrm{act}_2(s_2') \wedge \mathrm{sort}[\mathrm{act}_2(s_2')] \cap \Gamma' = \emptyset.$$

From this and (4.71) (i), it follows that $w \cdot \langle \delta(\Gamma') \rangle \in \mathcal{M}_2[\![s_2]\!]$, which contradicts (4.57) (ii). Summarizing, (4.69) does not hold, i.e., one has (4.66) (ii).
**Case 2.** Consider the case $\surd \in \Gamma$. This case is reduced to Case 1, as in part (1). ■

## 4.4 Concluding Remarks

We have presented denotational models $\mathcal{M}_1$ and $\mathcal{M}_2$ for two languages $\mathcal{L}_1$ and $\mathcal{L}_2$ for communicating processes, and established the full abstractness of each of $\mathcal{M}_1$ and $\mathcal{M}_2$ w.r.t. two linear models. The first language $\mathcal{L}_1$ is a simple concurrent language with predefined actions which are uninterpreted; the second language $\mathcal{L}_2$ is an *applicative* language which is based on $\mathcal{L}_1$ and supports *value-passing* in addition to the constructs of $\mathcal{L}_1$. It will be easy to see that when the set $\mathbf{V}$ of values is reduced to a singleton set, then $\mathcal{L}_2$ is reduced to $\mathcal{L}_1$. In that case, we can define, in an obvious way, two mappings $\phi : \mathcal{L}_2[\emptyset] \to \mathcal{L}_1$ and $\tilde{\phi} : \mathbf{P}_2 \to \mathbf{P}_1$ such that the following diagram commutes:

$$\boxed{\begin{array}{ccc} \mathcal{L}_2 & \xrightarrow{\phi} & \mathcal{L}_1 \\ \downarrow \mathcal{M}_2 & & \downarrow \mathcal{M}_1 \\ \mathbf{P}_2 & \xrightarrow{\tilde{\phi}} & \mathbf{P}_1 \end{array}} \tag{4.72}$$

In this chapter, we have treated so-called *strong models* which are strong in that they do not abstract from internal actions $\tau$. In Chapter 5, which is a minor revision of [Hor 92a], and Chapter 6, similar full abstractness problems for *weak* models for a language $\mathcal{L}_{\bar{1}}$, which is essentially the same as $\mathcal{L}_1$, are investigated; for example, the full abstractness of a failures model $\mathcal{C}_{\bar{1}}^{\mathrm{w}}$ w.r.t. two weak linear models $\mathcal{O}_{\bar{1}}^{\mathrm{w}}$ and $\mathcal{O}_{\bar{1}}^{\mathrm{wm}}$ is established in Chapter 5, where $\mathcal{C}_{\bar{1}}^{\mathrm{w}}$, $\mathcal{O}_{\bar{1}}^{\mathrm{w}}$, and $\mathcal{O}_{\bar{1}}^{\mathrm{wm}}$ are *weak* versions of $\mathcal{M}_1$, $\mathcal{O}_1$, and $\mathcal{O}_{\bar{1}}^{\mathrm{m}}$. (Roughly speaking, the weak models $\mathcal{C}_{\bar{1}}^{\mathrm{w}}$, $\mathcal{O}_{\bar{1}}^{\mathrm{w}}$, and $\mathcal{O}_{\bar{1}}^{\mathrm{wm}}$ are obtained from the strong models $\mathcal{M}_1$, $\mathcal{O}_1$, and $\mathcal{O}_{\bar{1}}^{\mathrm{m}}$, respectively, by abstracting from internal actions.)

In this chapter, the full abstractness of $\mathcal{M}_1$ w.r.t. $\mathcal{O}_{\bar{1}}^{\mathrm{m}}$ has been established under the assumption that the set $\mathbf{C}_2$ of communication actions is infinite. Somewhat surprisingly, however, a corresponding result for weak models, i.e., the full abstractness of $\mathcal{C}_{\bar{1}}^{\mathrm{w}}$ w.r.t. $\mathcal{O}_{\bar{1}}^{\mathrm{wm}}$, can be established without such an assumption (see Theorem 5.1). Note also that although in [BKO 88], a similar full abstractness result was established without such an assumption that $\mathbf{C}_2$ is infinite, the full abstractness result in [BKO 88] relies on a somewhat artificial *renaming* construct which maps *internal actions* to *communication actions* (see the proof of Theorem 7.2.1 of [BKO 88]).

Furthermore, in Chapter 5, a compositional model $\mathcal{C}_1^{\mathrm{wi}}$, which is obtained from $\mathcal{C}_1^{\mathrm{w}}$ by abstracting away from certain (redundant) information, was shown to be fully abstract w.r.t. a weak linear internal model $\mathcal{O}_1^{\mathrm{wi}}$, which is obtained from the strong version $\mathcal{O}_1^{\mathrm{i}}$ discussed in § 4.2.5.

In many respects, the treatment of weak models is more difficult than that of strong ones, but the order of difficulty seems opposite with the problem treated here. Roughly speaking, the reason why the treatment of strong models may be more difficult than that of weak models is as follows: Two processes distinguished in the strong model $\mathcal{M}_1$ may be so similar that, when composed with another process, the difference cannot be detected anymore in the more abstract strong model $\mathcal{O}_1^{\mathrm{m}}$, whereas such two processes are simply identified in the weak model $\mathcal{C}_1^{\mathrm{w}}$ corresponding to $\mathcal{M}_1$. An example of such two processes is given by:

**Example 4.8** Let

$$s_1 \equiv (c; \tau; ((c; \mathbf{0}) + (\tau; c; \mathbf{0}))) + (\tau; c; c; \mathbf{0}),$$

$$s_2 \equiv (c; ((\tau; c; \mathbf{0}) + (\tau; \tau; c; \mathbf{0}))) + (\tau; c; c; \mathbf{0}).$$

Then, $\mathcal{M}_1[\![s_1]\!] \neq \mathcal{M}_1[\![s_2]\!]$, since

$$\langle c, \tau, \delta(\emptyset)\rangle \notin \mathcal{M}_1[\![s_1]\!] \quad \text{but} \quad \langle c, \tau, \delta(\emptyset)\rangle \in \mathcal{M}_1[\![s_2]\!].$$

It is easy to check that

$$\mathcal{O}_1^{\mathrm{m}}[\![s_1]\!] \doteq \mathcal{O}_1^{\mathrm{m}}[\![s_2]\!] = \{\langle c, \tau, c\rangle, \langle c, \tau, \tau, c\rangle, \langle \tau, c, c\rangle\} \cdot \tilde{\mathbf{0}}_1,$$

and that

$$\begin{aligned} &\mathcal{C}_1^{\mathrm{w}}[\![s_1]\!] = \mathcal{C}_1^{\mathrm{w}}[\![s_2]\!] \\ &= \{\langle c, c\rangle\} \cdot \tilde{\mathbf{0}}_1 \cup \{\langle c, \delta(\Gamma)\rangle : \Gamma \in \wp(\mathbf{C}_1^{\checkmark}) \wedge c \notin \Gamma\} \\ &\quad \cup \{\langle \delta(\Gamma)\rangle : \Gamma \in \wp(\mathbf{C}_1^{\checkmark}) \wedge c \notin \Gamma\}. \end{aligned}$$

Without the assumption that $\mathbf{C}_1$ is infinite, we do not know whether or not it is possible to construct a suitable context $S_{\langle X\rangle} \in \mathcal{L}_1$ such that

$$\mathcal{O}_1^{\mathrm{m}}[\![S_{\langle X\rangle}[s_1]]\!] \neq \mathcal{O}_1^{\mathrm{m}}[\![S_{\langle X\rangle}[s_2]]\!]. \blacksquare$$

In Chapters 5 and 6, only the pure language $\mathcal{L}_{\bar{1}}$ is treated. We expect it will be possible to extend the result in Chapter 5 to construct a model for the applicative language $\mathcal{L}_2$ so that the extended model is fully abstract w.r.t. the weak linear models $\mathcal{O}_1^{\mathrm{w}}$ and $\mathcal{O}_1^{\mathrm{wm}}$; this extension also remains for future research.

As mentioned in the introduction, the result in [HI 90] is closely related to the result presented in this chapter; it also remains for future research to relate the *Acceptance Tree Model* ($\mathbf{AT}_{\mathrm{s}}^{\mathrm{v}}$) in [HI 90] and our model $\mathcal{M}_2$. Note that, for the pure language $\mathcal{L}_{\bar{1}}$, *the strong Acceptance tree model* $\mathbf{AT}_{\mathrm{s}}$ and a variant of $\mathcal{C}_1^{w}$ are shown to be isomorphic to each other, in § 6.6.3.

## 4.A Proof of Lemma 4.4

Lemma 4.4 can be proved along the lines of the proof of a similar result Theorem 4.3 in [Rut 89]. For the proof, we need a few preliminaries.

First, we observe that the languages $\mathcal{L}_1$ and $\mathcal{G}_1$ can be defined by explicit induction as follows:

**Definition 4.26 (1)** For $n \in \omega$, let $\hat{\mathcal{L}}_1(n)$ be defined as follows by induction on $n$:

(i) $\hat{\mathcal{L}}_1(0) = \{\mathbf{0}, \mathbf{e}\} \cup \mathbf{A}_1 \cup \mathcal{RV}$.

(ii)
$$\begin{aligned}\hat{\mathcal{L}}_1(n+1) = \hat{\mathcal{L}}_1(n) & \\ & \cup\{(s_1; s_2) : s_1, s_2 \in \hat{\mathcal{L}}_1(n)\} \\ & \cup\{(s_1 + s_2) : s_1, s_2 \in \hat{\mathcal{L}}_1(n)\} \\ & \cup\{(s_1 \parallel s_2) : s_1, s_2 \in \hat{\mathcal{L}}_1(n)\} \\ & \cup\{\partial_C(s) : C \subseteq \mathbf{C}_1 \wedge s \in \hat{\mathcal{L}}_1(n)\}.\end{aligned}$$

Then, let $\hat{\mathcal{L}}_1 = \bigcup\{\hat{\mathcal{L}}_1(n) : n \in \omega\}$.

**(2)** For $n \in \omega$, let $\mathcal{G}_1(n)$ be defined as follows by induction on $n$:

(i) $\mathcal{G}_1(0) = \{\mathbf{0}\} \cup \mathbf{A}_1$.

(ii)
$$\begin{aligned}\mathcal{G}_1(n+1) = \mathcal{G}_1(n) & \\ & \cup\{(g; s) : g \in \mathcal{G}_1(n) \wedge s \in \mathcal{L}_1[\emptyset]\} \\ & \cup\{(g_1 + g_2) : g_1, g_2 \in \mathcal{G}_1(n)\} \\ & \cup\{(g_1 \parallel g_2) : g_1, g_2 \in \mathcal{G}_1(n)\} \\ & \cup\{\partial_C(g) : C \subseteq \mathbf{C}_1 \wedge g \in \mathcal{G}_1(n)\}.\end{aligned}$$

Then, let $\hat{\mathcal{G}}_1 = \bigcup\{\mathcal{G}_1(n) : n \in \omega\}$. ∎

Then, it is easy to check that the following holds:

**Lemma 4.22 (1)** $\mathcal{L}_1 = \hat{\mathcal{L}}_1$. **(2)** $\mathcal{G}_1 = \hat{\mathcal{G}}_1$. ∎

The semantic operation $\tilde{;}_1$ can be characterized as a fixed-point as follows:

**Lemma 4.23** *For* $p_1, p_2 \in \mathbf{P}_1$, *let*

$$\begin{aligned}&\mathcal{R}_1^{;}(p_1, p_2) \\ &= \{\langle\delta(\Gamma)\rangle : \Gamma \subseteq \mathbf{C}_1^{\checkmark} \wedge \\ &\quad (\langle\delta(\Gamma \cup \{\surd\})\rangle \in p_1 \vee (\langle\delta(\Gamma \setminus \{\surd\})\rangle \in p_1 \wedge \langle\delta(\Gamma)\rangle \in p_2))\}.\end{aligned}$$

*Then, for* $p_1, p_2 \in \mathbf{P}_1$, *one has*

$$\begin{aligned}p_1 \tilde{;}_1 p_2 = \ & \mathrm{if}(\epsilon \in p_1, p_2 \setminus \{\langle\delta(\Gamma)\rangle : \Gamma \in \wp(\mathbf{C}_1^{\checkmark})\}, \emptyset) \\ & \cup \mathcal{R}_1^{;}(p_1, p_2) \\ & \cup \bigcup\{\langle a\rangle \cdot (p_1[\langle a\rangle] \tilde{;}_1 p_2) : a \in \mathbf{A}_1 \wedge p_1[\langle a\rangle] \neq \emptyset\}.\end{aligned}$$ ∎

**Proof.** Easily verified using the definition of $\tilde{;}_1$. ∎

One has the following lemma; from which Lemma 4.4 follows immediately.

**Lemma 4.24 (1)** *All the semantic operations* $\tilde{;}_1$, $\tilde{+}_1$, $\tilde{\|}_1$, $\tilde{\partial}^1_C(\cdot)$ *are nonexpansive.*

**(2)** $\forall s \in \mathcal{L}_1[\emptyset], \forall \tilde{\mathbf{p}}_1, \tilde{\mathbf{p}}_2 \in \mathbf{M}_1^{\mathcal{M}}[\ d_{\mathbf{P}_1}([\![s]\!]^{\mathcal{I}_1 \cup \dot{\mathbf{p}}_1}, [\![s]\!]^{\mathcal{I}_1 \cup \dot{\mathbf{p}}_2}) \le d_{\mathbf{M}_1^{\mathcal{M}}}(\tilde{\mathbf{p}}_1, \tilde{\mathbf{p}}_2)\ ]$

**(3)** *For* $p \in \mathbf{P}_1$, *let us write* $\Phi_1^{\mathrm{g}}(p)$ *to denote that*

$$\forall \Gamma \in \wp(\mathbf{C}_1)[\ \langle \delta(\Gamma) \rangle \in p \Rightarrow \langle \delta(\Gamma \cup \{\surd\}) \rangle \in p\ ] \wedge \epsilon \notin p.$$

*Then, one has* $\forall g \in \mathcal{G}_1, \forall \tilde{\mathbf{p}} \in \mathbf{M}_1^{\mathcal{M}}[\ \Phi_1^{\mathrm{g}}([\![g]\!]^{\mathcal{I}_1 \cup \dot{\mathbf{p}}})\ ]$.

**(4)** $\forall p_1, p_1', p_2, p_2' \in \mathbf{P}_1[\ \Phi_1^{\mathrm{g}}(p_1) \wedge \Phi_1^{\mathrm{g}}(p_1') \Rightarrow$
$d_{\mathbf{P}_1}((p_1 \tilde{;}_1 p_2), (p_1' \tilde{;}_1 p_2')) \le \max\{d_{\mathbf{P}_1}(p_1, p_1'), \kappa \cdot d_{\mathbf{P}_1}(p_2, p_2')\}\ ]$.

**(5)** $\forall g \in \mathcal{G}_1, \forall \tilde{\mathbf{p}}_1, \tilde{\mathbf{p}}_2 \in \mathbf{M}_1^{\mathcal{M}}[\ d_{\mathbf{P}_1}([\![g]\!]^{\mathcal{I}_1 \cup \dot{\mathbf{p}}_1}, [\![g]\!]^{\mathcal{I}_1 \cup \dot{\mathbf{p}}_2}) \le \kappa \cdot d_{\mathbf{M}_1^{\mathcal{M}}}(\tilde{\mathbf{p}}_1, \tilde{\mathbf{p}}_2)\ ]$. ∎

**Proof. (1)** Easily verified using the definition of the semantic operations.
**(2)** One can show the following holds for every $n \in \omega$, by induction on $n$:

$$\begin{aligned}&\forall s \in \hat{\mathcal{L}}_1(n)[\emptyset], \forall \tilde{\mathbf{p}}_1, \tilde{\mathbf{p}}_2 \in \mathbf{M}_1^{\mathcal{M}}[\\ &\quad d_{\mathbf{P}_1}([\![s]\!]^{\mathcal{I}_1 \cup \dot{\mathbf{p}}_1}, [\![s]\!]^{\mathcal{I}_1 \cup \dot{\mathbf{p}}_2}) \le d_{\mathbf{M}_1^{\mathcal{M}}}(\tilde{\mathbf{p}}_1, \tilde{\mathbf{p}}_2)\ ].\end{aligned} \tag{4.73}$$

The induction base is straightforward. In the induction step, part (1) is conveniently used. From (4.73) and Lemma 4.22 (1), the claim follows.
**(3)** One can show the following holds for every $n \in \omega$ by induction on $n$:

$$\forall g \in \mathcal{G}_1(n), \forall \tilde{\mathbf{p}} \in \mathbf{M}_1^{\mathcal{M}}[\ \Phi_1^{\mathrm{g}}([\![g]\!]^{\mathcal{I}_1 \cup \dot{\mathbf{p}}})\ ]. \tag{4.74}$$

The induction base is straightforward. In the induction step, we use the fact that all the semantic operations preserve the property $\Phi_1^{\mathrm{g}}$. From (4.74) and Lemma 4.22 (2), the claim follows.
**(4)** One can establish the claim by the definition of $d_{\mathbf{P}_1}$, exploiting the fixed-point characterization of $\tilde{;}_1$ stated by Lemma 4.23.
**(5)** One can show the following holds for every $n \in \omega$ by induction on $n$:

$$\begin{aligned}&\forall g \in \mathcal{G}_1(n), \forall \tilde{\mathbf{p}}_1, \tilde{\mathbf{p}}_2 \in \mathbf{M}_1^{\mathcal{M}}[\\ &\quad d_{\mathbf{P}_1}([\![g]\!]^{\mathcal{I}_1 \cup \dot{\mathbf{p}}_1}, [\![g]\!]^{\mathcal{I}_1 \cup \dot{\mathbf{p}}_2}) \le \kappa \cdot d_{\mathbf{M}_1^{\mathcal{M}}}(\tilde{\mathbf{p}}_1, \tilde{\mathbf{p}}_2)\ ].\end{aligned} \tag{4.75}$$

The induction base is straightforward. In the induction step, parts (1)–(4) are conveniently used. From (4.75) and Lemma 4.22 (2), the claim follows. ∎

**Proof of Lemma 4.4.** Let $\tilde{\mathbf{p}}_1, \tilde{\mathbf{p}}_1 \in \mathbf{M}_1^{\mathcal{M}}$. Then, by Lemma 4.24 (5), one has the following for every $P \in \mathcal{RV}$:

$$d_{\mathbf{P}_1}([\![g_P]\!]^{\mathcal{I}_1 \cup \dot{\mathbf{p}}_1}, [\![g_P]\!]^{\mathcal{I}_1 \cup \dot{\mathbf{p}}_2}) \le \kappa \cdot d_{\mathbf{M}_1^{\mathcal{M}}}(\tilde{\mathbf{p}}_1, \tilde{\mathbf{p}}_2).$$

Thus, one has

$$\begin{aligned}&d_{\mathbf{M}_1^{\mathcal{M}}}(\Pi_1(\tilde{\mathbf{p}}_1), \Pi_1(\tilde{\mathbf{p}}_2))\\ &= d_{\mathbf{M}_1^{\mathcal{M}}}((\lambda P \in \mathcal{RV}.\ [\![g_P]\!]^{\mathcal{I}_1 \cup \dot{\mathbf{p}}_1}), (\lambda P \in \mathcal{RV}.\ [\![g_P]\!]^{\mathcal{I}_1 \cup \dot{\mathbf{p}}_2}))\\ &= \sup\{d_{\mathbf{P}_1}([\![g_P]\!]^{\mathcal{I}_1 \cup \dot{\mathbf{p}}_1}, [\![g_P]\!]^{\mathcal{I}_1 \cup \dot{\mathbf{p}}_2}) : P \in \mathcal{RV}\}\\ &\le \kappa \cdot d_{\mathbf{M}_1^{\mathcal{M}}}(\tilde{\mathbf{p}}_1, \tilde{\mathbf{p}}_2).\end{aligned}$$

Since $\tilde{\mathbf{p}}_1$, $\tilde{\mathbf{p}}_2$ were arbitrary, $\Pi_1$ is a contraction from $\mathbf{M}_1^{\mathcal{M}}$ to $\mathbf{M}_1^{\mathcal{M}}$. ∎

## 4.B Proof of Lemma 4.15

Lemma 4.15 can be proved in a similar fashion to the proof of Lemma 4.4, with suitable adaptation for the setting of $\mathcal{L}_2$.

First, we observe that the languages $\mathcal{L}_2$ and $\mathcal{G}_2$ can also be defined by explicit induction as follows:

**Definition 4.27** **(1)** We will define $\hat{\mathcal{L}}_2(n)$ and $\hat{\mathcal{L}}_2^{(1)}(n)$ by induction on $n$ as follows:

**(i)**
$$\hat{\mathcal{L}}_2(0) = \{\mathbf{0}, \mathbf{e}, \tau\} \cup \{\mathbf{c}!(E) : E \in \tilde{\mathcal{E}}\} \cup \mathcal{RV}.$$
$$\hat{\mathcal{L}}_2^{(1)}(0) = \mathcal{RV}^{(1)}.$$

**(ii)**
$$\begin{aligned}\hat{\mathcal{L}}_2(n+1) = \; & \hat{\mathcal{L}}_2(n) \cup \{\mathbf{c}?(T) : T \in \hat{\mathcal{L}}_2(n)\} \\ & \cup\{(S_1; S_2) : S_1, S_2 \in \hat{\mathcal{L}}_2(n)\} \\ & \cup\{(S_1 + S_2) : S_1, S_2 \in \hat{\mathcal{L}}_2(n)\} \\ & \cup\{(S_1 \parallel S_2) : S_1, S_2 \in \hat{\mathcal{L}}_2(n)\} \\ & \cup\{\partial_C(S) : C \subseteq \mathbf{Chan} \wedge S \in \hat{\mathcal{L}}_2(n)\} \\ & \cup\{\mathbf{if}(E, S_1, S_2) : E \in \tilde{\mathcal{E}} \\ & \qquad \wedge S_1, S_2 \in \hat{\mathcal{L}}_2(n)\} \\ & \cup\{T(E) : T \in \hat{\mathcal{L}}_2^{(1)}(n) \wedge E \in \tilde{\mathcal{E}}\}.\end{aligned}$$

$$\begin{aligned}\hat{\mathcal{L}}_2^{(1)}(n+1) = \; & \hat{\mathcal{L}}_2^{(1)}(n) \\ & \cup\{(\lambda x.\ S) : S \in \hat{\mathcal{L}}_2(n)\}.\end{aligned}$$

Then, let

$$\hat{\mathcal{L}}_2 = \bigcup\{\hat{\mathcal{L}}_2(n) : n \in \omega\},$$

$$(\hat{\mathcal{L}}_2^{(1)}) = \bigcup\{\hat{\mathcal{L}}_2^{(1)}(n) : n \in \omega\},$$

and

$$\hat{\mathcal{L}}_2^* = \hat{\mathcal{L}}_2 \cup (\hat{\mathcal{L}}_2^{(1)}).$$

For each $n \in \omega$, let

$$\hat{\mathcal{L}}_2^*(n) = \hat{\mathcal{L}}_2(n) \cup \hat{\mathcal{L}}_2^{(1)}(n).$$

**(2)** For $n \in \omega$, let $\hat{\mathcal{G}}_2(n)$ and $\hat{\mathcal{G}}_2^{(1)}(n)$ be defined by induction on $n$ as follows:

**(i)**
$$\begin{aligned}\hat{\mathcal{G}}_2(0) = \; & \{\mathbf{0}\} \cup \{\tau\} \\ & \cup\{\mathbf{c}!(E) : \mathbf{c} \in \mathbf{Chan} \wedge E \in \tilde{\mathcal{E}}\} \\ & \cup\{\mathbf{c}?(T) : \mathbf{c} \in \mathbf{Chan} \wedge T \in (\hat{\mathcal{L}}_2^{(1)})\}.\end{aligned}$$

$$\hat{\mathcal{G}}_2^{(1)}(0) = \emptyset.$$

$$\textbf{(ii)}\qquad \begin{aligned}\hat{\mathcal{G}}_2(n+1) = \ & \hat{\mathcal{G}}_2(n) \cup \{(G;S) : G \in \hat{\mathcal{G}}_2(n) \wedge S \in (\hat{\mathcal{L}}_2^{(1)})\} \\ & \cup\{(G_1 + G_2) : G_1, G_2 \in \hat{\mathcal{G}}_2(n)\} \\ & \cup\{(G_1 \,\|\, G_2) : G_1, G_2 \in \hat{\mathcal{G}}_2(n)\} \\ & \cup\{\partial_C(G) :: C \in \wp(\mathbf{Chan}) \wedge G \in \hat{\mathcal{G}}_2(n)\} \\ & \cup\{\mathbf{if}(E, G_1, G_2) : E \in \tilde{\mathcal{E}} \wedge G_1, G_2 \in \hat{\mathcal{G}}_2(n)\} \\ & \cup\{H(E) : H \in \hat{\mathcal{G}}_2^{(1)}(n) \wedge E \in \tilde{\mathcal{E}}\}.\end{aligned}$$

$$\begin{aligned}\hat{\mathcal{G}}_2^{(1)}(n+1) = \ & \hat{\mathcal{G}}_2^{(1)}(n) \\ & \cup\{(\lambda x.\ G) : x \in \mathcal{X}_{\mathcal{V}} \wedge G \in \hat{\mathcal{G}}_2(n)\}.\end{aligned}$$

Then, let

$$\hat{\mathcal{G}}_2 = \bigcup\{\hat{\mathcal{G}}_2(n) : n \in \omega\},$$

$$\hat{\mathcal{G}}_2^{(1)} = \bigcup\{\hat{\mathcal{G}}_2^{(1)}(n) : n \in \omega\},$$

and

$$\hat{\mathcal{G}}_2^* = \hat{\mathcal{G}}_2 \cup \hat{\mathcal{G}}_2^{(1)}.$$

For each $n \in \omega$, let

$$\hat{\mathcal{G}}_2^*(n) = \hat{\mathcal{G}}_2(n) \cup \hat{\mathcal{G}}_2^*(n). \ \blacksquare$$

Then, it is easy to check that the following holds:

**Lemma 4.25 (1) (i)** $\mathcal{L}_2 = \hat{\mathcal{L}}_2$. **(ii)** $\mathcal{L}_2^{(1)} = \hat{\mathcal{L}}_2^{(1)}$.
**(2) (i)** $\mathcal{G}_2 = \hat{\mathcal{G}}_2$. **(ii)** $\mathcal{G}_2^{(1)} = \hat{\mathcal{G}}_2^{(1)}$. ■

As $\tilde{;}_1$, the semantic operation $\tilde{;}_2$ can be characterized as a fixed-point as follows:

**Lemma 4.26** *For* $p_1, p_2 \in \mathbf{P}_2$, *let*

$$\begin{aligned}&\mathcal{R}_2^{;}(p_1, p_2) \\ &= \{\langle\delta(\Gamma)\rangle : \Gamma \subseteq \mathbf{C}_2^{\checkmark} \wedge \\ &\quad ((\langle\delta(\Gamma \cup \{\surd\})\rangle \in p_1 \vee (\langle\delta(\Gamma \setminus \{\surd\})\rangle \in p_1 \wedge \langle\delta(\Gamma)\rangle \in p_2))\}.\end{aligned}$$

*Then, for* $p_1, p_2 \in \mathbf{P}_1$, *one has*

$$\begin{aligned}p_1 \tilde{;}_2 p_2 = \ & \mathrm{if}(\epsilon \in p_1, p_2 \setminus \{\langle\delta(\Gamma)\rangle : \Gamma \in \wp(\mathbf{C}_2^{\checkmark})\}, \emptyset) \\ & \cup\mathcal{R}_2^{;}(p_1, p_2) \\ & \cup\bigcup\{\langle a\rangle \cdot (p_1[\langle a\rangle] \tilde{;}_2 p_2) : a \in \mathbf{A}_2 \wedge p_1[\langle a\rangle] \neq \emptyset\}. \ \blacksquare\end{aligned}$$

**Proof.** Easily verified using the definition of $\tilde{;}_2$. ■

One has the following lemma, from which Lemma 4.15 follows immediately.

**Lemma 4.27 (1)** *All the semantic operations* $\tilde{;}_2$, $\tilde{+}_2$, $\tilde{\|}_2$, $\tilde{\partial}_C^2(\cdot)$, *and* $\mathrm{if}_2(v, \cdot, \cdot)$ $(v \in \mathbf{V})$ *are nonexpansive.*

**(2)** *Let $k \in \omega$, $\vec{x} \in (k \rightarrowtail \mathcal{X}_\mathcal{V})$. Then,*

$$\forall U \in \mathcal{L}_2^*[\mathrm{ran}(\vec{x})], \forall \tilde{\mathbf{p}}_1, \tilde{\mathbf{p}}_2 \in \mathbf{M}_2^{\mathcal{M}}, \forall \vec{v} \in \mathbf{V}^k[ \\ d_{\mathbf{P}_2^*}([\![U_{\vec{x}}]\!]^{\mathcal{I}_2 \cup \tilde{\mathbf{P}}_1}(\vec{v}), [\![U_{\vec{x}}]\!]^{\mathcal{I}_2 \cup \tilde{\mathbf{P}}_2}(\vec{v})) \leq d_{\mathbf{M}_2^{\mathcal{M}}}(\tilde{\mathbf{p}}_1, \tilde{\mathbf{p}}_2) \ ].$$

**(3)** *For $p \in \mathbf{P}_2^*$, let us write $\Phi_2^{\mathrm{g}}(p)$ to denote that*

$$\forall \Gamma \in \wp(\mathbf{C}_2)[ \ \langle \delta(\Gamma) \rangle \in p \Rightarrow \langle \delta(\Gamma \cup \{\surd\}) \rangle \in p \ ] \wedge \epsilon \notin p.$$

*Also, for $r \in \mathbf{P}_2^{(1)}$, let us write $\Phi_2^{\mathrm{g}}(r)$ to denote that $\forall v \in \mathbf{V}[ \ \Phi_2^{\mathrm{g}}(r(v)) \ ]$. Then, one has the following for every $k \in \omega$ and $\vec{x} \in (k \rightarrowtail \mathcal{X}_\mathcal{V})$:*

$$\forall \tilde{G} \in \mathcal{G}_2^*[\mathrm{ran}(\vec{x})], \forall \vec{v} \in \mathbf{V}^k, \forall \tilde{\mathbf{p}} \in \mathbf{M}_2^{\mathcal{M}}[ \ \Phi_2^{\mathrm{g}}([\![\tilde{G}_{\vec{x}}]\!]^{\mathcal{I}_2 \cup \tilde{\mathbf{P}}}(\vec{v})) \ ].$$

**(4)** $\forall p_1, p_1', p_2, p_2' \in \mathbf{P}_2[ \ \Phi_2^{\mathrm{g}}(p_1) \wedge \Phi_2^{\mathrm{g}}(p_1') \Rightarrow$
$d_{\mathbf{P}_2}((p_1 \tilde{;}_2 p_2), (p_1' \tilde{;}_2 p_2')) \leq \max\{d_{\mathbf{P}_2}(p_1, p_1'), \kappa \cdot d_{\mathbf{P}_2}(p_2, p_2')\} \ ]$.

**(5)** *Let $k \in \omega$, $\vec{x} \in (k \rightarrowtail \mathcal{X}_\mathcal{V})$. Then,*

$$\forall \tilde{G} \in \hat{\mathcal{G}}_2^*[\mathrm{ran}(\vec{x})], \forall \vec{v} \in \mathbf{V}^k, \forall \tilde{\mathbf{p}}_1, \tilde{\mathbf{p}}_2 \in \mathbf{M}_2^{\mathcal{M}}[ \\ d_{\mathbf{P}_2^*}([\![G_{\vec{x}}]\!]^{\mathcal{I}_2 \cup \tilde{\mathbf{P}}_1}(\vec{v}), [\![G_{\vec{x}}]\!]^{\mathcal{I}_2 \cup \tilde{\mathbf{P}}_2}(\vec{v})) \leq \kappa \cdot d_{\mathbf{M}_2^{\mathcal{M}}}(\tilde{\mathbf{p}}_1, \tilde{\mathbf{p}}_2) \ ]. \blacksquare$$

**Proof. (1)** Easily verified using the definition of the semantic operations.
**(2)** One can show that the following holds for every $n \in \omega$, by induction on $n$:

$$\forall k \in \omega, \forall \vec{x} \in (k \rightarrowtail \mathcal{X}_\mathcal{V})[ \\ \forall U \in \mathcal{L}_2^*(n)[\mathrm{ran}(\vec{x})], \forall \tilde{\mathbf{p}}_1, \tilde{\mathbf{p}}_2 \in \mathbf{M}_2^{\mathcal{M}}, \forall \vec{v} \in \mathbf{V}^k[ \\ d_{\mathbf{P}_2^*}([\![U_{\vec{x}}]\!]^{\mathcal{I}_2 \cup \tilde{\mathbf{P}}_1}(\vec{v}), [\![U_{\vec{x}}]\!]^{\mathcal{I}_2 \cup \tilde{\mathbf{P}}_2}(\vec{v})) \leq d_{\mathbf{M}_2^{\mathcal{M}}}(\tilde{\mathbf{p}}_1, \tilde{\mathbf{p}}_2) \ ]]. \tag{4.76}$$

The induction base is straightforward. In the induction step, part (1) is conveniently used. From (4.76) and Lemma 4.25 (1), the claim follows.
**(3)** One can show that the following holds for every $n \in \omega$ by induction on $n$:

$$\forall k \in \omega, \forall \vec{x} \in (k \rightarrowtail \mathcal{X}_\mathcal{V})[ \\ \forall \tilde{G} \in \hat{\mathcal{G}}_2^*(n)[\mathrm{ran}(\vec{x})], \forall \vec{v} \in \mathbf{V}^k, \forall \tilde{\mathbf{p}} \in \mathbf{M}_2^{\mathcal{M}}[ \ \Phi_2^{\mathrm{g}}([\![\tilde{G}_{\vec{x}}]\!]^{\mathcal{I}_2 \cup \tilde{\mathbf{P}}}(\vec{v})) \ ]]. \tag{4.77}$$

The induction base is straightforward. In the induction step, we use the fact that all the semantic operations preserve the property $\Phi_2^{\mathrm{g}}$. From (4.77) and Lemma 4.25 (2), the claim follows.
**(4)** One can establish the claim by the definition of $d_{\mathbf{P}_2}$, exploiting the fixed-point characterization of $\tilde{;}_2$ stated by Lemma 4.26.
**(5)** One can show the following holds for every $n \in \omega$ by induction on $n$:

$$\forall k \in \omega, \forall \vec{x} \in (k \rightarrowtail \mathcal{X}_\mathcal{V})[ \\ \forall \tilde{G} \in \hat{\mathcal{G}}_2^*(n)[\mathrm{ran}(\vec{x})], \forall \tilde{\mathbf{p}}_1, \tilde{\mathbf{p}}_2 \in \mathbf{M}_2^{\mathcal{M}}[ \\ d_{\mathbf{P}_2}([\![\tilde{G}_{\vec{x}}]\!]^{\mathcal{I}_2 \cup \tilde{\mathbf{P}}_1}(\vec{v}), [\![\tilde{G}_{\vec{x}}]\!]^{\mathcal{I}_2 \cup \tilde{\mathbf{P}}_2}(\vec{v})) \leq \kappa \cdot d_{\mathbf{M}_2^{\mathcal{M}}}(\tilde{\mathbf{p}}_1, \tilde{\mathbf{p}}_2) \ ]]. \tag{4.78}$$

The induction base is straightforward. In the induction step, parts (1)–(4) are conveniently used. From (4.78) and Lemma 4.25 (2), the claim follows. ■

**Proof of Lemma 4.15.** Let $\tilde{\mathbf{p}}_1, \tilde{\mathbf{p}}_1 \in \mathbf{M}_2^{;\mathcal{M}}$. Then, by Lemma 4.27 (5), one has the following for every $Z \in \mathcal{RV}^*$:

$$d_{\mathbf{P}_2^*}([\![\tilde{g}z]\!]^{\mathcal{I}_2 \cup \dot{\mathbf{p}}_1}, [\![\tilde{g}z]\!]^{\mathcal{I}_2 \cup \dot{\mathbf{p}}_2}) \leq \kappa \cdot d_{\mathbf{M}_2^{;\mathcal{M}}}(\tilde{\mathbf{p}}_1, \tilde{\mathbf{p}}_2).$$

Thus, one has

$$\begin{aligned}
& d_{\mathbf{M}_2^{;\mathcal{M}}}(\Pi_2(\tilde{\mathbf{p}}_1), \Pi_2(\tilde{\mathbf{p}}_2)) \\
&= d_{\mathbf{M}_1^{;\mathcal{M}}}((\lambda Z \in \mathcal{RV}^*.\ [\![\tilde{g}z]\!]^{\mathcal{I}_2 \cup \dot{\mathbf{p}}_1}), (\lambda Z \in \mathcal{RV}^*.\ [\![\tilde{g}z]\!]^{\mathcal{I}_2 \cup \dot{\mathbf{p}}_2})) \\
&= \sup\{d_{\mathbf{P}_2^*}([\![\tilde{g}z]\!]^{\mathcal{I}_2 \cup \dot{\mathbf{p}}_1}, [\![\tilde{g}z]\!]^{\mathcal{I}_2 \cup \dot{\mathbf{p}}_2}) : Z \in \mathcal{RV}^*\} \\
&\leq \kappa \cdot d_{\mathbf{M}_2^{;\mathcal{M}}}(\tilde{\mathbf{p}}_1, \tilde{\mathbf{p}}_2).
\end{aligned}$$

Since $\tilde{\mathbf{p}}_1$, $\tilde{\mathbf{p}}_2$ were arbitrary, $\Pi_2$ is a contraction from $\mathbf{M}_2^{;\mathcal{M}}$ to $\mathbf{M}_2^{;\mathcal{M}}$. ■

$$\begin{aligned}&\texttt{Philosophers}\\&\equiv \partial_C(P_{(0,1,1)} \| P_{(1,1,1)} \| P_{(2,1,1)} \| P_{(3,1,1)} \| P_{(4,1,1)}),\end{aligned}$$

where,

$$C = \{\overline{\texttt{get}_n}\}_{n\in 5} \cup \{\overline{\texttt{put}_n}\}_{n\in 5} \cup \{\texttt{get}_n\}_{n\in 5} \cup \{\texttt{put}_n\}_{n\in 5}.$$

and for each $(n, i, j) \in 5 \times 2 \times 2$,

$$\begin{aligned}P_{(n,i,j)} \Leftarrow\ & (\overline{\texttt{think}_n}; P'_{(n,i,j,0)})\\&+(\texttt{get}_{\imath(n)}; P_{(n,0,j)}) + (\texttt{get}_{\jmath(n)}; P_{(n,i,0)})\\&+(\texttt{put}_{\imath(n)}; P_{(n,1,j)}) + (\texttt{put}_{\jmath(n)}; P_{(n,i,1)}),\end{aligned}$$

$$\begin{aligned}P'_{(n,1,1,0)} \Leftarrow\ & (\overline{\texttt{get}_{\imath(n)}}; P'_{(n,1,1,1)})\\&+(\texttt{get}_{\imath(n)}; P'_{(n,0,1,0)}) + (\texttt{get}_{\jmath(n)}; P'_{(n,1,0,0)}),\end{aligned}$$

$$P'_{(n,1,1,1)} \Leftarrow (\overline{\texttt{get}_{\jmath(n)}}; \overline{\texttt{eat}_n}; \overline{\texttt{put}_{\imath(n)}}; P''_{(n,1,1)}),$$

$$\begin{aligned}P''_{(n,i,1)} \Leftarrow\ & (\overline{\texttt{put}_{\jmath(n)}}; P_{(n,i,1)})\\&+(\texttt{get}_{\imath(n)}; P''_{(n,0,1)}) + (\texttt{put}_{\imath(n)}; P''_{(n,1,1)}),\end{aligned}$$

$$P'_{(n,1,0,0)} \Leftarrow (\texttt{put}_{\jmath(n)}; P'_{(n,1,1,0)}) + (\texttt{get}_{\imath(n)}; P'_{(n,0,0,0)}),$$

$$P'_{(n,1,0,1)} \Leftarrow (\texttt{put}_{\jmath(n)}; P'_{(n,1,1,1)}),$$

$$P'_{(n,0,1,0)} \Leftarrow (\texttt{put}_{\imath(n)}; P'_{(n,1,1,k)}) + (\texttt{get}_{\jmath(n)}; P'_{(n,0,0,k)}),$$

$$P'_{(n,0,0,0)} \Leftarrow (\texttt{put}_{\imath(n)}; P'_{(n,1,0,k)}) + (\texttt{put}_{\jmath(n)}; P'_{(n,0,1,k)}).$$

Figure 4.2: Description of Dining Philosophers in $\mathcal{L}_1$

$$\texttt{Accumulating-Responder} \equiv Y(0),$$

where

$$\begin{aligned} Y \Leftarrow (\lambda x_0.\mathbf{c}_0?(\lambda x_1.\ \mathbf{if}(&x_0 + x_1 \leq N, \\ &(\mathbf{c}_1!(x_0 + x_1); Y(x_0 + x_1)), \\ &(c_2!(0); Y(0))\ ))). \end{aligned}$$

Figure 4.3: Description of Accumulating Responder in $\mathcal{L}_2$

# Chapter 5

# Fully Abstract Models for Communicating Processes with respect to Weak Linear Semantics with Divergence

The semantics of a language $\mathcal{L}_{\bar{1}}$ for communicating processes is investigated, and three full abstractness results for $\mathcal{L}_{\bar{1}}$ are established. The language $\mathcal{L}_{\bar{1}}$ is a minor variant of $\mathcal{L}_1$ treated in Chapter 4; both the two languages have *atomic actions*, *termination*, *inaction*, *sequential* and *parallel composition*, *nondeterministic choice*, *action restriction*, and a form of *guarded* recursion. (The guardedness restriction on recursion is necessary to establish one of the full abstractness results.) The minor difference between $\mathcal{L}_1$ and $\mathcal{L}_{\bar{1}}$ is that $\mathcal{L}_1$ treats recursion in terms of *declarations*, whereas $\mathcal{L}_{\bar{1}}$ treats it in terms of *$\mu$-notation*.

Three Plotkin-style operational semantics $\mathcal{O}_{\bar{1}}^{\mathrm{w}}$, $\mathcal{O}_{\bar{1}}^{\mathrm{wm}}$, and $\mathcal{O}_{\bar{1}}^{\mathrm{wi}}$ of the language are defined. These semantics are *linear* in that the meaning of each program in any of these semantics is a set of *action sequences* the program may perform, and are *weak* in that the action sequences are obtained by abstracting from (finite sequences of) *internal moves*. All the three semantics distinguish *divergence* (an infinite sequence of internal moves) from deadlock.

The semantics $\mathcal{O}_{\bar{1}}^{\mathrm{wi}}$ differs from the other two in that $\mathcal{O}_{\bar{1}}^{\mathrm{wi}}$ is a so-called *internal action semantics* taking into account only internal moves under the assumption that the environment allows no (external) communication actions, and hence, the only possible actions for processes are internal moves, whereas the other two semantics take into account *communication actions* in addition to internal moves. The two semantics $\mathcal{O}_{\bar{1}}^{\mathrm{w}}$ and $\mathcal{O}_{\bar{1}}^{\mathrm{wm}}$ differ from each other in that $\mathcal{O}_{\bar{1}}^{\mathrm{wm}}$ is a so-called *maximal trace semantics*, whereas $\mathcal{O}_{\bar{1}}^{\mathrm{w}}$ is not.

Then, two compositional models $\mathcal{C}_{\bar{1}}^{\mathrm{w}}$ and $\mathcal{C}_{\bar{1}}^{\mathrm{wi}}$ for the language are proposed, and the *full abstractness* of $\mathcal{C}_{\bar{1}}^{\mathrm{w}}$ (resp. of $\mathcal{C}_{\bar{1}}^{\mathrm{wi}}$) w.r.t. $\mathcal{O}_{\bar{1}}^{\mathrm{w}}$ and $\mathcal{O}_{\bar{1}}^{\mathrm{wm}}$ (resp. w.r.t. $\mathcal{O}_{\bar{1}}^{\mathrm{wi}}$), as expressed in the following, is established:

$$\begin{aligned} &\mathcal{C}[\![s_1]\!] = \mathcal{C}[\![s_2]\!] \Leftrightarrow \\ &\forall S[\cdot][\; S[\cdot] \text{ is a } \textit{context} \text{ of } \mathcal{L}_{\bar{1}} \Rightarrow \mathcal{O}[\![S[s_1]]\!] = \mathcal{O}[\![S[s_2]]\!]\;], \end{aligned} \tag{5.1}$$

where

$$(\mathcal{C}, \mathcal{O}) = (\mathcal{C}_{\bar{1}}^{\mathrm{w}}, \mathcal{O}_{\bar{1}}^{\mathrm{w}}), (\mathcal{C}_{\bar{1}}^{\mathrm{w}}, \mathcal{O}_{\bar{1}}^{\mathrm{wm}}), (\mathcal{C}_{\bar{1}}^{\mathrm{wi}}, \mathcal{O}_{\bar{1}}^{\mathrm{wi}}).$$

A similar full abstractness result has been established by Bergstra, Klop, and Olderog for a language without recursion or internal moves. Moreover, Rutten investigated the semantics of a language similar to $\mathcal{L}_{\bar{1}}$, in the framework of complete metric spaces, and showed that the failures model is fully abstract w.r.t. a *strong linear semantics* $\mathcal{O}_{\bar{1}}^{\mathrm{i}}$, where $\mathcal{O}_{\bar{1}}^{\mathrm{i}}$ is *strong* in that it does not abstract from internal moves. The full abstractness of $\mathcal{C}_{\bar{1}}^{\mathrm{w}}$ w.r.t. $\mathcal{O}_{\bar{1}}^{\mathrm{wm}}$, expressed in (5.1) with $(\mathcal{C}, \mathcal{O}) = (\mathcal{C}_{\bar{1}}^{\mathrm{w}}, \mathcal{O}_{\bar{1}}^{\mathrm{wm}})$, is an extension of the result of Bergstra et al. to a language with recursion and internal moves. Also, the full abstractness of $\mathcal{C}_{\bar{1}}^{\mathrm{wi}}$ w.r.t. $\mathcal{O}_{\bar{1}}^{\mathrm{wi}}$, expressed in (5.1) with $(\mathcal{C}, \mathcal{O}) = (\mathcal{C}_{\bar{1}}^{\mathrm{wi}}, \mathcal{O}_{\bar{1}}^{\mathrm{wi}})$, is an extension of Rutten's result, to the case of *weak linear semantics with divergence*.

## 5.1 Introduction

The semantics of a language $\mathcal{L}_{\bar{1}}$ for communicating processes is investigated, and three full abstractness results for $\mathcal{L}_{\bar{1}}$ are established. The language $\mathcal{L}_{\bar{1}}$, to be presented in § 5.2, is a minor extension of the language investigated in [Rut 89], and a minor variant of $\mathcal{L}_1$ treated in Chapter 4; both $\mathcal{L}_{\bar{1}}$ and $\mathcal{L}_1$ have *atomic actions*, *termination*, *inaction*, *sequential* and *parallel composition*, *nondeterministic choice*, *action restriction*, and a form of *guarded* recursion. (The guardedness restriction on recursion is necessary to establish one of the full abstractness results.) The minor difference between $\mathcal{L}_1$ and $\mathcal{L}_{\bar{1}}$ is that $\mathcal{L}_1$ treats recursion in terms of *declarations*, whereas $\mathcal{L}_{\bar{1}}$ treats it in terms of *μ-notation*.

First, in § 5.3, three operational semantics $\mathcal{O}_{\bar{1}}^{\mathrm{w}}$, $\mathcal{O}_{\bar{1}}^{\mathrm{wm}}$, and $\mathcal{O}_{\bar{1}}^{\mathrm{wi}}$ of $\mathcal{L}_{\bar{1}}$ are defined in terms of a labeled transition system, in the style of Plotkin ([Plo 81]). These semantics are *linear* in that the meaning of each program in any of these semantics is a set of *action sequences* the program may perform, and are *weak* in that the action sequences are obtained by abstracting from (finite sequences of) *internal moves* (denoted by $\tau$ in [Mil 80]).

There are two alternative approaches to abstracting from $\tau$'s: One is to distinguish *divergence* (an infinite sequence of $\tau$'s) from deadlock; the other is to identify them. In this chapter, the former approach is adopted. The semantics identifying divergence and deadlock are somewhat simpler than the ones distinguishing them, and are often useful in applications. However, it may be appropriate and useful to distinguish them in certain circumstances, even when it is necessary to ignore finite sequences of $\tau$'s. (For motivations for distinguishing divergence from deadlock, we refer to [AH 89], [Uch 90], [Wal 91], although all of them treat *branching-time* semantics instead of linear (time) semantics treated here.)

The semantics $\mathcal{O}_1^{wi}$ differs from the other two in that $\mathcal{O}_1^{wi}$ is a so-called *internal action semantics* taking into account only internal moves under the assumption that the environment allows no (external) communication actions, and hence, the only possible actions for processes are internal moves, whereas the other two semantics take into account *communication actions* in addition to internal moves. The two semantics $\mathcal{O}_1^{w}$ and $\mathcal{O}_1^{wm}$ differ from each other in that $\mathcal{O}_1^{wm}$ is a so-called *maximal trace semantics* based on the view that a process may deadlock only when there is no possible action for it, whereas $\mathcal{O}_1^{w}$is based on the different view that a process may deadlock when the only possible actions for it are (external) communication actions.

Next, in § 5.4, two compositional models $\mathcal{C}_1^{w}$ and $\mathcal{C}_1^{wi}$ are proposed, which are variants of the *failures model* proposed by Brookes, Hoare, and Roscoe ([BHR 85]) and later improved ([BR 84]). It is shown, in § 5.5, § 5.6 and Appendix 5.A, that $\mathcal{C}_1^{w}$ (resp. $\mathcal{C}_1^{wi}$) is *fully abstract* w.r.t. $\mathcal{O}_1^{w}$ and $\mathcal{O}_1^{wm}$ (resp. w.r.t. $\mathcal{O}_1^{wi}$). That is, for

$$(\mathcal{C}, \mathcal{O}) = (\mathcal{C}_1^{w}, \mathcal{O}_1^{w}), (\mathcal{C}_1^{w}, \mathcal{O}_1^{wm}), (\mathcal{C}_1^{wi}, \mathcal{O}_1^{wi}),$$

one has that $\mathcal{C}$ is the most abstract compositional model which is correct w.r.t. $\mathcal{O}$. Equivalently, one has the following for every $s_1, s_2 \in \mathcal{L}_{\bar{1}}$:[1]

$$\begin{aligned} &\mathcal{C}[\![s_1]\!] = \mathcal{C}[\![s_2]\!] \Leftrightarrow \\ &\forall S[\cdot][\ S[\cdot] \text{ is a } \textit{context} \text{ of } \mathcal{L}_{\bar{1}} \Rightarrow \mathcal{O}[\![S[s_1]]\!] = \mathcal{O}[\![S[s_2]]\!]\ ], \end{aligned} \tag{5.2}$$

where

$$(\mathcal{C}, \mathcal{O}) = (\mathcal{C}_1^{w}, \mathcal{O}_1^{w}), (\mathcal{C}_1^{w}, \mathcal{O}_1^{wm}), (\mathcal{C}_1^{wi}, \mathcal{O}_1^{wi}),$$

and for a *context* $S[\cdot]$ (a statement consisting of the language constructs and a place-holder $[\cdot]$) $S[s]$ denotes the result of substituting $s$ for $[\cdot]$ in $S[\cdot]$.

Bergstra, Klop, and Olderog have shown that the failures model for a language without recursion or internal moves is fully abstract w.r.t. a *maximal trace semantics* **trace** ([BKO 88]). Moreover, Rutten investigated the semantics of a language similar to $\mathcal{L}_{\bar{1}}$, in the framework of complete metric spaces, and showed that the failures model is fully abstract w.r.t. a *strong linear semantics* $\mathcal{O}_1^{i}$ ([Rut 89]), where $\mathcal{O}_1^{i}$ is *strong* in that it does not abstract from internal moves.[2]

The full abstractness of $\mathcal{C}_1^{w}$ w.r.t. $\mathcal{O}_1^{wm}$, expressed in (5.2) with $(\mathcal{C}, \mathcal{O}) = (\mathcal{C}_1^{w}, \mathcal{O}_1^{wm})$, is an extension of the result of Bergstra et al. to a language with recursion and internal moves, since both $\mathcal{O}_1^{wm}$ and **trace** are maximal trace semantics. Also, the full abstractness of $\mathcal{C}_1^{wi}$ w.r.t. $\mathcal{O}_1^{wi}$, expressed in (5.2) with $(\mathcal{C}, \mathcal{O}) = (\mathcal{C}_1^{wi}, \mathcal{O}_1^{wi})$, is an extension of Rutten's result, to the case of *weak linear semantics with divergence*, since $\mathcal{O}_1^{wi}$ is obtained from $\mathcal{O}_1^{w}$ by abstracting from (finite sequences of) $\tau$'s present in $\mathcal{O}_1^{w}$.

[1] For a semantic model $\mathcal{M}$ for a language $\mathcal{L}_{\bar{1}}$ and $s \in \mathcal{L}_{\bar{1}}$, the notation $\mathcal{M}[\![s]\!]$ is used to denote the value of $\mathcal{M}$ at $s$.

[2] As is described in § 4.2.5, there was an error in the original full abstractness proof in [Rut 89], but Franck van Breugel found a way to overcome this error under an additional assumption.

Brookes ([Bro 83]) studied the relation between two models of concurrent behavior, Milner's *synchronization trees* for CCS, and the *failures model* of TCSP. There, a mappings $\mathcal{T}$ from the set **CSP** of finite TCSP terms to the set **ST** of synchronization trees, and a function **failures** from **ST** to the domain of failures were defined. Then, it was shown that the composite function $\mathcal{C}_{\mathrm{f}} = \mathbf{failures} \circ \mathcal{T}$ coincides with a compositionally defined failures model **F** (cf. [Bro 83] Proposition 1.4.3). In other words, it was shown that:

$\mathcal{C}_{\mathrm{f}}$ *is compositional w.r.t. the TCSP combinators treated in [Bro 83].* (5.3)

As a topic for future work, Brookes raised the problem of how to extend the result (5.3) to the case of infinite processes (cf. [Bro 83] § 2). This chapter provides two possible ways of such extension as follows. It is easy to see that $\mathcal{C}_{\mathrm{f}}$ can be defined in terms of a *labeled transition system* (instead of a **ST**) with **CSP** as its state set, because labeled transition systems are mapped into synchronization trees by a natural translation (and vice versa). We define two variants $\mathcal{C}_{\mathrm{I}}^{\mathrm{w}}$ and $\mathcal{C}_{\mathrm{I}}^{\mathrm{wi}}$ of the failures model for infinite processes, in terms of a labeled transition system; these two models may be considered as extensions of $\mathcal{C}_{\mathrm{f}}$ to the case of infinite processes. Then, we establish the compositionality of $\mathcal{C}_{\mathrm{I}}^{\mathrm{w}}$ and $\mathcal{C}_{\mathrm{I}}^{\mathrm{wi}}$ (cf. Lemmas 5.4, 5.5 in § 5.4). Thus the result (5.3) is extended to the case of infinite processes.

Our models $\mathcal{C}_{\mathrm{I}}^{\mathrm{w}}$ and $\mathcal{C}_{\mathrm{I}}^{\mathrm{wi}}$ differ from the original failures model in [BHR 85] even for finite processes, because we treat '+', *the nondeterministic choice* of CCS, and the original model is not compositional w.r.t. this combinator even for finite processes. In [Bro 83], this modification is not needed, because the TCSP combinators treated there do not contain '+'.

Closely related to the study of the failures model (and hence to this chapter) is the work of De Nicola and Hennessy ([DH 84]), where a framework of extracting information about a system by *testing* is proposed, and three equivalences $\eqsim_i$ $(i = 1, 2, 3)$ for CCS are introduced based on a particular definition of *tests* (see Remark 5.4 in § 5.7 for the comparison between the testing equivalences and the equivalence induced by $\mathcal{C}_{\mathrm{I}}^{\mathrm{w}}$).

Although the models $\mathcal{C}_{\mathrm{I}}^{\mathrm{w}}$ and $\mathcal{C}_{\mathrm{I}}^{\mathrm{wi}}$ are compositional and the meaning of each recursive program in $\mathcal{C}_{\mathrm{I}}^{\mathrm{w}}$ or $\mathcal{C}_{\mathrm{I}}^{\mathrm{wi}}$ is a fixed-point of the associated function (the interpretation of the body of its defining equation), they are not *denotational* in the framework of *complete partial orders* (cpo's) or in the framework of *complete metric spaces* (cms's) (cf. [BZ 82]), where the meaning of a recursive program is defined as the *least fixed-point* (in the cpo framework) or as the *unique fixed-point* (in the cms framework) of the associated function. In this section, we intuitively explain why $\mathcal{C}_{\mathrm{I}}^{\mathrm{w}}$ (or $\mathcal{C}_{\mathrm{I}}^{\mathrm{wi}}$) cannot be constructed denotationally in the standard cpo or cms framework; for more precise explanation, see § 5.7.

First, $\mathcal{C}_{\mathrm{I}}^{\mathrm{w}}$ (or $\mathcal{C}_{\mathrm{I}}^{\mathrm{wi}}$) cannot be constructed in the standard cpo framework with the *reverse inclusion* $\supseteq$ being the order of the semantic domain, because the semantic operation $\tilde{\|}_1$ corresponding to the parallel composition combinator of $\mathcal{L}_{\mathrm{I}}$ is *not continuous* w.r.t. the order $\supseteq$ (for an example showing this, see Example 5.3). It also turns out that $\tilde{\|}_1$ is *not nonexpansive* w.r.t. the metric defined in terms truncation as in Chapter 4; thus $\mathcal{C}_{\bar{1}}^{w}$ (or $\mathcal{C}_{\bar{1}}^{wi}$) cannot be constructed in the standard cms framework based on the metric. (for an example showing this, see Example 5.4).

Such order-theoretic or metric topological construction of $\mathcal{C}_{\bar{1}}^{w}$ and $\mathcal{C}_{\bar{1}}^{wi}$ remains for future study. The characterization of $\mathcal{C}_{\bar{1}}^{w}$ and $\mathcal{C}_{\bar{1}}^{wi}$ in this chapter as fully abstract models is analogous to Milner's characterization of the so-called *observation congruence* in [Mil 83] and [Mil 85].

Note that such denotational construction of $\mathcal{C}_{\bar{1}}^{w}$ (or $\mathcal{C}_{\bar{1}}^{wi}$) as described above is not necessarily possible as is shown in [AP 86]. (In [HP 79], a fully abstract model for a parallel language was constructed in an order-theoretic framework. However, the concurrency treated there is different from the one treated here, because the language in [HP 79] contains a *coroutine* construct as well as the usual interleaving.)

We tackle this problem of denotational model construction in Chapter 6; there a denotational model $\mathcal{D}_{\bar{1}}^{wf}$ for $\mathcal{L}_{\bar{1}}$ is constructed in an order-theoretic setting, and its full abstractness w.r.t. a weak linear operational model $\mathcal{O}_{\bar{1}}^{mf}$ is established, where $\mathcal{O}_{\bar{1}}^{mf}$ is a weak linear operational model similar to $\mathcal{O}_{\bar{1}}^{wm}$, but slightly different from $\mathcal{O}_{\bar{1}}^{wm}$. Intuitively, the difference between the two models is that $\mathcal{O}_{\bar{1}}^{mf}$ is based on the view that *divergence* is disastrous, whereas $\mathcal{O}_{\bar{1}}^{wm}$ is based on another view that *divergence* is not particularly disastrous as deadlock is not.

## 5.2 A Language $\mathcal{L}_{\bar{1}}$ for Processes

The language $\mathcal{L}_{\bar{1}}$ treated in this section is a minor variant of the language $\mathcal{L}_1$ introduced in Chapter 4. (As described in § 1.3.1, the only difference between $\mathcal{L}_{\bar{1}}$ and $\mathcal{L}_1$ is in the way to treat recursion: $\mathcal{L}_{\bar{1}}$ (resp. $\mathcal{L}_1$) uses $\mu$-notation (resp. declarations) to treat recursion.) Just as $\mathcal{L}_1$, $\mathcal{L}_{\bar{1}}$ is a minor extension of the language investigated in [Rut 89], containing *atomic actions*, *termination*, *inaction*, *sequential* and *parallel composition*, *nondeterministic choice*, *action restriction*, and a form of recursion.[3]

**Definition 5.1** Let $(a \in)$ $\mathbf{A}_1$ be the set of *actions*. As in [Mil 89], it is assumed that $\mathbf{A}_1$ is partitioned into $\mathbf{A}_1 = \mathbf{C}_1 \uplus \{\tau\}$, where $(c \in)$ $\mathbf{C}_1$ is the set of *communicating actions*, and $\tau$ is the internal move, which represents *successful communication* and is *unobservable* by the environment. Further, let us use $C$ as a typical variable ranging over $\wp(\mathbf{C}_1)$, and let $(X, Y \in)$ $\mathcal{X}_{\mathcal{P}}$ be the set of *statement variables*.

First, a language $(S \in)$ $\tilde{\mathcal{L}}_{\bar{1}}$ without guardedness condition is defined by:

$$S ::= a \mid \mathbf{0} \mid \mathbf{e} \mid (S_1; S_2) \mid (S_1 + S_2) \mid (S_1 \parallel S_2) \mid \partial_C(S) \mid X \mid (\mu X.\ S), \tag{5.4}$$

[3] Except for the addition of action restriction, the difference between $\mathcal{L}_{\bar{1}}$ and the language in [Rut 89] is that $\mathcal{L}_{\bar{1}}$ contains the construct $\mathbf{e}$ representing termination, while the latter language does not. The construct $\mathbf{e}$ corresponds to the regular expression denoting $\{\epsilon\}$, **SKIP** in [BHR 85], and **exit** in [ISO 89].

where the constants: "$a$" ($a \in \mathbf{A}_1$), "**0**", and "**e**" represent an *atomic action*, the *inaction*, and the *termination*, respectively; the combinators: ";", " $\|$ ", "+", and "$\partial_C$" ($C \in \wp(\mathbf{C}_1)$) represent the *sequential composition*, the *parallel composition*, the *nondeterministic choice*, and the *action restriction*, respectively.

Intuitively $(\mu X.\ S)$ stands for a *solution* of the equation $X = S$. Syntactically the prefix "$\mu X$" *binds* each variable $X$, as "$\lambda x$" in $\lambda$-notation. For $S \in \tilde{\mathcal{L}}_{\bar{1}}$, let $\mathrm{FV}(S)$ be the set of *free variables* contained in $S$. ∎

Intuitively, the language $\mathcal{L}_{\bar{1}}$ is defined to be the set of $S \in \tilde{\mathcal{L}}_{\bar{1}}$ satisfying the following guardedness condition:[4]

*For each subexpression $(\mu Y.\ S')$ of $S$, each occurrence of $Y$ in $S'$ occurs in a subexpression of $S'$ of the form $(a; S'')$ $(a \in \mathbf{A}_1)$ or $(\mathbf{0}; S'')$.* (5.5)

Below, we will give a formal definition of the guardedness condition and the language $\mathcal{L}_{\bar{1}}$.

As a preliminary to the formal definition of $\mathcal{L}_{\bar{1}}$, we introduce a signature, and characterize $\tilde{\mathcal{L}}_{\bar{1}}$ as the set of terms generated by the signature:

**Definition 5.2** A single-sorted *signature* $\mathbf{Sig}_1 = (\mathbf{Fun}_1, \mathrm{arity}_1(\cdot))$ is defined as in Definition 4.2, with $(\mathrm{op} \in)\ \mathbf{Fun}_1$ being a set of *combinators* and $\mathrm{arity}_1(\cdot)$ a function which maps each combinator to its *arity*: (Here constant symbols are regarded as nullary combinators.)

$$\mathbf{Fun}_1 = \{\mathbf{0}, \mathbf{e}\} \cup \{a : a \in \mathbf{A}\} \cup \{\partial_C : C \in \wp(\mathbf{C})\} \cup \{+, \|, ;\},$$

$$\mathrm{arity}_1(\mathrm{op}) = \begin{cases} 0 & \text{if } \mathrm{op} \in \{\mathbf{0}, \mathbf{e}\} \cup \{a : a \in \mathbf{A}\}, \\ 1 & \text{if } \mathrm{op} \in \{\partial_C : C \in \wp(\mathbf{C})\}, \\ 2 & \text{if } \mathrm{op} \in \{+, \|, ;\}, \end{cases}$$

where the constant symbols: "$a$" ($a \in \mathbf{A}$), "**0**", and "**e**" represent an *atomic action*, *inaction*, and *termination*, respectively; the combinators: ";", " $\|$ ", "+", and "$\partial_C$" ($C \in \wp(\mathbf{C})$) represent *sequential composition*, *parallel composition*, *nondeterministic choice*, and *action restriction*, respectively. For $r \in 3$, we put $\mathbf{Fun}_1^{(r)} = \{\mathrm{op} \in \mathbf{Fun}_1 : \mathrm{arity}_1(\mathrm{op}) = r\}$. ∎

It is obvious that the language $\tilde{\mathcal{L}}_{\bar{1}}$ defined by (5.1) is characterized as the set of terms generated by the signature $\mathbf{Sig}_1$ and $\mathcal{X}_{\mathcal{P}}$ with $\mu$-notation.

In terms of the signature $\mathbf{Sig}_1$, we formally define the the language $\mathcal{L}_{\bar{1}}$ by:

**Definition 5.3 (1)** The language $\mathcal{L}_{\bar{1}}$ is defined to be the set of *guardedly defined statements*, i.e., statements $S \in \tilde{\mathcal{L}}_{\bar{1}}$ satisfying the following guardedness condition:

[4]The guardedness restriction is imposed so that the labeled transition system, to be defined in § 5.3, is *finitely branching*. The property of being finitely branching is necessary to establish the full abstractness of $\mathcal{C}_{\bar{1}}^{\mathrm{wi}}$ w.r.t. $\mathcal{O}_{\bar{1}}^{\mathrm{wi}}$, while it is not necessary for the full abstractness of $\mathcal{C}_{\bar{1}}^{\mathrm{w}}$ w.r.t. $\mathcal{O}_{\bar{1}}^{\mathrm{w}}$ and $\mathcal{O}_{\bar{1}}^{\mathrm{wm}}$ (cf. Lemma 5.1).

*For each subexpression $(\mu Y.\ S')$ of $S$, each occurrence of $Y$ in $S'$ is preceded by $a \in \mathbf{A}$ or by $\mathbf{0}$.* (5.6)

More formally, $\langle(\mathcal{L}_{\bar{1}}(n), \mathcal{G}(n))\rangle_{n\in\omega}$ are inductively defined as follows:

**(i)** $\mathcal{L}_{\bar{1}}(0) = \emptyset$, and $\mathcal{G}(0) = \emptyset$.

**(ii)**
$$\begin{aligned}\mathcal{L}_{\bar{1}}(n+1) = \ & \mathcal{X}_{\mathcal{P}} \\ & \cup\{\ \mathsf{op}(S_0,\ldots,S_{r-1}) : r \in 3 \wedge \mathsf{op} \in \mathbf{Fun}_{\bar{1}}^{(r)} \\ & \quad \wedge S_0,\ldots,S_{r-1} \in \mathcal{L}_{\bar{1}}(n)\} \\ & \cup\{(\mu X.\ S) : X \in \mathcal{X}_{\mathcal{P}} \wedge S \in \mathcal{G}(n)\},\end{aligned}$$

and

$$\begin{aligned}\mathcal{G}(n+1) = \ & \{\ \mathsf{op}(S_0,\ldots,S_{r-1}) : r \in 3 \wedge \mathsf{op} \in (\mathbf{Fun}_{\bar{1}}^{(r)} \setminus \{\mathsf{e}\}) \\ & \quad S_0,\ldots,S_{r-1} \in \mathcal{G}(n)\} \\ & \cup\{(S_0;S_1) : S_0 \in \mathcal{G}(n) \wedge S_1 \in \mathcal{L}_{\bar{1}}(n)\} \\ & \cup\{(\mu X.\ S) : X \in \mathcal{X}_{\mathcal{P}} \wedge S \in \mathcal{G}(n)\}.\end{aligned}$$

Then, let $\mathcal{L}_{\bar{1}} = \bigcup_{n\in\omega}[\mathcal{L}_{\bar{1}}(n)]$, and $\mathcal{G} = \bigcup_{n\in\omega}[\mathcal{G}(n)]$.

**(2)** For $\tilde{\mathcal{X}} \subseteq \mathcal{X}_{\mathcal{P}}$, let $\mathcal{L}_{\bar{1}}[\tilde{\mathcal{X}}] = \{S \in \mathcal{L}_{\bar{1}} :\ \mathrm{FV}(S) \subseteq \tilde{\mathcal{X}}\}$. Thus $\mathcal{L}_{\bar{1}}[\emptyset]$ denotes the set of *closed statements*. We write $\mathcal{L}_{\bar{1}}[X]$ for $\mathcal{L}_{\bar{1}}[\{X\}]$ $(X \in \mathcal{X}_{\mathcal{P}})$. ∎

**Remark 5.1** In [KR 90], a seemingly more general definition of guarded statements is given (cf. [KR 90] § 2.1). For example, the expression

$$s \equiv (\mu X.\ (a; X; (\mu Y.\ X; Y)))$$

is considered to be a guardedly well-defined statement in [KR 90], whereas $s \notin \mathcal{L}_{\bar{1}}$ because $(X;Y) \notin \mathcal{G}$ according to Definition 5.2. However, our language is not so restricted as it seems. For example, we can construct a statement $s' \in \mathcal{L}_{\bar{1}}$ which is bisimilar to $s$ as follows (for the notion of strong bisimulation, cf. [Mil 89]): Put

$$s' \equiv (\mu X.\ (a; X; (\mu Y.\ (a; X; Y); Y))).$$

Then one has $s \sim s'$ with $\sim$ representing strong bisimilarity as follows: First, putting $s_1 \equiv (\mu Y.\ s; Y)$, one has

$$s \sim (a; s; s_1) \wedge s_1 \sim (s; s_1),$$

and therefore,

$$s \sim (a; s; s_1) \wedge s_1 \sim (a; s; s_1); s_1,$$

On the other hand, putting, $s'_1 = (\mu Y.\ (a; s'; Y); Y)$, one has $s'_1 \in \mathcal{L}_{\bar{1}}$ and

$$s' \sim (a; s'; s'_1) \wedge s'_1 \sim (a; s'; s'_1); s'_1.$$

Thus, both $(s, s_1)$ and $(s', s'_1)$ are a solution of the system of guarded equations:

$$X = (a; X; Y),\ \ Y = (a; X; Y); Y.$$

Thus, applying Proposition 14 of [Mil 89], one has

$s \sim s' \wedge s_1 \sim s_1'$. ∎

The next proposition immediately follows from Part (3) of the above definition:

**Proposition 5.1 (1)** $\forall n \in \omega[\ \mathcal{L}_{\bar{1}}(n) \subseteq \mathcal{L}_{\bar{1}}(n+1)\ ]$.
**(2)** $\forall n \in \omega[\ \mathcal{G}(n) \subseteq \mathcal{G}(n+1)\ ]$.
**(3)** $\forall n \in \omega[\ \mathcal{G}(n) \subseteq \mathcal{L}_{\bar{1}}(n+1)\ ]$. ∎

The language $\mathcal{L}_{\bar{1}}$ is closed under substitution in the following sense:

**Proposition 5.2** *Let* $r \in \omega$, $\vec{X} \in (r \rightarrowtail \mathcal{X})$, *and* $\vec{s} \in (r \to \mathcal{L}_{\bar{1}}[\emptyset])$, *and* $S \in \mathcal{L}_{\bar{1}}$ *such that* $\mathrm{FV}(S) \subseteq \mathrm{ran}(\vec{X})$. *Then,* $S[\vec{s}/\vec{X}] \in \mathcal{L}_{\bar{1}}$. ∎

**Proof.** We can prove the following by induction on $n$:

$$\begin{aligned}\forall n \in \omega[\ &\forall S \in \mathcal{G}(n), \forall X \in \mathcal{X}, \forall s \in \mathcal{L}_{\bar{1}}[\emptyset][\ S[s/X] \in \mathcal{G}\ ] \\ &\wedge \forall S \in \mathcal{L}_{\bar{1}}(n), \forall X \in \mathcal{X}, \forall s \in \mathcal{L}_{\bar{1}}[\emptyset][\ S[s/X] \in \mathcal{L}_{\bar{1}}\ ]].\end{aligned} \tag{5.7}$$

From this, the claim immediately follows. ∎

**Notation 5.1** For $S, S' \in \mathcal{L}_{\bar{1}}$, we write $S \equiv S'$, to denote that $S$ and $S'$ are syntactically identical. For $S, S' \in \mathcal{L}_{\bar{1}}$, $X \in \mathcal{X}_{\mathcal{P}}$, we denote by $S[S'/X]$ the result of substituting $S'$ for all free occurrences of $X$ in $S$. ∎

## 5.3 Three Weak Linear Semantics $\mathcal{O}_1^{\mathrm{w}}$, $\mathcal{O}_1^{\mathrm{wm}}$, $\mathcal{O}_1^{\mathrm{wi}}$

In this section, three *weak linear* operational semantics $\mathcal{O}_1^{\mathrm{w}}$, $\mathcal{O}_1^{\mathrm{wm}}$, and $\mathcal{O}_1^{\mathrm{wi}}$ of $\mathcal{L}_{\bar{1}}$ are defined operationally in the style of Plotkin ([Plo 81]). Here the superscript "w" stands for *weak*, and "m" and "i" stand for *maximal* and *internal*, respectively. As a preliminary to the definition, a labeled transition system is defined as in Definition 4.3 follows, with a suitable adaptation in accordance with the minor difference between $\mathcal{L}_1$ and $\mathcal{L}_{\bar{1}}$.

**Definition 5.4** (Transition System) Let us use $w$ and $u$ as typical variables ranging over $(\mathbf{A}_1)^{\leq\omega}$ and $(\mathbf{C}_1)^{\leq\omega}$, respectively.

**(1)** Let $\surd$ ($\notin \mathbf{A}_1$) be a symbol representing the (successful) *termination*, and let $(\alpha \in)\ \mathbf{A}_1^{\surd} = \mathbf{A} \cup \{\surd\}$.

**(2)** A bijection $\bar{\cdot} : \mathbf{C}_1 \to \mathbf{C}_1$ is assumed to be given such that $\forall c \in \mathbf{C}_1[\ \bar{\bar{c}} = c\ ]$. For $c \in \mathbf{C}_1$, $\bar{c}$ is called the *complement* of $c$. Moreover, for $u = \langle c_i \rangle_{i \in k} \in (\mathbf{C}_1)^{<\omega}$, $C \subseteq \mathbf{C}_1$, let $\overline{u} = \langle \bar{c}_i \rangle_{i \in k}$, $\overline{C} = \{\bar{c} : c \in C\}$.

**(3)** A family of binary transition relations $\xrightarrow{a}_1 \subseteq \mathcal{L}_{\bar{1}}[\emptyset] \times \mathcal{L}_{\bar{1}}[\emptyset]$ $(a \in \mathbf{A})$, and a unary relation $\xrightarrow{\surd}_1 \subseteq \mathcal{L}_{\bar{1}}[\emptyset]$ are defined as the smallest sets satisfying the rules **(i)**–**(x)** below. For $s_1, s_2 \in \mathcal{L}_{\bar{1}}[\emptyset]$, $a \in \mathbf{A}$, we write $s_1 \xrightarrow{a}_1 s_2$ (resp. $s_1 \xrightarrow{\surd}_1$) for $(s_1 s_2) \in \xrightarrow{a}_1$ (resp. $s \in \xrightarrow{\surd}_1$), as usual. Intuitively, the expression "$s_1 \xrightarrow{a}_1 s_2$"

means that *the process* $s_1$ *may perform the action* $a$ *as a first step, resulting in the process* $s_2$, and "$s_1 \xrightarrow{\surd}_1$" means that $s_1$ *may terminate successfully* or that $s_1$ *is in a final state* (following the terminology of classical automata theory).[5]
In the sequel, we use the notational convention that *for* $\alpha \in \mathbf{A}_1^{\surd}$, *the phrase* "$s_1 \xrightarrow{\alpha}_1 \left(s_2\right)$" *means* $s_1 \xrightarrow{\surd}_1$, *if* $\alpha = \surd$; *otherwise, it means* $s_1 \xrightarrow{\alpha}_1 s_2$.

**(i)** $a \xrightarrow{a}_1 \mathbf{e} \quad (a \in \mathbf{A})$. **(ii)** $\mathbf{e} \xrightarrow{\surd}_1$ .

$$\textbf{(iii)}\ \frac{s_1 \xrightarrow{a}_1 s_1'}{(s_1; s_2) \xrightarrow{a}_1 (s_1'; s_2)} \quad (a \in \mathbf{A})$$

$$\textbf{(iv)}\ \frac{s_1 \xrightarrow{\surd}_1,\ s_2 \xrightarrow{\alpha}_1 \left(s_2'\right)}{(s_1; s_2) \xrightarrow{\alpha}_1 \left(s_2'\right)} \quad (\alpha \in \mathbf{A}_1^{\surd})$$

$$\textbf{(v)}\ \frac{s_1 \xrightarrow{\alpha}_1 \left(s_1'\right)}{\begin{array}{c}(s_1 + s_2) \xrightarrow{\alpha}_1 \left(s_1'\right)\\ (s_2 + s_1) \xrightarrow{\alpha}_1 \left(s_1'\right)\end{array}} \quad (\alpha \in \mathbf{A}_1^{\surd})$$

$$\textbf{(vi)}\ \frac{s_1 \xrightarrow{a}_1 s_1'}{\begin{array}{c}(s_1 \parallel s_2) \xrightarrow{a}_1 (s_1' \parallel s_2)\\ (s_2 \parallel s_1) \xrightarrow{a}_1 (s_2 \parallel s_1')\end{array}} \quad (a \in \mathbf{A})$$

$$\textbf{(vii)}\ \frac{s_1 \xrightarrow{c}_1 s_1',\ s_2 \xrightarrow{\bar{c}}_1 s_2'}{(s_1 \parallel s_2) \xrightarrow{\tau}_1 (s_1' \parallel s_2')} \quad (c \in \mathbf{C}_1)$$

$$\textbf{(viii)}\ \frac{s_1 \xrightarrow{\surd}_1,\ s_2 \xrightarrow{\surd}_1}{(s_1 \parallel s_2) \xrightarrow{\surd}_1}$$

$$\textbf{(ix)}\ \frac{s \xrightarrow{\alpha}_1 \left(s'\right)}{\partial_C(s) \xrightarrow{\alpha}_1 \left(\partial_C(s')\right)} \quad (\alpha \in (\mathbf{A}_1^{\surd} \setminus C))$$

[5]The rules presented here are all standard except for the treatment of termination and sequential composition. The rules for termination and sequential composition here are very similar to the ones in § 9.2 of [Mil 89] and to the ones in § 7.5 of [ISO 89], except that termination is treated here in terms of the *unary relation* $\xrightarrow{\surd}_1$ on processes, rather than in terms of a binary transition relation labeled by a special symbol such as $\surd$.

$$\text{(x)}\quad \frac{S[(\mu X.\ S)/X] \xrightarrow{\alpha}_1 \left(s'\right)}{(\mu X.\ S) \xrightarrow{\alpha}_1 \left(s'\right)} \quad (\alpha \in \mathbf{A}_1^{\checkmark})$$

The last rule is called the *recursion rule*

**(4)** For $w = \langle a_0, \cdots, a_{n-1}\rangle \in (\mathbf{A}_1)^{<\omega}$, a binary relation $\xrightarrow{w}_{1*}$ is defined by:

$$\xrightarrow{w}_{1*} = \xrightarrow{a_0}_1 \xrightarrow{a_1}_1 \cdots \xrightarrow{a_{n-1}} .$$

(Here and in the sequel, juxtaposition is used for denoting relational composition, e.g., $\xrightarrow{a}_1\xrightarrow{a'}_1 = \{(s,s') : \exists s''[\ s \xrightarrow{a}_1 s'' \xrightarrow{a'}_1 s'\ ]\}$.) For $a \in \mathbf{A}_1$, $w \in (\mathbf{A}_1)^{<\omega}$, we write $s \xrightarrow{a}_1$ and $s \xrightarrow{w}_{1*}$, to denote that $\exists s'[s \xrightarrow{a}_1 s']$ and $\exists s'[s \xrightarrow{w}_{1*} s']$, respectively.

**(5)** For $w = (\mathbf{A}_1)^{\omega}$, a unary relation $\xrightarrow{w}_{1\omega}$ is defined as follows: For $s \in \mathcal{L}_{\bar{1}}[\emptyset]$, $s \xrightarrow{w}_{1\omega}$ iff there exists $\langle s_i\rangle_{i\in\omega} \in (\mathcal{L}_{\bar{1}}[\emptyset])^{\omega}$ such that $s = s_0$ and $\forall i \in \omega[\ s_i \xrightarrow{w(i)}_1 s_{i+1}\ ]$.

**(6)** Let

$$\begin{aligned}(\rho \in)\ \mathbf{Q}_1^{\mathrm{wl}} = (\mathbf{C}_1)^{\omega} &\cup ((\mathbf{C}_1)^{<\omega} \cdot \{\langle\surd\rangle\}) \cup (\mathbf{C}_1)^{<\omega} \cdot \{\langle\,\delta\,\rangle\} \\ &\cup (\mathbf{C}_1)^{<\omega} \cdot \{\langle\bot\rangle\},\end{aligned}$$

where $\delta$ and $\bot$ are distinct symbols representing *deadlock* and *divergence*, respectively. Also, let

$$(\rho \in)\ (\mathbf{C}_1)_{\bot}^{\leq\omega} = (\mathbf{C}_1)^{\leq\omega} \cup ((\mathbf{C}_1)^{<\omega} \cdot \{\langle\bot\rangle\}).$$

A function $\theta : (\mathbf{A}_1)^{\leq\omega} \to (\mathbf{C}_1)_{\bot}^{\leq\omega}$ is defined as follows: For $w \in (\mathbf{A}_1)^{\leq\omega}$,

$$\theta(w) = \begin{cases}(\tilde{w} \setminus \tau)\cdot\langle\bot\rangle & \text{if } \exists\tilde{w} \in (\mathbf{A}_1)^{<\omega}[\ w = \tilde{w}\cdot\tau^{\omega}\ ],\\ (w \setminus \tau) & \text{otherwise,}\end{cases}$$

where $(w \setminus \tau)$ is the result of erasing $\tau$'s in $w$.

**(7)** For $u \in (\mathbf{C}_1)^{<\omega}$, a binary relation $\xRightarrow{u}_1$ is defined by:

$$s \xRightarrow{u}_1 s' \Leftrightarrow \exists w \in (\mathbf{A}_1)^{<\omega}[\ \theta(w) = u \wedge s \xrightarrow{w}_{1*} s'\ ].$$

For $\rho \in (\mathbf{C}_1)_{\bot}^{\leq\omega}$, a unary relation $\xRightarrow{\rho}_1$ is defined by:

$$\begin{aligned}s \xRightarrow{\rho}_1 \Leftrightarrow\ &\exists w \in (\mathbf{A}_1)^{<\omega}[\ \theta(w) = \rho \wedge s \xrightarrow{w}_{1*}\ ]\\ &\vee \exists w \in (\mathbf{A}_1)^{\omega}[\ \theta(w) = \rho \wedge s \xrightarrow{w}_{1\omega}\ ].\end{aligned}$$

**(8)** For $s \in \mathcal{L}_{\bar{1}}[\emptyset]$, let

$$\mathrm{act}_1(s) = \{\alpha \in \mathbf{A}_1^{\checkmark} : s \xrightarrow{\alpha}_1\}.$$

**(9)** For $s \in \mathcal{L}_{\bar{1}}[\emptyset]$, we call a pair $(a, s') \in \mathbf{A} \times \mathcal{L}_{\bar{1}}[\emptyset]$ an *immediate derivative* of $s$, when $s \xrightarrow{a}_1 s'$. Let $\mathrm{IDrv}(s)$ be the set of immediate derivatives of $s$, i.e., let

$$\mathrm{IDrv}(s) = \{(a, s') \in \mathbf{A} \times \mathcal{L}_{\bar{1}}[\emptyset] : s \xrightarrow{a}_1 s'\}. \blacksquare$$

By the guardedness condition (5.5), it can be shown that the labeled transition system $(\mathcal{L}_{\bar{1}}[\emptyset], \langle \xrightarrow{\alpha}_1 : \alpha \in \mathbf{A}_1^{\checkmark} \rangle)$ is *finitely branching* in the following sense:

**Lemma 5.1** *For every* $s \in \mathcal{L}_{\bar{1}}$, *the set* $\mathrm{IDrv}(s)$ *is finite.* ∎

**Proof.** By the definition of transition relations, one has the following properties (5.8)–(5.11):

$$\begin{aligned}&\forall r \in 3, \forall \mathrm{op} \in \mathbf{Fun}_1^{(r)}, \forall s_0, \ldots, s_{r-1} \in \mathcal{L}_{\bar{1}}[\emptyset][\\ &\quad \forall i \in r[\ \mathrm{IDrv}(s_i) \text{ is finite } ] \Rightarrow \mathrm{IDrv}(\mathrm{op}(s_0, \ldots, s_{r-1})) \text{ is finite } ].\end{aligned} \tag{5.8}$$

$$\begin{aligned}&\forall r \in 3, \forall \mathrm{op} \in \mathbf{Fun}_1^{(r)}, \forall s_0, \ldots, s_{r-1} \in \mathcal{L}_{\bar{1}}[\emptyset][\\ &\quad \forall i \in r[\ \surd \notin \mathrm{act}_1(s_i)\ ] \Rightarrow \surd \notin \mathrm{act}_1(\mathrm{op}(s_0, \ldots, s_{r-1}))\ ].\end{aligned} \tag{5.9}$$

$$\forall s_0, s_1 \in \mathcal{L}_{\bar{1}}[\emptyset][\ \surd \notin \mathrm{act}_1(s_0) \Rightarrow \mathrm{act}_1(s_0; s_1) = \mathrm{act}_1(s_0)\ ]. \tag{5.10}$$

$$\forall X \in \mathcal{X}, \forall S \in \mathcal{G}[\ \mathrm{IDrv}((\mu X.\ S)) = \mathrm{IDrv}(S[(\mu X.\ S)/X])\ ]. \tag{5.11}$$

It suffices to prove that the following holds for every $n \in \omega$:

$$\forall s \in \mathcal{L}_{\bar{1}}(n)[\ \mathrm{IDrv}(s) \text{ is finite } ]. \tag{5.12}$$

We will prove this by induction.

For $S \in \mathcal{L}_{\bar{1}}$, let us write $\mathrm{GStat}(S)$ to denote that

$$\begin{aligned}&\forall r \in \omega, \forall \vec{X} \in (r \rightarrowtail \mathcal{X}), \forall \vec{s} \in (r \to \mathcal{L}_{\bar{1}}[\emptyset])[\ \mathrm{FV}(S) \subseteq \mathrm{ran}(\vec{X})\\ &\quad \Rightarrow (\mathrm{IDrv}(S[\vec{s}/\vec{X}]) \text{ is finite} \wedge \surd \notin \mathrm{act}_1(S)\ )\ ].\end{aligned} \tag{5.13}$$

We will prove, by induction, that the following holds for every $n \in \omega$:

$$\forall S \in \mathcal{G}(n)[\ \mathrm{GStat}(S)\ ]. \tag{5.14}$$

For $n = 0$, (5.14) holds vacuously. Suppose (5.14) holds for $n = k$, and let $S \in \mathcal{G}(n)$. Then, one of the following propositions (5.15)–(5.17) holds:

$$S \equiv \mathrm{op}(S_0, \ldots, S_r) \text{ with } S_0, \ldots, S_r \in \mathcal{G}(n). \tag{5.15}$$

$$S \equiv (S_0; S_1) \text{ with } S_0 \in \mathcal{G}(n) \text{ and } S_1 \in \mathcal{L}_{\bar{1}}(n). \tag{5.16}$$

$$S \equiv (\mu X.\ \tilde{S}) \text{ with } \tilde{S} \in \mathcal{G}(n). \tag{5.17}$$

When (5.15) (resp. (5.16)) holds, one has $\mathrm{GStat}(S)$ by the induction hypothesis using (5.8) and (5.9) (resp. (5.10)).

Suppose (5.17) holds, and let $r \in \omega$, $\vec{X} \in (r \rightarrowtail \mathcal{X})$, $\vec{s} \in (r \to \mathcal{L}_{\bar{1}}[\emptyset])$. We can assume, without loss of generality, that $X \notin \mathrm{ran}(\vec{X})$. Then

$$S[\vec{s}/\vec{X}] \equiv (\mu X.\ \tilde{S}[\vec{s}/\vec{X}]).$$

Thus,

$$\begin{aligned}\mathrm{IDrv}(S[\vec{s}/\vec{X}]) &= \mathrm{IDrv}((\tilde{S}[\vec{s}/\vec{X}])[S[\vec{s}/\vec{X}]/X]) \\ &= \mathrm{IDrv}(\tilde{S}[(\vec{s}\cdot\langle S[\vec{s}/\vec{X}]\rangle)/(\vec{X}\cdot\langle X\rangle)]).\end{aligned}$$

By the induction hypothesis, $\mathrm{IDrv}(\tilde{S}[(\vec{s}\cdot\langle S[\vec{s}/\vec{X}]\rangle)/(\vec{X}\cdot\langle X\rangle)])$ is finite, and $\surd \notin \mathrm{act}_1(\tilde{S}[(\vec{s}\cdot\langle S[\vec{s}/\vec{X}]\rangle)/(\vec{X}\cdot\langle X\rangle)])$. Thus, $\mathrm{IDrv}(S[\vec{s}/\vec{X}])$ is finite, and $\surd \notin \mathrm{act}_1(S[\vec{s}/\vec{X}])$, i.e., $\mathrm{GStat}(S[\vec{s}/\vec{X}])$.

Summarizing, (5.14) holds for $n = k+1$. By induction, (5.14) holds for every $n$. Using this, one can show that if (5.12) holds for $n = k$, then it also holds for $n = k+1$. Thus, (5.12) holds for every $n$. ∎

**Notation 5.2** For $\alpha \in \mathbf{A}_1^{\surd}$, we use $s \not\xrightarrow{\alpha}_1$ as a shorthand for $\neg(s \xrightarrow{\alpha}_1)$. ∎

By means of transition relations $\xrightarrow{\alpha}_1$, three weak linear operational semantics $\mathcal{O}_1^{\mathrm{w}}$, $\mathcal{O}_1^{\mathrm{wm}}$, $\mathcal{O}_1^{\mathrm{wi}}$ are defined. First, the common semantic domain of these semantics is defined by:

**Definition 5.5** (Semantic Domain for $\mathcal{O}_1^{\mathrm{w}}$, $\mathcal{O}_1^{\mathrm{wm}}$, $\mathcal{O}_1^{\mathrm{wi}}$) We define $\mathbf{P}_1^{\mathrm{wl}}$, the common semantic domain for the three weak linear semantics, from $\mathbf{Q}_1^{\mathrm{wl}}$ by:

$$(p \in)\ \mathbf{P}_1^{\mathrm{wl}} = \wp(\mathbf{Q}_1^{\mathrm{wl}}). \blacksquare$$

The three weak linear semantics $\mathcal{O}_1^{\mathrm{w}}$, $\mathcal{O}_1^{\mathrm{wm}}$, $\mathcal{O}_1^{\mathrm{wi}}$: $\mathcal{L}_{\bar{1}}[\emptyset] \to \mathbf{P}_1^{\mathrm{wl}}$ are defined by:

**Definition 5.6** (Weak Linear Semantics) Let $s \in \mathcal{L}_{\bar{1}}[\emptyset]$.

**(1)** As preliminaries to the definition of the semantics, five auxiliary functions $\mathcal{T}_{\surd}$, $\mathcal{T}_{\delta}$, $\mathcal{T}_{\delta}^{\mathrm{m}}$, $\mathcal{T}_{\omega}$, and $\mathcal{T}_{\perp}$ are defined as follows:

**(i)** $$\mathcal{T}_{\surd}(s) = \{u\cdot\langle\surd\rangle : u \in (\mathbf{C}_1)^{<\omega} \wedge s \overset{u}{\Longrightarrow}_1 \xrightarrow{\surd}_1\}.$$

Each element $u \in \mathcal{T}_{\surd}(s)$ represents a finite, successfully terminating sequence of observable actions $s$ may perform.

**(ii)** $$\begin{aligned}\mathcal{T}_{\delta}(s) = \{u\cdot\langle\,\delta\,\rangle : u \in (\mathbf{C}_1)^{<\omega} \wedge \\ \exists s'[\, s \overset{u}{\Longrightarrow}_1 s' \wedge \tau, \surd \notin \mathrm{act}_1(s')\,]\}.\end{aligned}$$

Each element $u\cdot\langle\,\delta\,\rangle \in \mathcal{T}_{\delta}(s)$ represents a computation which first performs observable actions in $u$, and then results in $s'$ which may perform no internal moves or successful termination, and so, *deadlocks*, when the environment offers no complements of the communication actions $\mathrm{act}_1(s')$ which $s'$ may perform.[6]

**(iii)** $$\begin{aligned}\mathcal{T}_{\delta}^{\mathrm{m}}(s) = \{u\cdot\langle\,\delta\,\rangle : u \in (\mathbf{C}_1)^{<\omega} \wedge \\ \exists s'[\, s \overset{u}{\Longrightarrow}_1 s' \wedge \mathrm{act}_1(s') = \emptyset\,]\}.\end{aligned}$$

[6] Here, we assume that the interaction between a process and its environment is *synchronous* in the sense of CCS or TCSP, i.e., that a process $s$ cannot *wait* (or be *suspended*) unless $\tau \in \mathrm{act}_1(s)$.

**(iv)**

$$\mathcal{T}_\omega(s) = \{u \in (\mathbf{C}_1)^\omega : s \stackrel{u}{\Longrightarrow}_1\}.$$

Each element $u \in \mathcal{T}_\omega(s)$ represents an infinite sequence of observable actions $s$ may perform.

**(v)** $$\mathcal{T}_\perp(s) = \{u \cdot \langle\perp\rangle : u \in (\mathbf{C}_1)^{<\omega} \wedge s \stackrel{u\cdot\langle\perp\rangle}{\Longrightarrow}_1\}.$$

Each element $u \cdot \langle\perp\rangle \in \mathcal{T}_\perp(s)$ represents a computation which first performs observable actions in $u$, and then, diverges, i.e., performs an infinite number of $\tau$'s.

**(2)** $$\mathcal{O}_1^{\text{w}}[\![s]\!] = \mathcal{T}_{\surd}(s) \cup \mathcal{T}_\delta(s) \cup \mathcal{T}_\omega(s) \cup \mathcal{T}_\perp(s).$$

**(3)** $$\mathcal{O}_1^{\text{wm}}[\![s]\!] = \mathcal{T}_{\surd}(s) \cup \mathcal{T}_\delta^{\text{m}}(s) \cup \mathcal{T}_\omega(s) \cup \mathcal{T}_\perp(s).$$

**(4)** Let $\mathbf{Q}_1^{\text{wli}} = \{\epsilon, \langle\delta\rangle, \langle\perp\rangle\}$. We define $\mathcal{O}_1^{\text{wi}}[\![s]\!]$ as the subset of $\mathcal{O}_1^{\text{w}}[\![s]\!]$ consisting only of elements of $\mathbf{Q}_1^{\text{wli}}$ under the assumption that the environment allows no (external) communication actions, and hence, the only possible action for processes is $\tau$. That is, we define $\mathcal{O}_1^{\text{wi}}$ with the help of a restriction function $\Theta$ as follows:

**(i)** For $p \in \mathbf{P}_1^{\text{wl}}$, let $\Theta(p) = p \cap \mathbf{Q}_1^{\text{wli}}$.

**(ii)** $\mathcal{O}_1^{\text{wi}}[\![s]\!] = \Theta(\mathcal{O}_1^{\text{w}}[\![s]\!])$. ∎

As stated in the introduction, none of the three semantics is compositional, as is exhibited by the following example.

**Example 5.1** Let

$$s_1 \equiv (c_0; c_1) + (c_0; c_2), \quad s_2 \equiv c_0; (c_1 + c_2).$$

Then,

$$\mathcal{O}[\![s_1]\!] = \mathcal{O}[\![s_2]\!]$$

($\mathcal{O} = \mathcal{O}_1^{\text{w}}, \mathcal{O}_1^{\text{wm}}, \mathcal{O}_1^{\text{wi}}$). However, putting

$$s \equiv (\bar{c}_0; \bar{c}_1), \quad C = \{c_2, \bar{c}_1\},$$

one has

$$\langle\delta\rangle \in \mathcal{O}[\![s_1 \parallel s]\!] \setminus \mathcal{O}[\![s_2 \parallel s]\!]$$

for $\mathcal{O} = \mathcal{O}_1^{\text{w}}, \mathcal{O}_1^{\text{wi}}$, and

$$\langle\delta\rangle \in \mathcal{O}_1^{\text{wm}}[\![\partial_C(s_1 \parallel s)]\!] \setminus \mathcal{O}_1^{\text{wm}}[\![\partial_C(s_2 \parallel s)]\!].$$ ∎

## 5.4 Two Compositional Models $\mathcal{C}_1^{\mathrm{w}}$, $\mathcal{C}_1^{\mathrm{wi}}$

In this section, two compositional models $\mathcal{C}_1^{\mathrm{w}}$ and $\mathcal{C}_1^{\mathrm{wi}}$ for $\mathcal{L}_1$ are introduced. The models $\mathcal{C}_1^{\mathrm{w}}$ and $\mathcal{C}_1^{\mathrm{wi}}$ are mild variants of the failures model of [BHR 85]. Here superscripts 'w' and 'i' stand for *weak* and *internal*, respectively. The models are defined operationally in the style of Plotkin ([Plo 81]) first, then they are shown to be *compositional.*

First, the common domain of $\mathcal{C}_1^{\mathrm{w}}$ and $\mathcal{C}_1^{\mathrm{wi}}$, written $\mathbf{P}_1^{\mathrm{wf}}$, is defined by:

**Definition 5.7** (Semantic Domain for $\mathcal{C}_1^{\mathrm{w}}$ and $\mathcal{C}_1^{\mathrm{wi}}$) Let

$$(\gamma \in)\ \mathbf{C}_1^{\surd} = \mathbf{C}_1 \cup \{\surd\},$$

and let us use $\Gamma$ as a variable ranging over $\wp(\mathbf{C}_1^{\surd})$. Let $\mathit{\Delta}$ be a symbol distinct from $\delta$ and $\bot$. The symbol $\mathit{\Delta}$ represents *immediate deadlock* (see Remark 5.2 below for more explanation of $\mathit{\Delta}$). For $\Gamma \in \wp(\mathbf{C}_1^{\surd})$, let us write, for easier readability, $\delta(\Gamma)$ and $\Delta(\Gamma)$ for $(\delta, \Gamma)$ and $(\mathit{\Delta}, \Gamma)$, respectively.

First, let

$$\begin{aligned}(q \in)\ \mathbf{Q}_1^{\mathrm{wf}} = \ & ((\mathbf{C}_1)^{<\omega} \cdot \{\langle\surd\rangle\}) \\ & \cup ((\mathbf{C}_1)^{<\omega} \cdot \{\langle\delta(\Gamma)\rangle : \Gamma \subseteq \mathbf{C}_1^{\surd}\}) \\ & \cup \{\langle\Delta(\Gamma)\rangle : \Gamma \subseteq \mathbf{C}_1^{\surd}\} \\ & \cup (\mathbf{C}_1)^{\omega} \cup ((\mathbf{C}_1)^{<\omega} \cdot \{\langle\bot\rangle\}).\end{aligned}$$

From $\mathbf{Q}_1^{\mathrm{wf}}$, we define $\mathbf{P}_1^{\mathrm{wf}}$ by:

$$(p \in)\ \mathbf{P}_1^{\mathrm{wf}} = \wp(\mathbf{Q}_1^{\mathrm{wf}}).$$

For $p \in \mathbf{P}_1^{\mathrm{wf}}$, let

$$\tilde{\mathcal{T}}_{\surd}(p) = p \cap ((\mathbf{C}_1)^{<\omega} \cdot \{\langle\surd\rangle\}),$$

$$\tilde{\mathcal{F}}(p) = p \cap ((\mathbf{C}_1)^{<\omega} \cdot \{\langle\delta(\Gamma)\rangle : \Gamma \subseteq \mathbf{C}_1^{\surd}\}),$$

$$\tilde{\mathcal{R}}(p) = p \cap \{\langle\Delta(\Gamma)\rangle : \Gamma \subseteq \mathbf{C}_1^{\surd}\},$$

$$\tilde{\mathcal{T}}_{\omega}(p) = p \cap (\mathbf{C}_1)^{\omega},$$

$$\tilde{\mathcal{T}}_{\bot}(p) = p \cap ((\mathbf{C}_1)^{<\omega} \cdot \{\langle\bot\rangle\}). \blacksquare$$

The compositional models

$$\mathcal{C}_1^{\mathrm{w}}, \mathcal{C}_1^{\mathrm{wi}} : \mathcal{L}_1[\emptyset] \to \mathbf{P}_1^{\mathrm{wf}}$$

are defined by:

**Definition 5.8** (Compositional Models) Let $s \in \mathcal{L}_1[\emptyset]$.

**(1)** Let $\mathcal{F}(s)$ be the set of *failures* in the sense of [BHR 85], $\mathcal{R}(s)$ the set of refusals of $s$ (not of some $s'$ such that $s \xrightarrow{\tau}_1 \xRightarrow{\epsilon}_1 s'$). Here $\mathcal{F}(s)$ and $\mathcal{R}(s)$ are defined as follows:[7]

$$\begin{aligned}&\mathcal{F}(s) = \\ &\{u \cdot \langle \delta(\Gamma)\rangle : u \in (\mathbf{C}_1)^{<\omega} \wedge \Gamma \subseteq \mathbf{C}_1^{\checkmark} \wedge \\ &\quad \exists s'[\ s \xRightarrow{u}_1 s' \not\xrightarrow{\tau}_1 / \wedge \Gamma \cap \mathrm{act}_1(s') = \emptyset\ ]\},\end{aligned}$$

$$\mathcal{R}(s) = \{\langle \Delta(\Gamma)\rangle : \Gamma \subseteq \mathbf{C}_1^{\checkmark} \wedge s \not\xrightarrow{\tau}_1 \wedge \Gamma \cap \mathrm{act}_1(s) = \emptyset\}.$$

**(2)** $$\mathcal{C}_1^{\mathrm{w}}[\![s]\!] = \mathcal{T}_{\checkmark}(s) \cup \mathcal{F}(s) \cup \mathcal{R}(s) \cup \mathcal{T}_\omega(s) \cup \mathcal{T}_\perp(s).$$

**(3)** For defining the second model $\mathcal{C}_1^{\mathrm{wi}}$ from $\mathcal{C}_1^{\mathrm{w}}$, a *pruning operation* $\Lambda$ on elements of

$$\wp((\mathbf{C}_1)^\omega \cup ((\mathbf{C}_1)^{<\omega} \cdot \{\langle\perp\rangle\}))$$

is defined as follows For $p \in \wp((\mathbf{C}_1)^\omega \cup ((\mathbf{C}_1)^{<\omega} \cdot \{\langle\perp\rangle\}))$,

$$\begin{aligned}\Lambda(p) = &\{u \in p \cap (\mathbf{C}_1)^\omega : \forall u' <_{\mathrm{p}} u[\ u' \cdot \langle\perp\rangle \notin p\ ]\} \cup \\ &\{u \cdot \langle\perp\rangle \in p \cap ((\mathbf{C}_1)^{<\omega} \cdot \{\langle\perp\rangle\}) : \forall u' <_{\mathrm{p}} u[\ u' \cdot \langle\perp\rangle \notin p\ ]\}.\end{aligned}$$

Moreover, for $p \in \mathbf{P}_1^{\mathrm{wf}}$, let

$$\begin{aligned}\tilde{\Lambda}(p) = \ &(p \setminus ((\mathbf{C}_1)^\omega \cup ((\mathbf{C}_1)^{<\omega} \cdot \{\langle\perp\rangle\}))) \\ &\cup \Lambda(p \cap ((\mathbf{C}_1)^\omega \cup ((\mathbf{C}_1)^{<\omega} \cdot \{\langle\perp\rangle\}))).\end{aligned}$$

Then, $\mathcal{C}_1^{\mathrm{wi}}[\![s]\!]$ is defined by:

$$\mathcal{C}_1^{\mathrm{wi}}[\![s]\!] = \tilde{\Lambda}(\mathcal{C}_1^{\mathrm{w}}[\![s]\!]).$$

See Remark 5.3 below for an intuitive explanation of the notion of pruning.

**(4)** Let

$$\psi : \{\langle \Delta(\Gamma)\rangle : \Gamma \subseteq \mathbf{C}_1^{\checkmark}\} \to \{\langle \delta(\Gamma)\rangle : \Gamma \subseteq \mathbf{C}_1^{\checkmark}\}$$

be defined by:

$$\psi(\langle \Delta(\Gamma)\rangle) = \langle \delta(\Gamma)\rangle. \blacksquare$$

Note that, from $\mathcal{C}_1^{\mathrm{w}}[\![s]\!]$, the five parts: $\mathcal{T}_{\checkmark}(s)$, $\mathcal{F}(s)$, $\mathcal{R}(s)$, $\mathcal{T}_\omega(s)$, $\mathcal{T}_\perp(s)$ are represented by:

$$\mathcal{P}(s) = \tilde{\mathcal{P}}(\mathcal{C}_1^{\mathrm{w}}[\![s]\!]) \quad (\mathcal{P} = \mathcal{T}_{\checkmark}, \mathcal{F}, \mathcal{R}, \mathcal{T}_\omega, \mathcal{T}_\perp).$$

**Remark 5.2**

[7] Originally in [BHR 85], a *failure* is defined as a pair $(u, \Gamma)$ with $u \in (\mathbf{C}_1)^{<\omega}$, $\Gamma \subseteq \mathbf{C}_1^{\checkmark}$. The pair $(u, \Gamma)$ in [BHR 85] corresponds to the sequence $u \cdot \langle \delta(\Gamma)\rangle$ in this chapter. We adopt the above definition of failures for the convenience in defining the semantic operations presented in Definitions 5.9–5.12.

**(1)** For $s \in \mathcal{L}_{\bar{1}}[\emptyset]$, $\mathcal{R}(s)$ is included in $\mathcal{F}(s)$ is the sense that

$$\psi[\mathcal{R}(s)] \subseteq \mathcal{F}(s).$$

The difference between $\langle\Delta(\Gamma)\rangle$ and $\langle\delta(\Gamma)\rangle$ is that $\langle\Delta(\Gamma)\rangle$ is an immediate refusal of $s$ itself (not of $s'$ such that $s \xrightarrow{\tau}_1 \overset{\epsilon}{\Longrightarrow}_1 s'$), while $\langle\delta(\Gamma)\rangle$ is a refusal of some $s'$ such that $s \overset{\epsilon}{\Longrightarrow}_1 s'$.

**(2)** There are alternative formulations for $\mathcal{C}_1^{\mathrm{w}}$ and $\mathcal{C}_1^{\mathrm{wi}}$. For example, let us define $\mathcal{R}'$ as follows: For $s \in \mathcal{L}_{\bar{1}}[\emptyset]$,

$$\mathcal{R}'(s) = \begin{cases} \{\Delta\} & \text{if } s \not\xrightarrow{\tau}_1, \\ \emptyset & \text{otherwise.} \end{cases}$$

Then it is easy to see that for every $s_1, s_2 \in \mathcal{L}_{\bar{1}}[\emptyset]$, the following holds:

$$\begin{aligned} &\mathcal{F}(s_1) \cup \mathcal{R}(s_1) = \mathcal{F}(s_2) \cup \mathcal{R}(s_2) \\ &\Leftrightarrow \mathcal{F}(s_1) \cup \mathcal{R}'(s_1) = \mathcal{F}(s_2) \cup \mathcal{R}'(s_2). \end{aligned}$$

Thus the part $\mathcal{R}(s)$ can be replaced by a *1-bit* piece of information $\mathcal{R}'(s)$. Nevertheless, we prefer the present formulation for the convenience in defining the semantic operations to be presented in § 5.4.

**(3)** In [BP 91], an additional ingredient similar to $\mathcal{R}(s)$ is used to construct a fully abstract model for a concurrent constraint programming language based on *asynchronous communication*. Notice that the communication treated here is *synchronous* unlike the one in [BP 91]. ▮

**Remark 5.3** As mentioned in § 5.1, the second failures model $\mathcal{C}_1^{\mathrm{wi}}$ is intended to be fully abstract w.r.t. the internal action model $\mathcal{O}_1^{\mathrm{wi}}$. For this purpose, $\mathcal{C}_1^{\mathrm{wi}}[\![s]\!]$ ($s \in \mathcal{L}_{\bar{1}}[\emptyset]$) should contain just enough information in order that $\mathcal{C}_1^{\mathrm{wi}}[\![s]\!]$ contain all the information in $\mathcal{O}_1^{\mathrm{wi}}[\![s]\!]$ (i.e., $\mathcal{C}_1^{\mathrm{wi}} \leq_{\mathrm{abs}} \mathcal{O}_1^{\mathrm{wi}}$) and that $\mathcal{C}_1^{\mathrm{wi}}$ be compositional.

Let us say $q \in \mathcal{C}_1^{\mathrm{w}}[\![s]\!]$ is *omissible* when $q \in (\mathbf{C}_1)^\omega \cup (\mathbf{C}_1)^{<\omega} \cdot \{\langle\bot\rangle\}$ and there is $u \in (\mathbf{C}_1)^{<\omega}$ such that $u <_{\mathrm{p}} \mathrm{strip}(q)$, where $\mathrm{strip}(q) = q$ if $q \in (\mathbf{C}_1)^\omega$, and $\mathrm{strip}(q) = u'$ if $q = u' \cdot \langle\bot\rangle$; otherwise we say $q$ is *non-omissible*.

Only in order to have $\mathcal{C}_1^{\mathrm{wi}} \leq_{\mathrm{abs}} \mathcal{O}_1^{\mathrm{wi}}$, we may omit all elements of $\mathcal{C}_1^{\mathrm{w}}[\![s]\!] \setminus \mathbf{Q}_1^{\mathrm{wli}}$ (from $\mathcal{C}_1^{\mathrm{w}}[\![s]\!]$ for obtaining $\mathcal{C}_1^{\mathrm{wi}}[\![s]\!]$), since all elements of $\mathcal{O}_1^{\mathrm{wi}}[\![s]\!]$ are contained in $\mathbf{Q}_1^{\mathrm{wli}}$. However, if we omit all elements of $\mathcal{C}_1^{\mathrm{w}}[\![s]\!] \setminus \mathbf{Q}_1^{\mathrm{wli}}$ (from $\mathcal{C}_1^{\mathrm{w}}[\![s]\!]$ for obtaining $\mathcal{C}_1^{\mathrm{wi}}[\![s]\!]$), we will loose the compositionality of $\mathcal{C}_1^{\mathrm{wi}}$, since for some context $S[\cdot]$, some elements of $\mathcal{C}_1^{\mathrm{w}}[\![S[s]]\!] \cap \mathbf{Q}_1^{\mathrm{wli}}$ stem from elements of $\mathcal{C}_1^{\mathrm{w}}[\![s]\!] \setminus \mathbf{Q}_1^{\mathrm{wli}}$. (For example, consider the context $\cdot\|s'$ with $\langle\overline{c_i}\rangle_{i\in\omega} \in \mathcal{C}_1^{\mathrm{w}}[\![s']\!]$. We may or may not have $\langle\bot\rangle \in \mathcal{O}_1^{\mathrm{wi}}[\![s\|s']\!]$, according as $\langle c_i\rangle_{i\in\omega} \in \mathcal{C}_1^{\mathrm{w}}[\![s]\!]$ or not.) Thus, we cannot omit all elements of $\mathcal{C}_1^{\mathrm{w}}[\![s]\!] \setminus \mathbf{Q}_1^{\mathrm{wli}}$ (from $\mathcal{C}_1^{\mathrm{w}}[\![s]\!]$ for obtaining $\mathcal{C}_1^{\mathrm{wi}}[\![s]\!]$).

But, we can omit all the omissible elements of $\mathcal{C}_1^{\mathrm{w}}[\![s]\!]$ (from it for obtaining $\mathcal{C}_1^{\mathrm{wi}}[\![s]\!]$), because for every context $S[\cdot]$, all non-omissible elements of $\mathcal{C}_1^{\mathrm{w}}[\![S[s]]\!]$ are obtained from non-omissible elements of $\mathcal{C}_1^{\mathrm{w}}[\![s]\!]$. (For example, let us consider the above context again, under the assumption that $\langle c_i\rangle_{i\in\omega} \in \mathcal{C}_1^{\mathrm{w}}[\![s]\!]$ with the additional assumption that $\langle c_i\rangle_{i\in k} \cdot \langle\bot\rangle \in \mathcal{C}_1^{\mathrm{w}}[\![s]\!]$. Then, $\langle c_i\rangle_{i\in\omega}$ is omissible by definition, since $\langle c_i \rangle_{i \in k} <_{\mathrm{p}} \langle c_i \rangle_{i \in \omega} = \mathrm{strip}(\langle c_i \rangle_{i \in \omega})$. We obtain the element $\langle \bot \rangle$ of $\mathcal{C}_1^{\mathrm{w}}[\![s \| s']\!] \cap \mathbf{Q}_1^{\mathrm{wli}}$, without using the omissible element $\langle c_i \rangle_{i \in \omega}$, by using the non-omissible element $\langle c_i \rangle_{i \in k} \cdot \langle \bot \rangle$ instead of $\langle c_i \rangle_{i \in \omega}$.)

Thus we can omit all the omissible elements of $\mathcal{C}_1^{\mathrm{w}}[\![s]\!]$ from it for obtaining $\mathcal{C}_1^{\mathrm{wi}}[\![s]\!]$; such omissible elements of $\mathcal{C}_1^{\mathrm{w}}[\![s]\!]$ are *pruned away* by the function $\Theta$, in order to obtain $\mathcal{C}_1^{\mathrm{wi}}[\![s]\!]$.

Note that in Chapter 6, the notion of *flattening*, which is a slightly stronger version of of *pruning*, is used for constructing a fully abstract model in a cpo framework. ∎

From Definition 5.8, it immediately follows that $\mathcal{C}_1^{\mathrm{w}} \leq_{\mathrm{abs}} \mathcal{C}_1^{\mathrm{wi}}$, i.e., that $\mathcal{C}_1^{\mathrm{wi}}$ is as abstract as or more abstract than $\mathcal{C}_1^{\mathrm{w}}$. Moreover, it holds that $\mathcal{C}_1^{\mathrm{w}} <_{\mathrm{abs}} \mathcal{C}_1^{\mathrm{wi}}$, i.e., that $\mathcal{C}_1^{\mathrm{wi}}$ is strictly more abstract than $\mathcal{C}_1^{\mathrm{w}}$; this fact follows from the following example, where we give two statements $s_0$, $s_1$ such that $\mathcal{C}_1^{\mathrm{w}}[\![s_0]\!] \neq \mathcal{C}_1^{\mathrm{w}}[\![s_1]\!]$ but $\mathcal{C}_1^{\mathrm{wi}}[\![s_0]\!] = \mathcal{C}_1^{\mathrm{wi}}[\![s_1]\!]$.

**Example 5.2** Let

$$s_0 \equiv (\mu X.\ c + (\tau; X; c)), \quad s_1 \equiv (\mu X.\ c; X) + s_0$$

(see Figure 5.1 for pictorial representations of $s_0$ and $s_1$). Then,

$$\begin{aligned}\mathcal{C}_1^{\mathrm{w}}[\![s_0]\!] = \{c^{n+1} \cdot \langle \surd \rangle : n \in \omega\} \cup \{c^{n+1} \cdot \langle \delta(\Gamma) \rangle : n \in \omega \wedge \Gamma \in \wp(\mathbf{C}_1)\} \\ \{c^{n+1} \cdot \langle \delta(\Gamma) \rangle : n \in \omega \wedge \Gamma \in \wp(\mathbf{C}_1^{\surd} \setminus \{c\})\} \cup \{\langle \bot \rangle\},\end{aligned}$$

and $\mathcal{C}_1^{\mathrm{w}}[\![s_1]\!] = \{c^{\omega}\} \cup \mathcal{C}_1^{\mathrm{w}}[\![s_0]\!]$. Thus, $\mathcal{C}_1^{\mathrm{w}}[\![s_0]\!] \neq \mathcal{C}_1^{\mathrm{w}}[\![s_1]\!]$.

On the other hand,

$$\mathcal{C}_1^{\mathrm{wi}}[\![s_0]\!] = \mathcal{C}_1^{\mathrm{w}}[\![s_0]\!] = \mathcal{C}_1^{\mathrm{w}}[\![s_1]\!] \setminus \{c^{\omega}\} = \mathcal{C}_1^{\mathrm{wi}}[\![s_1]\!]. \blacksquare$$

The following lemma, whose first part states a kind of *closedness* of $\mathcal{C}_1^{\mathrm{wi}}[\![s]\!]$ ($s \in \mathcal{L}_{\bar{1}}[\emptyset]$), will play a key role in the proof of the full abstractness of $\mathcal{C}_1^{\mathrm{wi}}$ w.r.t. $\mathcal{O}_1^{\mathrm{wi}}$.

**Lemma 5.2**

**(1)** *Let $s \in \mathcal{L}_{\bar{1}}[\emptyset]$ and $u \in (\mathbf{C}_1)^{\omega}$. Then*

$$\begin{aligned}&\forall u' <_{\mathrm{p}} u[\ u' \cdot \langle \bot \rangle \notin \mathcal{C}_1^{\mathrm{w}}[\![s]\!] \wedge \mathcal{C}_1^{\mathrm{w}}[\![s]\!][u'] \neq \emptyset\ ] \\ &\Rightarrow u \in \mathcal{C}_1^{\mathrm{w}}[\![s]\!].\end{aligned}$$

**(2)** *Let $s_1, s_2 \in \mathcal{L}_{\bar{1}}[\emptyset]$. Then*

$$\begin{aligned}&(\mathcal{C}_1^{\mathrm{wi}}[\![s_1]\!] \setminus (\mathbf{C}_1)^{\omega}) = (\mathcal{C}_1^{\mathrm{wi}}[\![s_2]\!] \setminus (\mathbf{C}_1)^{\omega}) \\ &\Rightarrow (\mathcal{C}_1^{\mathrm{wi}}[\![s_1]\!] \cap (\mathbf{C}_1)^{\omega}) = (\mathcal{C}_1^{\mathrm{wi}}[\![s_2]\!] \cap (\mathbf{C}_1)^{\omega}). \blacksquare\end{aligned}$$

**Proof.** **(1)** By the fact the transition system is finitely branching (Lemma 5.1), with the help of König's Lemma (cf., e.g., [Kun 80] Lemma 5.6).
**(2)** Let $u \in (\mathcal{C}_1^{\mathrm{wi}}[\![s_1]\!] \cap (\mathbf{C}_1)^{\omega})$ and $n \in \omega$. Then, by the definition of $\mathcal{C}_1^{\mathrm{wi}}$, one has $u_{[n]} \cdot \langle \bot \rangle \notin \mathcal{C}_1^{\mathrm{w}}[\![s_1]\!]$, and hence,

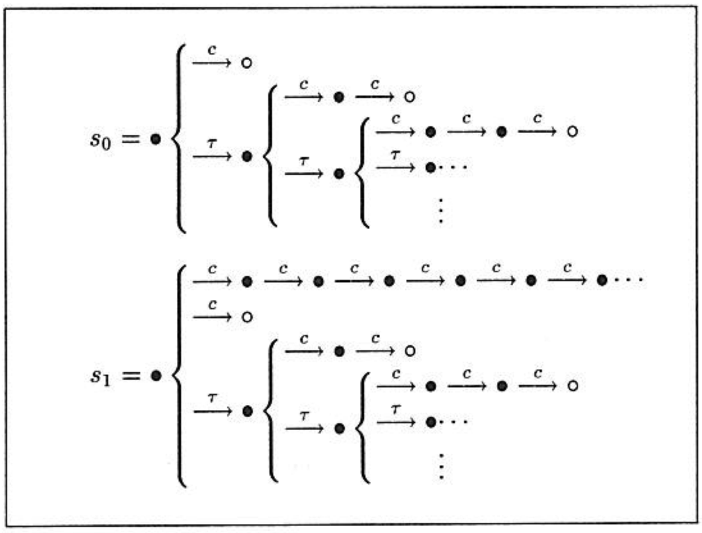

Figure 5.1: Synchronization Trees of Statements $s_0$, $s_1$ in $\mathcal{L}_{\bar{1}}$

$$u_{[n]} \cdot \langle \perp \rangle \notin \mathcal{C}_{\bar{1}}^{\mathrm{w}}[\![s_2]\!]. \tag{5.18}$$

Clearly there exists $s_1'$ such that $s_1 \stackrel{u_{[n]}}{\Longrightarrow}_1 s_1'$. Since $u_{[n]} \cdot \langle \perp \rangle \notin \mathcal{C}_{\bar{1}}^{\mathrm{w}}[\![s_1]\!]$, there exists $s_1''$ such that

$$s_1' \stackrel{\epsilon}{\Longrightarrow}_1 s_1'' \stackrel{\tau}{\not\longrightarrow}_1 .$$

Thus

$$u_{[n]} \cdot \langle \delta(\Gamma) \rangle \in \mathcal{C}_{\bar{1}}^{\mathrm{w}}[\![s_1]\!]$$

for some $\Gamma \subseteq \mathbf{C}_{\bar{1}}^{\checkmark}$. Thus, one has

$$u_{[n]} \cdot \langle \delta(\Gamma) \rangle \in (\mathcal{C}_{\bar{1}}^{\mathrm{wi}}[\![s_1]\!] \setminus (\mathbf{C}_1)^{\omega}) = (\mathcal{C}_{\bar{1}}^{\mathrm{wi}}[\![s_2]\!] \setminus (\mathbf{C}_1)^{\omega}),$$

and hence,

$$u_{[n]} \cdot \langle \delta(\Gamma) \rangle \in \mathcal{C}_{\bar{1}}^{\mathrm{w}}[\![s_2]\!].$$

By this and (5.18), applying Part **(1)**, one has $u \in \mathcal{C}_{\bar{1}}^{\mathrm{w}}[\![s_2]\!]$. By this and (5.18), recalling the definition of $\mathcal{C}_{\bar{1}}^{\mathrm{wi}}$, one has

$$u \in (\mathcal{C}_{\bar{1}}^{\mathrm{wi}}[\![s_2]\!] \cap (\mathbf{C}_1)^{\omega}). \blacksquare$$

It can be shown that both $\mathcal{C}_{\bar{1}}^{\mathrm{w}}$ and $\mathcal{C}_{\bar{1}}^{\mathrm{wi}}$ are *compositional* w.r.t. all the combinators of $\mathcal{L}_{\bar{1}}$. In order to show this, we will define semantic operations corresponding to the syntactic combinators of $\mathcal{L}_{\bar{1}}$.

First, two binary semantic operations $\bar{;}$ and $\hat{;}$ corresponding to ';' are defined by:

**Definition 5.9** With two auxiliary operations:

$$\mathcal{F}^{;}, \mathcal{R}^{;} : (\mathbf{P}_1^{\mathrm{wf}} \times \mathbf{P}_1^{\mathrm{wf}}) \to \mathbf{P}_1^{\mathrm{wf}},$$

the operations $\tilde{;}$ and $\hat{;}$ are defined as follows: For $p_1, p_2 \in \mathbf{P}_1^{\mathrm{wf}}$,

$$\begin{aligned}
&\mathcal{F}^{;}(p_1,p_2) = \\
&\{u \cdot \langle \delta(\Gamma) \rangle : u \cdot \langle \delta(\Gamma \cup \{\surd\}) \rangle \in \tilde{\mathcal{F}}(p_1) \lor \\
&\quad (u \cdot \langle \delta(\Gamma \setminus \{\surd\}) \rangle \in \tilde{\mathcal{F}}(p_1) \land \langle \Delta(\Gamma) \rangle \in \tilde{\mathcal{R}}(p_2))\} \\
&\cup \{w \cdot q : w \cdot \langle \surd \rangle \in \tilde{\mathcal{T}}_{\surd}(p_1) \land q \in (\tilde{\mathcal{F}}(p_2) \setminus \psi[\tilde{\mathcal{R}}(p_2)])\},
\end{aligned}$$

$$\begin{aligned}
&\mathcal{R}^{;}(p_1,p_2) = \\
&\{\langle \Delta(\Gamma) \rangle : \langle \Delta(\Gamma \cup \{\surd\}) \rangle \in \tilde{\mathcal{R}}(p_1) \lor \\
&\quad (\langle \Delta(\Gamma \setminus \{\surd\}) \rangle \in \tilde{\mathcal{R}}(p_1) \land \langle \Delta(\Gamma) \rangle \in \tilde{\mathcal{R}}(p_2))\},
\end{aligned}$$

$$\begin{aligned}
&p_1 \,\tilde{;}\, p_2 = \\
&\{w \cdot q : w \cdot \langle \surd \rangle \in \tilde{\mathcal{T}}_{\surd}(p_1) \land q \in \tilde{\mathcal{T}}_{\surd}(p_2)\} \\
&\cup \mathcal{F}^{;}(p_1,p_2) \cup \mathcal{R}^{;}(p_1,p_2) \\
&\cup (\tilde{\mathcal{T}}_{\omega}(p_1) \cup \{w \cdot q : w \cdot \langle \surd \rangle \in \tilde{\mathcal{T}}_{\surd}(p_1) \land q \in \tilde{\mathcal{T}}_{\omega}(p_2)\}) \\
&\cup (\tilde{\mathcal{T}}_{\perp}(p_1) \cup \{w \cdot q : w \cdot \langle \surd \rangle \in \tilde{\mathcal{T}}_{\surd}(p_1) \land q \in \tilde{\mathcal{T}}_{\perp}(p_2)\}),
\end{aligned}$$

$$p_1 \,\hat{;}\, p_2 = \tilde{\Lambda}(p_1 \,\tilde{;}\, p_2). \blacksquare$$

Next, two binary semantic operations $\tilde{+}$ and $\hat{+}$ corresponding to '+' are defined by:

**Definition 5.10** With an auxiliary operation $\mathcal{F}^{+}$, the operations $\tilde{+}$ and $\hat{+}$ are defined as follows: For $p_1, p_2 \in \mathbf{P}_1^{\mathrm{wf}}$,

$$\begin{aligned}
&\mathcal{F}^{+}(p_1,p_2) = \\
&(\tilde{\mathcal{F}}(p_1) \setminus \psi[\tilde{\mathcal{R}}(p_1)]) \cup (\tilde{\mathcal{F}}(p_2) \setminus \psi[\tilde{\mathcal{R}}(p_2)]) \\
&\cup \psi[\tilde{\mathcal{R}}(p_1) \cap \tilde{\mathcal{R}}(p_2)],
\end{aligned}$$

$$\begin{aligned}
&p_1 \tilde{+} p_2 = \\
&(\tilde{\mathcal{T}}_{\surd}(p_1) \cup \tilde{\mathcal{T}}_{\surd}(p_2)) \\
&\cup \mathcal{F}^{+}(p_1,p_2) \cup (\tilde{\mathcal{R}}(p_1) \cap \tilde{\mathcal{R}}(p_2)) \\
&\cup (\tilde{\mathcal{T}}_{\omega}(p_1) \cup \tilde{\mathcal{T}}_{\omega}(p_2)) \cup (\tilde{\mathcal{T}}_{\perp}(p_1) \cup \tilde{\mathcal{T}}_{\perp}(p_2)),
\end{aligned}$$

$$p_1 \hat{+} p_2 = \tilde{\Lambda}(p_1 \tilde{+} p_2). \blacksquare$$

Moreover, two binary semantic operations $\tilde{\|}$ and $\hat{\|}$ corresponding to '$\|$' are defined. As a preliminary to the definition, a function

$$\mathrm{mrg} : ((\mathbf{A}_1)^{\leq\omega} \times (\mathbf{A}_1)^{\leq\omega}) \to \wp((\mathbf{A}_1)^{\leq\omega})$$

is defined as follows:[8]

[8] It is easy to see that the above definition of the function mrg is a natural extension of the definition of *shuffle* of two (possibly infinite) sequences in [Mey 86], so as to allow *synchronization* of communicating actions with their complements (cf. [Mey 86] § 2.2).

**Definition 5.11** (Merging of Sequences) Let $w_1, w_2 \in (\mathbf{A}_1)^{\leq\omega}$.

**(1)** First, the set of merged sequences of $w_1$ and $w_2$ with *extra information* on the *origin* of its elements, written $\mathrm{mrg}^*(w_1, w_2)$, is defined. Let $L$, $R$, and $S$ be distinct symbols standing for "*from left-hand side*", "*from right-hand side*", and "*from both sides by synchronization*", respectively. Let

$$\mathbf{H} = \{\ \eta \in (\mathbf{A}_1 \times \{L, R, S\})^{\leq\omega} : \forall i \in \mathrm{lgt}(\eta)[\\ \pi_1^2(\eta(i)) = S \Rightarrow \pi_0^2(\eta(i)) \in \mathbf{C}_1\ ]\}.$$

For $a \in \mathbf{A}_1$, let us write simply $a^L$, $a^R$, $a^S$ for $(a, L)$, $(a, R)$, $(a, S)$, respectively. Two homomorphisms $\pi_L, \pi_R : \mathbf{H} \to (\mathbf{A}_1)^{\leq\omega}$ are defined as follows: For $a \in \mathbf{A}_1$, $c \in \mathbf{C}_1$,

$$\begin{array}{l} \textbf{(i)}\ \pi_L(\langle a^L\rangle) = \langle a\rangle,\ \ \pi_R(\langle a^L\rangle) = \epsilon, \\ \textbf{(ii)}\ \pi_L(\langle a^R\rangle) = \epsilon,\ \ \pi_R(\langle a^R\rangle) = \langle a\rangle, \\ \textbf{(iii)}\ \pi_L(\langle c^S\rangle) = \langle c\rangle,\ \ \pi_R(\langle c^S\rangle) = \langle \bar{c}\rangle. \end{array} \tag{5.19}$$

(Note that $\pi_L$ and $\pi_R$ in (5.19) (iii) can be exchanged without changing the meaning of $\mathbf{H}$, because $\bar{\bar{c}} = c$, by Definition 5.4 (2).)
Then, let $\mathrm{mrg}^*(w_1, w_2)$ be the set of elements $\eta \in \mathbf{H}$ satisfying the following conditions:

**(i)** If both $\mathrm{lgt}(w_1)$ and $\mathrm{lgt}(w_2)$ are finite, then $\pi_L(\eta) = w_1$ and $\pi_R(\eta) = w_2$. (In this case, one has $\mathrm{lgt}(\eta) \leq \mathrm{lgt}(w_1) + \mathrm{lgt}(w_2)$, by the definitions of $\pi_L$ and $\pi_R$.)

**(ii)** If either $\mathrm{lgt}(w_1)$ or $\mathrm{lgt}(w_2)$ is infinite, then $\mathrm{lgt}(\eta) = \omega$, $\pi_L(\eta) \leq_{\mathrm{p}} w_1$ and $\pi_R(\eta) \leq_{\mathrm{p}} w_2$. (In this case, one has $\pi_j(\eta) \leq_{\mathrm{p}} w_j$ instead of $\pi_j(\eta) = w_j$, because the actions in the merged sequence might stem only from $w_i$ from some stage, if $w_i$ is infinite $((i, j) = (1, 2), (2, 1))$. Note that no fairness constraint for merging is imposed here.)

**(2)** Another homomorphism $\pi : \mathbf{H} \to (\mathbf{A}_1)^{\leq\omega}$ is defined as follows: For $a \in \mathbf{A}_1$, $c \in \mathbf{C}_1$,

$$\pi(\langle a^L\rangle) = \pi(\langle a^R\rangle) = \langle a\rangle,\ \ \pi(\langle c^S\rangle) = \langle\tau\rangle.$$

Then, let

$$\mathrm{mrg}(w_1, w_2) = \pi[\mathrm{mrg}^*(w_1, w_2)],$$

$$\mathrm{mrg}_{\mathrm{w}}(w_1, w_2) = \theta[\mathrm{mrg}(w_1, w_2)].$$

**(3)** For $p_1, p_2 \subseteq (\mathbf{C}_1)^{\leq\omega}$, let

$$\mathrm{Mrg}_{\mathrm{w}}(p_1, p_2) = \bigcup\{\mathrm{mrg}_{\mathrm{w}}(w_1, w_2) : w_1 \in p_1 \wedge w_2 \in p_2\}. \blacksquare$$

From the homomorphisms defined above, we have the following lemma:

**Lemma 5.3 (1)** *Let* $s_1, s_2, s' \in \mathcal{L}_1^{\mathrm{i}}[\emptyset]$, $w \in (\mathbf{A}_1)^{<\omega}$. *Then,*

$$s_1 \,\|\, s_2 \xrightarrow{w}_{1*} s' \Leftrightarrow$$
$$\exists w_1, w_2 \in (\mathbf{A}_1)^{<\omega}, \exists s_1', s_2' \in \mathcal{L}_{\bar{1}}[\emptyset][\ w \in \mathrm{mrg}(w_1, w_2)$$
$$\wedge\, s' \equiv (s_1' \,\|\, s_2') \wedge\ s_1 \xrightarrow{w_1}_{1*} s_1' \ \wedge\ s_2 \xrightarrow{w_2}_{1*} s_2'\ ].$$

**(2)** *Let* $w_1, w_2 \in (\mathbf{A}_1)^{<\omega}$. *Then*

$$\exists w \in \{\tau\}^{<\omega}[\ w \in \mathrm{mrg}(w_1, w_2)\ ] \Rightarrow \theta(w_1) = \overline{\theta(w_2)}.$$

**Proof.** The part (1) (resp. (2)) can be shown by easy induction on $\mathrm{lgt}(w)$ (resp. on $\mathrm{lgt}(w_1) + \mathrm{lgt}(w_2)$). ■

From $\mathrm{Mrg_w}$, the semantic operations $\widetilde{\|}$ and $\widehat{\|}$ are defined by:

**Definition 5.12** For $p \in \mathbf{P}_1^{\mathrm{wf}}$, let

$$\tilde{\mathcal{T}}_1^*(p) = \{u \in (\mathbf{C}_1)^{<\omega} :\ p[u] \neq \emptyset\},$$

$$\mathrm{Pref}_\perp(p) = \{u \in (\mathbf{C}_1)^{<\omega} :\ u \cdot \langle \perp \rangle \in \mathcal{T}_\perp(p)\}.$$

First, four auxiliary operations

$$\mathcal{F}^{\|}, \mathcal{R}^{\|}, \mathcal{T}_\omega^{\|}, \mathcal{T}_\perp^{\|} : (\mathbf{P}_1^{\mathrm{wf}} \times \mathbf{P}_1^{\mathrm{wf}}) \to \mathbf{P}_1^{\mathrm{wf}}$$

are defined as follows: For $p_1, p_2 \in \mathbf{P}_1^{\mathrm{wf}}$,

$$\begin{aligned}
&\mathcal{F}^{\|}(p_1, p_2) = \\
&\{\theta(w) \cdot \langle \delta(\Gamma) \rangle : \\
&\ \exists u_1 \cdot \langle \delta(\Gamma_1) \rangle \in \tilde{\mathcal{F}}(p_1), \exists u_2 \cdot \langle \delta(\Gamma_1) \rangle \in \tilde{\mathcal{F}}(p_2) \\
&\ [\ w \in \mathrm{mrg}(u_1, u_2) \\
&\ \wedge (\mathbf{C}_1 \setminus \Gamma_1) \cap \overline{(\mathbf{C}_1 \setminus \Gamma_2)} = \emptyset \\
&\ \wedge ((\Gamma \setminus \{\surd\} \subseteq \Gamma_1 \ \wedge\ \Gamma \subseteq \Gamma_2) \\
&\qquad \vee (\Gamma \subseteq \Gamma_1 \ \wedge\ \Gamma \setminus \{\surd\} \subseteq \Gamma_2))\ ]\},
\end{aligned}$$

$$\begin{aligned}
&\mathcal{R}^{\|}(p_1, p_2) = \\
&\{\langle \Delta(\Gamma) \rangle : \\
&\ \exists \langle \Delta(\Gamma_1) \rangle \in \tilde{\mathcal{R}}(p_1), \exists \langle \Delta(\Gamma_2) \rangle \in \tilde{\mathcal{R}}(p_2) \\
&\ [\ (\mathbf{C}_1 \setminus \Gamma_1) \cap \overline{(\mathbf{C}_1 \setminus \Gamma_2)} = \emptyset \\
&\quad \wedge ((\Gamma \setminus \{\surd\} \subseteq \Gamma_1 \ \wedge\ \Gamma \subseteq \Gamma_2) \\
&\qquad \vee (\Gamma \subseteq \Gamma_1 \ \wedge\ \Gamma \setminus \{\surd\} \subseteq \Gamma_2))\ ]\},
\end{aligned}$$

$$\begin{aligned}
&\mathcal{T}_\omega^{\|}(p_1, p_2) = \\
&\ \mathrm{Mrg_w}(\tilde{\mathcal{T}}_1^*(p_1), \tilde{\mathcal{T}}_\omega(p_2)) \\
&\ \cup \mathrm{Mrg_w}(\tilde{\mathcal{T}}_1^*(p_2), \tilde{\mathcal{T}}_\omega(p_1)) \\
&\ \cup ((\mathbf{C}_1)^\omega \ \cap\ \mathrm{Mrg_w}(\tilde{\mathcal{T}}_\omega(p_1), \tilde{\mathcal{T}}_\omega(p_2))),
\end{aligned}$$

$$\begin{aligned}
&\mathcal{T}_\perp^{\|}(p_1, p_2) = \\
&\ \mathrm{Mrg_w}(\tilde{\mathcal{T}}_1^*(p_1), \mathrm{Pref}_\perp(p_2)) \cdot \{\langle \perp \rangle\} \\
&\ \cup \mathrm{Mrg_w}(\tilde{\mathcal{T}}_1^*(p_2), \mathrm{Pref}_\perp(p_1)) \cdot \{\langle \perp \rangle\} \\
&\ \cup (((\mathbf{C}_1)^{<\omega} \cdot \{\langle \perp \rangle\}) \ \cap\ \mathrm{Mrg_w}(\tilde{\mathcal{T}}_\omega(p_1), \tilde{\mathcal{T}}_\omega(p_2))).
\end{aligned}$$

From these auxiliary operations, $\widetilde{\|}$ and $\widehat{\|}$ are defined by:

$$\begin{array}{rl} p_1 \widetilde{\|} p_2 = & \mathrm{Mrg_w}(\{w : w \cdot \langle\surd\rangle \in \tilde{\mathcal{T}}_{\surd}(p_1)\}, \{w : w \cdot \langle\surd\rangle \in \tilde{\mathcal{T}}_{\surd}(p_1)\}) \\ & \cup \mathcal{F}^{\|}(p_1, p_2) \cup \mathcal{R}^{\|}(p_1, p_2) \\ & \cup \mathcal{T}_{\omega}^{\|}(p_1, p_2) \cup \mathcal{T}_{\perp}^{\|}(p_1, p_2). \end{array}$$

$$p_1 \widehat{\|} p_2 = \tilde{\Lambda}(p_1 \widetilde{\|} p_2). \blacksquare$$

Finally, a unary semantic operation $\widetilde{\partial}_C$ ($C \subseteq \mathbf{C}_1$) is defined by:

**Definition 5.13** Let $C \subseteq \mathbf{C}_1$. For $p \in \mathbf{Q}_1^{\mathrm{wf}}$, let

$$\begin{array}{l} \widetilde{\partial}_C(p) = \\ \{u \cdot \langle\surd\rangle \in \tilde{\mathcal{T}}_{\surd}(p) : \mathrm{ran}(u) \cap C = \emptyset\} \\ \cup \{u \cdot \langle\delta(\Gamma)\rangle : u \in (\mathbf{C}_1)^{<\omega} \wedge \Gamma \subseteq \mathbf{C}_1^{\surd} \wedge \\ \quad \mathrm{ran}(u) \cap C = \emptyset \wedge u \cdot \langle\delta(\Gamma \setminus C)\rangle \in \tilde{\mathcal{F}}(p)\} \\ \cup \{\langle\Delta(\Gamma)\rangle : \Gamma \subseteq \mathbf{C}_1^{\surd} \wedge \langle\Delta(\Gamma \setminus C)\rangle \in \tilde{\mathcal{R}}(p)\} \\ \cup \{u \in \tilde{\mathcal{T}}_{\omega}(p) : \mathrm{ran}(u) \cap C = \emptyset\} \\ \cup \{u \cdot \langle\perp\rangle \in \tilde{\mathcal{T}}_{\perp}(p) : \mathrm{ran}(u) \cap C = \emptyset\}. \blacksquare \end{array}$$

By means of the semantic operations $\widetilde{\mathsf{op}}$ and $\widehat{\mathsf{op}}$ ($\mathsf{op} \equiv$ ';', '+', '$\|$') and $\widetilde{\partial}_C$, the compositionality of $\mathcal{C}_1^{\mathrm{w}}$ and $\mathcal{C}_1^{\mathrm{wi}}$ w.r.t. the combinators is shown:

**Lemma 5.4** (Compositionality w.r.t. Combinators) *The models $\mathcal{C}_1^{\mathrm{w}}$ and $\mathcal{C}_1^{\mathrm{wi}}$ are compositional w.r.t. all the combinators of $\mathcal{L}_{\bar{1}}$. That is, the following propositions hold for every $s_1, s_2 \in \mathcal{L}_{\bar{1}}[\emptyset]$ and $C \subseteq \mathbf{C}_1$:*

**(i)** $\mathcal{C}_1^{\mathrm{w}}[\![s_1 \ \mathsf{op}\ s_2]\!] = \mathcal{C}_1^{\mathrm{w}}[\![s_1]\!] \ \widetilde{\mathsf{op}}\ \mathcal{C}_1^{\mathrm{w}}[\![s_2]\!]$,
**(ii)** $\mathcal{C}_1^{\mathrm{wi}}[\![s_1 \ \mathsf{op}\ s_2]\!] = \mathcal{C}_1^{\mathrm{wi}}[\![s_1]\!] \ \widehat{\mathsf{op}}\ \mathcal{C}_1^{\mathrm{wi}}[\![s_2]\!]$,
**(iii)** $\mathcal{C}[\![\partial_C(s_1)]\!] = \widetilde{\partial}_C(\mathcal{C}[\![s_1]\!])$,

*where* $\mathsf{op} \equiv$ ';', '+', '$\|$', *and* $\mathcal{C} = \mathcal{C}_1^{\mathrm{w}}, \mathcal{C}_1^{\mathrm{wi}}$. ∎

**Proof.** By case analysis on the types of elements of $\mathcal{C}_1^{\mathrm{w}}[\![s_1 \ \mathsf{op}\ s_2]\!]$ and $\mathcal{C}_1^{\mathrm{wi}}[\![s_1 \ \mathsf{op}\ s_2]\!]$ ($\mathsf{op} \equiv$ ';', '+', '$\|$'), using the definitions of $\mathcal{C}_1^{\mathrm{w}}$ and $\mathcal{C}_1^{\mathrm{wi}}$, together with the definitions of the semantic operations. ∎

A semantic model $\mathcal{M} : \mathcal{L}_{\bar{1}}[\emptyset] \to \mathbf{D}(\mathcal{M})$ is called *compositional w.r.t. all contexts* of $\mathcal{L}_{\bar{1}}$ iff it satisfies:[9]

$$\begin{array}{l} \forall X \in \mathcal{X}_{\mathcal{P}}, \forall S \in \mathcal{L}_{\bar{1}}[X], \exists f \in (\mathbf{D}(\mathcal{M}) \to \mathbf{D}(\mathcal{M}))[ \\ \quad \forall s \in \mathcal{L}_{\bar{1}}[\emptyset][\ \mathcal{M}[\![S[s/X]]\!] = f(\mathcal{M}[\![s]\!])\ ]], \end{array}$$

which is obviously equivalent to the following:

$$\begin{array}{l} \forall s_1, s_2 \in \mathcal{L}_{\bar{1}}[\emptyset][\ \mathcal{M}[\![s_1]\!] = \mathcal{M}[\![s_2]\!] \Rightarrow \\ \quad \forall X \in \mathcal{X}_{\mathcal{P}}, \forall S \in \mathcal{L}_{\bar{1}}[X][\ \mathcal{M}[\![S[s_1/X]]\!] = \mathcal{M}[\![S[s_2/X]]\!]\ ]]. \end{array} \tag{5.20}$$

[9] In many of the references, the notion of *compositionality* is defined only in terms of certain sets of combinators. The above definition is due to Apt and Plotkin (cf. [AP 86] Definition 3.3).

It can be shown that both $\mathcal{C}_{\bar{1}}^{\mathrm{w}}$ and $\mathcal{C}_{\bar{1}}^{\mathrm{wi}}$ are compositional w.r.t. all $\mathcal{L}_{\bar{1}}$ contexts (possibly containing $\mu$-expressions) as stated by the next Lemma:

**Lemma 5.5** (Compositionality of $\mathcal{C}_{\bar{1}}^{\mathrm{w}}$ and $\mathcal{C}_{\bar{1}}^{\mathrm{wi}}$) *Let $\mathcal{C}$ be $\mathcal{C}_{\bar{1}}^{\mathrm{w}}$ or $\mathcal{C}_{\bar{1}}^{\mathrm{wi}}$. Then, for every $s_1, s_2 \in \mathcal{L}_{\bar{1}}[\emptyset]$, the following holds:*

$$\begin{array}{l}\mathcal{C}[\![s_1]\!] = \mathcal{C}[\![s_2]\!] \Rightarrow \\ \forall X \in \mathcal{X}_{\mathcal{P}}, \forall S \in \mathcal{L}_{\bar{1}}[X][\ \mathcal{C}[\![S[s_1/X]]\!] = \mathcal{C}[\![S[s_2/X]]\!]\ ].\ \blacksquare\end{array} \tag{5.21}$$

**Proof.** See Appendix 5.A. ■

The proof of this lemma is rather more involved than that of Lemma 5.4 because of (possibly nested) $\mu$-expressions contained in the contexts; we relegate the proof to the appendix, and first establish the full abstractness of $\mathcal{C}_{\bar{1}}^{\mathrm{w}}$ (resp. $\mathcal{C}_{\bar{1}}^{\mathrm{wi}}$) w.r.t. $\mathcal{O}_{\bar{1}}^{\mathrm{w}}$ and $\mathcal{O}_{\bar{1}}^{\mathrm{wm}}$ (resp. w.r.t. $\mathcal{O}_{\bar{1}}^{\mathrm{wi}}$), using Lemma 5.5, in the next two sections.

## 5.5 Correctness of Models

The correctness of $\mathcal{C}_{\bar{1}}^{\mathrm{w}}$ (resp. of $\mathcal{C}_{\bar{1}}^{\mathrm{wi}}$) w.r.t. $\mathcal{O}_{\bar{1}}^{\mathrm{w}}$ and $\mathcal{O}_{\bar{1}}^{\mathrm{wm}}$ (resp. w.r.t. $\mathcal{O}_{\bar{1}}^{\mathrm{wi}}$) can be shown by means of three abstraction functions $\mathcal{A}_{\bar{1}}^{\mathrm{w}}, \mathcal{A}_{\bar{1}}^{\mathrm{wm}}, \mathcal{A}_{\bar{1}}^{\mathrm{wi}} : \mathbf{P}_{\bar{1}}^{\mathrm{wf}} \to \mathbf{P}_{\bar{1}}^{\mathrm{wl}}$, which are defined by:

**Definition 5.14** Let $p \in \mathbf{P}_{\bar{1}}^{\mathrm{wf}}$.

**(1)** $\mathcal{A}_{\bar{1}}^{\mathrm{w}}(p) = \{u \cdot \langle\, \delta\,\rangle : \exists \Gamma \in \wp(\mathbf{C}_{\bar{1}}^{\checkmark})[\ \checkmark \in \Gamma \wedge u \cdot \langle \delta(\Gamma)\rangle \in p\ ]\} \cup \tilde{\mathcal{T}}_{\checkmark}(p) \cup \tilde{\mathcal{T}}_{\omega}(p) \cup \tilde{\mathcal{T}}_{\perp}(p).$

**(2)** $\mathcal{A}_{\bar{1}}^{\mathrm{wm}}(p) = \{u \cdot \langle\, \delta\,\rangle : u \cdot \langle \delta(\mathbf{C}_{\bar{1}}^{\checkmark})\rangle \in p\ \} \cup \tilde{\mathcal{T}}_{\checkmark}(p) \cup \tilde{\mathcal{T}}_{\omega}(p) \cup \tilde{\mathcal{T}}_{\perp}(p).$

**(3)** $\mathcal{A}_{\bar{1}}^{\mathrm{wi}}(p) = \mathcal{A}_{\bar{1}}^{\mathrm{w}}(p) \cap \{\epsilon, \langle\, \delta\,\rangle, \langle \perp \rangle\}.$ ■

The following lemma follows immediately from the definitions of the linear semantics, the compositional models, and the abstraction functions.

**Lemma 5.6** *Let $(\mathcal{C}, \mathcal{O}, \mathcal{A})$ be $(\mathcal{C}_{\bar{1}}^{\mathrm{w}}, \mathcal{O}_{\bar{1}}^{\mathrm{w}}, \mathcal{A}_{\bar{1}}^{\mathrm{w}})$, $(\mathcal{C}_{\bar{1}}^{\mathrm{w}}, \mathcal{O}_{\bar{1}}^{\mathrm{wm}}, \mathcal{A}_{\bar{1}}^{\mathrm{wm}})$, or $(\mathcal{C}_{\bar{1}}^{\mathrm{wi}}, \mathcal{O}_{\bar{1}}^{\mathrm{wi}}, \mathcal{A}_{\bar{1}}^{\mathrm{wi}})$. Then,*

$$\forall s \in \mathcal{L}_{\bar{1}}[\emptyset][\ \mathcal{O}[\![s]\!] = \mathcal{A}(\mathcal{C}[\![s]\!])\ ].\ \blacksquare$$

By this and the *compositionality* of $\mathcal{C}_{\bar{1}}^{\mathrm{w}}$ and $\mathcal{C}_{\bar{1}}^{\mathrm{wi}}$ w.r.t. all contexts of $\mathcal{L}_{\bar{1}}$, we can show the correctness of $\mathcal{C}_{\bar{1}}^{\mathrm{w}}$ (resp. $\mathcal{C}_{\bar{1}}^{\mathrm{wi}}$) w.r.t. $\mathcal{O}_{\bar{1}}^{\mathrm{w}}$ and $\mathcal{O}_{\bar{1}}^{\mathrm{wm}}$ (resp. w.r.t. $\mathcal{O}_{\bar{1}}^{\mathrm{wi}}$).

**Lemma 5.7** (Correctness of Models) *Let $(\mathcal{C}, \mathcal{O})$ be $(\mathcal{C}_{\bar{1}}^{\mathrm{w}}, \mathcal{O}_{\bar{1}}^{\mathrm{w}})$, $(\mathcal{C}_{\bar{1}}^{\mathrm{w}}, \mathcal{O}_{\bar{1}}^{\mathrm{wm}})$, or $(\mathcal{C}_{\bar{1}}^{\mathrm{wi}}, \mathcal{O}_{\bar{1}}^{\mathrm{wi}})$. Then, for every $s_1, s_2 \in \mathcal{L}_{\bar{1}}[\emptyset]$, the following holds:*

$$\begin{array}{l}\mathcal{C}[\![s_1]\!] = \mathcal{C}[\![s_2]\!] \Rightarrow \\ \forall X \in \mathcal{X}_{\mathcal{P}}, \forall S \in \mathcal{L}_{\bar{1}}[X][\ \mathcal{O}[\![S[s_1/X]]\!] = \mathcal{O}[\![S[s_2/X]]\!]\ ].\ \blacksquare\end{array} \tag{5.22}$$

$$\begin{array}{ccc}
\mathcal{C}_{\bar{1}}^{\mathrm{wi}} & \xrightarrow{\mathcal{A}_{\bar{1}}^{\mathrm{wi}}\ (\mathrm{f.a.})} & \mathcal{O}_{\bar{1}}^{\mathrm{wi}} \\
\uparrow_{\tilde{\mathrm{A}}} & & \uparrow_{\Theta} \\
\mathcal{C}_{\bar{1}}^{\mathrm{w}} & \xrightarrow{\mathcal{A}_{\bar{1}}^{\mathrm{w}}\ (\mathrm{f.a.})} & \mathcal{O}_{\bar{1}}^{\mathrm{w}} \\
\| & & \\
\mathcal{C}_{\bar{1}}^{\mathrm{w}} & \xrightarrow{\mathcal{A}_{\bar{1}}^{\mathrm{wm}}\ (\mathrm{f.a.})} & \mathcal{O}_{\bar{1}}^{\mathrm{wm}}
\end{array}$$

Figure 5.2: Connections between Operational Models for $\mathcal{L}_{\bar{1}}$

**Proof.** Let us prove (5.22) for $(\mathcal{C}, \mathcal{O}) = (\mathcal{C}_{\bar{1}}^{\mathrm{w}}, \mathcal{O}_{\bar{1}}^{\mathrm{w}})$ (for $(\mathcal{C}, \mathcal{O}) = (\mathcal{C}_{\bar{1}}^{\mathrm{wi}}, \mathcal{O}_{\bar{1}}^{\mathrm{wi}}), (\mathcal{C}_{\bar{1}}^{\mathrm{w}}, \mathcal{O}_{\bar{1}}^{\mathrm{wm}})$, (5.22) can be shown in a similar fashion). Let $s_1, s_2 \in \mathcal{L}_{\bar{1}}[\emptyset]$ such that

$$\mathcal{C}_{\bar{1}}^{\mathrm{w}}[\![s_1]\!] = \mathcal{C}_{\bar{1}}^{\mathrm{w}}[\![s_2]\!],$$

and let $X \in \mathcal{X}_{\mathcal{P}}$, $S \in \mathcal{L}_{\bar{1}}[X]$. Then,

$$\begin{array}{ll}
\mathcal{O}_{\bar{1}}^{\mathrm{w}}[\![S[s_1/X]]\!] & \\
= \mathcal{A}_{\bar{1}}^{\mathrm{w}}(\mathcal{C}_{\bar{1}}^{\mathrm{w}}[\![S[s_1/X]]\!]) & \text{(by Lemma 5.6)} \\
= \mathcal{A}_{\bar{1}}^{\mathrm{w}}(\mathcal{C}_{\bar{1}}^{\mathrm{w}}[\![S[s_2/X]]\!]) & \text{(by Lemma 5.5)} \\
= \mathcal{O}_{\bar{1}}^{\mathrm{w}}[\![S[s_2/X]]\!] & \text{(by Lemma 5.6).} \blacksquare
\end{array}$$

## 5.6 Full Abstractness of Models

In this section, we will prove the full abstractness of $\mathcal{C}_{\bar{1}}^{\mathrm{w}}$ (resp. of $\mathcal{C}_{\bar{1}}^{\mathrm{wi}}$) w.r.t. $\mathcal{O}_{\bar{1}}^{\mathrm{w}}$ and $\mathcal{O}_{\bar{1}}^{\mathrm{wm}}$ (resp. w.r.t. $\mathcal{O}_{\bar{1}}^{\mathrm{wi}}$).

The connections, including the full abstractness results stated above, between the three weak linear semantics and the two compositional models are summarized in Figure 5.2. where (unlike usual commutative diagrams)

$$\mathcal{C} \xrightarrow{\mathcal{A}\ (\mathrm{f.a.})} \mathcal{O}$$

denotes that $\mathcal{C}$ is fully abstract w.r.t. $\mathcal{O}$ with $\mathcal{A} \circ \mathcal{C} = \mathcal{O}$.

We need a few preliminaries to the full abstractness proof. First, let us define the *sort* of $s \in \mathcal{L}_{\bar{1}}[\emptyset]$, written $\mathcal{S}(s)$, by:

**Definition 5.15**

$$\mathcal{S}(s) = \{c \in \mathbf{C}_1 : \exists w \in (\mathbf{A}_1)^{<\omega}[\ s \xrightarrow{w}_{1*} \xrightarrow{c}_1\ ]\}. \blacksquare$$

Then, the *sort-finiteness* stated by the following lemma immediately follows from the definition of $\mathcal{L}_{\bar{1}}$.

**Lemma 5.8** $\forall s \in \mathcal{L}_{\bar{1}}[\emptyset][\ \mathcal{S}(s)$ *is finite* $]$. ∎

The following two lemmas, to be used in the full abstractness proof, follow immediately from the definitions of $\mathcal{C}_{\bar{1}}^{\mathrm{w}}$ and $\mathcal{C}_{\bar{1}}^{\mathrm{wi}}$.

**Lemma 5.9** *Let* $s \in \mathcal{L}_{\bar{1}}[\emptyset]$, $u \in (\mathbf{C}_1)^{<\omega}$, $\Gamma \subseteq \mathbf{C}_1$, *and let* $\mathcal{C}$ *be* $\mathcal{C}_{\bar{1}}^{\mathrm{w}}$ *or* $\mathcal{C}_{\bar{1}}^{\mathrm{wi}}$. *Then,*

**(1)** $u \cdot \langle \delta(\Gamma) \rangle \in \mathcal{C}[\![s]\!] \Leftrightarrow u \cdot \langle \delta(\Gamma \cup \{\surd\}) \rangle \in \mathcal{C}[\![s \parallel \mathbf{0}]\!]$,

**(2)** $\langle \Delta(\Gamma) \rangle \in \mathcal{C}[\![s]\!] \Leftrightarrow \langle \Delta(\Gamma \cup \{\surd\}) \rangle \in \mathcal{C}[\![s \parallel \mathbf{0}]\!]$. ∎

**Lemma 5.10** *Let* $s \in \mathcal{L}_{\bar{1}}[\emptyset]$, $\Gamma \subseteq \mathbf{C}_1^{\surd}$, *and* $C \subseteq \mathbf{C}_1$. *Then,*

**(1)** $\langle \delta(\Gamma) \rangle \in \mathcal{C}_{\bar{1}}^{\mathrm{w}}[\![s]\!] \Leftrightarrow \langle \delta(C \cup \Gamma) \rangle \in \mathcal{C}_{\bar{1}}^{\mathrm{w}}[\![\partial_C(s)]\!]$.

**(2)** $\langle \Delta(\Gamma) \rangle \in \mathcal{C}_{\bar{1}}^{\mathrm{w}}[\![s]\!] \Leftrightarrow \langle \Delta(C \cup \Gamma) \rangle \in \mathcal{C}_{\bar{1}}^{\mathrm{w}}[\![\partial_C(s)]\!]$. ∎

**Theorem 5.1** (Full Abstractness of Models) *Let* $(\mathcal{C}, \mathcal{O})$ *be* $(\mathcal{C}_{\bar{1}}^{\mathrm{w}}, \mathcal{O}_{\bar{1}}^{\mathrm{w}})$, $(\mathcal{C}_{\bar{1}}^{\mathrm{w}}, \mathcal{O}_{\bar{1}}^{\mathrm{wm}})$, *or* $(\mathcal{C}_{\bar{1}}^{\mathrm{wi}}, \mathcal{O}_{\bar{1}}^{\mathrm{wi}})$. *Then, for every* $s_1, s_2 \in \mathcal{L}_{\bar{1}}[\emptyset]$, *the following holds:*

$$\begin{array}{l} \mathcal{C}[\![s_1]\!] = \mathcal{C}[\![s_2]\!] \Leftrightarrow \\ \forall X \in \mathcal{X}_{\mathcal{P}}, \forall S \in \mathcal{L}_{\bar{1}}[X][\ \mathcal{O}[\![S[s_1/X]]\!] = \mathcal{O}[\![S[s_2/X]]\!]\ ]. \ ∎ \end{array} \tag{5.23}$$

**Proof. Part 1.** First, let us prove (5.23) for

$$(\mathcal{C}, \mathcal{O}) = (\mathcal{C}_{\bar{1}}^{\mathrm{w}}, \mathcal{O}_{\bar{1}}^{\mathrm{w}}).$$

The $\Rightarrow$-part is the claim of Lemma 5.7. For establishing the $\Leftarrow$-part, it suffices to show that if $\mathcal{C}_{\bar{1}}^{\mathrm{w}}[\![s_1]\!] \neq \mathcal{C}_{\bar{1}}^{\mathrm{w}}[\![s_2]\!]$, then

$$\exists X, \exists S[\ \mathcal{O}_{\bar{1}}^{\mathrm{w}}[\![S[s_1/X]]\!] \neq \mathcal{O}_{\bar{1}}^{\mathrm{w}}[\![S[s_2/X]]\!]\ ]. \tag{5.24}$$

Suppose $\mathcal{C}_{\bar{1}}^{\mathrm{w}}[\![s_1]\!] \neq \mathcal{C}_{\bar{1}}^{\mathrm{w}}[\![s_2]\!]$. Let us show (5.24).

When $\mathcal{T}_{\surd}(s_1) \neq \mathcal{T}_{\surd}(s_2)$, $\mathcal{T}_{\omega}(s_1) \neq \mathcal{T}_{\omega}(s_2)$, or $\mathcal{T}_{\perp}(s_1) \neq \mathcal{T}_{\perp}(s_2)$, it immediately follows that

$$\mathcal{O}_{\bar{1}}^{\mathrm{w}}[\![s_1]\!] \neq \mathcal{O}_{\bar{1}}^{\mathrm{w}}[\![s_2]\!].$$

Otherwise, there are two cases.

**Case 1.** Suppose $\mathcal{F}(s_1) \neq \mathcal{F}(s_2)$. Let us construct an appropriate statement $T$ called a *tester* such that $\mathcal{O}_{\bar{1}}^{\mathrm{w}}[\![s_1 \parallel T]\!] \neq \mathcal{O}_{\bar{1}}^{\mathrm{w}}[\![s_2 \parallel T]\!]$.[10] Since $\mathcal{F}(s_1) \neq \mathcal{F}(s_2)$, one can assume without loss of generality that there exist $u, \Gamma$ such that

$$u \cdot \langle \delta(\Gamma) \rangle \in \mathcal{F}(s_1) \setminus \mathcal{F}(s_2). \tag{5.25}$$

Two subcases are distinguished according to whether $\surd \in \Gamma$ or not.

**Subcase 1.1.** Suppose $\surd \in \Gamma$. One can assume without loss of generality that $\Gamma$ is finite by Lemma 5.8; let $\Gamma = \{\surd, c_1, \cdots, c_n\}$, and $u = \langle c'_1, \cdots, c'_k \rangle$.

Set $\Omega \equiv (\mu X.\ \tau; X)$, and

[10] The variable $T$ is used to denote a statement when it is considered a tester, while the typical variable ranging over the set of statements is $s$.

$$\tilde{T} \equiv \mathbf{e} + (\bar{c}_1;\Omega) + \cdots + (\bar{c}_n;\Omega),$$
$$T \equiv (\overline{c'_1} + \Omega);\cdots;(\overline{c'_k} + \Omega);\tilde{T}.$$

Then, it immediately follows from (5.25) and the definition of $T$ that

$$\langle\,\delta\,\rangle \in \mathcal{O}_1^{\mathrm{w}}[\![s_1 \,\|\, T]\!]. \tag{5.26}$$

Let us show, by contradiction, that

$$\langle\,\delta\,\rangle \notin \mathcal{O}_1^{\mathrm{w}}[\![s_2 \,\|\, T]\!]. \tag{5.27}$$

Assume that this does not hold. Then, by the definition of $\mathcal{O}_1^{\mathrm{w}}$, there are $s' \in \mathcal{L}_{\bar{1}}[\emptyset]$ and $w \in \{\tau\}^{<\omega}$ such that

$$\textbf{(i)}\ \ (s_2 \,\|\, T) \xrightarrow{w}_{1*} s', \quad \textbf{(ii)}\ \ \tau, \surd \notin \mathrm{act}_1(s'). \tag{5.28}$$

Let us show that there is $s'_2$ such that

$$\textbf{(i)}\ \ s_2 \overset{u}{\Longrightarrow}_1 s'_2, \quad \textbf{(ii)}\ \ s' \equiv (s'_2 \,\|\, \tilde{T}). \tag{5.29}$$

By Lemma 5.3 (1), there are $w_1, w_2 \in (\mathbf{A}_1)^{<\omega}$ and $s'_2, T' \in \mathcal{L}_{\bar{1}}[\emptyset]$ such that

$$\begin{array}{ll}\textbf{(i)}\ \ w \in \mathrm{mrg}(w_1, w_2), & \textbf{(ii)}\ \ s' \equiv (s'_2 \,\|\, T'),\\ \textbf{(iii)}\ \ s_2 \xrightarrow{w_1}_{1*} s'_2, & \textbf{(iv)}\ \ T \xrightarrow{w_2}_{1*} T'.\end{array} \tag{5.30}$$

First, let us show, by contradiction, that

$$w_2 = \langle \overline{c'_1}, \cdots, \overline{c'_k} \rangle \ \wedge\ T' \equiv \tilde{T}. \tag{5.31}$$

Assume this does not hold. Then, one has either

$$\exists i \in \bar{k}[\ T' \equiv (\overline{c'_i} + \Omega);\cdots;(\overline{c'_k} + \Omega);\tilde{T}\ ]$$

(when $w_2$ is a proper prefix of $\langle \overline{c'_1}, \cdots, \overline{c'_k} \rangle$), or

$$T' \equiv \Omega$$

(when $\tau$ is contained in $w_2$ or $w_2 = \langle \overline{c'_1}, \cdots, \overline{c'_k}, \bar{c}_j \rangle$ for some $j \in \bar{n}$). In both cases, it follows that

$$\tau \in \mathrm{act}_1(T') \subseteq \mathrm{act}_1(s'),$$

which contradicts (5.28) (ii). Hence one has (5.31). By this and Lemma 5.3 (2), one has

$$\theta(w_1) = \overline{w_2} = \overline{\langle \overline{c'_1}, \cdots, \overline{c'_k} \rangle} = u.$$

Hence, by (5.30) (ii) and (iii), one has (5.29).

By (5.29) (ii) and (5.28) (ii), one has $\tau, \surd \notin \mathrm{act}_1(s'_2)$ and

$$\begin{aligned}\emptyset &= (\mathbf{C}_1 \cap \mathrm{act}_1(s'_2)) \cap \overline{(\mathbf{C}_1 \cap \mathrm{act}_1(\tilde{T}))}\\ &= \mathrm{act}_1(s'_2) \cap \{c_1, \cdots, c_n\}.\end{aligned}$$

Thus,

$$\mathrm{act}_1(s_2') \cap \Gamma = \mathrm{act}_1(s_2') \cap \{\surd, c_1, \cdots, c_n\} = \emptyset.$$

It follows from this and (5.29) (i) that $u \cdot \langle \delta(\Gamma) \rangle \in \mathcal{F}(s_2)$, which contradicts (5.25).

Thus one has (5.27). From (5.26) and (5.27), the desired consequence (5.24) follows.

**Subcase 1.2.** Suppose $\surd \notin \Gamma$. Then, by Lemma 5.9 (1) with $\mathcal{C} = \mathcal{C}_1^{\mathrm{w}}$, one has

$$u \cdot \langle \delta(\Gamma \cup \{\surd\}) \rangle \in \mathcal{C}_1^{\mathrm{w}}[\![s_1 \,\|\, \mathbf{0}]\!] \setminus \mathcal{C}_1^{\mathrm{w}}[\![s_2 \,\|\, \mathbf{0}]\!].$$

Thus, this subcase is reduced to Subcase 1.1.

**Case 2.** Suppose

$$\text{(i) } \mathcal{F}(s_1) = \mathcal{F}(s_2), \quad \text{(ii) } \mathcal{R}(s_1) \neq \mathcal{R}(s_2). \tag{5.32}$$

Then one can assume, without loss of generality, that there exists $\Gamma \subseteq \mathbf{C}_1^{\surd}$ such that

$$\langle \Delta(\Gamma) \rangle \in \mathcal{R}(s_1) \setminus \mathcal{R}(s_2). \tag{5.33}$$

As in Case 1, two subcases are distinguished according to whether $\surd \in \Gamma$ or not.

**Subcase 2.1.** Suppose $\surd \in \Gamma$. Then, since $\langle \Delta(\Gamma) \rangle \in \mathcal{R}(s_1)$, one has

$$s_1 \overset{\tau}{\not\longrightarrow}_1 . \tag{5.34}$$

Moreover, by the fact stated in Remark 5.2 (1), one has $\langle \delta(\Gamma) \rangle \in \mathcal{F}(s_1)$, and hence, $\langle \delta(\Gamma) \rangle \in \mathcal{F}(s_2)$, by (5.32) (i).

Thus there must exist $s_2'$ such that

$$s_2 \overset{\epsilon}{\Longrightarrow}_1 s_2' \overset{\tau}{\not\longrightarrow}_1 \wedge \Gamma \cap \mathrm{act}_1(s_2') = \emptyset.$$

From this and the fact $\surd \in \Gamma$, it follows that

$$\tau, \surd \notin \mathrm{act}_1(s_2'). \tag{5.35}$$

By (5.33), $s_2'$ cannot be $s_2$ itself; thus

$$s_2 \overset{\tau}{\longrightarrow}_1 \overset{\epsilon}{\Longrightarrow}_1 s_2'. \tag{5.36}$$

Let us take the context $S \equiv X + \mathbf{e}$ with $X \in \mathcal{X}_{\mathcal{P}}$ arbitrary.

First, one has

$$\langle\, \delta \,\rangle \notin \mathcal{O}_1^{\mathrm{w}}[\![S[s_1/X]]\!] = \mathcal{O}_1^{\mathrm{w}}[\![s_1 + \mathbf{e}]\!], \tag{5.37}$$

since by (5.34), there is no $s_1'$ such that

$$(s_1 + \mathbf{e}) \overset{\epsilon}{\Longrightarrow}_1 s_1' \wedge \tau, \surd \notin \mathrm{act}_1(s_1').$$

On the other hand, one has

$$\langle\, \delta \,\rangle \in \mathcal{O}_1^{\mathrm{w}}[\![S[s_2/X]]\!] = \mathcal{O}_1^{\mathrm{w}}[\![s_2 + \mathbf{e}]\!], \tag{5.38}$$

since it follows from (5.35) and (5.36) that

$$(s_2 + \mathbf{e}) \overset{\epsilon}{\Longrightarrow}_1 s_2' \ \wedge \ \tau, \surd \notin \mathrm{act}_1(s_2').$$

From (5.37) and (5.38), the desired consequence (5.24) follows.

**Subcase 2.2.** Suppose $\surd \notin \Gamma$. Then, by Lemma 5.9 (2) with $\mathcal{C} = \mathcal{C}_1^{\mathrm{w}}$, one has

$$\langle \Delta(\Gamma \cup \{\surd\}) \rangle \in \mathcal{C}_1^{\mathrm{w}}[\![s_1 \,\|\, \mathbf{0}]\!] \setminus \mathcal{C}_1^{\mathrm{w}}[\![s_2 \,\|\, \mathbf{0}]\!].$$

Thus, this subcase is reduced to Subcase 2.1.

**Part 2.** Next, let us prove (5.23) for

$$(\mathcal{C}, \mathcal{O}) = (\mathcal{C}_1^{\mathrm{w}}, \mathcal{O}_1^{\mathrm{wm}}).$$

As in Part 1, it suffices for this purpose to show that if $\mathcal{C}_1^{\mathrm{w}}[\![s_1]\!] \neq \mathcal{C}_1^{\mathrm{w}}[\![s_2]\!]$, then

$$\exists X, \exists S[\ \mathcal{O}_1^{\mathrm{wm}}[\![S[s_1/X]]\!] \neq \mathcal{O}_1^{\mathrm{wm}}[\![S[s_2/X]]\!]\ ]. \tag{5.39}$$

Suppose $\mathcal{C}_1^{\mathrm{w}}[\![s_1]\!] \neq \mathcal{C}_1^{\mathrm{w}}[\![s_2]\!]$. Let us show (5.39).

As can be seen from Part 1, one can assume, without loss of generality, that $\mathcal{F}(s_1) \neq \mathcal{F}(s_2)$ or $\mathcal{R}(s_1) \neq \mathcal{R}(s_2)$.

**Case 1.** Suppose $\mathcal{F}(s_1) \neq \mathcal{F}(s_2)$. Then, one can assume, without loss of generality, that there exist $u$, $\Gamma$ such that

$$u \cdot \langle \delta(\Gamma) \rangle \in \mathcal{F}(s_1) \setminus \mathcal{F}(s_2).$$

Two subcases are distinguished as in Part 1.

**Subcase 1.1.** Suppose $\surd \in \Gamma$. Then, setting $T$ as in Subcase 1.1 in Part 1, one can show that the following holds, in a similar fashion to Subcase 1.1 of Part 1 using Lemma 5.10 (1):

$$\langle\, \delta \,\rangle \in \mathcal{O}_1^{\mathrm{wm}}[\![\partial_{\mathbf{C}_1}(s_1 \,\|\, T)]\!] \setminus \mathcal{O}_1^{\mathrm{wm}}[\![\partial_{\mathbf{C}_1}(s_2 \,\|\, T)]\!].$$

**Subcase 1.2.** Suppose $\surd \notin \Gamma$. This subcase is reduced to Subcase 1.1, as in Part 1.

**Case 2.** Suppose $\mathcal{F}(s_1) = \mathcal{F}(s_2)$ and $\mathcal{R}(s_1) \neq \mathcal{R}(s_2)$. Then one can assume, without loss of generality, that there exists $\Gamma \subseteq \mathbf{C}_1^{\surd}$ such that

$$\langle \Delta(\Gamma) \rangle \in \mathcal{R}(s_1) \setminus \mathcal{R}(s_2).$$

Two subcases are distinguished as in Part 1.

**Subcase 2.1.** Suppose $\surd \in \Gamma$. Then, it can be shown in a similar fashion to Subcase 2.1 of Part 1 using Lemma 5.10, that

$$\langle\, \delta \,\rangle \in \mathcal{O}_1^{\mathrm{wm}}[\![\partial_{\mathbf{C}_1}(s_2 + \mathbf{e})]\!] \setminus \mathcal{O}_1^{\mathrm{wm}}[\![\partial_{\mathbf{C}_1}(s_1 + \mathbf{e})]\!].$$

**Subcase 2.2.** Suppose $\surd \notin \Gamma$. This subcase is reduced to Subcase 2.1, as in Part 1.

**Part 3.** Finally let us prove (5.23) for

$$(\mathcal{C}, \mathcal{O}) = (\mathcal{C}_1^{\mathrm{wi}}, \mathcal{O}_1^{\mathrm{wi}}).$$

As in Part 1, it suffices for this purpose to show that if $\mathcal{C}_1^{\mathrm{wi}}[\![s_1]\!] \neq \mathcal{C}_1^{\mathrm{wi}}[\![s_2]\!]$, then

$$\exists X, \exists S[\ \mathcal{O}_1^{\mathrm{wi}}[\![S[s_1/X]]\!] \neq \mathcal{O}_1^{\mathrm{wi}}[\![S[s_2/X]]\!]\ ]. \tag{5.40}$$

Suppose $\mathcal{C}_1^{\mathrm{wi}}[\![s_1]\!] \neq \mathcal{C}_1^{\mathrm{wi}}[\![s_2]\!]$. Let us show (5.40).

By Lemma 5.2 (2), one has

$$(\mathcal{C}_1^{\mathrm{wi}}[\![s_1]\!] \setminus (\mathbf{C}_1)^{\omega}) \neq (\mathcal{C}_1^{\mathrm{wi}}[\![s_2]\!] \setminus (\mathbf{C}_1)^{\omega}).$$

Thus, there are four possible cases, i.e., one of the following four propositions holds:

$$\begin{array}{ll}\textbf{(i)}\ \mathcal{F}(s_1) \neq \mathcal{F}(s_2), & \textbf{(ii)}\ \mathcal{R}(s_1) \neq \mathcal{R}(s_2), \\ \textbf{(iii)}\ \mathcal{T}_{\surd}(s_1) \neq \mathcal{T}_{\surd}(s_2), & \textbf{(iv)}\ \tilde{\mathcal{T}}_{\perp}(\mathcal{C}_1^{\mathrm{wi}}[\![s_1]\!]) \neq \tilde{\mathcal{T}}_{\perp}(\mathcal{C}_1^{\mathrm{wi}}[\![s_2]\!]).\end{array} \tag{5.41}$$

In the cases (i), (ii), one can show (5.40) in the same way as in Part 1. Let us show (5.40) in the cases (iii), (iv).

**Case 1.** Suppose (5.41) (iii) holds. Then one can assume, without loss of generality, that there exists $\langle c_1, \cdots, c_k\rangle \in (\mathbf{C}_1)^{<\omega}$ such that

$$\langle c_1, \cdots, c_k, \surd\rangle \in \mathcal{T}_{\surd}(s_1) \setminus \mathcal{T}_{\surd}(s_2).$$

Then, setting $T \equiv (\bar{c}_1; \cdots; \bar{c}_k)$, it can be shown, in a similar fashion to Subcase 1.1 of Part 1, that

$$\langle\surd\rangle \in (\mathcal{O}_1^{\mathrm{wi}}[\![s_1 \parallel T]\!] \setminus \mathcal{O}_1^{\mathrm{wi}}[\![s_2 \parallel T]\!]).$$

**Case 2.** Suppose (5.41) (iv) holds. Then one can assume, without loss of generality, that there exists

$$u = \langle c_1, \cdots, c_k\rangle \in (\mathbf{C}_1)^{<\omega}$$

such that

$$\textbf{(i)}\ u \cdot \langle\perp\rangle \in \tilde{\mathcal{T}}_{\perp}(\mathcal{C}_1^{\mathrm{wi}}[\![s_1]\!]), \quad \textbf{(ii)}\ u \cdot \langle\perp\rangle \notin \tilde{\mathcal{T}}_{\perp}(\mathcal{C}_1^{\mathrm{wi}}[\![s_2]\!]). \tag{5.42}$$

Then, by the definition of $\mathcal{C}_1^{\mathrm{wi}}$ and (5.42) (i), one has

$$\forall u' <_{\mathrm{p}} u[\ u' \cdot \langle\perp\rangle \notin \mathcal{T}_{\perp}(s_1)\ ]. \tag{5.43}$$

Two subcases are distinguished according to whether the following holds or not:

$$\exists u' <_{\mathrm{p}} u[\ u' \cdot \langle\perp\rangle \in \mathcal{T}_{\perp}(s_2)\ ]. \tag{5.44}$$

**Subcase 2.1.** Suppose (5.44) holds with $\mathrm{lgt}(u') = i$. Then, $i < k$. We set

$$T \equiv (\overline{c_1}; \cdots; \overline{c_i}).$$

Then it follows from (5.43) and (5.44), in a similar fashion to Subcase 1.1 of Part 1, that

$$\langle\perp\rangle \in \mathcal{O}_1^{\mathrm{wi}}[\![s_2 \parallel T]\!] \setminus \mathcal{O}_1^{\mathrm{wi}}[\![s_1 \parallel T]\!].$$

**Subcase 2.2.** Suppose (5.44) does not hold. Then

$$u \cdot \langle \bot \rangle \notin \mathcal{T}_\bot(s_2),$$

since otherwise it follows that

$$u \cdot \langle \bot \rangle \in \tilde{\mathcal{T}}_\bot(\mathcal{C}_{\bar{1}}^{\mathrm{wi}}[\![s_2]\!]),$$

which contradicts (5.42) (ii).

We set

$$T \equiv (\overline{c_1}; \cdots; \overline{c_k}).$$

Then it can be shown, in a similar fashion to Subcase 2.1, that

$$\langle \bot \rangle \in \mathcal{O}_{\bar{1}}^{\mathrm{wi}}[\![s_1 \,\|\, T]\!] \setminus \mathcal{O}_{\bar{1}}^{\mathrm{wi}}[\![s_2 \,\|\, T]\!].$$

Summarizing, in both subcases, one has (5.40). ■

## 5.7 Concluding Remarks

We conclude this chapter with remarks about future work.

There are two directions for possible extensions of the reported full abstractness results.

One is to investigate the same full abstractness problem for other languages that are extensions of $\mathcal{L}_{\bar{1}}$. The set of combinators of $\mathcal{L}_{\bar{1}}$ is rather restricted; the *abstraction* combinator of $\mathrm{ACP}_\tau$ (cf. [BK 85]), as well as the *relabeling* combinator of CCS (cf. [Mil 80]) are good candidates to add to $\mathcal{L}_{\bar{1}}$. (However, it should be noted that the failures model and its variants are not compositional w.r.t. some combinators defined in the style of Plotkin (cf. [BIM 88]).) A similar full abstractness problem for *nonuniform languages* such as the ones treated in [HBR 90], also remains for future study.

The other direction is to investigate denotational constructions of $\mathcal{C}_{\bar{1}}^{\mathrm{w}}$ or $\mathcal{C}_{\bar{1}}^{\mathrm{wi}}$ in the order-theoretic or metric topological framework; an extension in this direction is provided in Chapter 6, as described in § 5.1.

*Equational theories* where "$s_1 = s_2$" is interpreted as $\mathcal{C}_{\bar{1}}^{\mathrm{w}}[\![s_1]\!] = \mathcal{C}_{\bar{1}}^{\mathrm{w}}[\![s_2]\!]$ and *algorithms* for deciding whether $\mathcal{C}_{\bar{1}}^{\mathrm{w}}[\![s_1]\!] = \mathcal{C}_{\bar{1}}^{\mathrm{w}}[\![s_2]\!]$ for arbitrarily given two elements $s_1, s_2$ of some subclass of $\mathcal{L}_{\bar{1}}[\emptyset]$ (and the same issues for $\mathcal{C}_{\bar{1}}^{\mathrm{wi}}$) are also to be investigated; the first candidate class for which this issue is to be investigated is the class of *serial statements* (cf. [Mil 89] § 7.4), where seriality is a syntactical restriction yielding the set of *finite state processes.*

**Remark 5.4** It is known that on a class of simple expressions (at least $\tau$-free finite processes), the second testing equivalence $\eqsim_2$ defined in [DH 84], coincides with the equivalence $\equiv_{\mathcal{F}}$ induced by the failures model (cf. Remark 7.3.3 in [BKO 88]). Although the general framework of testing is very appealing, there seems to be a fairly large arbitrariness in the definition of *tests* (cf. [Abr 87]), and we expect that, on the intersection of CCS and our language $\mathcal{L}_{\bar{1}}$, all the equivalences $\eqsim_i$

($i = 1, 2, 3$) are strictly coarser than the equivalence $\equiv_{\mathcal{C}_1^w}$ induced by $\mathcal{C}_1^w$, mainly because the tests used in [DH 84] do not pay, in nature, so much attention to the distinction between divergence and deadlock. Actually, it is shown that both $\equiv_{\mathcal{C}_1^w}$ and $\equiv_{\mathcal{C}_1^{wi}}$ is strictly finer than $\eqsim_i$ ($i = 1, 2, 3$) for a sublanguage $\mathcal{L}_{\hat{1}}$ (see Figure 6.2 in § 6.6.4). We expect there will be no difficulty in extending this result for $\mathcal{L}_{\hat{1}}$ to $\mathcal{L}_{\tilde{1}}$. ∎

As mentioned in § 5.1, $\mathcal{C}_1^w$ (or $\mathcal{C}_1^{wi}$) cannot be constructed in the standard cpo framework with the *reverse inclusion* $\supseteq$ being the order of the semantic domain,$\mathcal{C}_1^w$ (or $\mathcal{C}_1^{wi}$) cannot be constructed in the standard cpo framework with the *reverse inclusion* $\supseteq$ being the order of the semantic domain, because the semantic operation $\tilde{\|}_1$ corresponding to the parallel composition combinator of $\mathcal{L}_{\tilde{1}}$ is *not continuous* w.r.t. the order $\supseteq$:

**Example 5.3** (Discontinuity of $\tilde{\|}_1$) Let $c \in \mathbf{C}_1$. We define a sequence $\langle \tilde{s}_0 \rangle_{n \in \omega}$ of statements by induction as follows: First, we define $\tilde{s}_0$ as in Example 5.2 by:

$$\tilde{s}_0 \equiv (\mu X.\ c + (\tau; X; c));$$

then for every $n \in \omega$, we define $\tilde{s}_{n+1}$ by:

$$\tilde{s}_{n+1} \equiv \tilde{s}_n; c.$$

Note that Figure 5.1 depicts $\tilde{s}_0$, and $\tilde{s}_n$ in general is depicted by Figure 5.3 For each $n \in \omega$, let $p_n = \mathcal{C}_1^w[\![\tilde{s}_n]\!]$. Then, clearly the sequence $\langle p_n \rangle_{n \in \omega}$ is a $\supseteq$-chain, i.e.,

$$p_0 \supseteq p_1 \supseteq p_2 \cdots.$$

Next, we define another statement $\bar{s}$ by:

$$\bar{s}_0 \equiv (\mu X.\ \bar{c} + (\tau; X; c));$$

we put $\bar{p} = \mathcal{C}_1^w[\![\bar{s}]\!]$. Then, one has

$$\forall n \in \omega[\ \langle \surd \rangle \in p_n \tilde{\|}_1 \bar{p}\ ],$$

and so,

$$\langle \surd \rangle \in \bigcap_{n \in \omega} [p_n \tilde{\|}_1 \bar{p}].$$

However, for every $n \in \omega$,

$$c^{n+1} \cdot \langle \surd \rangle \notin \bigcap_{n \in \omega} [p_n],$$

and so,

$$\langle \surd \rangle \notin (\bar{p} \tilde{\|}_1 \bigcap_{n \in \omega} [p_n]).$$

Thus,

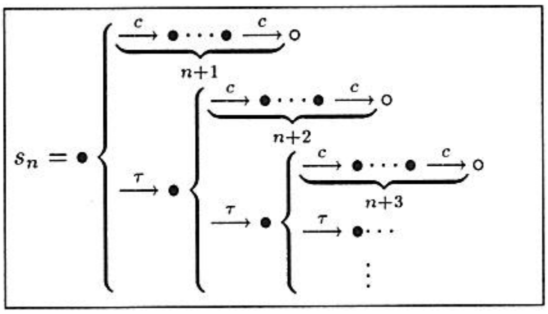


Figure 5.3: Synchronization Trees of Statements $\tilde{s}_n$ in $\mathcal{L}_{\tilde{1}}$

$$\bigcap_{n\in\omega}[p_n \,\tilde{\|}_1\, \bar{p}] \neq (\bar{p} \,\tilde{\|}_1 \bigcap_{n\in\omega}[p_n]),$$

and so, the semantic operation $\tilde{\|}_1$ is not continuous w.r.t. $\supseteq$. ∎

Also, the next example shows that $\tilde{\|}_1$ is *not nonexpansive* w.r.t. the pseudo-metric defined on $\mathbf{P}_1^{\mathrm{wf}}$ in terms truncation as in Chapter 4; thus $\mathcal{C}_1^{\mathrm{w}}$ (or $\mathcal{C}_1^{\mathrm{wi}}$) cannot be constructed in the cms framework based on this metric.

**Example 5.4** (Lack of Nonexpansiveness of $\tilde{\|}_1$) Let $d_P$ be the pseudo-metric between elements of $\mathbf{P}_1^{\mathrm{wf}}$ defined in terms of truncation as in in Chapter 4. (Note that $d_P$ is not necessarily a metric on $\mathbf{P}_1^{\mathrm{wf}}$, because not all elements of $\mathbf{P}_1^{\mathrm{wf}}$ are closed.) Let $c \in \mathbf{C}_1$. Putting $\Omega \equiv (\mu X.\ \tau; X)$, we let

$$s_0 \equiv c, \quad s_0' \equiv c;\Omega, \quad os_1 \equiv \bar{c}, \quad s_1' \equiv \bar{c};\Omega.$$

Then, obviously,

$$d_P(\mathcal{C}_1^{\mathrm{w}}[\![s_0]\!], \mathcal{C}_1^{\mathrm{w}}[\![s_0']\!]) = d_P(\mathcal{C}_1^{\mathrm{w}}[\![s_1]\!], \mathcal{C}_1^{\mathrm{w}}[\![s_1']\!]) = \kappa,$$

but

$$d_P((\mathcal{C}_1^{\mathrm{w}}[\![s_0]\!] \,\tilde{\|}_1\, \mathcal{C}_1^{\mathrm{w}}[\![s_0']\!]), (\mathcal{C}_1^{\mathrm{w}}[\![s_1]\!] \,\tilde{\|}_1\, \mathcal{C}_1^{\mathrm{w}}[\![s_1']\!])) = 1.$$

Thus, $\tilde{\|}_1$ is not nonexpansive w.r.t. $d_P$. ∎

## 5.A Proof of Lemma 5.5

We will prove (5.21) for $\mathcal{C} = \mathcal{C}_1^{\mathrm{w}}$ by induction on the *depth* of *inference trees* through which transitions $s \xrightarrow{a}_1 s'$ are proved by the transition rules (For $\mathcal{C} = \mathcal{C}_1^{\mathrm{wi}}$, (5.21) can be shown in a similar fashion). This proof is analogous to Milner's proof of the fact that *the largest equivalence which respects weak bisimulation and is preserved by '+' is indeed a congruence* (cf. [Mil 85] §2.2).

We need a few notational preliminaries to the proof.

**Notation 5.3** For $s, s' \in \mathcal{L}_\emptyset$, $\alpha \in \mathbf{A}_1^{\checkmark}$, $n \in \omega$, let us write $\vdash_{(n)} s \xrightarrow{\alpha}_1 \left(s'\right)$, to denote that there is an inference tree, with depth $n$, through which $s \xrightarrow{\alpha}_1 \left(s'\right)$ is proved by the transition rules. We write $\vdash_{(<n)} s \xrightarrow{\alpha}_1 \left(s'\right)$, to denote that

$$\exists k < n[\vdash_{(k)} s \xrightarrow{\alpha}_1 \left(s'\right)].$$

Further, we use "$\vdash_{(n)} s \xrightarrow{\alpha}_1$" and "$\vdash_{(<n)} s \xrightarrow{\alpha}_1$" for shorthands for

$$\exists s'[\vdash_{(n)} s \xrightarrow{\alpha}_1 \left(s'\right)]$$

and

$$\exists s'[\vdash_{(<n)} s \xrightarrow{\alpha}_1 \left(s'\right)],$$

respectively. ∎

One obtains immediately, by the definition of transition relations $\xrightarrow{\alpha}_1$ ($\alpha \in \mathbf{A}_1^{\checkmark}$), that

$$s \xrightarrow{a}_1 s' \Leftrightarrow \exists n \in \omega[\vdash_{(n)} s \xrightarrow{a}_1 s'].$$

Let $s_1, s_2 \in \mathcal{L}_\emptyset$ with $\mathcal{C}_1^{\mathrm{w}}[\![s_1]\!] = \mathcal{C}_1^{\mathrm{w}}[\![s_2]\!]$. In order to prove (5.21) for $\mathcal{C} = \mathcal{C}_1^{\mathrm{w}}$, it suffices to prove (5.45) below for $\mathcal{P} = \mathcal{T}_{\surd}, \mathcal{R}, \mathcal{F}, \mathcal{T}_\omega, \mathcal{T}_\perp$:

$$\forall S \in \mathcal{L}_X[\mathcal{P}(S[s_1/X]) = \mathcal{P}(S[s_2/X])]. \tag{5.45}$$

The proposition (5.45) for $\mathcal{P} = \mathcal{T}_{\surd}, \mathcal{R}, \mathcal{F}, \mathcal{T}_\omega, \mathcal{T}_\perp$ will be proved in this order.

**Proof of (5.45) for $\mathcal{P} = \mathcal{T}_{\surd}$.** In order to prove (5.45) for $\mathcal{P} = \mathcal{T}_{\surd}$, it suffices to prove the following for every $S \in \mathcal{L}_X$:

$$\begin{array}{ll} \textbf{(i)} & S[s_1/X] \xrightarrow{\surd}_1 \Rightarrow S[s_2/X] \overset{\epsilon}{\Longrightarrow}_1 \xrightarrow{\surd}_1, \\ \textbf{(ii)} & \forall a \in \mathbf{A}_1, \forall w \in (\mathbf{A}_1)^{<\omega}, \forall S \in \mathcal{L}_X \\ & [\ S[s_1/X] \xrightarrow{a}_1 \xrightarrow{w}_{1*} \xrightarrow{\surd}_1 \\ & \quad \Rightarrow S[s_2/X] \xRightarrow{\theta(\langle a\rangle \cdot w)}_1 \xrightarrow{\surd}_1\ ]. \end{array} \tag{5.46}$$

We will prove part **(ii)** (part **(i)** can be proved more easily in a similar fashion).

For $\langle n, k\rangle \in \omega \times \omega$, let us write $\Phi_{\mathcal{T}}^{\surd}(n, k)$ to denote that

$$\begin{array}{l} \forall a \in \mathbf{A}_1, \forall w \in (\mathbf{A}_1)^n, \forall S \in \mathcal{L}_X \\ [\ \exists s_1'[\vdash_{(k)} S[s_1/X] \xrightarrow{a}_1 s_1' \xrightarrow{w}_{1*} \xrightarrow{\surd}_1\ ] \\ \quad \Rightarrow S[s_2/X] \xRightarrow{\theta(\langle a\rangle \cdot w)}_1 \xrightarrow{\surd}_1\ ]. \end{array}$$

The proof is achieved by transfinite induction on the well-ordered set $(\omega \times \omega, \prec)$, where $\prec$ is the lexicographical ordering defined as follows: For $(n, k), (n', k') \in \omega \times \omega$,

$$(n,k) \prec (n',k') \Leftrightarrow$$
$$n < n' \vee (n = n' \wedge k < k').$$

**Induction Base:** The propositions $\Phi^{\surd}_{\mathcal{T}}(n,0)$ $(n \in \omega)$, especially the base case: $\Phi^{\surd}_{\mathcal{T}}(0,0)$ can be shown straightforwardly.
**Induction Step:** Let $(n,k) \succ (0,0)$ with $k > 0$, and assume

$$\forall (n',k') \prec (n,k)[\ \Phi^{\surd}_{\mathcal{T}}(n',k')\ ].$$

Let us prove $\Phi^{\surd}_{\mathcal{T}}(n,k)$.

Let $a \in \mathbf{A}_1$, $w \in (\mathbf{A}_1)^n$, $S \in \mathcal{L}_X$, and suppose

$$\exists s_1'[\ \vdash_{(k)} S[s_1/X] \xrightarrow{a}_1 s_1' \xrightarrow{w}_{1*} \xrightarrow{\surd}_1\ ]. \tag{5.47}$$

It suffices to show

$$S[s_2/X] \xRightarrow{\theta(\langle a\rangle \cdot w)}_1 \xrightarrow{\surd}_1 . \tag{5.48}$$

If $S \equiv X$, then (5.48) immediately follows. Otherwise, we distinguish five cases according to the form of $S$, i.e., one of the following five propositions holds: $S \equiv (S';S'')$, $S \equiv (S' + S'')$, $S \equiv (S' \parallel S'')$, $S \equiv \partial_C(S')$, or $S \equiv (\mu Y.\ S')$. Out of the five cases, we consider the the case $S \equiv (S';S'')$ (in the other three cases, the same result is obtained more easily in a similar fashion; in particular in the case $S \equiv (\mu Y.\ S')$, the result is obtained in a similar fashion to the induction step in the proof of (5.45) for $\mathcal{P} = \mathcal{R}$ below).

By the definition of $\xrightarrow{\alpha}_1$ $(\alpha \in \mathbf{A}_1^{\surd})$, there are two possible cases, i.e., either of the following holds:

$$\begin{array}{ll}
\text{(i)} & \exists \tilde{s}', \exists w' \in (\mathbf{A}_1)^+, \exists w'' \in (\mathbf{A}_1)^{<\omega} \\
& [\ w = w' \cdot w'' \\
& \quad \wedge \vdash_{(<k)} S'[s_1/X] \xrightarrow{a}_1 \tilde{s}' \xrightarrow{w'}_{1*} \xrightarrow{\surd}_1 \\
& \quad \wedge S''[s_1/X] \xrightarrow{w''}_{*} \xrightarrow{\surd}_1 \\
& \quad \wedge s_1' \equiv (\tilde{s}'; S''[s_1/X])\ ] \\
\text{(ii)} & S'[s_1/X] \xrightarrow{\surd}_1 \wedge \vdash_{(<k)} S''[s_1/X] \xrightarrow{a}_1 s_1'.
\end{array}$$

We consider the case (i) (in the case (ii), the same result is obtained more easily in a similar fashion). By the induction hypothesis and (5.46) (i), one has

$$S'[s_2/X] \xRightarrow{\theta(\langle a\rangle \cdot w')}_1 \xrightarrow{\surd}_1 \wedge S''[s_2/X] \xRightarrow{\theta(w'')}_1 \xrightarrow{\surd}_1 .$$

Thus, one has

$$\begin{array}{l}
S[s_2/X] \equiv (S'[s_2/X]; S''[s_2/X]) \\
\xRightarrow{\theta(\langle a\rangle \cdot w')}_1 \xRightarrow{\theta(w'')}_1 \xrightarrow{\surd}_1,
\end{array}$$

and hence (5.48), since

$$\theta(\langle a\rangle \cdot w') \cdot \theta(w'') = \theta(\langle a\rangle \cdot w). \blacksquare$$

**Proof of (5.45) for $\mathcal{P} = \mathcal{R}$.** In order to prove (5.45) for $\mathcal{P} = \mathcal{R}$, it suffices, by the definition of $\mathcal{R}$ (cf. Definition 5.8), to prove the following for every $S \in \mathcal{L}_X$:

$$\begin{array}{ll} \textbf{(i)} & S[s_2/X] \not\xrightarrow{\tau}_1 \Rightarrow \\ & \forall\gamma \in \mathbf{C}_1^{\checkmark}[\ S[s_1/X] \xrightarrow{\gamma}_1 \Rightarrow S[s_2/X] \xrightarrow{\gamma}_1\ ], \\ \textbf{(ii)} & S[s_1/X] \xrightarrow{\tau}_1 \Rightarrow S[s_2/X] \xrightarrow{\tau}_1 . \end{array}$$

We will prove part **(i)** (part **(ii)** can be proved in a similar fashion using **(i)**).

For $n \in \omega$, let us write $\Phi_{\mathcal{R}}(n)$ to denote that

$$\begin{array}{l} \forall S \in \mathcal{L}_X[\ S[s_2/X] \not\xrightarrow{\tau}_1 \Rightarrow \\ \qquad \forall\gamma \in \mathbf{C}_1^{\checkmark}[\ \vdash_{(n)} S[s_1/X] \xrightarrow{\gamma}_1 \\ \qquad\qquad \Rightarrow S[s_2/X] \xrightarrow{\gamma}_1\ ]]. \end{array}$$

Let us prove, by induction, that $\forall n[\ \Phi_{\mathcal{R}}(n)\ ]$.

**Induction Base:** First, we will prove $\Phi_{\mathcal{R}}(0)$. Let $S \in \mathcal{L}_X$, and suppose $S[s_2/X] \not\xrightarrow{\tau}_1$. Further, let $\gamma \in \mathbf{C}_1^{\checkmark}$, and suppose $\vdash_{(0)} S[s_1/X] \xrightarrow{\gamma}_1$. If $S \equiv X$, then one has immediately that $S[s_1/X] \equiv s_1 \xrightarrow{\gamma}_1$, and hence, $S[s_2/X] \equiv s_2 \xrightarrow{\gamma}_1$.

Otherwise, $S$ is either $\gamma$ (if $\gamma \in \mathbf{C}_1$) or $\mathbf{e}$ (if $\gamma = \epsilon$). Thus, $S[s_2/X] \equiv S[s_1/X]$, and hence, $S[s_2/X] \xrightarrow{\gamma}_1$.

**Induction Step:** Let $k \in \omega$, and assume that $\Phi_{\mathcal{R}}(i)$ holds for every $i \leq k$. We will prove $\Phi_{\mathcal{R}}(k+1)$.

Let $S \in \mathcal{L}_X$, $\gamma \in \mathbf{C}_\epsilon$, and suppose

$$\textbf{(i)}\ S[s_2/X] \not\xrightarrow{\tau}_1, \quad \textbf{(ii)}\ \vdash_{(k+1)} S[s_1/X] \xrightarrow{\gamma}_1 . \tag{5.49}$$

Let us show

$$S[s_2/X] \xrightarrow{\gamma}_1 . \tag{5.50}$$

If $S \equiv X$, then (5.50) is obtained immediately. Otherwise, we distinguish five cases according to the form of $S$, i.e., one of the following five propositions holds: $S \equiv (S'; S'')$, $S \equiv (S' + S'')$, $S \equiv (S' \parallel S'')$, $S \equiv \partial_C(S')$, or $S \equiv (\mu Y.\ S')$.

Out of the five cases, we consider the case $S \equiv (\mu Y.\ S')$ (in the other three cases, the same result is obtained similarly). If $Y \equiv X$, then (5.50) is obtained immediately. Otherwise, by (5.49) (ii), one has

$$\vdash_{(k+1)} (\mu Y.\ S')[s_1/X] \equiv (\mu Y.\ S'[s_1/X]) \xrightarrow{\gamma}_1 .$$

By this and the definition of $\xrightarrow{\gamma}_1$ (cf. the *recursion rule* in Definition 5.4 (3)), one has

$$\begin{array}{l} \vdash_{(k)} S'[s_1/X][(\mu Y. S'[s_1/X])/Y] \\ \qquad \equiv S'[(\mu Y.\ S')/Y][s_1/X] \xrightarrow{\gamma}_1 . \end{array}$$

By this, (5.49) (i), and the induction hypothesis, one has

$$S'[(\mu Y.\ S')/Y][s_2/X] \equiv$$
$$S'[s_2/X][(\mu Y.\ S'[s_2/X])/Y] \xrightarrow{\gamma}_1 .$$

Thus, by the recursion rule, one has

$$(\mu Y.\ S'[s_2/X]) \xrightarrow{\gamma}_1,$$

and hence, (5.50). ■

**Proof of (5.45) for $\mathcal{P} = \mathcal{F}$.** In order to prove (5.45) for $\mathcal{P} = \mathcal{F}$, it suffices to prove that the proposition (5.51) defined below holds for every $(n, k) \in \omega \times \omega$, since (5.45) for $\mathcal{P} = \mathcal{R}$ has already been shown.

$$\begin{array}{l} \forall a \in \mathbf{A}_1, \forall \tilde{w}_1 \in (\mathbf{A}_1)^n, \forall \Gamma \subseteq \mathbf{C}_1^{\checkmark}, \forall S \in \mathcal{L}_X \\ [\ \exists s_2'', s_1' [\ \vdash_{(k)} S[s_1/X] \xrightarrow{a}_1 s_2'' \xrightarrow{\tilde{w}_1}_{1*} s_1' \xrightarrow{\tau}\!\!\!\!\!/_1 \\ \qquad \wedge \operatorname{act}(s_1') \cap \Gamma = \emptyset\ ] \Rightarrow \\ \quad \exists w_2 \in (\mathbf{A}_1)^+, \exists s_2' \\ \quad [\ \theta(w_2) = \theta(\langle a \rangle \cdot \tilde{w}_1) \\ \quad\ \wedge S[s_2/X] \xrightarrow{w_2}_{1*} s_2' \xrightarrow{\tau}\!\!\!\!\!/_1 \\ \quad\ \wedge \operatorname{act}(s_2') \cap \Gamma = \emptyset\ ]]. \end{array} \tag{5.51}$$

This is achieved by transfinite induction on the well-ordered set $(\omega \times \omega, \prec)$, in a similar fashion to the proof of (5.45) for $\mathcal{P} = \mathcal{T}_{\surd}$. ■

Let us prove (5.45) for $\mathcal{P} = \mathcal{T}_\omega, \mathcal{T}_\perp$. We need a few preliminaries to the proof.

**Definition 5.16 (1)** Let $\sqsubseteq\ \subseteq \mathcal{L}_\emptyset \times \mathcal{L}_\emptyset$ be the largest relation satisfying the following condition:

$$\begin{array}{l} \forall s_1, s_2 \in \mathcal{L}_\emptyset \\ [\ s_1 \sqsubseteq s_2 \Rightarrow \\ \quad \forall a \in \mathbf{A}_1, \forall s_1' \\ \quad [s_1 \xrightarrow{a}_1 s_1' \Rightarrow \exists s_2' [\ s_2 \xrightarrow{a}_1 s_2' \wedge s_1' \sqsubseteq s_2'\ ]]]. \end{array} \tag{5.52}$$

**(2)** For $w \in \mathbf{A}_1$, and $s_1, s_2 \in \mathcal{L}_{\bar{1}}[\emptyset]$, we write

$$s_1 \xrightarrow{w}_{1*} \sqsubseteq s_2$$

to denote that

$$\exists s' \in \mathcal{L}_{\bar{1}}[\emptyset][\ s_1 \xrightarrow{w}_{1*} s' \sqsubseteq s_2\ ]. \blacksquare$$

The existence of such a relation $\sqsubseteq$ can be shown in a similar fashion to the way that so-called *strong bisimulation* is defined (cf., e.g., [Mil 89] §4).

**Definition 5.17** Let $\langle w_n \rangle_{n \in \omega} \in ((\mathbf{A}_1)^+)^\omega$. The *concatenation* of $\langle w_n \rangle_{n \in \omega}$, written $\operatorname{cat}(\langle w_n \rangle_{n \in \omega})$, is defined to be

$$w_0 \cdot w_1 \cdot w_2 \cdot \cdots.$$

Formally, this is defined to be

$$(\lambda i \in \omega.\ w_{m(i)}(i - \Sigma_{j \in m(i)} \mathrm{lgt}(w_j))),$$

where $m(i) = \min\{k : i < \Sigma_{j \in (k+1)} \mathrm{lgt}(w_j)\}$. ■

The following proposition, to be used in the proof of (5.45) for $\mathcal{P} = \mathcal{T}_\omega, \mathcal{T}_\perp$ immediately follows from property (5.52):

**Lemma 5.11** *Let $\langle w_n \rangle_{n\in\omega} \in ((\mathbf{A}_1)^+)^\omega$, $s \in \mathcal{L}_\emptyset$, and $\langle s_n \rangle_{n\in\omega} \in (\mathcal{L}_\emptyset)^\omega$. Then*

$$s \sqsupseteq s_0 \wedge \forall n \in \omega[\ s_n \xrightarrow{w_n}_{1*} \sqsupseteq s_{n+1}\ ] \Rightarrow s \xrightarrow{\mathrm{cat}(\langle w_n \rangle_{n\in\omega})}_{*}. ■$$

The following lemma plays a key role in the proof of (5.45) for $\mathcal{P} = \mathcal{T}_\omega, \mathcal{T}_\perp$.

**Lemma 5.12** *Let $s_1, s_2 \in \mathcal{L}_\emptyset$ with $\mathcal{C}_1^{\mathrm{w}}[\![s_1]\!] = \mathcal{C}_1^{\mathrm{w}}[\![s_2]\!]$. Then for $w \in (\mathbf{A}_1)^+$, $\rho \in (\mathbf{C}_1)_{\mathrm{I}}^{\leq\omega}$, and $S \in \mathcal{L}_X$, the following holds:*

$$\begin{aligned}
&\theta(w) \neq \epsilon \wedge S[s_1/X] \xrightarrow{\theta(w)\cdot\rho}_{1*} \Rightarrow \\
&\exists \tilde{s} \in \mathcal{L}_\emptyset, \exists \tilde{S} \in \mathcal{L}_X \\
&[\ \theta(\tilde{w}) = \theta(\tilde{w}') \wedge (\tilde{s} \,\|\, \tilde{S}[s_1/X]) \stackrel{\rho}{\Longrightarrow}_1 \\
&\quad \wedge S[s_2/X] \stackrel{\theta(w)}{\Longrightarrow} \sqsupseteq (\tilde{s} \,\|\, \tilde{S}[s_2/X])\ ]. ■
\end{aligned}$$

**Proof.** In order to prove this, it suffices to prove the proposition (5.53) defined below, holds for every $(n, k) \in \omega \times \omega$:

$$\begin{aligned}
&\forall a \in \mathbf{A}_1, \forall w \in (\mathbf{A}_1)^n, \forall \rho \in (\mathbf{C}_1)_{\mathrm{I}}^{\leq\omega}, \forall S \in \mathcal{L}_X \\
&[\ \theta(\langle a \rangle \cdot w) \neq \epsilon \wedge \\
&\quad \exists s_1' [\ \vdash_{(k)} S[s_1/X] \xrightarrow{a}_1 s_1' \xrightarrow{w}_1 \stackrel{\rho}{\Longrightarrow}_1\ ] \Rightarrow \\
&\quad \exists \tilde{s} \in \mathcal{L}_\emptyset, \exists \tilde{S} \in \mathcal{L}_X \\
&\quad [\ (\tilde{s} \,\|\, \tilde{S}[s_1/X]) \stackrel{\rho}{\Longrightarrow}_1 \\
&\qquad \wedge S[s_2/X] \stackrel{\theta(\langle a \rangle \cdot w)}{\Longrightarrow}_1 \sqsupseteq (\tilde{s} \,\|\, \tilde{S}[s_2/X])\ ]].
\end{aligned} \tag{5.53}$$

This is achieved by transfinite induction on the well-ordered set $(\omega \times \omega, \prec)$, in a similar fashion to the proof of (5.45) for $\mathcal{P} = \mathcal{F}$. ■

By means of Lemma 5.12, let us prove (5.45) for $\mathcal{P} = \mathcal{T}_\omega$.

**Proof of (5.45) for $\mathcal{P} = \mathcal{T}_\omega$.** Let $S \in \mathcal{L}_X$. In order to prove (5.45) for $\mathcal{P} = \mathcal{T}_\omega$, it suffices to prove

**(i)** $\mathcal{T}_\omega(S[s_1/X]) \subseteq \mathcal{T}_\omega(S[s_2/X])$,
**(ii)** $\mathcal{T}_\omega(S[s_2/X]) \subseteq \mathcal{T}_\omega(S[s_1/X])$.

Let us show the first part (the second part can be shown in the same way).

Let $\langle c_n \rangle_{n\in\omega} \in (\mathbf{C}_1)^\omega$, and suppose $S[s_1/X] \stackrel{\langle c_n \rangle_{n\in\omega}}{\Longrightarrow}_1$. Let us prove $S[s_2/X] \stackrel{\langle c_n \rangle_{n\in\omega}}{\Longrightarrow}_1$. In order to show this, it suffices, by Lemma 5.11, to prove that there exists $\langle (\tilde{s}_n, \tilde{S}_n) \rangle_{n\in\omega} \in (\mathcal{L}_\emptyset \times \mathcal{L}_X)^\omega$ such that

$$S[s_2/X] \sqsupseteq (\tilde{s}_0 \,\|\, \tilde{S}_0[s_2/X]) \wedge \\ \forall n \in \omega[\; (\tilde{s}_n \,\|\, \tilde{S}_n[s_2/X]) \stackrel{\langle c_n\rangle}{\Longrightarrow}_1 \sqsupseteq (\tilde{s}_{n+1} \,\|\, \tilde{S}_{n+1}[s_2/X]) \;]. \tag{5.54}$$

First, let $\langle \tilde{s}_0, \tilde{S}_0\rangle = \langle \mathbf{0}, S\rangle$. Next, let us define $(\tilde{s}_n, \tilde{S}_n)$ $(n > 0)$ inductively as follows: For $n \in \omega$, suppose $(\tilde{s}_n, \tilde{S}_n)$ has been defined. If

$$(\tilde{s}_n, \tilde{S}_n) \notin \mathcal{L}_\emptyset \times \mathcal{L}_X,$$

let

$$(\tilde{s}_{n+1}, \tilde{S}_{n+1}) = (\bot, \bot).$$

Otherwise, putting

$$N = \{(\tilde{s}, \tilde{S}) \in \mathcal{L}_\emptyset \times \mathcal{L}_X : (\tilde{s} \,\|\, \tilde{S}[s_1/X]) \stackrel{\langle c_{n+1+i}\rangle_{i\in\omega}}{\Longrightarrow}_1 \\ \wedge\; (\tilde{s}_n \,\|\, \tilde{S}_n[s_2/X]) \stackrel{\langle c_n\rangle}{\Longrightarrow}_1 \sqsupseteq (\tilde{s} \,\|\, \tilde{S}[s_2/X])\},$$

let

$$(\tilde{s}_{n+1}, \tilde{S}_{n+1}) = \begin{cases} (\bot, \bot) & \text{if } N = \emptyset, \\ \text{any member of } N & \text{otherwise,} \end{cases}$$

where $(\tilde{s}_{n+1}, \tilde{S}_{n+1}) = (\bot, \bot)$ indicates that $(\tilde{s}_{n+1}, \tilde{S}_{n+1})$ is not properly defined. Then, one can show that

$$\langle(\tilde{s}_n, \tilde{S}_n)\rangle_{n\in\omega} \in (\mathcal{L}_\emptyset \times \mathcal{L}_X)^\omega$$

and (5.54), by induction on $n$ using Lemma 5.12. ■

Finally, let us prove (5.45) for $\mathcal{P} = \mathcal{T}_\bot$. In order to prove it, the following lemma is employed.

**Lemma 5.13** *Let* $s_1, s_2 \in \mathcal{L}_\emptyset$ *with*

$$\mathcal{C}_1^{\mathrm{w}}[\![s_1]\!] = \mathcal{C}_1^{\mathrm{w}}[\![s_2]\!].$$

*Then for every* $S \in \mathcal{L}_X$*, the following holds:*

$$S[s_1/X] \stackrel{\bot}{\Longrightarrow}_1 \Rightarrow \\ \exists \tilde{s} \in \mathcal{L}_\emptyset, \exists \tilde{S} \in \mathcal{L}_X[\; (\tilde{s} \,\|\, \tilde{S}[s_1/X]) \stackrel{\bot}{\Longrightarrow}_1 \\ \wedge\; S[s_2/X] \stackrel{\tau}{\to}_1 \stackrel{\epsilon}{\Longrightarrow}_1 \sqsupseteq (\tilde{s} \,\|\, \tilde{S}[s_2/X]) \;]. \;■$$

**Proof.** This lemma is proved by induction on the number $k$ such that $\exists s_1'[\; \vdash_{(k)} S[s_1/X] \stackrel{\tau}{\to}_1 s_1' \stackrel{\bot}{\Longrightarrow}_1 \;]$, using Lemma 5.12. ■

By means of Lemma 5.13, one can prove (5.45) for $\mathcal{P} = \mathcal{T}_\bot$, in a similar fashion to the way (5.45) for $\mathcal{P} = \mathcal{T}_\omega$, is proved by means of Lemma 5.12.

Summarizing, we obtain (5.45) for

$$\mathcal{P} = \mathcal{T}_{\surd}, \mathcal{R}, \mathcal{F}, \mathcal{T}_\omega, \mathcal{T}_\bot,$$

and hence, (5.21) for $\mathcal{C} = \mathcal{C}_1^{\mathrm{w}}$.

# Chapter 6

# A Fully Abstract Model for Communicating Processes Based on the Smyth Powerdomain of Failures

## 6.1 Introduction

In this chapter, we construct a fully abstract model $\mathcal{D}_1^{\text{wf}}$ for the language $\mathcal{L}_1$ treated in Chapter 5, but w.r.t. a new operational model $\mathcal{O}_1^{\text{mf}}$. The model $\mathcal{O}_1^{\text{mf}}$ is weak linear and maximal as the one $\mathcal{O}_1^{\text{wm}}$ treated in Chapter 5, but also it is *flattened* in the following sense: If a sequence $w \cdot \langle\bot\rangle$ is contained in the meaning $\mathcal{O}_1^{\text{mf}}[\![s]\!]$ of a given statement $s$ under $\mathcal{O}_1^{\text{mf}}$, then there is no element $w' \cdot \langle x \rangle$ in $\mathcal{O}_1^{\text{mf}}[\![s]\!]$ such that $w'$ is an extension of $w$ and $x \neq \bot$. The model $\mathcal{O}_1^{\text{mf}}$ is obtained from $\mathcal{O}_1^{\text{wm}}$ by *flattening*, i.e., by removing all elements of the form $w' \cdot \langle x \rangle$ with $w'$ an extension of $w$ and $x \neq \bot$. when $w \cdot \langle\bot\rangle \in \mathcal{O}_1^{\text{wm}}[\![s]\!]$. The flattening operation reflects the view that an element of the form $w \cdot \langle\bot\rangle$ indicates that something disastrous can happen after the execution of $w$, and thus, all elements of the form $w' \cdot \langle x \rangle$ with $w'$ an extension of $w$ are of no interest.

First, in § 6.2, a semantic domain $\mathbf{P}_{\text{cl}}$ on which $\mathcal{D}_1^{\text{wf}}$ is based is given. We call $\mathbf{P}_{\text{cl}}$ *the Smyth powerdomain of failures* because $\mathbf{P}_{\text{cl}}$ is a cpo ordered by the Smyth ordering $\sqsubseteq_{\text{S}}$. The domain $\mathbf{P}_{\text{cl}}$ is also a cms with a metric $\tilde{d}$ defined in Chapter 2. Further $\mathbf{P}_{\text{cl}}$ has the following useful property:

*Let $\langle p_n \rangle_{n\in\omega}$ be a chain consisting of elements of $\mathbf{P}_{\text{cl}}$. Then, $\langle p_n \rangle_{n\in\omega}$ is a Cauchy sequence w.r.t. the metric $d$, and the lub of $\langle p_n \rangle_{n\in\omega}$ in $(\mathbf{P}_{\text{cl}}, \sqsubseteq_{\text{s}})$ and its limit in $(\mathbf{P}_{\text{cl}}, d)$ coincide, i.e.,* (6.1)

$$\bigsqcup_{n\in\omega}[p_n] = \lim_{n\in\omega}[p_n].$$

(This property is established in § 2.4, where certain results for the space of compact sets of streams, established by de Bakker et al. ([BM 87], [MV 88], and [MO 90]) are generalized to the space of closed sets.) The property (6.1) plays a key role in the full abstractness proof, as is explained below.

In § 6.3, we introduce three operational models; the LTS $(\mathcal{L}_{\bar{1}}[\emptyset], \langle \xrightarrow{\alpha}_1 : \alpha \in \mathbf{A}_1^{\checkmark} \rangle)$ defined in Chapter 5 is also used for defining these operational models. First, a weak, linear, maximal, and flattened model $\mathcal{O}_1^{\text{mf}}$ is defined in terms of the LTS. Further, two failure models $\mathcal{C}_1^{\text{sr}}$ and $\mathcal{C}_1^{\text{wf}}$ are defined operationally. Both $\mathcal{C}_1^{\text{sr}}$ and $\mathcal{C}_1^{\text{wf}}$ are flattened in the sense explained above. A purely operational argument shows that $\mathcal{C}_1^{\text{wf}}$ is *complete* w.r.t. $\mathcal{O}_1^{\text{mf}}$, i.e. that

For every two statements $s_0$, $s_1$ of $\mathcal{L}_{\bar{1}}$ with $\mathcal{C}_1^{\text{wf}}[\![s_0]\!] \neq \mathcal{C}_1^{\text{wf}}[\![s_1]\!]$, there is a context $S$ of $\mathcal{L}_{\bar{1}}$ such that $\mathcal{O}_1^{\text{mf}}[\![S[s_0]]\!] \neq \mathcal{O}_1^{\text{mf}}[\![S[s_1]]\!]$. (6.2)

In § 6.4, semantic operations $\widetilde{\text{op}}$ corresponding to the combinators op of $\mathcal{L}_{\bar{1}}$ are defined on the basis of $\mathbf{P}_{\text{cl}}$. Metric and order-theoretic models $\mathcal{M}_1^{\text{sr}}$ and $\mathcal{D}_1^{\text{sr}}$ are defined in terms of the semantic operations. Using (6.1), one has

$$\forall s[\ \mathcal{M}_1^{\text{sr}}[\![s]\!] = \mathcal{D}_1^{\text{sr}}[\![s]\!]\ ]. \tag{6.3}$$

Moreover, another order-theoretic model $\mathcal{D}_1^{\text{wf}}$ is defined in terms of *weak* semantic operations $\widehat{\text{op}}$, which are obtained by applying a *hiding function* $H$ to the original semantic operations $\widetilde{\text{op}}$. The continuity of the operations $\widehat{\text{op}}$ follows from that of $H$. A purely order-theoretic argument shows that

$$\forall s[\ \mathcal{D}_1^{\text{wf}}[\![s]\!] = H(\mathcal{D}_1^{\text{sr}}[\![s]\!])\ ]. \tag{6.4}$$

Then, in § 6.5, the models defined in the previous section are related; in particular the full abstractness of $\mathcal{D}_1^{\text{wf}}$ w.r.t. $\mathcal{O}_1^{\text{mf}}$ (the main result of this chapter) is established: The correctness of $\mathcal{D}_1^{\text{wf}}$ w.r.t. $\mathcal{O}_1^{\text{mf}}$ can easily be shown. For establishing the full abstractness, it suffices, by (6.2), to show that

$$\forall s[\ \mathcal{C}_1^{\text{wf}}[\![s]\!] = \mathcal{D}_1^{\text{wf}}[\![s]\!]\ ], \tag{6.5}$$

which is shown as follows: For every statement $s$,

$$\mathcal{C}_1^{\text{wf}}[\![s]\!] = H(\mathcal{C}_1^{\text{sr}}[\![s]\!]) = H(\mathcal{M}_1^{\text{sr}}[\![s]\!]) = H(\mathcal{D}_1^{\text{sr}}[\![s]\!]) = \mathcal{D}_1^{\text{wf}}[\![s]\!].$$

In this chapter, we are primarily concerned with the two models $\mathcal{O}_1^{\text{mf}}$ and $\mathcal{D}_1^{\text{wf}}$ and the connection between them. All the other models, however, are necessary for connecting $\mathcal{O}_1^{\mathrm{mf}}$ and $\mathcal{D}_1^{\mathrm{wf}}$, and for establishing the full abstractness. In particular, note that the metric framework is essential in connecting the operational and denotational models, although the denotational model $\mathcal{D}_1^{\mathrm{wf}}$, which we are primarily concerned with, is constructed order-theoretically: The equivalence between the metric ($\mathcal{M}_1^{\mathrm{sr}}$) and operational ($\mathcal{C}_1^{\mathrm{sr}}$) models is conveniently established in the metric framework, by means of Banach's Fixed-Point Theorem. It seems much more difficult than this to show the equivalence between $\mathcal{D}_1^{\mathrm{sr}}$ and $\mathcal{C}_1^{\mathrm{sr}}$ in the order-theoretic framework, although the metric ($\mathcal{M}_1^{\mathrm{sr}}$) and order-theoretic ($\mathcal{D}_1^{\mathrm{sr}}$) models are equivalent.

Finally, in § 6.6, a slightly simplified variant $\mathcal{D}_*^{\mathrm{wf}}$ of the fully abstract model $\mathcal{D}_1^{\mathrm{wf}}$ is compared with two well-known models for communicating processes: the *improved failures model* $\mathcal{N}$ of Brookes and Roscoe ([BR 84]) and the *strong Acceptance tree model* $\mathbf{AT}_s$ of Hennessy ([Hen 88]); these three models $\mathcal{D}_*^{\mathrm{wf}}$, $\mathcal{N}$, and $\mathbf{AT}_S$ are shown to be isomorphic. We also give a comparison between these models and the two models $\mathcal{C}_1^{\mathrm{w}}$ and $\mathcal{C}_1^{\mathrm{wi}}$ introduced in Chapter 5.

Some of the tedious mathematical proofs are relegated to the appendices.

## 6.2 The Smyth Powerdomain of Failures

In this section, we define the common semantic domain $\mathbf{P}_{\mathrm{cl}}$ of the several models to be presented in the following sections. Roughly speaking, $\mathbf{P}_{\mathrm{cl}}$ consists of closed sets of failures and is endowed with an order $\sqsubseteq_s$ due to Smyth; we call $\mathbf{P}_{\mathrm{cl}}$ *the Smyth Powerdomain of failures.*

First, we present some preliminary definitions.

**Definition 6.1 (1)** Let $\mathbf{C}_1$ be a nonempty set, $\tau$ a symbol with $\tau \notin \mathbf{C}_1$. (Elements of $\mathbf{C}_1$ represent *communication actions*, and $\tau$ represents an *internal move* (or a *silent action*).) Set $\mathbf{A}_1 = \mathbf{C}_1 \cup \{\tau\}$.

**(2)** Let $\mathtt{D}$, $\mathit{\Delta}$, and $\delta$ be distinct symbols standing for *direct deadlock*, *indirect deadlock*, and *deadlock*, respectively. Let $\mathbf{C}_1^{\checkmark} = \mathbf{C}_1 \cup \{\checkmark\}$, and let $D : \wp(\mathbf{C}_1^{\checkmark}) \to \{\mathtt{D}\} \times \wp(\mathbf{C}_1^{\checkmark})$ and $\Delta : \wp(\mathbf{C}_1^{\checkmark}) \to \{\mathit{\Delta}\} \times \wp(\mathbf{C}_1^{\checkmark})$ be defined as follows: For $\Gamma \in \wp(\mathbf{C}_1^{\checkmark})$,

$$D(\Gamma) = (\mathit{\Delta}, \Gamma), \quad \Delta(\Gamma) = (\mathtt{D}, \Gamma).$$

**(3)** Let

$$\mathbf{B} = D[\wp(\mathbf{C}_1^{\checkmark})] \cup \Delta[\wp(\mathbf{C}_1^{\checkmark})] \cup \{\checkmark\}$$

and $\mathbf{B}_\perp = \mathbf{B} \cup \{\perp\}$. Let $\mathbf{Q}$ be defined just as in Definition 2.3 with $A = \mathbf{A}_1$ and $B = \mathbf{B}$. That is,

$$\mathbf{Q} = ((\mathbf{A}_1)^{<\omega} \cdot \mathbf{B}_\perp) \cup (\mathbf{A}_1)^{\omega}.$$

**(4)** For $p \in \wp(\mathbf{Q})$, let $\mathcal{B}(p) = p \cap \mathbf{B}_\perp$ and $\oplus(p) = p \cap ((\mathbf{A}_1)^1 \cdot \mathbf{Q})$. ∎

From $\mathbf{Q}$, the domain $\mathbf{P}_{\mathrm{cl}}$ is defined by:

**Definition 6.2** (Semantic Domain $\mathbf{P}_{\mathrm{cl}}$) Let $p \in \wp(\mathbf{Q})$.

**(1)** **(i)** $\mathrm{act}(p) = \{a \in \mathbf{A}_1 : p[\langle a \rangle] \neq \emptyset\}$.

**(ii)** $\mathrm{cact}(p) = \mathrm{act}(p) \cap \mathbf{C}_1$.

**(2)** Let $\mathbf{P}_{\mathrm{cl}}$ be the set of elements $p$ of $\wp_{+\mathrm{fcl}}(\mathbf{Q})$ satisfying the following conditions (i)–(v):

**(i)** $p$ is *bounded in the action part* in the sense that

$$\forall n \in \omega[\ \mathrm{strip}[p^{[n]}] \text{ is finite }]. \tag{6.6}$$

**(ii)** $p$ is *downward closed* in the sense that

$$\forall w \in (\mathbf{A}_1)^{<\omega}, \forall \Gamma \subseteq \mathbf{C}_1^{\checkmark}[\ w \cdot \langle \Upsilon(\Gamma) \rangle \in p \Rightarrow \forall \Gamma' \in \wp(\Gamma)[\ w \cdot \langle \Upsilon(\Gamma') \rangle \in p\ ]],$$

where $\Upsilon = D, \Delta$.

**(iii)** $p$ is *upward closed w.r.t. disabled actions* in the sense that

$$\forall w \in (\mathbf{A}_1)^{<\omega}, \forall \Gamma \subseteq \mathbf{C}_1^{\checkmark}[\ w \cdot \langle \Upsilon(\Gamma) \rangle \in p \Rightarrow w \cdot \langle \Upsilon(\Gamma \cup (\mathbf{C}_1^{\checkmark} \setminus \mathrm{act}(p[w])))\rangle \in p\ ],$$

where $\Upsilon = D, \Delta$.

**(iv)** $\forall w \in (\mathbf{A}_1)^{+}, \forall \Gamma[\ w \cdot \langle \Delta(\Gamma) \rangle \notin p\ ]$. (We call this condition the $\Delta$-*condition.*)

**(v)** For every $w \in (\mathbf{A}_1)^{<\omega}$, if $p[w] \neq \emptyset$, $\langle \bot \rangle \notin p[w]$, and $\tau \notin \mathrm{act}(p[w])$, then $\exists \Gamma[\ \langle D(\Gamma) \rangle \in p[w]\ ]$ or $\exists \Gamma[\ \langle \Delta(\Gamma) \rangle \in p[w]\ ]$ (We call this condition the *disabled-$\tau$ condition* (*DTC*).)

We call $\mathbf{P}_{\mathrm{cl}}$ *the domain of processes.*

**(3)** Let

$$\tilde{\delta}(p) = \{w \cdot \langle \delta(\Gamma) \rangle : w \cdot \langle \Delta(\Gamma) \rangle \in p \vee w \cdot \langle D(\Gamma) \rangle \in p\},$$

where the notation $\delta(\Gamma)$ is used to denote $(\delta, \Gamma)$. ∎

For $\mathbf{P}_{\mathrm{cl}}$, we have the following lemma which is analogous to Lemma 2.8 for $\wp_{\mathrm{fco}}(\mathbf{Q})$.

**Lemma 6.1** *Every* $\langle p_n \rangle_{n \in \omega} \in \mathrm{Chain}(\mathbf{P}_{\mathrm{cl}}, \sqsubseteq_{\mathrm{s}})$ *is also a CS in* $\mathbf{P}_{\mathrm{cl}}$, *and*

$$\mathrm{Lub}(\langle p_n \rangle_{n \in \omega}) = \bigsqcup(\langle p_n \rangle_{n \in \omega}, \mathbf{P}_{\mathrm{cl}}) = \lim(\langle p_n \rangle_{n \in \omega}, \mathbf{P}_{\mathrm{cl}}). \ \blacksquare$$

This lemma essentially follows from Lemma 2.7, but we need some preliminaries for the proof.

**Definition 6.3** **(1)** Let $\mathbf{P}_{\mathrm{co}}$ (with 'co' standing for 'compact') be the set of elements of $\wp_{+\mathrm{fco}}(\mathbf{Q})$ satisfyfing the conditions (ii), (iv), (v) in Definition 6.2 (2) and the following condition:

$$\forall w, \forall \Gamma[\ w \cdot \langle D(\Gamma)\rangle \in p \lor w \cdot \langle \Delta(\Gamma)\rangle \in p \\ \Rightarrow \Gamma \subseteq \mathrm{act}(p[w])\ ]. \tag{6.7}$$

**(2)** Let $\Lambda : \mathbf{P}_{\mathrm{cl}} \to \mathbf{P}_{\mathrm{co}}$ be defined as follows: For every $p \in \mathbf{P}_{\mathrm{cl}}$,

$$\begin{aligned}\Lambda(p) = &\{\langle \Delta(\Gamma \cap \mathrm{act}(p))\rangle : \langle \Delta(\Gamma)\rangle \in p\}\\ &\cup \{w \cdot \langle D(\Gamma \cap \mathrm{act}(p[w]))\rangle : w \cdot \langle D(\Gamma)\rangle \in p\}\\ &\cup (p \setminus ((\mathbf{A}_1)^{<\omega} \cdot (\Delta[\wp(\mathbf{C}_1^{\checkmark})] \cup D[\wp(\mathbf{C}_1^{\checkmark})]))).\end{aligned}$$

**(3)** Let $\Lambda' : \mathbf{P}_{\mathrm{co}} \to \mathbf{P}_{\mathrm{cl}}$ be defined as follows: For every $p \in \mathbf{P}_{\mathrm{co}}$,

$$\begin{aligned}\Lambda'(p) = &\{\langle \Delta(\Gamma \cup \Gamma')\rangle : \langle \Delta(\Gamma)\rangle \in p \land \Gamma' \subseteq \mathbf{C}_1^{\checkmark} \setminus \mathrm{act}(p[w])\}\\ &\cup \{w \cdot \langle D(\Gamma \cup \Gamma')\rangle : w \cdot \langle D(\Gamma)\rangle \in p\\ &\quad \land \Gamma' \subseteq \mathbf{C}_1^{\checkmark} \setminus \mathrm{act}(p[w])\}\\ &\cup (p \setminus ((\mathbf{A}_1)^{<\omega} \cdot (\Delta[\wp(\mathbf{C}_1^{\checkmark})] \cup D[\wp(\mathbf{C}_1^{\checkmark})]))).\ \blacksquare\end{aligned}$$

The mappings $\Lambda$, $\Lambda'$ have the following properties:

**Lemma 6.2** **(1)** *The mapping $\Lambda$ is monotonic in the sense that*

$$\forall p_1, p_2 \in \mathbf{P}_{\mathrm{cl}}[\ p_1 \sqsubseteq_{\mathrm{s}} p_2 \Rightarrow \Lambda(p_1) \sqsubseteq_{\mathrm{s}} \Lambda(p_2)\ ].$$

**(2)** $\Lambda' \circ \Lambda = \mathrm{id}$ *and* $\Lambda \circ \Lambda' = \mathrm{id}'$, *where* id *and* id$'$ *are the identity mappings on* $\mathbf{P}_{\mathrm{cl}}$ *and* $\mathbf{P}_{\mathrm{co}}$, *respectively.*

**(3)** **(i)** *The mapping $\Lambda$ is an isometry from $\mathbf{P}_{\mathrm{cl}}$ onto $\mathbf{P}_{\mathrm{co}}$.*

**(ii)** *The mapping $\Lambda'$ is an isometry from $\mathbf{P}_{\mathrm{co}}$ onto $\mathbf{P}_{\mathrm{cl}}$.* $\blacksquare$

**Proof.** **(1)** Let $p_1, p_2$ be elements of $\mathbf{P}_{\mathrm{cl}}$ such that $p_1 \sqsubseteq_{\mathrm{s}} p_2$. We will show that $\Lambda(p_1) \sqsubseteq_{\mathrm{s}} \Lambda(p_2)$, i.e., that

$$\forall q_2 \in \Lambda(p_2), \exists q_1 \in \Lambda(p_1)[\ q_1 \sqsubseteq q_2\ ]. \tag{6.8}$$

Fix $q_2 \in \Lambda(p_2)$. We distinguish two cases according to whether

$$q_2 \in (\mathbf{A}_1)^{<\omega} \cdot (D[\wp(\mathbf{C}_1^{\checkmark})] \cup \Delta[\wp(\mathbf{C}_1^{\checkmark})])$$

or not.

**Case 1.** Suppose $q_2 \in (\mathbf{A}_1)^{<\omega} \cdot (D[\wp(\mathbf{C}_1^{\checkmark})] \cup \Delta[\wp(\mathbf{C}_1^{\checkmark})])$ and $q_2 = w \cdot \langle \Upsilon(\Gamma)\rangle$ with $\Upsilon = D, \Delta$. By the definition of $\Lambda$, one has $\Gamma \subseteq \mathrm{act}(p_2[w])$. Let

$$\tilde{q}_2 = w \cdot \langle \Upsilon(\Gamma \cup (\mathbf{C}_1^{\checkmark} \setminus \mathrm{act}(p_2[w])))\rangle.$$

One has $\tilde{q}_2 \in p_2$, since $p_2$ is upward closed w.r.t. disabled actions. Since $p_1 \sqsubseteq_{\mathrm{s}} p_2$, there exists $\tilde{q}_1 \in p_1$ such that $\tilde{q}_1 \sqsubseteq \tilde{q}_2$. By the definition $\sqsubseteq$, either $\tilde{q}_1 = \tilde{q}_2$ or $\exists \tilde{w} \leq_{\mathrm{p}} w[\ \tilde{w} \cdot \langle \perp \rangle = \tilde{q}_1\ ]$.

**Subcase 1.1.** Consider the case $\tilde{q}_1 = \tilde{q}_2$. Then, from the fact that $p_1 \sqsubseteq_{\mathrm{s}} p_2$, it follows that

$$\mathrm{act}(p_1[w]) \supseteq \mathrm{act}(p_2[w]). \tag{6.9}$$

Thus, $(\mathbf{C}_1^{\checkmark} \setminus \mathrm{act}(p_1[w])) \subseteq (\mathbf{C}_1^{\checkmark} \setminus \mathrm{act}(p_2[w]))$, and therefore,

$$\Gamma \cup (\mathbf{C}_1^{\checkmark} \setminus \mathrm{act}(p_1[w])) \subseteq \Gamma \cup (\mathbf{C}_1^{\checkmark} \setminus \mathrm{act}(p_2[w])).$$

From this and the fact that $w \cdot \langle \Upsilon(\Gamma \cup (\mathbf{C}_1^{\checkmark} \setminus \mathrm{act}(p_2[w]))) \rangle \in p_1$, it follows that $w \cdot \langle \Upsilon(\Gamma \cup (\mathbf{C}_1^{\checkmark} \setminus \mathrm{act}(p_1[w]))) \rangle \in p_1$, since $p_1$ is downward closed. Thus, putting $q_1 = w \cdot \langle \Upsilon(\Gamma \cap \mathrm{act}(p_1[w])) \rangle$, one has

$$q_1 \in \Lambda(p_1). \tag{6.10}$$

Since $\Gamma \subseteq \mathrm{act}(p_2[w]) \subseteq \mathrm{act}(p_1[w])$, one has $\Gamma \cap \mathrm{act}(p_1[w]) = \Gamma$. By this and (6.10), one has $q_1 = w \cdot \langle \Upsilon(\Gamma) \rangle = q_2 \in \Lambda(p_1)$, and therefore, $\exists q_1 \in \Lambda(p_1)[\ q_1 \sqsubseteq q_2\ ]$.
**Subcase 1.2.** Consider the case $\exists \tilde{w} \leq_{\mathrm{p}} w[\ \tilde{w} \cdot \langle \bot \rangle = \tilde{q}_1\ ]$. By the definition of $\Lambda(\cdot)$, one has $\tilde{w} \cdot \langle \bot \rangle \in \Lambda(p_1)$. Moreover $\tilde{w} \cdot \langle \bot \rangle \sqsubseteq w \cdot \langle \Upsilon(\Gamma) \rangle = q_2$. Thus, $\exists q_1 \in \Lambda(p_1)[\ q_1 \sqsubseteq q_2\ ]$.
**Case 2.** Suppose $q_2 \notin (\mathbf{A}_1)^{<\omega} \cdot (D[\wp(\mathbf{C}_1^{\checkmark})] \cup \Delta[\wp(\mathbf{C}_1^{\checkmark})])$. Then, by the definition of $\Lambda$, one has $q_2 \in p_2$. By this and the fact that $p_1 \sqsubseteq_{\mathrm{s}} p_2$, there exists $q_1 \in p_1$ such that $q_1 \sqsubseteq q_2$. By the definition of $\sqsubseteq$, either $q_1 = q_2$ or

$$\exists w \in (\mathbf{A}_1)^{<\omega}[\ w \leq_{\mathrm{p}} \mathrm{strip}(q_2) \wedge q_1 = w \cdot \langle \bot \rangle\ ].$$

In both cases, one has $q_1 \in \Lambda(p_1)$, and therefore, $\exists q_1 \in \Lambda(p_1)[\ q_1 \sqsubseteq q_2\ ]$.
Summarizing we have (6.8).
**(2)** This part immediately follows from the definition of $\Lambda$ and $\Lambda'$; the proof is omitted.
**(3)** We will show only part (i); part (ii) is obtained in a similar fashion. From the definition of $\Lambda$, it immediately follows that

$$\forall n \in \omega, \forall p \in \mathbf{P}_{\mathrm{cl}}[\ (\Lambda(p))^{[n]} = \Lambda(p^{[n]})\ ].$$

By this, one has

$$\forall n \in \omega, \forall p_1, p_2 \in \mathbf{P}_{\mathrm{cl}}[\ (p_1)^{[n]} = (p_2)^{[n]} \Leftrightarrow (\Lambda(p_1))^{[n]} = (\Lambda(p_2))^{[n]}\ ].$$

From this and (2.10), it immediately follows that $\Lambda$ is an isometry. Also, from part (2) it immediately follows that $\mathrm{ran}(\Lambda) = \mathbf{P}_{\mathrm{co}}$. ■

Using the above lemma, we can prove Lemma 6.1 as follows:

**Proof of Lemma 6.1** Let $\langle p_n \rangle_{n\in\omega} \in \mathrm{Chain}(\mathbf{P}_{\mathrm{cl}}, \sqsubseteq_{\mathrm{s}})$. First, we will prove that

$$\langle p_n \rangle_{n\in\omega} \in \mathrm{CS}(\mathbf{P}_{\mathrm{cl}}, d). \tag{6.11}$$

Since $\Lambda$ is monotonic, $\langle \Lambda(p_n) \rangle_{n\in\omega} \in \mathrm{Chain}(\mathbf{P}_{\mathrm{co}}, \sqsubseteq_{\mathrm{s}})$. Thus, by Lemma 2.8, $\langle \Lambda(p_n) \rangle_{n\in\omega} \in \mathrm{CS}(\mathbf{P}_{\mathrm{co}}, d)$. From this and the fact $\Lambda'$ is an isometry, it follows that $\langle \Lambda'(\Lambda(p_n)) \rangle_{n\in\omega} \in \mathrm{CS}(\mathbf{P}_{\mathrm{cl}}, d)$. Since $\Lambda'(\Lambda(p_n)) = p_n$ by Lemma 6.2 (2), one has (6.11).

Thus, by Lemma 2.7,

$$\mathrm{Lub}(\langle p_n \rangle_{n\in\omega}) = \bigsqcup(\langle p_n \rangle_{n\in\omega}, \wp_{\mathrm{fcl}}(\mathbf{Q})) = \lim(\langle p_n \rangle_{n\in\omega}, \wp_{\mathrm{fcl}}(\mathbf{Q})). \tag{6.12}$$

Since the property of *being nonempty and flat and satisfying (i)–(v) in Definition 6.2 (2)* is finitely characterized (see Chapter 2), $\mathbf{P}_{\mathrm{cl}}$ is closed in $\wp_{\mathrm{cl}}(\mathbf{Q})$. Thus, $\lim(\langle p_n\rangle_{n\in\omega}, \wp_{\mathrm{fcl}}(\mathbf{Q})) \in \mathbf{P}_{\mathrm{cl}}$. From this and (6.12), it follows that

$$\mathrm{Lub}(\langle p_n\rangle_{n\in\omega}) = \bigsqcup(\langle p_n\rangle_{n\in\omega}, \mathbf{P}_{\mathrm{cl}}) = \lim(\langle p_n\rangle_{n\in\omega}, \mathbf{P}_{\mathrm{cl}}). \blacksquare$$

From Lemma 2.7, one obtains the following lemma:

**Lemma 6.3 (1)** $(\mathbf{P}_{\mathrm{cl}}, d)$ *is a complete ultra-metric space.*

**(2)** $(\mathbf{P}_{\mathrm{cl}}, \sqsubseteq_{\mathrm{s}}, \{\langle\bot\rangle\})$ *is a cpo.* ∎

**Proof. (1)** The set $\mathbf{P}_{\mathrm{cl}} \subseteq \wp_{\mathrm{cl}}(\mathbf{Q})$ is closed in the complete metric space $\wp_{\mathrm{cl}}(\mathbf{Q})$, for the reason explained in the proof of Lemma 2.7. Thus, $\mathbf{P}_{\mathrm{cl}}$ is also a complete metric space with the metric $d$ restricted to it.
**(2)** It suffices to show that every $\langle p_n\rangle_{n\in\omega} \in \mathrm{Chain}(\mathbf{P}_{\mathrm{cl}}, \sqsubseteq_{\mathrm{s}})$ has a lub in $\mathbf{P}_{\mathrm{cl}}$. This immediately follows from Lemma 6.1. ∎

## 6.3 Operational Models

In this section, three models for $\mathcal{L}_{\bar{1}}$ are defined operationally, i.e., defined in terms of a transition system $\langle \xrightarrow{\alpha}_1 : \alpha \in \mathbf{A}_1^{\checkmark}\rangle$. The relations $\xrightarrow{\alpha}_1$ are the same as those defined in Definition 5.4. First, we need some preliminary definitions.

**Definition 6.4** Let $s \in \mathcal{L}_{\bar{1}}[\emptyset]$.

**(1)** We call the pair $(a, s') \in \mathbf{A}_1 \times \mathcal{L}_{\bar{1}}[\emptyset]$ an *immediate derivative* of $s$, when $s \xrightarrow{a}_1 s'$. Let $\mathrm{IDrv}(s)$ be the set of immediate derivatives of $s$, i.e., let

$$\mathrm{IDrv}(s) = \{(a, s') \in \mathbf{A}_1 \times \mathcal{L}_{\bar{1}}[\emptyset] : s \xrightarrow{a}_1 s'\}.$$

**(2)** $\mathrm{act}_1(s) = \{\alpha \in \mathbf{A}_1^{\checkmark} : (\alpha \in \mathbf{A}_1 \wedge \exists s'[\, s \xrightarrow{a}_1 s' \,]) \vee (\alpha = \surd \wedge s \xrightarrow{\surd}_1)\}$. ∎

The labeled transition system $(\mathcal{L}_{\bar{1}}[\emptyset], \langle \xrightarrow{\alpha}_1 : \alpha \in \mathbf{A}_1^{\checkmark}\rangle)$ is *finitely branching* as stated in Lemma 5.1.

### 6.3.1 A Flattened Weak Linear Model $\mathcal{O}_1^{\mathrm{mf}}$

In this subsection, we define a *flattened weak linear operational model* $\mathcal{O}_1^{\mathrm{mf}}$; this is obtained from an auxiliary model $\mathcal{O}_1^{\mathrm{m}}$ by applying a hiding operation $H_1$.

First, we define the auxiliary model $\mathcal{O}_1^{\mathrm{m}}$; this model is the same as the one defined in Definition 4.5.

**Definition 6.5 (1)** Let $(\rho \in)\ \mathbf{B}_1 = (\mathbf{A}_1)^{<\omega} \cdot \{\langle\surd\rangle, \langle\delta\rangle\} \cup (\mathbf{A}_1)^{\omega}$.

**(2)** The model $\mathcal{O}_1^{\mathrm{m}} : \mathcal{L}_{\bar{1}} \to \wp(\mathbf{B}_1)$ is defined as follows: Let $s \in \mathcal{L}_{\bar{1}}[\emptyset]$ and $\rho \in \mathbf{B}_1$, we put $\rho \in \mathcal{O}_1^{\mathrm{m}}[\![s]\!]$ iff one of the following propositions (6.13), (6.14), (6.15) holds:

$$\exists\langle s_i\rangle_{i\in\omega}, \exists\langle a_i\rangle_{i\in\omega}[\ \rho = \langle a_i\rangle_{i\in\omega} \wedge s_0 \equiv s \wedge \forall i \in \omega[s_i \xrightarrow{a_i}_1 s_{i+1}]\ ]. \tag{6.13}$$

$$\exists n \in \omega, \exists\langle s_i\rangle_{i\in(n+1)}, \exists\langle a_i\rangle_{i\in n}[\ \rho = \langle a_i\rangle_{i\in n}\cdot\langle\surd\rangle \wedge s_0 \equiv s \wedge \forall i \in n[\ s_i \xrightarrow{a_i}_1 s_{i+1}\ ] \wedge \surd \in \mathrm{act}_1(s_n)\ ]. \tag{6.14}$$

$$\exists n \in \omega, \exists\langle s_i\rangle_{i\in(n+1)}, \exists\langle a_i\rangle_{i\in n}[\rho = \langle a_i\rangle_{i\in n}\cdot\langle\,\delta\,\rangle \wedge s_0 \equiv s \wedge \forall i \in n[\ s_i \xrightarrow{a_i}_1 s_{i+1}\ ] \wedge \mathrm{act}_1(s_n) = \emptyset\ ]. \blacksquare \tag{6.15}$$

Next, we define the hiding operation $H_1$:

**Definition 6.6** **(1)** For $w \in (\mathbf{A}_1)^{<\omega}$, let $(w \setminus \tau)$ denote the result of erasing $\tau$'s in $w$.

**(2)** Let $(\eta \in)\ \mathbf{B}_{\mathrm{wl}} = (\mathbf{C}_1)^{<\omega}\cdot\{\langle\bot\rangle, \langle\surd\rangle, \langle\,\delta\,\rangle\} \cup (\mathbf{C}_1)^{\omega}$.

**(3)** A function $h_1 : \mathbf{B}_1 \to \mathbf{B}_{\mathrm{wl}}$ is defined as follows: Let $\rho \in \mathbf{B}_1$. When $\rho = w\cdot\langle x\rangle$ with $(w, x) \in (\mathbf{A}_1)^{<\omega} \times \{\bot, \surd, \delta\}$ let

$$h_1(q) = (w \setminus \tau)\cdot\langle x\rangle;$$

otherwise $q \in (\mathbf{A}_1)^{\omega}$, and let

$$h_1(q) = \begin{cases} (\tilde{w} \setminus \tau)\cdot\langle\bot\rangle & \text{if } \exists\tilde{w}[\ w = \tilde{w}\cdot\langle\tau\rangle_{i\in\omega}\ ], \\ (w \setminus \tau) & \text{otherwise.} \end{cases}$$

**(4)** A hiding function $H_1 : \wp(\mathbf{B}_1) \to \wp(\mathbf{B}_{\mathrm{wl}})$ is defined by:

$$H_1(p) = (\lambda p \in \mathbf{P}_{\mathrm{cl}}.\ \mathrm{mini}(h_1[p])),$$

where mini is defined as in Definition 2.12. $\blacksquare$

From $\mathcal{O}_1^{\mathrm{m}}$ and $H_1$, the model $\mathcal{O}_1^{\mathrm{mf}}$ is defined by:

**Definition 6.7** The model $\mathcal{O}_1^{\mathrm{mf}} : \mathcal{L}_1 \to \wp(\mathbf{Q})$ is defined as follows: For $s \in \mathcal{L}_1[\emptyset]$,

$$\mathcal{O}_1^{\mathrm{mf}}[\![s]\!] = H_1(\mathcal{O}_1^{\mathrm{m}}[\![s]\!]). \blacksquare$$

**Example 6.1** Let $c_0, c_1 \in \mathbf{C}_1$ and put

$$s \equiv (c_0; c_1) + (c_0; (\mu X.\ \tau; X)) + (c_1; c_0).$$

Then,

$$\mathcal{O}_1^{\mathrm{m}}[\![s]\!] = \{\langle c_0, c_1, \surd\rangle, \langle c_0\rangle\cdot\tau^{\omega}, \langle c_1, c_0, \surd\rangle\}.$$

Thus,

$$\begin{aligned}\mathcal{O}_1^{\mathrm{mf}}[\![s]\!] &= \mathrm{mini}(\{\langle c_0, c_1, \surd\rangle, \langle c_0, \bot\rangle, \langle c_1, c_0, \surd\rangle\}) \\ &= \{\langle c_0, \bot\rangle, \langle c_1, c_0, \surd\rangle\},\end{aligned}$$

where $\langle c_0, c_1, \surd\rangle$ is pruned away by $\mathrm{mini}(\cdot)$. $\blacksquare$

An equivalence relation $\equiv_1^{\mathrm{mf}}$ on $\mathcal{L}_1[\emptyset]$ is induced by $\mathcal{O}_1^{\mathrm{mf}}$ as usual:

**Definition 6.8** For every $s_0, s_1 \in \mathcal{L}_1[\emptyset]$,

$$s_0 \equiv_1^{\mathrm{mf}} s_1 \Leftrightarrow \mathcal{O}_1^{\mathrm{mf}}[\![s_0]\!] = \mathcal{O}_1^{\mathrm{mf}}[\![s_1]\!]. \blacksquare$$

### 6.3.2 A Strong Failures Model $\mathcal{C}_1^{\mathrm{sr}}$

In this subsection, we define a *strong failures model* $\mathcal{C}_1^{\mathrm{sr}}$ operationally. The model $\mathcal{C}_1^{\mathrm{sr}}$ is essentially the same as $\mathcal{C}_1$ defined in Definition 4.6. First, $\mathcal{C}_1^{\mathrm{sr}}$ is defined nonrecursively:

**Definition 6.9** The $\mathcal{C}_1^{\mathrm{sr}} : \mathcal{L}_{\bar{1}}[\emptyset] \to \wp(\mathbf{Q}_1)$ is defined as follows: Let $s \in \mathcal{C}_1[\emptyset]$. For $q \in \mathbf{Q}$, we put $q \in \mathcal{C}_1^{\mathrm{sr}}[\![s]\!]$ iff one of the following four conditions (6.16)–(6.19) is satisfied:

$$\exists \langle s_i \rangle_{i \in \omega}, \exists \langle a_i \rangle_{i \in \omega}[\ q = \langle a_i \rangle_{i \in \omega} \ \wedge\ s_0 = s \ \wedge\ \forall i \in \omega[\ s_i \xrightarrow{a_i}_1 s_{i+1}\ ]]; \tag{6.16}$$

$$\begin{aligned} &\exists n \in \omega, \exists \langle s_i \rangle_{i \in (n+1)}, \exists \langle a_i \rangle_{i \in n}[q = \langle a_i \rangle_{i \in n} \cdot \langle \surd \rangle \ \wedge\ s_0 = s \\ &\quad \wedge \forall i \in n[\ s_i \xrightarrow{a_i}_1 s_{i+1}\ ] \ \wedge\ \surd \in \mathrm{act}_1(s_n)\ ]; \end{aligned} \tag{6.17}$$

$$\begin{aligned} &\exists n \in (\omega \setminus \{0\}), \exists \langle s_i \rangle_{i \in (n+1)}, \exists \langle a_i \rangle_{i \in n}, \exists \Gamma \in \wp(\mathbf{C}_1^{\surd})[ \\ &\quad q = \langle a_i \rangle_{i \in n} \cdot \langle D(\Gamma) \rangle \ \wedge\ s_0 = s \ \wedge\ \forall i \in n[\ s_i \xrightarrow{a_i} s_{i+1}\ ] \\ &\quad \wedge \tau \notin \mathrm{act}_1(s_n) \ \wedge\ \Gamma \cap \mathrm{act}_1(s_n) = \emptyset\ ]; \end{aligned} \tag{6.18}$$

$$\exists \Gamma \in \wp(\mathbf{C}_1^{\surd})[\ q = \langle \Delta(\Gamma) \rangle \ \wedge\ \tau \notin \mathrm{act}_1(s) \ \wedge\ \Gamma \cap \mathrm{act}_1(s) = \emptyset\ ]. \blacksquare \tag{6.19}$$

**Example 6.2** Let $c_0, c_1 \in \mathbf{C}_1$ and put $s \equiv (c_0; c_1)$. Then it is easity checked that

$$\begin{aligned} \mathcal{C}_1^{\mathrm{sr}}[\![s]\!] = &\{\langle \Delta(\Gamma) \rangle : c_0 \notin \Gamma\} \cup \{\langle c_0, D(\Gamma) \rangle : c_1 \notin \Gamma\} \\ &\cup \{\langle c_0, c_1, D(\Gamma) \rangle : \surd \notin \Gamma \rangle\} \cup \{\langle c_0, c_1, \surd \rangle\}. \blacksquare \end{aligned}$$

From Lemma 5.1, it follows that for every statement, its meaning under $\mathcal{C}_1^{\mathrm{sr}}$ is in $\mathbf{P}_{\mathrm{cl}}$:

**Proposition 6.1** $\forall s \in \mathcal{L}_{\bar{1}}[\emptyset][\ \mathcal{C}_1^{\mathrm{sr}}[\![s]\!] \in \mathbf{P}_{\mathrm{cl}}\ ]$. $\blacksquare$

The model $\mathcal{C}_1^{\mathrm{sr}}$ can also defined *recursively*, i.e., as the fixed-point of a higher-order contraction; we will define another model $\hat{\mathcal{C}}_1^{\mathrm{sr}} : \mathbf{P}_{\mathrm{cl}} \to \mathbf{P}_{\mathrm{cl}}$ recursively, and show that $\mathcal{C}_1^{\mathrm{sr}} = \hat{\mathcal{C}}_1^{\mathrm{sr}}$:

**Definition 6.10** **(1)** First, an auxiliary function $\psi : \mathbf{Q} \to \mathbf{Q}$ is defined as follows: For $q \in \mathbf{Q}$,

$$\psi(q) = \begin{cases} \langle D(\Gamma) \rangle & \text{if } q = \langle \Delta(\Gamma) \rangle, \\ q & \text{otherwise.} \end{cases} \tag{6.20}$$

**(2)** For $s \in \mathcal{L}_{\bar{1}}$, let

$$\mathrm{cact}(s) = \mathrm{act}_1(s) \cap \mathbf{C}_1.$$

**(3)** The model $\hat{\mathcal{C}}_1^{\mathrm{sr}} : \mathcal{L}_{\bar{1}} \to \mathbf{P}_{\mathrm{cl}}$ is defined recursively as follows: For $s \in \mathcal{L}_{\bar{1}}[\emptyset]$,

$$\begin{aligned} \hat{\mathcal{C}}_1^{\mathrm{sr}}[\![s]\!] = &\ \mathrm{if}(\surd \in \mathrm{act}_1(s), \{\langle \surd \rangle\}, \emptyset) \\ &\cup \{\langle \Delta(\Gamma) \rangle : \tau \notin \mathrm{act}_1(s) \ \wedge\ \Gamma \cap \mathrm{act}_1(s) = \emptyset\} \\ &\cup \bigcup\{\langle a \rangle \cdot \psi[\hat{\mathcal{C}}_1^{\mathrm{sr}}[\![s']\!]] : s \xrightarrow{a}_1 s'\}. \blacksquare \end{aligned} \tag{6.21}$$

**Remark 6.1** In Definition 6.1 (3), $\hat{\mathcal{C}}_{\bar{1}}^{\mathrm{sr}}$ is defined recursively. To be precise, $\hat{\mathcal{C}}_{\bar{1}}^{\mathrm{sr}}$ is defined as the unique fixed-point of an appropriate higher-order contraction from $(\mathcal{L}_{\bar{1}} \to \mathbf{P}_{\mathrm{cl}})$ to itself; the precise definition is similar to the definition of $\hat{\mathcal{C}}_1$ in Chapter 4 (cf. Definition 4.7). ∎

**Example 6.3** Let $s$ be defined as in Example 6.2. Then

$$\hat{\mathcal{C}}_{\bar{1}}^{\mathrm{sr}}[\![s]\!] = \{\langle\Delta(\Gamma)\rangle : c_0 \notin \Gamma\} \cup \langle c_0\rangle \cdot \psi[\hat{\mathcal{C}}_{\bar{1}}^{\mathrm{sr}}[\![c_1]\!]]. \tag{6.22}$$

First, $\hat{\mathcal{C}}_{\bar{1}}^{\mathrm{sr}}[\![c_1]\!]$ is calculated by:

$$\begin{aligned}&\hat{\mathcal{C}}_{\bar{1}}^{\mathrm{sr}}[\![c_1]\!]\\ &= \{\langle\Delta(\Gamma)\rangle : c_1 \notin \Gamma\} \cup \langle c_1\rangle \cdot \psi[\{\langle\Delta(\Gamma)\rangle : \surd \notin \Gamma\}]\\ &\cup\{\langle c_1, \surd\rangle\} = \{\langle\Delta(\Gamma)\rangle : c_1 \notin \Gamma\} \cup \{\langle c_1, D(\Gamma)\rangle : \surd \notin \Gamma\} \cup \{\langle c_1, \surd\rangle\}.\end{aligned}$$

Hence,

$$\begin{aligned}&\langle c_0\rangle \cdot \psi[\hat{\mathcal{C}}_{\bar{1}}^{\mathrm{sr}}[\![c_1]\!]]\\ &= \{\langle c_0, D(\Gamma)\rangle : c_1 \notin \Gamma\} \cup \{\langle c_0, c_1, D(\Gamma)\rangle : \surd \notin \Gamma\rangle\} \cup \{\langle c_0, c_1, \surd\rangle\}.\end{aligned}$$

Thus by (6.22), one has

$$\begin{aligned}\hat{\mathcal{C}}_{\bar{1}}^{\mathrm{sr}}[\![s]\!] = &\{\langle\Delta(\Gamma)\rangle : c_0 \notin \Gamma\} \cup \{\langle c_0, D(\Gamma)\rangle : c_1 \notin \Gamma\}\\ &\cup\{\langle c_0, c_1, D(\Gamma)\rangle : \surd \notin \Gamma\rangle\} \cup \{\langle c_0, c_1, \surd\rangle\}.\end{aligned}$$

By this and the result of Example 6.2, one has

$$\hat{\mathcal{C}}_{\bar{1}}^{\mathrm{sr}}[\![s]\!] = \mathcal{C}_{\bar{1}}^{\mathrm{sr}}[\![s]\!]. \;∎$$

The two models $\mathcal{C}_{\bar{1}}^{\mathrm{sr}}$ and $\hat{\mathcal{C}}_{\bar{1}}^{\mathrm{sr}}$ coincide as expected:

**Proposition 6.2** (Nonrecursive Characterization of $\mathcal{C}_{\bar{1}}^{\mathrm{sr}}$)

$$\mathcal{C}_{\bar{1}}^{\mathrm{sr}} = \hat{\mathcal{C}}_{\bar{1}}^{\mathrm{sr}}. \;∎$$

**Proof.** It is easily checked that (6.21) with $\hat{\mathcal{C}}_{\bar{1}}^{\mathrm{sr}}$ replaced by $\mathcal{C}_{\bar{1}}^{\mathrm{sr}}$ holds, using Proposition 6.1. Thus both $\mathcal{C}_{\bar{1}}^{\mathrm{sr}}$ and $\hat{\mathcal{C}}_{\bar{1}}^{\mathrm{sr}}$ are the fixed-point of a higher-order contraction, and therefore, they coincide. ∎

### 6.3.3 A Flattened Weak Failures Model $\mathcal{C}_{\bar{1}}^{\mathrm{wf}}$

In this subsection, we define a *flattened weak failures model* $\mathcal{C}_{\bar{1}}^{\mathrm{wf}}$ from $\mathcal{C}_{\bar{1}}^{\mathrm{sr}}$ by applying a hiding function $H$.

First, we define the hiding function $H$ by:

**Definition 6.11** **(1)** For $w \in (\mathbf{A}_1)^{<\omega}$, let $(w \setminus \tau)$ denote the result of erasing $\tau$'s in $w$.

**(2)** A function $h : \mathbf{Q} \to \mathbf{Q}$ is defined as follows: Let $q \in \mathbf{Q}$. When $q = w \cdot \langle x\rangle$ with $(w, x) \in (\mathbf{A}_1)^{<\omega} \times (D[\wp(\mathbf{C}_{\bar{1}}^{\surd})] \cup \Delta[\wp(\mathbf{C}_{\bar{1}}^{\surd})] \cup \{\surd, \bot\})$, let

$$h(q) = (w \setminus \tau) \cdot \langle x\rangle;$$

otherwise $q \in (\mathbf{A}_1)^\omega$, and let

$$h(q) = \begin{cases} (\tilde{w} \setminus \tau) \cdot \langle \bot \rangle & \text{if } \exists \tilde{w}[\ w = \tilde{w} \cdot \langle \tau \rangle_{i \in \omega}\ ], \\ (w \setminus \tau) & \text{otherwise.} \end{cases}$$

**(3)** A hiding function $H : \wp(\mathbf{Q}) \to \wp(\mathbf{Q})$ is defined by:

$$H(p) = (\lambda p \in \mathbf{P}_{\text{cl}}.\ \text{mini}(h[p])). \blacksquare$$

**Example 6.4** Let $c_0, c_1 \in \mathbf{C}_1$ and put

$$p = \{\langle \Delta(\Gamma) \rangle : c_0 \notin \Gamma\} \cup \{\langle c_0, D(\Gamma) \rangle : c_1 \notin \Gamma\}$$
$$\cup \{\langle c_0, c_1, \surd \rangle\} \cup \{\langle c_0 \rangle \cdot \tau^\omega\}.$$

(It will be easy to check that $p = \mathcal{C}_1^{\text{sr}}[\![(c_0; c_1) + (c_0; (\mu X.\ \tau; X))]\!]$.) Then,

$$h[p] = \{\langle \Delta(\Gamma) \rangle : c_0 \notin \Gamma\} \cup \{\langle c_0, D(\Gamma) \rangle : c_1 \notin \Gamma\}$$
$$\cup \{\langle c_0, c_1, \surd \rangle\} \cup \{\langle c_0, \bot \rangle\}.$$

Thus,

$$H(p) = \text{mini}(h[p]) = \{\langle \Delta(\Gamma) \rangle : c_0 \notin \Gamma\} \cup \{\langle c_0, \bot \rangle\}. \blacksquare$$

**Remark 6.2** It will be easy to check that

$$\forall s \in \mathcal{L}_{\bar{1}}[\emptyset][\ \mathcal{C}_1^{\text{w}}[\![s]\!] = h[\mathcal{C}_1^{\text{sr}}[\![s]\!]\ ],$$

where $\mathcal{C}_1^{\text{w}}$ is the weak model defined in Chapter 5. Thus, $\mathcal{C}_1^{\text{wf}}$ is obtained from $\mathcal{C}_1^{\text{w}}$ by flattening:

$$\forall s \in \mathcal{L}_{\bar{1}}[\emptyset][\ \mathcal{C}_1^{\text{wf}}[\![s]\!] = \text{mini}(\mathcal{C}_1^{\text{w}}[\![s]\!])\ ]. \blacksquare$$

Then we define the *flattened weak failures model* $\mathcal{C}_1^{\text{wf}} : \mathcal{L}_{\bar{1}} \to \wp(\mathbf{Q})$ by:

**Definition 6.12** For $s \in \mathcal{L}_{\bar{1}}[\emptyset]$, let $\mathcal{C}_1^{\text{wf}}[\![s]\!] = H(\mathcal{C}_1^{\text{sr}}[\![s]\!])$. $\blacksquare$

The range of $\mathcal{C}_1^{\text{wf}}$ is contained in $\mathbf{P}_{\text{cl}}$:

**Proposition 6.3** $\forall s \in \mathcal{L}_{\bar{1}}[\emptyset][\ \mathcal{C}_1^{\text{wf}}[\![s]\!] \in \mathbf{P}_{\text{cl}}\ ]$. $\blacksquare$

### 6.3.4 Completeness of $\mathcal{C}_1^{\text{wf}}$ w.r.t. $\mathcal{O}_1^{\text{mf}}$

In this subsection, we will show that *the equivalence relation induced by $\mathcal{C}_1^{\text{wf}}$ is coarser than the congruence induced by $\mathcal{O}_1^{\text{mf}}$, provided that the set $\mathbf{C}_1$ of communication actions is infinite.* That is,

**Lemma 6.4** *If $\mathbf{C}_1$ is infinite, then*

$$\begin{array}{l} \forall s_0, s_1 \in \mathcal{L}_{\bar{1}}[\emptyset][ \\ \quad \forall X, \forall S \in \mathcal{L}_{\bar{1}}[X][\ \mathcal{O}_1^{\text{mf}}[\![S[s_0/X]]\!] = \mathcal{O}_1^{\text{mf}}[\![S[s_1/X]]\!]\ ] \\ \quad \Rightarrow \quad \mathcal{C}_1^{\text{wf}}[\![s_0]\!] = \mathcal{C}_1^{\text{wf}}[\![s_1]\!]\ ]. \blacksquare \end{array} \tag{6.23}$$

We call this property *the completeness of* $\mathcal{C}_1^{\mathrm{wf}}$ *w.r.t.* $\mathcal{O}_1^{\mathrm{mf}}$ *(or w.r.t.* $\equiv_1^{\mathrm{mf}}$*)*, following the terminology of Apt and Plotkin (cf. [AP 86] § 3.3).

For the proof of Lemma 6.4, we need a few preliminaries.

**Definition 6.13** For $s \in \mathcal{L}_{\bar{1}}[\emptyset]$, let

$$\mathcal{S}(s) = \{c \in \mathbf{C}_1 : \exists w \in (\mathbf{A}_1)^{<\omega}, \exists s'[\ s \xrightarrow{w}_{1*} s' \land c \in \mathrm{act}_1(s')\ ]\},$$

and $\mathcal{S}^{\pm}(s) = \mathcal{S}(s) \cup \overline{\mathcal{S}(s)}$. ∎

**Example 6.5** Let $s$ be defined as in Example 6.1. Then, $\mathcal{S}(s) = \{c_0, c_1\}$, and $\mathcal{S}^{\pm}(s) = \{c_0, c_1, \overline{c_0}, \overline{c_1}\}$. ∎

**Definition 6.14** Let $p \in \wp(\mathbf{Q})$.

**(1)** $\tilde{\mathcal{S}}(p) = \{c \in \mathbf{C}_1 : \exists w \in (\mathbf{A}_1)^{<\omega}[\ c \in \mathrm{act}(p[w])\ ]\}$.

**(2)** $\tilde{\mathcal{T}}_{\perp}(p) = p \cap ((\mathbf{A}_1)^{<\omega} \cdot \{\langle \perp \rangle\})$.

**(3)** $\tilde{\mathcal{T}}_{\surd}(p) = p \cap ((\mathbf{A}_1)^{<\omega} \cdot \{\langle \surd \rangle\})$.

**(4)** $\tilde{\mathcal{T}}_{\omega}(p) = p \cap (\mathbf{A}_1)^{\omega}$.

**(5)** $\tilde{\mathcal{F}}(p) = \{w \cdot \langle \delta(\Gamma) \rangle : w \cdot \langle D(\Gamma) \rangle \in p \lor w \cdot \langle \Delta(\Gamma) \rangle \in p\}$.

**(6)** $\tilde{\mathcal{R}}(p) = p \cap \{\langle \Delta(\Gamma) \rangle : \Gamma \in \wp(\mathbf{C}_1^{\surd})\}$.

**(7)** $$\mathcal{A}_1^{\mathrm{wf}}(p) = \tilde{\mathcal{T}}_{\perp}(p) \cup \tilde{\mathcal{T}}_{\surd}(p) \cup \tilde{\mathcal{T}}_{\omega}(p) \cup \{w \cdot \langle\ \delta\ \rangle : w \cdot \langle \delta(\mathbf{C}_1^{\surd}) \rangle \in \tilde{\mathcal{F}}(p)\}.$$ ∎

Then, the next proposition immediately follows from the definitions of $\mathcal{O}_1^{\mathrm{mf}}$ and $\mathcal{C}_1^{\mathrm{wf}}$.

**Proposition 6.4** *For* $s \in \mathcal{L}_{\bar{1}}[\emptyset]$, $\mathcal{O}_1^{\mathrm{mf}}[\![s]\!] = \mathcal{A}_1^{\mathrm{wf}}(\mathcal{C}_1^{\mathrm{wf}}[\![s]\!])$. ∎

**Proof of Lemma 6.4** Suppose $\mathbf{C}_1$ is infinite. Obviously (6.23) is equivalent to the following:

$$\begin{array}{l}\forall s_0, s_1 \in \mathcal{L}_{\bar{1}}[\emptyset][\ \mathcal{C}_1^{\mathrm{wf}}[\![s_0]\!] \neq \mathcal{C}_1^{\mathrm{wf}}[\![s_1]\!] \Rightarrow \\ \quad \exists X, \exists S \in \mathcal{L}_{\bar{1}}[X][\ \mathcal{O}_1^{\mathrm{mf}}[\![S[s_0/X]]\!] \neq \mathcal{O}_1^{\mathrm{mf}}[\![S[s_1/X]]\!]\ ]].\end{array} \tag{6.24}$$

We will prove this. Let $s_0, s_2 \in \mathcal{L}_{\bar{1}}[\emptyset]$ such that $\mathcal{C}_1^{\mathrm{wf}}[\![s_0]\!] \neq \mathcal{C}_1^{\mathrm{wf}}[\![s_1]\!]$. Let us fix $X \in \mathcal{X}_{\mathcal{P}}$. It suffices to show that

$$\exists S \in \mathcal{L}_{\bar{1}}[X][\ \mathcal{O}_1^{\mathrm{mf}}[\![S[s_0/X]]\!] \neq \mathcal{O}_1^{\mathrm{mf}}[\![S[s_1/X]]\!]\ ]. \tag{6.25}$$

Since $\mathcal{C}_1^{\mathrm{wf}}[\![s_0]\!] \neq \mathcal{C}_1^{\mathrm{wf}}[\![s_1]\!]$, one the following propositions (6.26)–(6.30) holds:

$$\tilde{\mathcal{T}}_{\perp}(\mathcal{C}_1^{\mathrm{wf}}[\![s_0]\!]) \neq \tilde{\mathcal{T}}_{\perp}(\mathcal{C}_1^{\mathrm{wf}}[\![s_1]\!]), \tag{6.26}$$

$$\tilde{\mathcal{T}}_{\surd}(\mathcal{C}_1^{\mathrm{wf}}[\![s_0]\!]) \neq \tilde{\mathcal{T}}_{\surd}(\mathcal{C}_1^{\mathrm{wf}}[\![s_1]\!]), \tag{6.27}$$

$$\tilde{\mathcal{T}}_{\omega}(\mathcal{C}_1^{\mathrm{wf}}[\![s_0]\!]) \neq \tilde{\mathcal{T}}_{\omega}(\mathcal{C}_1^{\mathrm{wf}}[\![s_1]\!]), \tag{6.28}$$

$$\tilde{\mathcal{F}}(\mathcal{C}_1^{\mathrm{wf}}[\![s_0]\!]) \neq \tilde{\mathcal{F}}(\mathcal{C}_1^{\mathrm{wf}}[\![s_1]\!]), \tag{6.29}$$

$$\tilde{\mathcal{R}}(\mathcal{C}_1^{\mathrm{wf}}[\![s_0]\!]) \neq \tilde{\mathcal{R}}(\mathcal{C}_1^{\mathrm{wf}}[\![s_1]\!]). \tag{6.30}$$

In the cases (6.26), (6.27), or (6.28) holds, (6.25) follows immediately from Proposition 6.4. Let us consider the other two cases.

**Case 1.** Suppose (6.29) holds. Then, we can assume, without loss of generality, that there are $w \in (\mathbf{C}_1)^{<\omega}$ and $\Gamma \in \wp(\mathbf{C}_1^{\checkmark})$ such that

$$w \cdot \langle \delta(\Gamma) \rangle \in \tilde{\mathcal{F}}(\mathcal{C}_1^{\mathrm{wf}}[\![s_0]\!]) \land w \cdot \langle \delta(\Gamma) \rangle \notin \tilde{\mathcal{F}}(\mathcal{C}_1^{\mathrm{wf}}[\![s_1]\!]). \tag{6.31}$$

Let $\Gamma' = \Gamma \cap (\mathcal{S}(s_0) \cup \mathcal{S}(s_1))$. Then, obviously $\Gamma'$ is finite. Further, by (6.31) and the definition of $\mathcal{C}_1^{\mathrm{wf}}[\![s_0]\!]$, one has

$$\text{(i) } w \cdot \langle \delta(\Gamma') \rangle \in \tilde{\mathcal{F}}(\mathcal{C}_1^{\mathrm{wf}}[\![s_0]\!]), \quad \text{(ii) } w \cdot \langle \delta(\Gamma') \rangle \notin \tilde{\mathcal{F}}(\mathcal{C}_1^{\mathrm{wf}}[\![s_1]\!]). \tag{6.32}$$

We distinguish two cases according to whether $\surd \in \Gamma'$ or not.

**Subcase 1.1.** Suppose $\surd \in \Gamma'$. Since $\Gamma'$ is finite, one can put

$$\Gamma' = \{c_0, \ldots, c_{n-1}, \surd\} \tag{6.33}$$

for some $c_0, \ldots, c_{n-1} \in \mathbf{C}_1$. Also, $w = \langle c_0', \ldots, c_{k-1}' \rangle$ for some $k \in \omega$ and $c_0', \ldots, c_{k-1}' \in \mathbf{C}_1$. Fix $c \in \mathbf{C}_1$ so that

$$c \notin (\mathcal{S}^{\pm}(s_0) \cup \mathcal{S}^{\pm}(s_1)), \tag{6.34}$$

and put

$$\hat{s} \equiv (\overline{c_0'} + c); \cdots; (\overline{c_{k-1}'} + c); \mathbf{e}.$$

Finally, put $C = (\mathcal{S}(s_0) \cup \mathcal{S}(s_1)) \setminus \{c_0, \cdots, c_{n-1}\}$. We will show that

$$\text{(i) } \langle \delta \rangle \in \mathcal{O}_1^{\mathrm{mf}}[\![\partial_C(s_0 \| \hat{s})]\!], \quad \text{(ii) } \langle \delta \rangle \notin \mathcal{O}_1^{\mathrm{mf}}[\![\partial_C(s_1 \| \hat{s})]\!]. \tag{6.35}$$

First, let us show (6.35) (i). It follows straightforwardly from (6.32) (i) that

$$\langle \delta \rangle \in h_1[\mathcal{O}_1^{\mathrm{m}}[\![\partial_C(s_0 \| \hat{s})]\!]]. \tag{6.36}$$

Thus, it suffices, for establishing (6.35) (i), to show that

$$\langle \bot \rangle \notin h_1[\mathcal{O}_1^{\mathrm{m}}[\![\partial_C(s_0 \| \hat{s})]\!]]. \tag{6.37}$$

We will show (6.37) by contradiction.

Assume, for the sake of contradiction, that (6.37) does not hold, i.e., that $\langle \bot \rangle \in h_1[\mathcal{O}_1^{\mathrm{m}}[\![\partial_C(s_0 \| \hat{s})]\!]]$. Then, from the form of $\hat{s}$, it follows that for some $i \in k$,

$$\langle c_0', \ldots, c_i', \bot \rangle \in h_1[\mathcal{O}_1^{\mathrm{m}}[\![s_0]\!]].$$

This is, however, impossible by (6.32) (i). Thus (6.37) must hold. Summarizing the above, one has (6.35) (i).

Next, let us show (6.35) (ii). Assume, for the sake of contradiction, that (6.35) (ii) does not hold, i.e., that

$$\langle \delta \rangle \in \mathcal{O}_1^{\mathrm{mf}}[\![\partial_C(s_1 \| \hat{s})]\!]. \tag{6.38}$$

Then, by the definition of $\mathcal{O}_1^{\mathrm{mf}}$, there is $s' \in \mathcal{L}_{\bar{1}}[\emptyset]$ such that

$$\text{(i) } s_1 \| \hat{s} \stackrel{\epsilon}{\Longrightarrow}_1 s', \quad \text{(ii) } \mathrm{act}_1(s') \subseteq C. \tag{6.39}$$

By (6.39) (i), there are $i \in k$ and $s_1'$ such that

$$s_1 \xrightarrow{\langle c_0', \cdots, c_i' \rangle}_1 s_1' \wedge s' \equiv s_1' \| ((\overline{c_{i+1}'} + c); \cdots ; (\overline{c_{k-1}'} + c); \mathrm{e}). \tag{6.40}$$

By (6.39) (ii), $i$ must be $k-1$. Thus, $s' \equiv s_1' \| \mathrm{e}$, and therefore, $\mathrm{act}_1(s') = \mathrm{act}_1(s_1')$. By this and (6.39) (ii), one has $\mathrm{act}_1(s_1') \subseteq C$, and therefore,

$$\tau \notin \mathrm{act}_1(s_1'), \quad \mathrm{act}_1(s_1') \cap \Gamma' = \emptyset. \tag{6.41}$$

By (6.40) and (6.41), one has $w \cdot \langle \delta(\Gamma') \rangle \in \tilde{\mathcal{F}}(h[\mathcal{C}_1^{\mathrm{sr}}[\![s_1]\!]])$. Further there is no $w' \leq_{\mathrm{p}} w$ such that $w' \cdot \langle \perp \rangle \in h[\mathcal{C}_1^{\mathrm{sr}}[\![s_1]\!]]$, because if there exists such a sequence $w'$, then one has $\tau^\ell \cdot \langle \perp \rangle \in \mathcal{C}_1^{\mathrm{sr}}[\![s_1 \| \hat{s}]\!]$ with $\ell = \mathrm{lgt}(w')$, which contradicts (6.38). Thus, one has

$$w \cdot \langle \delta(\Gamma') \rangle \in \tilde{\mathcal{F}}(\mathrm{mini}(h[\mathcal{C}_1^{\mathrm{sr}}[\![s_1]\!]])) = \tilde{\mathcal{F}}(H(\mathcal{C}_1^{\mathrm{sr}}[\![s_1]\!])) = \tilde{\mathcal{F}}(\mathcal{C}_1^{\mathrm{wf}}[\![s_1]\!]).$$

This contradicts (6.32) (ii). Thus, (6.35) (ii) must hold.

Summarizing the above, one has (6.35).

**Subcase 1.2.** Suppose the case $\surd \notin \Gamma'$. Then, putting $\tilde{s}_j \equiv s_j \| \mathbf{0}$ ($j \in 2$) and $\tilde{\Gamma}' = \Gamma' \cup \{\surd\}$, one has $\surd \in \tilde{\Gamma}'$ and

$$\text{(i) } w \cdot \langle \delta(\tilde{\Gamma}') \rangle \in \tilde{\mathcal{F}}(\mathcal{C}_1^{\mathrm{wf}}[\![\tilde{s}_0]\!]), \quad \text{(ii) } w \cdot \langle \delta(\tilde{\Gamma}') \rangle \notin \tilde{\mathcal{F}}(\mathcal{C}_1^{\mathrm{wf}}[\![\tilde{s}_1]\!]).$$

Thus, this subcase is reduced to Subcase 1.1 by taking $\tilde{s}_j$ as $s_j$ in Subcase 1.1 ($j \in 2$).

**Case 2** Suppose that (6.29) does not hold, and (6.30) holds. Then, we can assume, without loss of generality, that

$$\tilde{\mathcal{R}}(\mathcal{C}_1^{\mathrm{wf}}[\![s_1]\!]) \setminus \tilde{\mathcal{R}}(\mathcal{C}_1^{\mathrm{wf}}[\![s_0]\!]) \neq \emptyset. \tag{6.42}$$

Thus, one has

$$\text{(i) } \tau \in \mathrm{act}_1(s_0), \quad \text{(ii) } \tau \notin \mathrm{act}_1(s_1). \tag{6.43}$$

Since (6.29) does not hold,

$$\tilde{\mathcal{F}}(\mathcal{C}_1^{\mathrm{wf}}[\![s_0]\!]) = \tilde{\mathcal{F}}(\mathcal{C}_1^{\mathrm{wf}}[\![s_1]\!]). \tag{6.44}$$

By (6.42), $\tilde{\mathcal{R}}(\mathcal{C}_1^{\mathrm{wf}}[\![s_1]\!]) \neq \emptyset$. Thus, by (6.43) (i) and (6.44), there exists $s_0'$ such that

$$s_0 \xrightarrow{\tau}_1 \stackrel{\epsilon}{\Longrightarrow}_1 s_0' \wedge \tau \notin \mathrm{act}_1(s_0').$$

Let us choose $c \in \mathbf{C}_1$ so that $c \notin \mathcal{S}^{\pm}(s_0) \cup \mathcal{S}^{\pm}(s_1)$. Then, one has

$$\langle \delta(\{c\}) \rangle \in \tilde{\mathcal{F}}(\mathcal{C}_1^{\mathrm{wf}}[\![s_0 + c]\!]),$$

because $(s_0 + c) \overset{\epsilon}{\Rightarrow}_1 s'_0$ with $\tau \notin \mathrm{act}_1(s'_0)$ and $c \notin \mathrm{act}_1(s'_0)$. Also,

$$\langle \delta(\{c\}) \rangle \notin \tilde{\mathcal{F}}(\mathcal{C}_1^{\mathrm{wf}}[\![s_1 + c]\!]),$$

because for every $s'$, if $(s_1 + c) \overset{\epsilon}{\Rightarrow} s'$ then $s' \equiv (s_1 + c)$, and therefore, $c \in \mathrm{act}_1(s')$. Thus, this case is reduced to Case 1 by taking $(s_j + c)$ as $s_j$ in Case 1 $(j \in 2)$.

Thus in all cases, one has (6.25). ∎

# 6.4 Metric and Order-Theoretic Models

## 6.4.1 Definition of Semantic Operations

In this subsection, we define a semantic operation

$$\widetilde{\mathsf{op}} : (\mathbf{P}_{\mathrm{cl}})^r \to \mathbf{P}_{\mathrm{cl}}$$

for each combinator $\mathsf{op} \in \mathbf{Fun}_1$ with arity $r$.

First, we define interpretations of constant symbols of $\mathcal{L}_{\bar{1}}$ simply as their meanings under $\mathcal{C}_1^{\mathrm{sr}}$:

**Definition 6.15** For $\mathsf{op} \in \mathbf{Fun}_1^{(0)} = \{\mathbf{0}, \mathsf{e}\} \cup \{a : a \in \mathbf{A}_1\}$, let $\widetilde{\mathsf{op}} = \mathcal{C}_1^{\mathrm{sr}}[\![\mathsf{op}]\!]$. ∎

### Semantic Operation for Alternative Choice

We define a semantic operation $\tilde{+}$ corresponding to the combinator '+' for choice with several auxiliary functions:

**Definition 6.16 (1)** For $p \in \wp(\mathbf{Q})$, let

$$\tilde{D}(p) = p \cap (D[\wp(\mathbf{C}_1^{\checkmark})])^1, \quad \tilde{\Delta}(p) = p \cap (\Delta[\wp(\mathbf{C}_1^{\checkmark})])^1.$$

**(2)** For $p_1, p_2 \in \wp(\mathbf{Q})$, let

$$\begin{aligned} \tilde{+}_{\mathrm{b}}(p_1, p_2) = {} & (\mathcal{B}(p_1) \setminus \tilde{\Delta}(p_1)) \,\hat{\cup}\, (\mathcal{B}(p_2) \setminus \tilde{\Delta}(p_2)) \\ & \hat{\cup}\, (\tilde{\Delta}(p_1) \cap \tilde{\Delta}(p_2)), \end{aligned}$$

where $\hat{\cup}$ is the operation defined in Definition 2.12.

**(3)** For $p_1, p_2 \in \wp(\mathbf{Q})$, let

$$\tilde{+}(p_1, p_2) = \tilde{+}_{\mathrm{b}}(p_1, p_2) \,\hat{\cup}\, \oplus (p_1) \,\hat{\cup}\, \oplus (p_2). \blacksquare$$

The following proposition is easily obtained from the above definition of $\tilde{+}$:

**Proposition 6.5** $\forall p_1, p_2 \in \mathbf{P}_{\mathrm{cl}}[\ p_1 \tilde{+} p_2 \in \mathbf{P}_{\mathrm{cl}}\ ]$.[1] ∎

[1] We use binary semantic operations $\widetilde{\mathsf{op}}$ both in prefix form $\widetilde{\mathsf{op}}(p_0, p_1)$ and in infix form $p_0 \ \widetilde{\mathsf{op}}\ p_1$ $(\mathsf{op} \in \mathbf{Fun}_1^{(2)})$.

**Definition of Semantic Operation for Parallel Composition**

First we state a few useful properties of the domain $\mathbf{P}_{\mathrm{cl}}$.

The domain $\mathbf{P}_{\mathrm{cl}}$ is closed under the operation $\hat{\cup}$:

**Proposition 6.6** $\forall p_1, p_2 \in \mathbf{P}_{\mathrm{cl}}[\ p_1 \hat{\cup} p_2 \in \mathbf{P}_{\mathrm{cl}}\ ].$ ∎

The domain $\mathbf{P}_{\mathrm{cl}}$ is closed uder taking remainders:

**Proposition 6.7** $\forall p \in \mathbf{P}_{\mathrm{cl}}, \forall w \in (\mathbf{A}_1)^{<\omega}[\ p[w] \neq \emptyset \Rightarrow p[w] \in \mathbf{P}_{\mathrm{cl}}\ ].$ ∎

From this, we obtain the following:

**Proposition 6.8** $\forall p \in \tilde{\mathbf{P}}_{\mathrm{cl}}, \forall a \in \mathrm{act}(p)[\ p[\langle a\rangle] \in \mathbf{P}_{\mathrm{cl}}\ ].$ ∎

As a preliminary to the definition of the semantic operation $\tilde{\|}$ corresponding to $\|$, we introduce a higher-order mapping $\Omega_{\|}$; the operation $\tilde{\|}$ is defined as the unique fixed-point of $\Omega_{\|}$.

**Definition 6.17** **(1)** For $p_1, p_2 \in \wp(\mathbf{Q})$, let

**(i)** $\tilde{\|}_{\perp}(p_1, p_2) = \{\langle\perp\rangle\} \cap (p_1 \cup p_2)$,

**(ii)** $\tilde{\|}_{\surd}(p_1, p_2) = \{\langle\surd\rangle\} \cap (p_1 \cap p_2)$,

**(iii)** $\tilde{\|}_D(p_0, p_1)$
$$= \{\langle D(\Gamma)\rangle : \bigvee_{(j,k)\in\{(0,1),(1,0)\}}[\ \exists\langle D(\Gamma_1)\rangle \in p_j, \exists\langle\delta(\Gamma_2)\rangle \in \tilde{\delta}(p_k)[$$
$$((\Gamma \setminus \{\surd\} \subseteq \Gamma_1 \wedge \Gamma \subseteq \Gamma_2) \vee (\Gamma \subseteq \Gamma_1 \wedge \Gamma \setminus \{\surd\} \subseteq \Gamma_2))$$
$$\wedge (\mathbf{C}_1 \setminus \Gamma_1) \cap \overline{(\mathbf{C}_1 \setminus \Gamma_2)} = \emptyset\ ]]\},$$

**(iv)** $\tilde{\|}_{\Delta}(p_1, p_2)$
$$= \{\langle\Delta(\Gamma)\rangle : \Gamma \in \wp(\mathbf{C}_1) \wedge \exists\langle\Delta(\Gamma_1)\rangle \in p_1, \exists\langle\Delta(\Gamma_2)\rangle \in p_2[$$
$$((\Gamma \setminus \{\surd\} \subseteq \Gamma_1 \wedge \Gamma \subseteq \Gamma_2) \vee (\Gamma \subseteq \Gamma_1 \wedge \Gamma \setminus \{\surd\} \subseteq \Gamma_2))$$
$$\wedge (\mathbf{C}_1 \setminus \Gamma_1) \cap \overline{(\mathbf{C}_1 \setminus \Gamma_2)} = \emptyset\ ]\},$$

**(v)** $\tilde{\|}_{\mathrm{b}}(p_1, p_2) = \tilde{\|}_{\perp}(p_1, p_2) \hat{\cup} \tilde{\|}_{\surd}(p_1, p_2) \hat{\cup} \tilde{\|}_D(p_1, p_2) \hat{\cup} \tilde{\|}_{\Delta}(p_1, p_2)$.

**(2)** A mapping $\Omega_{\|} : ((\mathbf{P}_{\mathrm{cl}})^2 \to \mathbf{P}_{\mathrm{cl}}) \to ((\mathbf{P}_{\mathrm{cl}})^2 \to \wp(\mathbf{Q}))$ is defined as follows: For $F \in ((\mathbf{P}_{\mathrm{cl}})^2 \to \mathbf{P}_{\mathrm{cl}})$ and $p_1, p_2 \in \mathbf{P}_{\mathrm{cl}}$, let

$$\tilde{\Omega}_{\|}(F)(p_1, p_2) = \tilde{\|}_{\mathrm{b}}(p_1, p_2) \hat{\cup} \Omega^{+}_{\|}(F)(p_1, p_2),$$

where $\Omega^{+}_{\|}(F)(p_1, p_2)$ is defined by:

$$\Omega^{+}_{\|}(F)(p_1, p_2) = \Omega_{\|\!\!\lfloor}(F)(p_1, p_2) \hat{\cup} \Omega_{\|\!\!\lfloor}(F)(p_2, p_1) \hat{\cup} \Omega_{|}(F)(p_2, p_1),$$

with

$$\Omega_{\|\!\!\lfloor}(F)(p_1, p_2) = \bigcup\{\langle a\rangle \cdot \psi[F(p_1[\langle a\rangle], p_2)] : a \in \mathrm{act}(p_1)\}$$

and

$$\Omega_{|}(F)(p_1, p_2)$$
$$= \hat{\bigcup}\{\langle\tau\rangle \cdot \psi[F(p_1[\langle c\rangle], p_2[\langle\bar{c}\rangle])] : c \in \mathrm{act}(p_1) \wedge \bar{c} \in \mathrm{act}(p_2)\}. \blacksquare$$

Then, $((\mathbf{P}_{\mathrm{cl}})^2 \to \mathbf{P}_{\mathrm{cl}})$ is closed under $\Omega_{\|}$:

**Lemma 6.5** $\Omega_{\|}[((\mathbf{P}_{\mathrm{cl}})^2 \to \mathbf{P}_{\mathrm{cl}})] \subseteq ((\mathbf{P}_{\mathrm{cl}})^2 \to \mathbf{P}_{\mathrm{cl}})$. $\blacksquare$

Thus,

$$\Omega_{\|} : ((\mathbf{P}_{\mathrm{cl}})^2 \to \mathbf{P}_{\mathrm{cl}}) \to ((\mathbf{P}_{\mathrm{cl}})^2 \to \mathbf{P}_{\mathrm{cl}}).$$

Moreover, the mapping $\Omega_{\|}$ has the following useful properties:

**Lemma 6.6 (1)** *The mapping $\Omega_{\|}$ preserves nonexpansiveness, i.e., for every $F \in ((\mathbf{P}_{\mathrm{cl}})^2 \to \mathbf{P}_{\mathrm{cl}})$, if $F$ is nonexpansive, then $\Omega_{\|}(F)$ is also nonexpansive.*

**(2)** *The mapping $\Omega_{\|}$ preserves monotonicity w.r.t. $\sqsubseteq_{\mathrm{s}}$, i.e., for every $F \in ((\mathbf{P}_{\mathrm{cl}})^2 \to \mathbf{P}_{\mathrm{cl}})$, if $F$ is monotonic w.r.t. $\sqsubseteq_{\mathrm{s}}$, then $\Omega_{\|}(F)$ is also monotonic w.r.t. $\sqsubseteq_{\mathrm{s}}$.*

**(3)** *The mapping $\Omega_{\|}$ is a contraction: More specifically, for every $F_1, F_2 \in ((\mathbf{P}_{\mathrm{cl}})^2 \to \mathbf{P}_{\mathrm{cl}})$,*

$$\tilde{d}_{\mathrm{F}}(\Omega_{\|}(F_1), \Omega_{\|}(F_2)) \leq \kappa \cdot \tilde{d}_{\mathrm{F}}(F_1, F_2),$$

*where $\tilde{d}_{\mathrm{F}}$ is a metric on $((\mathbf{P}_{\mathrm{cl}})^2 \to \mathbf{P}_{\mathrm{cl}})$ defined in Chapter 2, and $\kappa$ is the positive real number introduced just after Definition 2.1.* $\blacksquare$

We define $\tilde{\|}$ as the unique fixed-point of the contraction $\Omega_{\|}$:

**Definition 6.18** (Semantic Operation for Parallel Composition)

**(1)** $\tilde{\|} = \mathrm{fix}(\Omega_{\|})$.

**(2)** $\tilde{\|\!\_} = \Omega_{\|\!\_}(\tilde{\|}) = (\lambda(p_1, p_2) \in (\mathbf{P}_{\mathrm{cl}})^2.\ \hat{\bigcup}\{\langle a\rangle \cdot (p_1[\langle a\rangle]\tilde{\|}p_2) : a \in \mathrm{act}(p_1)\})$.

**(3)** $\tilde{|} = \Omega_{|}(\tilde{\|}) = (\lambda(p_1, p_2) \in (\mathbf{P}_{\mathrm{cl}})^2.$
$\hat{\bigcup}\{\langle\tau\rangle \cdot (p_1[\langle c\rangle]\tilde{\|}p_2[\langle\bar{c}\rangle]) : c \in \mathrm{act}(p_1) \wedge \bar{c} \in \mathrm{act}(p_2)\})$. $\blacksquare$

Then, we have the following proposition by definition:

**Proposition 6.9** *For every $p_1, p_2 \in \mathbf{P}_{\mathrm{cl}}$,*

$$p_1 \tilde{\|} p_2 = \tilde{\|}_{\mathrm{b}}(p_1, p_2) \,\hat{\cup}\, \tilde{\|\!\_}(p_1, p_2) \,\hat{\cup}\, \tilde{\|\!\_}(p_2, p_1) \,\hat{\cup}\, \tilde{|}(p_1, p_2). \blacksquare$$

Then, we obtain the following two lemmas using Lemma 6.6 (1) and (2):

**Lemma 6.7** *The mapping $\tilde{\|}$ is nonexpansive.* $\blacksquare$

**Proof.** Let $F = (\lambda(p_1, p_2) \in (\mathbf{P}_{\mathrm{cl}})^2.\ \{\langle\perp\rangle\})$, and for $n \in \omega$, let $F_n = (\Omega_{\|})^n(F)$, where $(\Omega_{\|})^n$ is the $n$-th iteration of $\Omega_{\|}$. Then, by Definition 6.18 and Banach's Fixed-Point Theorem (cf. Chapter 2), one has $\tilde{\|} = \lim_{n\in\omega}[(\Omega_{\|})^n(F)]$. It is easily verified that *the set of nonexpansive functions is closed in* $((\mathbf{P}_{\mathrm{cl}})^2 \to \mathbf{P}_{\mathrm{cl}})$. From this and the fact that $F$ is nonexpansive, it follows that $\tilde{\|} = \lim_{n\in\omega}[(\Omega_{\|})^n(F)]$ is also nonexpansive. $\blacksquare$

**Lemma 6.8** *The mapping* $\tilde{\|}$ *is monotonic w.r.t.* $\sqsubseteq_s$. ∎

**Proof.** First, let us show that

$$\text{the set of monotonic functions is closed in } ((\mathbf{P}_{\text{cl}})^2 \to \mathbf{P}_{\text{cl}}). \tag{6.45}$$

(From this, the desired result follows easily as in Lemma 6.7.)

Let $\langle G_n \rangle_{n \in \omega}$ be a converging sequence of elements of $((\mathbf{P}_{\text{cl}})^2 \to \mathbf{P}_{\text{cl}})$ such that $\forall n \in \omega[\ G_n$ is monotonic ]; let $G = \lim_{n\in\omega}[G_n]$. Let us show that $G$ is also monotonic.

Let $p_1, p_1', p_2, p_2' \in \mathbf{P}_{\text{cl}}$ such that $p_1 \sqsubseteq_s p_2$ and $p_1' \sqsubseteq_s p_2'$. It suffices to show

$$G(p_1, p_1') \sqsubseteq_s G(p_2, p_2'). \tag{6.46}$$

For every $n \in \omega$, one has $G_n(p_1, p_1') \sqsubseteq_s G_n(p_2, p_2')$, because $G_n$ is monotonic. From this and Lemma 2.5 (3), it follows that

$$G(p_1, p_1') = \lim_{n\in\omega}[G_n(p_1, p_1')] \sqsubseteq_s \lim_{n\in\omega}[G_n(p_2, p_2')] = G(p_2, p_2'),$$

i.e., (6.46) holds. Thus, $G$ is also monotonic. Summarizing the above, one has (6.45).

Putting $F = (\lambda(p_1, p_2) \in (\mathbf{P}_{\text{cl}})^2.\ \{\langle\bot\rangle\})$ as in the proof of Lemma 6.7, one has $\tilde{\|} = \lim_{n\in\omega}[(\Omega_{\|})^n(F)]$. From (6.45) and the fact that $F$ is monotonic, it follows that $\tilde{\|} = \lim_{n\in\omega}[(\Omega_{\|})^n(F)]$ is also monotonic. ∎

### Definition of Semantic Operation for Sequential Composition

We define a semantic operation $\tilde{;}$ corresponding to the combinator ';' for sequential composition with several auxiliary functions:

**Definition 6.19 (1)** For $p_1, p_2 \in \wp(\mathbf{Q})$, let

$$\tilde{;}_{\bot}(p_1, p_2) = \begin{cases} \{\langle\bot\rangle\} & \text{if } \langle\bot\rangle \in p_1 \ \vee\ (\langle\surd\rangle \in p_1 \ \wedge\ \langle\bot\rangle \in p_2), \\ \emptyset & \text{otherwise.} \end{cases}$$

**(2)** For $p_1, p_2 \in \wp(\mathbf{Q})$, let

$$\tilde{;}_{\surd}(p_1, p_2) = \begin{cases} \{\langle\surd\rangle\} & \text{if } \langle\surd\rangle \in p_1 \ \wedge\ \langle\surd\rangle \in p_2, \\ \emptyset & \text{otherwise.} \end{cases}$$

**(3)** For $p_1, p_2 \in \wp(\mathbf{Q})$, let

$$\begin{aligned} \tilde{;}_{D}(p_1, p_2) = \{\langle D(\Gamma)\rangle : &\ \langle D(\Gamma \cup \{\surd\})\rangle \in p_1 \\ &\vee (\langle D(\Gamma \setminus \{\surd\})\rangle \in p_1 \ \wedge\ \langle\Delta(\Gamma)\rangle \in p_2) \\ &\vee (\langle\surd\rangle \in p_1 \ \wedge\ D(\Gamma) \in p_2)\ \}. \end{aligned}$$

**(4)** For $p_1, p_2 \in \wp(\mathbf{Q})$, let

$$\begin{aligned} \tilde{;}_{\Delta}(p_1, p_2) = \{\langle\Delta(\Gamma)\rangle : &\ \langle\Delta(\Gamma \cup \{\surd\})\rangle \in p_1 \ \vee \\ &(\langle\Delta(\Gamma \setminus \{\surd\})\rangle \in p_1 \ \wedge\ \langle\Delta(\Gamma)\rangle \in p_2)\}. \end{aligned}$$

**(5)** For $p_1, p_2 \in \wp(\mathbf{Q})$, let

$$\tilde{;}_{\mathrm{b}}(p_1,p_2) = \tilde{;}_{\perp}(p_1,p_2) \,\hat{\cup}\, \tilde{;}_{\surd}(p_1,p_2) \,\hat{\cup}\, \tilde{;}_{D}(p_1,p_2) \,\hat{\cup}\, \tilde{;}_{\Delta}(p_1,p_2).$$

**(6)** For $p_1, p_2 \in \mathbf{P}_{\mathrm{cl}}$, let

$$\tilde{;}'_{+}(p_1,p_2) = \begin{cases} \oplus(p_2) & \text{if } \langle\surd\rangle \in p_1, \\ \emptyset & \text{otherwise.} \end{cases}$$ ∎

**(7)** A function $\tilde{;} : (\mathbf{P}_{\mathrm{cl}})^2 \to \mathbf{P}_{\mathrm{cl}}$ is defined as follows: For $p_1, p_2 \in \mathbf{P}_{\mathrm{cl}}$,

$$\begin{aligned} &\tilde{;}(p_1,p_2) \\ &= \tilde{;}_{\mathrm{b}}(p_1,p_2) \,\hat{\cup}\, \bigcup_{a \in \mathrm{act}(p_1)} [\langle a\rangle \cdot \psi[\tilde{;}(p_1[\langle a\rangle], p_2)]] \,\hat{\cup}\, \tilde{;}'_{+}(p_1,p_2). \end{aligned}$$ ∎

The above definition of $\tilde{;}$ is recursive, since $\tilde{;}(p_1,p_2)$ is defined in terms of $\tilde{;}(p_1[\langle a\rangle], p_2)$; rigorously the operation $\tilde{;}$ is defined as the unique fixed-point of an appropriate higher-order mapping as $\tilde{\|}$ has been defined in Definition 6.18.

### Definition of Semantic Operation for Action Restriction

For each $C \in \wp(\mathbf{C}_1)$, we define a semantic operation $\tilde{\partial}_C : \mathbf{P}_{\mathrm{cl}} \to \mathbf{P}_{\mathrm{cl}}$ corresponding to the combinator '$\partial_C$' by:

**Definition 6.20** Let $C \in \wp(\mathbf{C}_1)$.

**(1)** For $p \in \wp(\mathbf{Q})$, let

$$\begin{aligned} \tilde{\partial}^{\mathrm{b}}_C(p) = {} & (p \cap \{\langle\perp\rangle, \langle\surd\rangle\}) \\ & \hat{\cup}\, \{\langle\Delta(\Gamma)\rangle : \langle\Delta(\Gamma \setminus C)\rangle \in p\} \\ & \hat{\cup}\, \{\langle D(\Gamma)\rangle : \langle D(\Gamma \setminus C)\rangle \in p\}. \end{aligned}$$

**(2)** For $p \in \mathbf{P}_{\mathrm{cl}}$, let

$$\tilde{\partial}_C(p) = \tilde{\partial}^{\mathrm{b}}_C(p) \,\hat{\cup}\, \hat{\bigcup}_{a \in (\mathrm{act}(p) \setminus C)} [\langle a\rangle \cdot \psi[\tilde{\partial}_C(p[\langle a\rangle])]].$$ ∎

The next proposition immediately follows from the definition of the semantic operations $\widetilde{\mathrm{op}}$ ($\mathrm{op} \in \mathbf{Fun}_1$) given so far in this section:

**Proposition 6.10** *For every* $r \in 3$ *and* $\mathrm{op} \in \mathbf{Fun}_1^{(r)}$, *one has*

$$\widetilde{\mathrm{op}} \in ((\mathbf{P}_{\mathrm{cl}})^r \to^1 \mathbf{P}_{\mathrm{cl}}),$$

*where* $((\mathbf{P}_{\mathrm{cl}})^r \to^1 \mathbf{P}_{\mathrm{cl}})$ *is the sapce defined in Definition 2.1 (8).* ∎

As a preliminary to the definition of metric and order-theoretic models in the next section, the notion of a *valuation* is introduced by:

**Definition 6.21** (Semantic Valuations)

**(1)** Let $(\zeta \in)$ SeVal $= (\mathcal{X}_{\mathcal{P}} \to \mathbf{P}_{\mathrm{cl}})$. Elements of SeVal are called *(semantic) valuations*, which are mappings assigning an element of $\mathbf{P}_{\mathrm{cl}}$ to each statement variable.

**(2)** Let $\tilde{d}_{\mathrm{f}}$ be the metric on SeVal defined from $\tilde{d}$ as follows: For $f, g \in$ SeVal,

$$\tilde{d}_{\mathrm{f}}(f,g) = \sup\{\tilde{d}(f(X), g(X)) : X \in \mathcal{X}_{\mathcal{P}}\}.$$

(Note that $\tilde{d}_{\mathrm{f}}$ is the metric defined in Definition 2.1 (3).)

**(3)** Let $\tilde{\sqsubseteq}_{\mathrm{s}}$ be the pointwise order on SeVal defined in Proposition 2.1, and let $\tilde{\perp} = (\lambda X \in \mathcal{X}_{\mathcal{P}}.\ \{\perp\})$. ∎

As is stated in Chapter 2, $(\mathrm{SeVal}, \tilde{d}_{\mathrm{f}})$ is a cms (cf. Definition 2.1), and $(\mathrm{SeVal}, \tilde{\sqsubseteq}_{\mathrm{s}}, \tilde{\perp})$ is a cpo (cf. Proposition 2.1).

### 6.4.2 A Strong Metric Model $\mathcal{M}_{\bar{1}}^{\mathrm{sr}}$

In this subsection, we define a model $\mathcal{M}_{\bar{1}}^{\mathrm{sr}}$ on the basis of the cms $(\mathbf{P}_{\mathrm{cl}}, \tilde{d})$ and in terms of the semantic operations $\widetilde{\mathrm{op}}$ defined in § 6.4.1; the definition is given using the following lemma:

**Lemma 6.9** *There is a unique function $\mathcal{M} : \mathcal{L}_{\bar{1}} \to (\mathrm{SeVal} \to^1 \mathbf{P}_{\mathrm{cl}})$ satisfying the following conditions:*

**(i)** $\mathcal{M}[\![s]\!](\zeta) = \tilde{s}$, *for* $s \in \mathbf{A}_1 \cup \{\mathbf{0}, \mathbf{e}\}$.

**(ii)** $\mathcal{M}[\![\partial_C(S)]\!](\zeta) = \tilde{\partial}_C(\mathcal{M}[\![S]\!](\zeta))$, *for* $C \in \wp(\mathbf{C}_1)$ *and* $S \in \mathcal{L}_{\bar{1}}$.

**(iii)** $\mathcal{M}[\![\mathrm{op}(S_0, S_1)]\!](\zeta) = \widetilde{\mathrm{op}}(\mathcal{M}[\![S_0]\!](\zeta), \mathcal{M}[\![S_0]\!](\zeta))$, *for* op $\in$ {';', '+', '‖'} *and* $S_0, S_1 \in \mathcal{L}_{\bar{1}}$.

**(iv)** $\mathcal{M}[\![X]\!](\zeta) = \zeta(X)$, *for* $X \in \mathcal{X}_{\mathcal{P}}$.

**(v)** *For every* $(\mu X.\ S) \in \mathcal{L}_{\bar{1}}$, *the function* $(\lambda p \in \mathbf{P}_{\mathrm{cl}}.\ \mathcal{M}[\![S]\!](\zeta[p/X]))$ *is a contraction on* $\mathbf{P}_{\mathrm{cl}}$, *and*

$$\mathcal{M}[\![(\mu X.\ S)]\!](\zeta) = \mathrm{fix}(\lambda p \in \mathbf{P}_{\mathrm{cl}}.\ \mathcal{M}[\![S]\!](\zeta[p/X])). \blacksquare \tag{6.47}$$

The proof of this lemma will be given later; first, $\mathcal{M}_{\bar{1}}^{\mathrm{sr}}$ is defined by:

**Definition 6.22** Let $\mathcal{M}_{\bar{1}}^{\mathrm{sr}} : \mathcal{L}_{\bar{1}} \to (\mathrm{SeVal} \to^1 \mathbf{P}_{\mathrm{cl}})$ be the unique function $\mathcal{M} : \mathcal{L}_{\bar{1}} \to (\mathrm{SeVal} \to^1 \mathbf{P}_{\mathrm{cl}})$ satisfying the conditions (i)–(v) in Lemma 6.9. ∎

By this definition, one has the following:

**Proposition 6.11** *The conditions (i)–(v) in Lemma 6.9 are satisfied with $\mathcal{M}$ replaced by $\mathcal{M}_{\bar{1}}^{\mathrm{sr}}$.* ∎

We need a few preliminaries to the proof of Lemma 6.9.

**Definition 6.23** Let $p \in \mathbf{P}_{\mathrm{cl}}$. We say $p$ is a *guard process*, when $\langle\surd\rangle \notin p$. Let $\mathbf{P}_{\mathrm{cl}}^{\mathrm{g}}$ be the set of guard processes. ∎

Obviously, $\mathbf{P}_{\mathrm{cl}}^{\mathrm{g}}$ is a finitely characterized subset of $\wp_{\mathrm{cl}}(\mathbf{Q})$, and therefore, closed by Lemma 2.2:

**Proposition 6.12** $\forall \langle p_n \rangle_{n\in\omega} \in \mathrm{CS}(\mathbf{P}^{\mathrm{g}}_{\mathrm{cl}}, \tilde{d})[\ \lim_{n\in\omega}[p_n] \in \mathbf{P}^{\mathrm{g}}_{\mathrm{cl}}\ ].$ ∎

We easily obtain the following two propositions.

**Proposition 6.13** **(1)** *For every $p \in \mathbf{P}_{\mathrm{cl}}$, if $p$ is a guard process, then*

$$\forall \Gamma \in \wp(\mathbf{C}^{\checkmark}_{\bar{1}})[\ \langle \Upsilon(\Gamma) \rangle \in p \Rightarrow \langle \Upsilon(\Gamma \cup \{\surd\}) \rangle \in p\ ],$$

*where $\Upsilon = \Delta, D$.*

**(2)** *For every $p \in \mathbf{P}^{\mathrm{g}}_{\mathrm{cl}}$ and $p_1, p_1' \in \mathbf{P}_{\mathrm{cl}}$, one has*

$$\tilde{d}(\tilde{;}(p, p_1), \tilde{;}(p, p_1')) \leq \kappa \cdot \tilde{d}(p_1, p_1').$$

**(3)** *For every $p_0, p_0' \in \mathbf{P}^{\mathrm{g}}_{\mathrm{cl}}$ and $p_1, p_1' \in \mathbf{P}_{\mathrm{cl}}$, one has*

$$\tilde{d}(\tilde{;}(p_0, p_1), \tilde{;}(p_0', p_1')) \leq \max\{\tilde{d}(p_0, p_0'), \kappa \cdot \tilde{d}(p_1, p_1')\}.$$ ∎

**Proof.** **(1)** Straightforward from the upward closedness of $p$ w.r.t. disabled actions.
**(2)** This part immediately follows from the definition of $\tilde{;}$ with the help of part (1).
**(3)** We obtain this part from part (2) as follows:

$$\begin{aligned}
&\tilde{d}(\tilde{;}(p_0, p_1), \tilde{;}(p_0', p_1')) \\
&\leq \max\{\tilde{d}(\tilde{;}(p_0, p_1), \tilde{;}(p_0', p_1)), \tilde{d}(\tilde{;}(p_0', p_1), \tilde{;}(p_0', p_1'))\} \\
&\qquad \text{(since } \tilde{d} \text{ is an ultra-metric)} \\
&\leq \max\{\tilde{d}(p_0, p_0'), \kappa \cdot \tilde{d}(p_1, p_1')\} \\
&\qquad \text{(by the fact that } \tilde{;} \text{ is nonexpansive and by part (2)).}
\end{aligned}$$ ∎

**Proposition 6.14** **(1)** $\forall r \in 3, \forall \mathrm{op} \in (\mathbf{Fun}^{(r)}_{\bar{1}} \setminus \{\mathrm{e}\})[\ \widetilde{\mathrm{op}}[(\mathbf{P}^{\mathrm{g}}_{\mathrm{cl}})^r] \subseteq \mathbf{P}^{\mathrm{g}}_{\mathrm{cl}}\ ].$
**(2)** $\forall p_0 \in \mathbf{P}^{\mathrm{g}}_{\mathrm{cl}}, \forall p_1 \in \mathbf{P}_{\mathrm{cl}}[\ \tilde{;}(p_0, p_1) \in \mathbf{P}^{\mathrm{g}}_{\mathrm{cl}}\ ].$ ∎

**Proof.** Straightforward from the definitions of the semantic operations. ∎

**Proof of Lemma 6.9** The uniqueness can be easily shown by induction on the structure of statements. For showing the existence, we will first define a family of mappings $\langle \mathcal{M}^{(n)} \rangle_{n\in\omega}$ satisfying the following conditions (6.48) and (6.49) for every $n \in \omega$, and then put $\mathcal{M} = \bigcup_{n\in\omega}[\mathcal{M}^{(n)}]$.

$$\mathcal{M}^{(n)} : \mathcal{L}_{\bar{1}}(n) \to (\mathrm{SeVal} \to^1 \mathbf{P}_{\mathrm{cl}}), \tag{6.48}$$

$$\begin{aligned}
&\forall X \in \mathcal{X}_{\mathcal{P}}, \forall S \in \mathcal{G}(n)[ \\
&\quad \forall \zeta \in \mathrm{SeVal}[\ \mathcal{M}^{(n)}[\![S]\!](\zeta) \in \mathbf{P}^{\mathrm{g}}_{\mathrm{cl}}] \\
&\quad \wedge (\lambda p \in \mathbf{P}_{\mathrm{cl}}.\ \mathcal{M}^{(n)}[\![S]\!](\zeta[p/X])) \in (\mathbf{P}_{\mathrm{cl}} \to^{\kappa} \mathbf{P}_{\mathrm{cl}})\ ].
\end{aligned} \tag{6.49}$$

(For the definition of $\mathcal{L}_{\bar{1}}(n)$ and $\mathcal{G}(n)$, see Definition 5.3.)
**Step 1.** Since $\mathcal{L}_{\bar{1}}(0) = \emptyset$, we simply define $\mathcal{M}^{(n)}$ to be the empty function.
**Step 2.** Let $n \in \omega$ and suppose that $\mathcal{M}^{(n)}$ has been defined so that the conditions (6.48) and (6.49) are satisfied.

**Step 2.1.** First, let us define $\mathcal{M}^{(n+1)}[\![S]\!](\zeta)$ for $S \in \mathcal{L}_{\bar{1}}(n+1)$ so that (6.48) is satisfied with $n$ replaced by $n+1$.

By the definition of $\mathcal{L}_{\bar{1}}(n+1)$, one of the following three conditions is satisfied (cf. Definition 5.2 (3)):

$$S \in \mathcal{X}_{\mathcal{P}}; \tag{6.50}$$

$$S \equiv \mathrm{op}(S_0, \ldots, S_{r-1}) \quad \text{for some } r \in 3,\ \mathrm{op} \in \mathbf{Fun}_{\bar{1}}^{(r)}, \text{ and } S_0, \ldots, S_{r-1} \in \mathcal{L}_{\bar{1}}(n); \tag{6.51}$$

$$S \equiv (\mu X.\ \tilde{S}) \text{ for some } X \in \mathcal{X}_{\mathcal{P}} \text{ and } \tilde{S} \in \mathcal{G}(n). \tag{6.52}$$

We distinguish three cases according to which of (6.50)–(6.52) holds.
**Case 1.** When (6.50) holds, let

$$\mathcal{M}^{(n+1)}[\![S]\!] = (\lambda\zeta.\ \zeta(S)).$$

Obviously, $\mathcal{M}^{(n+1)}[\![S]\!] \in (\mathrm{SeVal} \to^1 \mathbf{P}_{\mathrm{cl}})$.
**Case 2.** When (6.51) holds, let

$$\mathcal{M}^{(n+1)}[\![S]\!] = (\lambda\zeta.\ \widetilde{\mathrm{op}}(\mathcal{M}^{(n)}[\![S_0]\!](\zeta), \ldots, \mathcal{M}^{(n)}[\![S_{r-1}]\!](\zeta))).$$

It follows from the induction hypothesis and Proposition 6.10 that $\mathcal{M}^{(n+1)}[\![S]\!] \in (\mathrm{SeVal} \to^1 \mathbf{P}_{\mathrm{cl}})$.
**Case 3.** Suppose (6.52) holds. Then, by the induction hypothesis, the following holds for every $\zeta \in \mathrm{SeVal}$:

$$(\lambda p.\ \mathcal{M}^{(n)}[\![\tilde{S}]\!](\zeta[p/X])) \in (\mathbf{P}_{\mathrm{cl}} \to^{\kappa} \mathbf{P}_{\mathrm{cl}}).$$

Thus, $(\lambda p.\ \mathcal{M}^{(n)}[\![\tilde{S}]\!](\zeta[p/X]))$ has a unique fixed-point, and we put

$$\begin{aligned}\mathcal{M}^{(n+1)}[\![S]\!] &= (\lambda\zeta \in \mathrm{SeVal}.\ \mathrm{fix}(\phi(\zeta))) \\ &= (\lambda\zeta \in \mathrm{SeVal}.\ \lim_{n\in\omega}[(\phi(\zeta))^n(p_0)]),\end{aligned}$$

where $\phi(\tilde{\zeta}) = (\lambda p.\ \mathcal{M}^{(n)}[\![\tilde{S}]\!](\tilde{\zeta}[p/X]))$ for each $\tilde{\zeta}$, with $p_0$ being an arbitrary element of $\mathbf{P}_{\mathrm{cl}}$. For every $\zeta_1, \zeta_2 \in \mathrm{SeVal}$, one obtains the following, by induction, from the induction hypothesis:

$$\forall n \in \omega[\ \tilde{d}((\phi(\zeta_1))^n(p_0), (\phi(\zeta_2))^n(p_0),) \le \tilde{d}_{\mathrm{F}}(\zeta_1, \zeta_2)\ ];$$

therefore

$$\tilde{d}(\mathcal{M}^{(n+1)}[\![S]\!](\zeta_1), \mathcal{M}^{(n+1)}[\![S]\!](\zeta_2)) \le \tilde{d}(\zeta_1, \zeta_2).$$

Thus, one has $\mathcal{M}^{(n+1)}[\![S]\!] \in (\mathrm{SeVal} \to^1 \mathbf{P}_{\mathrm{cl}})$.
**Step 2.2.** Having defined $\mathcal{M}^{(n+1)}$ so that (6.48) is satisfied with $n$ replaced by $n+1$, let us show that (6.49) is satisfied with $n$ replaced by $n+1$. Let $X \in \mathcal{X}_{\mathcal{P}}$, $S \in \mathcal{G}(n+1)$, and $\zeta \in \mathrm{SeVal}$. It suffices to show (6.60) and (6.54) below.

$$\forall\zeta \in \mathrm{SeVal}[\ \mathcal{M}^{(n+1)}[\![S]\!](\zeta) \in \mathbf{P}_{\mathrm{cl}}^{\mathrm{g}}\ ]. \tag{6.53}$$

$$(\lambda p \in \mathbf{P}_{\rm cl}.\ \mathcal{M}^{(n+1)}[\![S]\!](\zeta[p/X])) \in (\mathbf{P}_{\rm cl} \to^{\kappa} \mathbf{P}_{\rm cl}). \tag{6.54}$$

By the definition of $\mathcal{G}(n+1)$, one of the following three propositions holds:

$$S \equiv \mathsf{op}(S_0, \ldots, S_{r-1}) \textit{ for some } r \in 3,\ \mathsf{op} \in (\mathbf{Fun}_1^{(r)} \setminus \{\mathsf{e}\}), \textit{ and } S_0, \ldots, S_{r-1} \in \mathcal{G}(n); \tag{6.55}$$

$$S \equiv (S_0; S_1) \textit{ for some } S_0 \in \mathcal{G}(n) \textit{ and } S_1 \in \mathcal{L}_{\bar{1}}(n); \tag{6.56}$$

$$S \equiv (\mu Y.\ \tilde{S}) \text{ for some } Y \in \mathcal{X}_{\mathcal{P}} \text{ and } \tilde{S} \in \mathcal{G}(n). \tag{6.57}$$

We distinguish three cases according to which of (6.55)–(6.57) holds.
**Case 1.** Suppose (6.55) holds. Then, we obtain (6.53) from the induction hypothesis and Proposition 6.14 (1). Also, we immediately obtain (6.54) from the induction hypothesis by applying Proposition 6.10.
**Case 2.** Suppose (6.56) holds. Then, we obtain (6.53) from the induction hypothesis and Proposition 6.14 (2). Also, we immediately obtain (6.54) from the induction hypothesis by applying Proposition 6.13 (3).
**Case 3.** Suppose (6.57) holds. First we will show (6.53). One has

$$\mathcal{M}^{(n+1)}[\![S]\!] = (\lambda \zeta \in \mathrm{SeVal}.\ \lim_{k \in \omega}[(\phi(\zeta))^k(p_0)]), \tag{6.58}$$

with $\phi(\zeta) = (\lambda p \in \mathbf{P}_{\rm cl}.\ \mathcal{M}^{(n)}[\![\tilde{S}]\!](\zeta[p/Y]))$, $(\phi(\zeta))^n$ the $n$-th iteration of $\phi(\zeta)$, and $p_0$ an arbitrary element of $\mathbf{P}_{\rm cl}$. By induction, we obtain the following from the induction hypothesis:

$$\forall k \in \omega[\ (\phi(\zeta))^k(p_0) \in \mathbf{P}_{\rm cl}^{\rm g}\ ]. \tag{6.59}$$

By applying Proposition 6.12, we obtain (6.53) from (6.59) and (6.58).

Next let us show (6.54). Let $p_1, p_2 \in \mathbf{P}_{\rm cl}$. We will show that the following holds for every $k \in \omega$ by induction:

$$\tilde{d}((\phi(\zeta[p_1/X]))^k(p_0), (\phi(\zeta[p_2/X]))^k(p_0)) \le \kappa \cdot \tilde{d}(p_1, p_2). \tag{6.60}$$

For $n = 0$, (6.60) obviously holds. Fix $k' \in \omega$ and suppose (6.60) holds for $k = k'$. We can show that (6.60) holds for $k = k' + 1$ as follows:

$$\begin{aligned}
&\tilde{d}((\phi(\zeta[p_1/X]))^{k'+1}(p_0), (\phi(\zeta[p_2/X]))^{k'+1}(p_0)) \\
&= \tilde{d}((\phi(\zeta[p_1/X]))((\phi(\zeta[p_1/X]))^{k'}(p_0)), \\
&\qquad (\phi(\zeta[p_2/X]))((\phi(\zeta[p_1/X]))^{k'}(p_0))) \\
&= \tilde{d}(\mathcal{M}^{(n)}[\![\tilde{S}]\!](\zeta[p_1/X][(\phi(\zeta[p_1/X]))^{k'}(p_0)/Y]), \\
&\qquad \mathcal{M}^{(n)}[\![\tilde{S}]\!](\zeta[p_2/Y][(\phi(\zeta[p_2/X]))^{k'}(p_0)/Y])) \\
&\le \max\{\tilde{d}(\mathcal{M}^{(n)}[\![\tilde{S}]\!](\zeta[p_1/X][(\phi(\zeta[p_1/X]))^{k'}(p_0)/Y]), \\
&\qquad\qquad \mathcal{M}^{(n)}[\![\tilde{S}]\!](\zeta[p_1/X][(\phi(\zeta[p_2/X]))^{k'}(p_0)/Y])) \\
&\qquad \tilde{d}(\mathcal{M}^{(n)}[\![\tilde{S}]\!](\zeta[p_1/X][(\phi(\zeta[p_2/X]))^{k'}(p_0)/Y]), \\
&\qquad\qquad \mathcal{M}^{(n)}[\![\tilde{S}]\!](\zeta[p_2/X][(\phi(\zeta[p_2/X]))^{k'}(p_0)/Y]))\ \}
\end{aligned}$$

(since $\tilde{d}$ is an ultra-metric)

$\leq \max\{\kappa^2 \cdot \tilde{d}(p_1, p_2), \kappa \cdot \tilde{d}(p_1, p_2)\}$

(by the induction hypothesis stating that (6.60) holds for $k = k'$)

$\leq \kappa \cdot \tilde{d}(p_1, p_2)$.

Thus, by induction (6.60) holds for every $k$, and therefore

$$\tilde{d}(\lim_{k\in\omega}[(\phi(\zeta[p_1/X]))^k(p_0)], \lim_{k\in\omega}[(\phi(\zeta[p_2/X]))^k(p_0)]) \leq \kappa \cdot \tilde{d}(p_1, p_2).$$

By this and (6.58), one has

$$\tilde{d}(\mathcal{M}^{(n+1)}[\![S]\!](\zeta[p_1/X]), \mathcal{M}^{(n+1)}[\![S]\!](\zeta[p_2/X])) \leq \kappa \cdot \tilde{d}(p_1, p_2).$$

Since $\zeta$, $p_1$, and $p_2$ were arbitrary, one has (6.54). ■

The following proposition is intuitively natural, and in fact follows from Proposition 6.11:

**Proposition 6.15** *For every* $S \in \mathcal{L}_{\bar{1}}$ *and* $\zeta_0, \zeta_1 \in$ SeVal *with* $\zeta_0 \upharpoonright \mathrm{FV}(S) = \zeta_1 \upharpoonright \mathrm{FV}(S)$, *one has*

$$\mathcal{M}_{\bar{1}}^{\mathrm{sr}}[\![S]\!](\zeta_0) = \mathcal{M}_{\bar{1}}^{\mathrm{sr}}[\![S]\!](\zeta_1). \blacksquare$$

**Proof.** This can be shown by induction of the structure of $S \in \mathcal{L}_{\bar{1}}$, using Proposition 6.11. ■

Using this proposition, we define a model $\dot{\mathcal{M}}_{\bar{1}}^{\mathrm{sr}} : \mathcal{L}_{\bar{1}}[\emptyset] \to \mathbf{P}_{\mathrm{cl}}$ from $\mathcal{M}_{\bar{1}}^{\mathrm{sr}}$ by:

**Definition 6.24** Putting $\zeta_0 = (\lambda X \in \mathcal{X}_{\mathcal{P}}.\ \tilde{\mathbf{0}})$, we define $\dot{\mathcal{M}}_{\bar{1}}^{\mathrm{sr}} : \mathcal{L}_{\bar{1}}[\emptyset] \to \mathbf{P}_{\mathrm{cl}}$ as follows: For every $s \in \mathcal{L}_{\bar{1}}[\emptyset]$,

$$\dot{\mathcal{M}}_{\bar{1}}^{\mathrm{sr}}[\![s]\!] = \mathcal{M}_{\bar{1}}^{\mathrm{sr}}[\![s]\!](\zeta_0). \blacksquare$$

### 6.4.3 A Strong Order-Theoretic Model $\mathcal{D}_{\bar{1}}^{\mathrm{sr}}$

In this subsection, we define a model $\mathcal{D}_{\bar{1}}^{\mathrm{sr}}$ on the basis of the cpo $(\mathbf{P}_{\mathrm{cl}}, \sqsubseteq_{\mathrm{s}})$ and in terms of the semantic operations $\widetilde{\mathrm{op}}$ defined in § 6.4.1; the definition is given using the following proposition, which is standard in (cpo-based) denotational semantics:

**Proposition 6.16** *Let* $\mathcal{D} \in (\tilde{\mathcal{L}}_{\bar{1}} \to [\mathrm{SeVal} \to \mathbf{P}_{\mathrm{cl}}])$ *with* $[\mathrm{SeVal} \to \mathbf{P}_{\mathrm{cl}}]$ *being the space of continuous functions defined in Definition 2.6. Then for every* $S \in \tilde{\mathcal{L}}_{\bar{1}}$, $\zeta \in$ SeVal, *and* $X \in \mathcal{X}_{\mathcal{P}}$, *the function* $(\lambda p \in \mathbf{P}_{\mathrm{cl}}.\ \mathcal{D}[\![S]\!](\zeta[p/X]))$ *is a continuous function from* $(\mathbf{P}_{\mathrm{cl}}, \sqsubseteq_{\mathrm{s}})$ *to itself.* ■

**Proof.** Straightforward from Proposition 2.1 (2). ■

Using the above proposition, the model $\mathcal{D}_{\bar{1}}^{\mathrm{sr}}$ is defined by:

**Definition 6.25** We define $\mathcal{D}_{\bar{1}}^{\mathrm{sr}} : \tilde{\mathcal{L}}_{\bar{1}} \to [\mathrm{SeVal} \to \mathbf{P}_{\mathrm{cl}}]$ by induction on the structure of $S$ as follows:

**(i)** $\mathcal{D}[\![s]\!](\zeta) = \widetilde{s}$, for $s \in \mathbf{A}_1 \cup \{\mathbf{0}, \mathbf{e}\}$ and $\zeta \in$ SeVal.

**(ii)** $\mathcal{D}[\![\partial_C(S)]\!](\zeta) = \widetilde{\partial}_C(\mathcal{D}[\![S]\!](\zeta))$, for $C \in \wp(\mathbf{C}_1)$, $S \in \tilde{\mathcal{L}}_{\bar{1}}$, and $\zeta \in$ SeVal.

**(iii)** $$\mathcal{D}[\![\mathsf{op}(S_0, S_1)]\!](\zeta) = \widetilde{\mathsf{op}}(\mathcal{D}[\![S_0]\!](\zeta), \mathcal{D}[\![S_0]\!](\zeta)),$$ for op $\in \{$';', '+', '$\|$'$\}$, $S_0, S_1 \in \tilde{\mathcal{L}}_{\bar{1}}$, and $\zeta \in$ SeVal.

**(iv)** $\mathcal{D}[\![X]\!](\zeta) = \zeta(X)$, for $X \in \mathcal{X}_{\mathcal{P}}$ and $\zeta \in$ SeVal.

**(v)** $$\mathcal{D}[\![(\mu X.\ S)]\!](\zeta) = \mathrm{fix}_1((\lambda p \in \mathbf{P}_{\mathrm{cl}}.\ \mathcal{D}[\![S]\!](\zeta[p/X]))),$$ for $(\mu X.\ S) \in \tilde{\mathcal{L}}_{\bar{1}}$ and $\zeta \in$ SeVal. (Note that this clause makes sense by Proposition 6.16.) ∎

We have the following proposition for $\mathcal{D}_{\bar{1}}^{\mathrm{sr}}$ corresponding to Proposition 6.15 for $\mathcal{M}_{\bar{1}}^{\mathrm{sr}}$:

**Proposition 6.17** *For every* $S \in \mathcal{L}_{\bar{1}}$ *and* $\zeta_0, \zeta_1 \in$ SeVal *with* $\zeta_0 \upharpoonright \mathrm{FV}(S) = \zeta_1 \upharpoonright \mathrm{FV}(S)$, *one has*

$$\mathcal{D}_{\bar{1}}^{\mathrm{sr}}[\![S]\!](\zeta_0) = \mathcal{D}_{\bar{1}}^{\mathrm{sr}}[\![S]\!](\zeta_1). \blacksquare$$

**Proof.** Similar to the proof of Proposition 6.15. ∎

Using this proposition, we define a model $\dot{\mathcal{D}}_{\bar{1}}^{\mathrm{sr}} : \mathcal{L}_{\bar{1}}[\emptyset] \to \mathbf{P}_{\mathrm{cl}}$ from $\mathcal{D}_{\bar{1}}^{\mathrm{sr}}$ by:

**Definition 6.26** Putting $\zeta_0 = (\lambda X \in \mathcal{X}_{\mathcal{P}}.\ \tilde{\mathbf{0}})$, we define $\dot{\mathcal{D}}_{\bar{1}}^{\mathrm{sr}} : \mathcal{L}_{\bar{1}}[\emptyset] \to \mathbf{P}_{\mathrm{cl}}$ as follows: For every $s \in \mathcal{L}_{\bar{1}}[\emptyset]$,

$$\dot{\mathcal{D}}_{\bar{1}}^{\mathrm{sr}}[\![s]\!] = \mathcal{D}_{\bar{1}}^{\mathrm{sr}}[\![s]\!](\zeta_0). \blacksquare$$

### 6.4.4 Continuity of the Hiding Operation $H$

In this subsection, we show the continuity of the hiding operation $H$ on the domain $\mathbf{P}_{\mathrm{cl}}$. We need a few preliminaries for this purpose.

By the *lifting method*, Theorem 2.2 and Corollary 2.1 presented in § 2.3, we immediately obtain the following lemma:

**Lemma 6.10** (Continuity of $H$ on $\wp_{\mathrm{fco}}(\mathbf{Q})$)

**(1)** $H[\wp_{\mathrm{fco}}(\mathbf{Q})] \subseteq \wp_{\mathrm{fco}}(\mathbf{Q})$.

**(2)** *The function* $H \upharpoonright \wp_{\mathrm{fco}}(\mathbf{Q})$ *is continuous w.r.t.* $\sqsubseteq_{\mathrm{s}}$. ∎

We cannot generalize Theorem 2.2 to the domain $\mathbf{P}_{\mathrm{cl}}$ in any straightforward way, because the proof of the lifting method (Theorem 2.2) depends crucially on the compactness of the underlying domain. As is shown below, however, a similar result for $\mathbf{P}_{\mathrm{cl}}$ (the continuity of $H$ on $\mathbf{P}_{\mathrm{cl}}$) can be obtained using Lemma 6.10, with the help of the close connection between $\mathbf{P}_{\mathrm{cl}}$ and $\mathbf{P}_{\mathrm{co}}$.

**Definition 6.27 (1)**
$$\wp_{\mathrm{fcl}}^{*}(\mathbf{Q}) = \{p \in \wp_{\mathrm{fcl}}(\mathbf{Q}) : p \text{ is bounded in the action part}\}.$$

**(2)** Let $(R \in)\ \mathbf{R}_{+} = \Delta[\wp_{+}(\mathbf{C}_{\bar{1}}^{\checkmark})] \cup D[\wp_{+}(\mathbf{C}_{\bar{1}}^{\checkmark})]$.

**(3)** For $p \in \wp^*_{\mathrm{fcl}}(\mathbf{Q})$, let

$$\tilde{\Lambda}(p) = p \setminus ((\mathbf{A}_1)^{<\omega} \cdot \mathbf{R}_+).$$

**(4)** For $p \in \wp^*_{\mathrm{fcl}}(\mathbf{Q})$, let

$$\tilde{H}(p) = \{h(w \cdot \langle\Gamma\rangle) : w \cdot \langle\Gamma\rangle \in p \cap ((\mathbf{A}_1)^{<\omega} \cdot \mathbf{R}_+) \wedge \neg\exists u \leq_{\mathrm{p}} w[\ u \cdot \langle\bot\rangle \in H(\tilde{\Lambda}(p))\ ]\}. \blacksquare$$

For $p \in \mathbf{P}_{\mathrm{cl}}$, $H(p)$ is represented in terms of $\tilde{\Lambda}(p)$ and $\tilde{H}(p)$ as follows:

**Proposition 6.18** $\forall p \in \mathbf{P}_{\mathrm{cl}}[\ H(p) = H(\tilde{\Lambda}(p)) \cup \tilde{H}(p)\ ]$. ∎

The following lemma follows straightforwardly from the definition of $\tilde{\Lambda}$:

**Proposition 6.19** *The function* $\tilde{\Lambda} : \mathbf{P}_{\mathrm{cl}} \to \wp_{\mathrm{fco}}(\mathbf{Q})$ *is continuous w.r.t.* $\sqsubseteq_{\mathrm{s}}$. ∎

We obtain the following lemma from Lemma 6.10 and Propositions 6.18 and 6.19.

**Lemma 6.11** (Continuity of $H$ on $\mathbf{P}_{\mathrm{cl}}$)

**(1)** $H[\mathbf{P}_{\mathrm{cl}}] \subseteq \mathbf{P}_{\mathrm{cl}}$.

**(2)** *The function* $H$ $\mathbf{P}_{\mathrm{cl}}$ *is continuous w.r.t.* $\sqsubseteq_{\mathrm{s}}$. ∎

**Proof. (1)** We will show this part using Lemma 6.10 (1) and Proposition 6.18. Let $p \in \mathbf{P}_{\mathrm{cl}}$. We will show $H(p) \in \mathbf{P}_{\mathrm{cl}}$. By the definition of $H$, obviously $H(p)$ is nonempty and flat. It suffices to show that $H(p)$ is closed and satisfies the conditions (i)–(v) in Definition 6.2 (2).

First, let us show that $H(p)$ is closed. By Proposition 6.18,

$$H(p) = H(\tilde{\Lambda}(p)) \cup \tilde{H}(p). \tag{6.61}$$

Let $q \in (H(p))^{\mathrm{cls}}$. If $q \notin (\mathbf{A}_1)^\omega$, then $q \in H(p)$. Suppose $q \in (\mathbf{A}_1)^\omega$. By the definitions of $\tilde{\Lambda}$ and $\tilde{H}$ and the fact $p$ is downward closed, one has

$$\forall w \cdot \langle D(\Gamma)\rangle \in \tilde{H}(p)[\ w \cdot \langle D(\emptyset)\rangle \in H(\tilde{\Lambda}(p))\ ]. \tag{6.62}$$

Thus, $q \in (H(\tilde{\Lambda}(p)))^{\mathrm{cls}}$, since for every approximation $w \cdot \langle D(\Gamma)\rangle \in \tilde{H}(p)$ of $q$, there is another approximation $w \cdot \langle D(\emptyset)\rangle \in H(\tilde{\Lambda}(p))$ such that $w \cdot \langle D(\Gamma)\rangle$ and $w \cdot \langle D(\emptyset)\rangle$ approximate $q$ to the same extent, i.e.,

$$\tilde{d}(q, w \cdot \langle D(\emptyset)\rangle) = \tilde{d}(q, w \cdot \langle D(\Gamma)\rangle).$$

By this and the fact $H(\tilde{\Lambda}(p))$ is closed, one has $q \in H(\tilde{\Lambda}(p)) \subseteq H(p)$. Thus, $\forall q \in (H(p))^{\mathrm{cls}}[\ q \in H(p)\ ]$, i.e., $H(p)$ is closed.

Next, let us show that $H(p)$ satisfies the condition (i) in Definition 6.2 (2), i.e., that $H(p)$ is bounded in the action part. Let $n \in \omega$. Then, by (6.61) and (6.62) one has

$$\mathrm{strip}[(H(p))^{[n]}] = \mathrm{strip}[(H(\tilde{\Lambda}(p)))^{[n]}].$$

Since $H(\tilde{\Lambda}(p))$ is compact by Lemma 6.10 (1), strip$[(H(\tilde{\Lambda}(p)))^{[n]}]$ is finite. Hence, strip$[(H(p))^{[n]}]$ is also finite. Thus, strip$[(H(p))^{[n]}]$ is finite for every $n$, i.e., $H(p)$ is bounded in the action part.

It is easily checked that $H(p)$ satisfies the conditions (ii)–(v) in Definition 6.2 (2). Thus $H(p) \in \mathbf{P}_{\mathrm{cl}}$, by Definition 6.2 (2).

**(2)** First, it immediately follows that $H$ is monotonic w.r.t. $\sqsubseteq_{\mathrm{s}}$, since $h$ is monotonic w.r.t. $\sqsubseteq$.

Let us show that $H$ is continuous on $\mathbf{P}_{\mathrm{cl}}$ w.r.t. $\sqsubseteq_{\mathrm{s}}$, i.e., that the following holds for every $\langle p_n \rangle_{n\in\omega} \in \mathrm{Chain}(\mathbf{P}_{\mathrm{cl}}, \sqsubseteq_{\mathrm{s}})$:

$$H(\bigsqcup_{n\in\omega}[p_n]) = \bigsqcup_{n\in\omega}[H(p_n)]. \tag{6.63}$$

Let $\langle p_n \rangle_{n\in\omega} \in \mathrm{Chain}(\mathbf{P}_{\mathrm{cl}}, \sqsubseteq_{\mathrm{s}})$. Since $H$ is monotonic w.r.t. $\sqsubseteq_{\mathrm{s}}$, it immediately follows that

$$\bigsqcup_{n\in\omega}[H(p_n)] \sqsubseteq_{\mathrm{s}} H(\bigsqcup_{n\in\omega}[p_n]). \tag{6.64}$$

We will show the other direction, i.e. that

$$H(\bigsqcup_{n\in\omega}[p_n]) \sqsubseteq_{\mathrm{s}} \bigsqcup_{n\in\omega}[H(p_n)]. \tag{6.65}$$

Let $q \in \bigsqcup_{n\in\omega}[H(p_n)]$. Then, by Proposition 6.18,

$$q \in \bigsqcup_{n\in\omega}[H(\tilde{\Lambda}(p_n)) \cup \tilde{H}(p_n)]. \tag{6.66}$$

We distinguish two cases according to whether $q \in (\mathbf{A}_1)^{<\omega} \cdot \mathbf{R}_+$ or not.

**Case 1.** Suppose $q \in (\mathbf{A}_1)^{<\omega} \cdot \mathbf{R}_+$. Then, by (6.66), one has $q \in \bigsqcup_{n\in\omega}[\tilde{H}(p_n)]$. Thus, $q = w \cdot \langle R \rangle$ for some $w \in (\mathbf{A}_1)^{<\omega}$ and $R \in \mathbf{R}_+$, and there exists $N$ such that

$$\forall n \geq N[\ q \in \tilde{H}(p_n)\ ]. \tag{6.67}$$

Fix such a number $N$. Obviously, the set

$$\hat{p}_N = \{\tilde{w} \cdot \langle R \rangle \in p_N : h(\tilde{w} \cdot \langle R \rangle) = w \cdot \langle R \rangle\}$$

is nonempty. Further $\hat{p}_N$ must be finite, since if it is infinite then there must $u \in (\mathbf{A}_1)^{<\omega}$ such that

$$u \leq_{\mathrm{p}} w \wedge u \cdot \langle \bot \rangle \in H(\tilde{\Lambda}(p_N)),$$

which contradicts the fact that $w \cdot \langle R \rangle \in \tilde{H}(p_N)$.

For each $\tilde{q} \in \hat{p}_N$, let

$$I(\tilde{q}) = \{n \geq N : \tilde{q} \in p_n\}.$$

Let us show

$$\bigcup\{I(\tilde{q}) : \tilde{q} \in \hat{p}_N\} = \{n \in \omega : n \geq N\}. \tag{6.68}$$

Fix $n \geq N$. Then

$$q = w \cdot \langle R \rangle \in \tilde{H}(p_n), \tag{6.69}$$

by (6.67). Therefore there exists $\tilde{w} \cdot \langle R \rangle \in p_n$ such that $h(\tilde{w} \cdot \langle R \rangle) = w \cdot \langle R \rangle$. Since $p_N \sqsubseteq_s p_n$, there is $\tilde{q} \in p_N$ such that $\tilde{q} \sqsubseteq \tilde{w} \cdot \langle R \rangle$. It must hold that $\tilde{q} = \tilde{w} \cdot \langle R \rangle$, since if $\tilde{q} \mathbin{@} \tilde{w} \cdot \langle R \rangle$, then $\tilde{q} = \tilde{w}' \cdot \langle \bot \rangle$ for some $\tilde{w}' \sqsubseteq \tilde{w}$, and therefore,

$$h[p_N] \ni h(\tilde{w}' \cdot \langle \bot \rangle) \mathbin{@} w \cdot \langle R \rangle = q,$$

which contradicts the minimality of $q$ in $h[p_N]$. Thus $\tilde{q} = \tilde{w} \cdot \langle R \rangle \in p_n$, and therefore, $n \in I(\tilde{q})$. Since $n$ was arbitrary, one has (6.68).

From (6.68) and the fact that $\hat{p}_N$ is finite, it follows that $I(\tilde{q})$ is infinite for some $\tilde{q} \in \hat{p}_N$. Fix such an element $\tilde{q}$. Then, using Proposition 2.4, one has $\forall n \geq N[\ \tilde{q} \in p_n\ ]$, and therefore, $\tilde{q} \in \bigsqcup_{n \in \omega}[p_n]$. Thus,

$$h(\tilde{q}) = q \in h[\bigsqcup_{n \in \omega}[p_n]]. \tag{6.70}$$

Also, since (6.69) holds for every $n \geq N$, one has

$$\neg \exists u \cdot \langle \bot \rangle \in h[\bigsqcup_{n \in \omega}[p_n]][\ u \cdot \langle \bot \rangle \mathbin{@} q\ ]. \tag{6.71}$$

By (6.70) and (6.71),

$$q \in \mathrm{mini}(h[\bigsqcup_{n \in \omega}[p_n]]) = H(\bigsqcup_{n \in \omega}[p_n]).$$

**Case 2.** Suppose $q \notin (\mathbf{A}_1)^{<\omega} \cdot \mathbf{R}_+$. Then, by (6.66), one has

$$q \in \bigsqcup_{n \in \omega}[H(\tilde{\Lambda}(p_n))]. \tag{6.72}$$

Moreover,

$$\begin{aligned} & q \in H(\textstyle\bigsqcup_{n \in \omega}[\tilde{\Lambda}(p_n)]) \quad \text{(by Lemma 6.10 (2))} \\ & = H(\tilde{\Lambda}(\textstyle\bigsqcup_{n \in \omega}[p_n])) \quad \text{(by Proposition 6.19)} \\ & \subseteq H(\textstyle\bigsqcup_{n \in \omega}[p_n]) \quad \text{(by Proposition 6.18).} \end{aligned}$$

Thus, $q \in H(\bigsqcup_{n \in \omega}[p_n])$.

Summarizing the above, in both cases one has $q \in H(\bigsqcup_{n \in \omega}[p_n])$. Since $q$ was an arbitrary element of $\bigsqcup_{n \in \omega}[H(p_n)]$, one has

$$H(\bigsqcup_{n \in \omega}[p_n]) \supseteq \bigsqcup_{n \in \omega}[H(p_n)], \text{ and therefore, } H(\bigsqcup_{n \in \omega}[p_n]) \sqsubseteq_s \bigsqcup_{n \in \omega}[H(p_n)].$$

By this and (6.64), one has the desired consequence (6.63). ■

### 6.4.5 A Weak Order-Theoretic Model $\mathcal{D}_{\mathrm{\bar{1}}}^{\mathrm{wf}}$

In this subsection, we define a *weak* order-theoretic model $\mathcal{D}_{\mathrm{\bar{1}}}^{\mathrm{wf}}$.

As a preliminary to the definition, we define weak versions of *semantic operations* $\widehat{\mathrm{op}}$ from the strong ones $\widetilde{\mathrm{op}}$ ($\mathrm{op} \in \mathbf{Fun}_{\mathrm{\bar{1}}}$), by applying the hiding function $H$:

**Definition 6.28** For $\mathrm{op} \in \mathbf{Fun}_{\mathrm{\bar{1}}}$, let $\widehat{\mathrm{op}} = H \circ \widetilde{\mathrm{op}}$. That is, we define $\widehat{\mathrm{op}}$ as follows: For $p_0, \ldots, p_{r-1} \in \mathbf{P}_{\mathrm{cl}}$ with $r = \mathrm{arity}(\mathrm{op})$,

$$\widehat{\mathrm{op}}(p_0, \ldots, p_{r-1}) = H(\widetilde{\mathrm{op}}(p_0, \ldots, p_{r-1})). \blacksquare$$

The next proposition immediately follows from Lemma 6.11:

**Proposition 6.20** *For every* $r \in \{1, 2\}$ *and* $\mathrm{op} \in \mathbf{Fun}_{\mathrm{\bar{1}}}^{(r)}$, *one has*

$$\widehat{\mathrm{op}} \in [(\mathbf{P}_{\mathrm{cl}})^r \to \mathbf{P}_{\mathrm{cl}}]. \blacksquare$$

We define the weak model $\mathcal{D}_{\mathrm{\bar{1}}}^{\mathrm{wf}}$ in the same way as $\mathcal{D}_{\mathrm{\bar{1}}}^{\mathrm{sr}}$, but in terms of *weak* semantic operation $\widehat{\mathrm{op}}$ instead of strong ones:

**Definition 6.29** The function $\mathcal{D}_{\mathrm{\bar{1}}}^{\mathrm{wf}} : \mathcal{L}_{\mathrm{\bar{1}}} \to (\mathrm{SeVal} \to \mathbf{P}_{\mathrm{cl}})$ is defined by the following clauses:

**(i)** For $X \in \mathcal{X}_{\mathcal{P}}$ and $\zeta \in \mathrm{SeVal}$, $\mathcal{D}_{\mathrm{\bar{1}}}^{\mathrm{wf}}[\![X]\!](\zeta) = \zeta(X)$.

**(ii)** For $r \in 3$, $\mathrm{op} \in \mathbf{Fun}_{\mathrm{\bar{1}}}^{(r)}$, $S_0, \ldots, S_{r-1} \in \mathcal{L}_{\mathrm{\bar{1}}}$, and $\zeta \in \mathrm{SeVal}$,

$$\mathcal{D}_{\mathrm{\bar{1}}}^{\mathrm{wf}}[\![\mathrm{op}(S_0, \ldots, S_{r-1})]\!](\zeta) = \widehat{\mathrm{op}}(\mathcal{D}_{\mathrm{\bar{1}}}^{\mathrm{wf}}[\![S_0]\!](\zeta), \ldots, \mathcal{D}_{\mathrm{\bar{1}}}^{\mathrm{wf}}[\![S_{r-1}]\!](\zeta)).$$

**(iii)** For $X \in \mathcal{X}_{\mathcal{P}}$, $S \in \mathcal{G}_X$, and $\zeta \in \mathrm{SeVal}$,

$$\mathcal{D}_{\mathrm{\bar{1}}}^{\mathrm{wf}}[\![(\mu X.\ S)]\!](\zeta) = \mathrm{fix}_{\mathrm{l}}(\lambda p \in \mathbf{P}_{\mathrm{cl}}.\ \mathcal{D}_{\mathrm{\bar{1}}}^{\mathrm{wf}}[\![(\mu X.\ S)]\!](\zeta[p/X])). \blacksquare$$

We have the following proposition for $\mathcal{D}_{\mathrm{\bar{1}}}^{\mathrm{wf}}$ corresponding to Proposition 6.15 for $\mathcal{M}_{\mathrm{\bar{1}}}^{\mathrm{sr}}$ and Proposition 6.17 for $\mathcal{D}_{\mathrm{\bar{1}}}^{\mathrm{sr}}$:

**Proposition 6.21** *For every* $S \in \mathcal{L}_{\mathrm{\bar{1}}}$ *and* $\zeta_0, \zeta_1 \in \mathrm{SeVal}$ *with* $\zeta_0 \quad \mathrm{FV}(S) = \zeta_1 \quad \mathrm{FV}(S)$, *one has*

$$\mathcal{D}_{\mathrm{rf}}^{\mathrm{w}}[\![S]\!](\zeta_0) = \mathcal{D}_{\mathrm{rf}}^{\mathrm{w}}[\![S]\!](\zeta_1). \blacksquare$$

**Proof.** Similar to the proof of Proposition 6.15. $\blacksquare$

Using this proposition, we define a model $\dot{\mathcal{D}}_{\mathrm{\bar{1}}}^{\mathrm{wf}} : \mathcal{L}_{\mathrm{\bar{1}}}[\emptyset] \to \mathbf{P}_{\mathrm{cl}}$ from $\mathcal{D}_{\mathrm{\bar{1}}}^{\mathrm{wf}}$ by:

**Definition 6.30** Putting $\zeta_0 = (\lambda X \in \mathcal{X}_{\mathcal{P}}.\ \tilde{\mathbf{0}})$, we define $\dot{\mathcal{D}}_{\mathrm{\bar{1}}}^{\mathrm{wf}} : \mathcal{L}_{\mathrm{\bar{1}}}[\emptyset] \to \mathbf{P}_{\mathrm{cl}}$ as follows: For every $s \in \mathcal{L}_{\mathrm{\bar{1}}}[\emptyset]$,

$$\dot{\mathcal{D}}_{\mathrm{\bar{1}}}^{\mathrm{wf}}[\![s]\!] = \mathcal{D}_{\mathrm{\bar{1}}}^{\mathrm{wf}}[\![s]\!](\zeta_0). \blacksquare$$

A model $\mathcal{D}$ for $\mathcal{L}_{\mathrm{\bar{1}}}$ is said to be *compositional*, when the meaning of a composite statement under $\mathcal{D}$ is determined by the meanings of its components. More precisely, we give the following definition:

**Definition 6.31** (Compositionality of Models) Let $\mathcal{D}$ be a model for $\mathcal{L}_{\bar{1}}$, i.e., let $\mathcal{D} : \mathcal{L}_{\bar{1}} \to ((\mathcal{X}_{\mathcal{P}} \to \mathbf{P}) \to \mathbf{P})$ with $\mathbf{P}$ being some set. The model $\mathcal{D}$ is *compositional*, when for every $s_0, s_1 \in \mathcal{L}_{\bar{1}}[\emptyset]$, $X \in \mathcal{X}_{\mathcal{P}}$, and $S \in \mathcal{L}_{\bar{1}}$, the following holds:

$$\mathcal{D}[\![s_0]\!] = \mathcal{D}[\![s_1]\!] \Rightarrow \mathcal{D}[\![S[s_0/X]]\!] = \mathcal{D}[\![S[s_1/X]]\!]. \blacksquare \tag{6.73}$$

Compositionality in the above sense is an intrinsic property of metric or order-theoretic models defined in terms of semantic operations using some fixed-point construction for recursive statements. Thus, all the three models defined in this section are compositional, whose proof is standard in denotational semantics. Here we need the compositionality of $\mathcal{D}_{\bar{1}}^{\mathrm{wf}}$ for establishing its full abstractness in § 6.5:

**Lemma 6.12** (Compositionality of $\mathcal{D}_{\bar{1}}^{\mathrm{wf}}$) *The model $\mathcal{D}_{\bar{1}}^{\mathrm{wf}}$ is compositional.* ■

**Proof.** By Proposition 6.21, it suffices to show that the following holds for every $s \in \mathcal{L}_{\bar{1}}[\emptyset]$, $X \in \mathcal{L}_{\bar{1}}$, $S \in \mathcal{L}_{\bar{1}}$:

$$\mathcal{D}_{\bar{1}}^{\mathrm{wf}}[\![S[s/X]]\!](\zeta) = \mathcal{D}_{\bar{1}}^{\mathrm{wf}}[\![S]\!](\zeta[\dot{\mathcal{D}}_{\bar{1}}^{\mathrm{wf}}[\![s]\!]/X]). \blacksquare \tag{6.74}$$

For $S \in \mathcal{L}_{\bar{1}}$, let us write Compos($S$) to denote that

$$\begin{aligned}&\forall X \in \mathcal{X}_{\mathcal{P}}, \forall s \in \mathcal{L}_{\bar{1}}[\emptyset], \forall \zeta \in \mathrm{SeVal}[\\ &\quad \mathcal{D}_{\bar{1}}^{\mathrm{wf}}[\![S[s/X]]\!](\zeta) = \mathcal{D}_{\bar{1}}^{\mathrm{wf}}[\![S]\!](\zeta[\dot{\mathcal{D}}_{\bar{1}}^{\mathrm{wf}}[\![s]\!]/X])\ ].\end{aligned} \tag{6.75}$$

We will prove, by induction, that the following holds for every $n \in \omega$:

$$\forall S \in \mathcal{L}_{\bar{1}}(n)[\ \mathrm{Compos}(S)\ ]. \tag{6.76}$$

For $n = 0$, (6.76) holds vacuously. Suppose (6.76) holds for $n = k$. Let us prove (6.76) for $n = k + 1$. Let $S \in \mathcal{L}_{\bar{1}}(n + 1)$. Then, one of the following three propositions (6.77)-(6.79) holds:

$$S \in \mathcal{X}_{\mathcal{P}}. \tag{6.77}$$

$$\begin{aligned}&S \equiv \mathrm{op}(S_0, \ldots, S_{r-1}) \text{ with some } r \in 3,\ \mathrm{op} \in \mathbf{Fun}_{1}^{(r)}, \text{ and}\\ &S_0, \ldots, S_{r-1} \in \mathcal{L}_{\bar{1}}(n).\end{aligned} \tag{6.78}$$

$$S \equiv (\mu Y.\ \tilde{S}) \text{ with some } Y \in \mathcal{X}_{\mathcal{P}} \text{ and } \tilde{S} \in \mathcal{L}_{\bar{1}}(n). \tag{6.79}$$

When (6.77) or (6.78) holds, (6.75) immediately follows from the induction hypothesis.

Suppose (6.79) holds. Let $X \in \mathcal{X}_{\mathcal{P}}$. When $Y \equiv X$, one immediately obtains (6.75) from the induction hypothesis using Proposition 6.21. When $Y \not\equiv X$, one obtains (6.75) as follows: For every $s \in \mathcal{L}_{\bar{1}}[\emptyset]$ and $\zeta \in \mathrm{SeVal}$,

$$\begin{aligned}&\mathcal{D}_{\bar{1}}^{\mathrm{wf}}[\![S[s/X]]\!](\zeta) = \mathcal{D}_{\bar{1}}^{\mathrm{wf}}[\![(\mu Y.\ \tilde{S}[s/X]]\!](\zeta)\\ &= \mathrm{fix}_{\mathrm{I}}(\lambda p.\ \mathcal{D}_{\bar{1}}^{\mathrm{wf}}[\![\tilde{S}[s/X]]\!](\zeta[p/Y]))\\ &= \mathrm{fix}_{\mathrm{I}}(\lambda p.\ \mathcal{D}_{\bar{1}}^{\mathrm{wf}}[\![\tilde{S}]\!]((\zeta[\dot{\mathcal{D}}_{\bar{1}}^{\mathrm{wf}}[\![s]\!]/X])[p/Y]))\end{aligned}$$

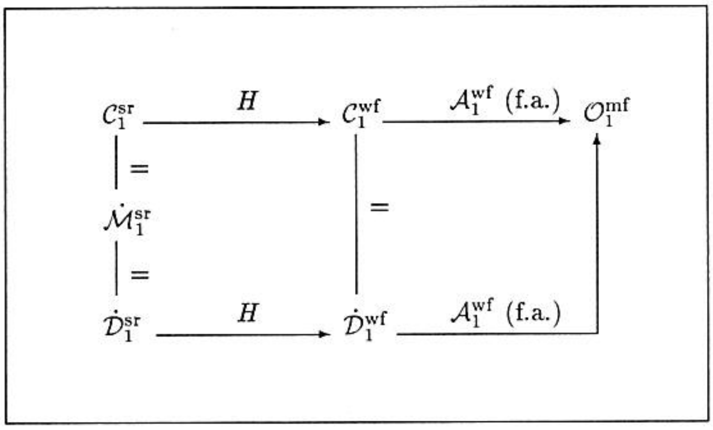


Figure 6.1: Connections between Semantic Models for $\mathcal{L}_1$

(since, by the induction hypothesis, one has

$$\begin{aligned}&\mathcal{D}_1^{\mathrm{wf}}[\![\tilde{S}[s/X]]\!](\zeta[p/Y])\\ &= \mathcal{D}_1^{\mathrm{wf}}[\![\tilde{S}]\!]((\zeta[p/Y])[\dot{\mathcal{D}}_1^{\mathrm{wf}}[\![s]\!]/X])\\ &= \mathcal{D}_1^{\mathrm{wf}}[\![\tilde{S}]\!]((\zeta[\dot{\mathcal{D}}_1^{\mathrm{wf}}[\![s]\!]/X])[p/Y])\ )\end{aligned}$$

$$= \mathcal{D}_1^{\mathrm{wf}}[\![(\mu Y.\ \tilde{S})]\!](\zeta[\dot{\mathcal{D}}_1^{\mathrm{wf}}[\![s]\!]/X]) = \mathcal{D}_1^{\mathrm{wf}}[\![S]\!](\zeta[\dot{\mathcal{D}}_1^{\mathrm{wf}}[\![s]\!]/X]). \blacksquare$$

## 6.5 Relating Semantic Models

In this section, we will relate the operational and denotational models defined in §§ 6.3 and 6.4, and thereby establish the full abstractness of $\mathcal{D}_1^{\mathrm{wf}}$ w.r.t. $\mathcal{O}_1^{\mathrm{mf}}$. The connections, including the full abstractness result stated above, between the semantic models are summarized in Figure 6.5, where (unlike the usual commutative diagrams) $\mathcal{C} \xrightarrow{\mathcal{A}\ (\mathrm{f.a.})}_1 \mathcal{O}$ denotes that $\mathcal{C}$ is fully abstract w.r.t. $\mathcal{O}$ with $\mathcal{A} \circ \mathcal{C} = \mathcal{O}$.

As a preliminary to the proofs of this section, the notion of a *syntactical valuation* is defined by:

**Definition 6.32** Partial functions from $\mathcal{X}_{\mathcal{P}}$ to $\mathcal{L}_1[\emptyset]$ are called *syntactical valuations.* Let

$$(\theta \in)\ \mathrm{SyVal} = (\mathcal{X}_{\mathcal{P}} \nrightarrow \mathcal{L}_1[\emptyset]). \blacksquare$$

### 6.5.1 Equivalence between $\mathcal{C}_{\bar{1}}^{\text{sr}}$ and $\mathcal{M}_{\bar{1}}^{\text{sr}}$

In this subsection, we will establish the equivalence between $\mathcal{C}_{\bar{1}}^{\text{sr}}$ and $\mathcal{M}_{\bar{1}}^{\text{sr}}$. In fact, we will prove the following lemma which is slightly more general than the equivalence:

**Lemma 6.13** **(1)** *For* $S \in \mathcal{L}_{\bar{1}}$, $\zeta \in \text{SeVal}$, *one has*

$$\begin{aligned}&\forall \theta \in \text{SyVal}[\ \text{FV}(S) \subseteq \text{dom}(\theta) \\ &\quad \Rightarrow \mathcal{C}_{\bar{1}}^{\text{sr}}[\![S[\theta]]\!] = \mathcal{M}_{\bar{1}}^{\text{sr}}[\![S]\!](\zeta[\mathcal{C}_{\bar{1}}^{\text{sr}} \circ \theta])\ ].\end{aligned} \tag{6.80}$$

**(2)** *For every* $s \in \mathcal{L}_{\bar{1}}[\emptyset]$, *one has* $\mathcal{C}_{\bar{1}}^{\text{sr}}[\![s]\!] = \mathcal{M}_{\bar{1}}^{\text{sr}}[\![s]\!]$. ∎

In the proof of the above lemma, the following lemma, which states the compositionality of $\mathcal{C}_{\bar{1}}^{\text{sr}}$ w.r.t. combinators in $\mathbf{Fun}_1$, will play a key role.

**Lemma 6.14** *For every* $r \in \{1,2\}$ *and* $\text{op} \in \mathbf{Fun}_1^{(r)}$, *one has*

$$\begin{aligned}&\forall s_0,\ldots,s_{r-1} \in \mathcal{L}_{\bar{1}}[\emptyset][\ \mathcal{C}_{\bar{1}}^{\text{sr}}[\![\text{op}(s_0,\ldots,s_{r-1})]\!] \\ &\qquad\qquad = \widetilde{\text{op}}(\mathcal{C}_{\bar{1}}^{\text{sr}}[\![s_0]\!],\ldots,\mathcal{C}_{\bar{1}}^{\text{sr}}[\![s_{r-1}]\!])\ ].\ \blacksquare\end{aligned} \tag{6.81}$$

We first prove Lemma 6.13 using Lemma 6.14, postponing the proof of Lemma 6.14.

**Proof of Lemma 6.13** **(1)** We will prove, by induction, that (6.82) below holds for every $n \in \omega$:

$$\textit{(6.80) holds for every } S \in \mathcal{L}_{\bar{1}}(n) \textit{ and } \zeta \in \text{SeVal}. \tag{6.82}$$

**Induction Base.** For $n = 0$, one has (6.82) vacuously, since $\mathcal{L}_{\bar{1}}(0) = \emptyset$.
**Induction Step.** Let $n \in \omega$, and suppose that (6.80) holds for every $S \in \mathcal{L}_{\bar{1}}(n)$. Fix $S \in \mathcal{L}_{\bar{1}}(n+1)$ and $\zeta \in \text{SeVal}$. Let us show (6.80) by distinguishing three cases according to the form of $S$.
**Case 1.** Suppose $S \in \mathcal{X}_{\mathcal{P}}$. Then one obtains (6.80) as follows: For every $\theta \in \text{SyVal}$ with $\text{dom}(\theta) \subseteq \text{FV}(S) = \{S\}$,

$$\mathcal{C}_{\bar{1}}^{\text{sr}}[\![S[\theta]]\!] = \mathcal{C}_{\bar{1}}^{\text{sr}}[\![\theta(S)]\!] = \mathcal{M}_{\bar{1}}^{\text{sr}}[\![S]\!](\zeta[\mathcal{C}_{\bar{1}}^{\text{sr}} \circ \theta]).$$

**Case 2.** Suppose $S \equiv \text{op}(S_0,\ldots,S_{r-1})$ with $r \in 3$, $\text{op} \in \mathbf{Fun}_1^{(r)}$, and $S_0,\ldots,S_{r-1} \in \mathcal{L}_{\bar{1}}(n)$. Then one has (6.80) as follows:

$$\begin{aligned}&\mathcal{C}_{\bar{1}}^{\text{sr}}[\![S[\theta]]\!] = \mathcal{C}_{\bar{1}}^{\text{sr}}[\![\text{op}(S_0[\theta],\ldots,S_{r-1}[\theta])]\!] \\ &= \widetilde{\text{op}}(\mathcal{C}_{\bar{1}}^{\text{sr}}[\![S_0[\theta]]\!],\ldots,\mathcal{C}_{\bar{1}}^{\text{sr}}[\![S_{r-1}[\theta]]\!]) \quad \text{(by Lemma 6.14)} \\ &= \widetilde{\text{op}}(\mathcal{M}_{\bar{1}}^{\text{sr}}[\![S_0]\!](\zeta[\mathcal{C}_{\bar{1}}^{\text{sr}} \circ \theta]),\ldots,\mathcal{M}_{\bar{1}}^{\text{sr}}[\![S_{r-1}]\!](\zeta[\mathcal{C}_{\bar{1}}^{\text{sr}} \circ \theta])) \\ &\qquad \text{(by the induction hypothesis)} \\ &= \mathcal{M}_{\bar{1}}^{\text{sr}}[\![S]\!](\zeta[\mathcal{C}_{\bar{1}}^{\text{sr}} \circ \theta]).\end{aligned}$$

**Case 3.** Suppose $S \equiv (\mu X.\ \tilde{S})$ for some $X \in \mathcal{X}_{\mathcal{P}}$ and $\tilde{S} \in \mathcal{G}(n) \subseteq \mathcal{L}_{\bar{1}}(n)$. Then, first one has

$$\begin{aligned}
&\mathcal{C}_{\bar{1}}^{\mathrm{sr}}[\![S[\theta]]\!] = \mathcal{C}_{\bar{1}}^{\mathrm{sr}}[\![(\mu X.\ \tilde{S}[\tilde{\theta}])]\!] \quad (\text{where } \tilde{\theta} = \theta \quad \mathrm{FV}(\tilde{S}))\\
&= \mathcal{C}_{\bar{1}}^{\mathrm{sr}}[\![(\tilde{S}[\tilde{\theta}])[S[\theta]/X]]\!]\\
&= \mathcal{C}_{\bar{1}}^{\mathrm{sr}}[\![\tilde{S}[\theta[S[\theta]/X]]]\!]\\
&= \mathcal{M}_{\bar{1}}^{\mathrm{sr}}[\![\tilde{S}]\!](\zeta[\mathcal{C}_{\bar{1}}^{\mathrm{sr}} \circ \theta[S[\theta]/X]]) \quad (\text{by the induction hypothesis})\\
&= \mathcal{M}_{\bar{1}}^{\mathrm{sr}}[\![\tilde{S}]\!]((\zeta[\mathcal{C}_{\bar{1}}^{\mathrm{sr}} \circ \theta])[\mathcal{C}_{\bar{1}}^{\mathrm{sr}}[\![S[\theta]]\!]/X]).
\end{aligned}$$

Thus,

$$\mathcal{C}_{\bar{1}}^{\mathrm{sr}}[\![S[\theta]]\!] = \mathcal{M}_{\bar{1}}^{\mathrm{sr}}[\![\tilde{S}]\!]((\zeta[\mathcal{C}_{\bar{1}}^{\mathrm{sr}} \circ \theta])[\mathcal{C}_{\bar{1}}^{\mathrm{sr}}[\![S[\theta]]\!]/X]).$$

By this and the fact (6.49), one has

$$\begin{aligned}
&\mathcal{C}_{\bar{1}}^{\mathrm{sr}}[\![S[\theta]]\!] = \mathrm{fix}(\lambda p.\ \mathcal{M}_{\bar{1}}^{\mathrm{sr}}[\![\tilde{S}]\!]((\zeta[\mathcal{C}_{\bar{1}}^{\mathrm{sr}} \circ \theta])[p/X]))\\
&= \mathcal{M}_{\bar{1}}^{\mathrm{sr}}[\![\tilde{S}]\!](\zeta[\mathcal{C}_{\bar{1}}^{\mathrm{sr}} \circ \theta])
\end{aligned}$$

(since (6.47) holds with $\mathcal{M}$ replaced by $\mathcal{M}_{\bar{1}}^{\mathrm{sr}}$ by Proposition 6.11).

**(2)** This part immediately follows from part (1). ■

Now let us prove Lemma 6.14. For the proof, we need a kind of *distributivity* of semantic operations $\tilde{\|}$, $\tilde{;}$, and $\tilde{\partial}_C$; first, we will establish the distributivity.

The operation $\tilde{\|}$ is *distributive* w.r.t. $\hat{\cup}$:

**Proposition 6.22** $\forall p, p_1, p_2 \in \mathbf{P}_{\mathrm{cl}}[\ p \tilde{\|} (p_1 \hat{\cup} p_2) = (p \tilde{\|} p_1) \hat{\cup} (p \tilde{\|} p_2)\ ]$. ■

**Proof.** See § 6.A.1. ■

The functions $\tilde{;}_{\mathrm{b}}$ and $\tilde{;}$ are distributive w.r.t. $\hat{\cup}$ in the first argument:

**Proposition 6.23** **(1)** *For* $p_0, p_1, p' \in \wp(\mathbf{Q})$,

$$\tilde{;}_{\mathrm{b}}(p_0 \hat{\cup} p_1, p') = \bigcup_{j \in 2} [\tilde{;}_{\mathrm{b}}(p_j, p')].$$

**(2)** *For* $p_0, p_1, p' \in \mathbf{P}_{\mathrm{cl}}$,

$$\tilde{;}(p_0 \hat{\cup} p_1, p') = \bigcup_{j \in 2} [\tilde{;}(p_j, p')]. \ ■$$

**Proof.** See § 6.A.2. ■

For each $C \in \wp(\mathbf{C}_1)$, the semantic operation $\tilde{\partial}_C$ is distributive w.r.t. $\hat{\cup}$:

**Proposition 6.24** *Let* $C \in \wp(\mathbf{C}_1)$.

**(1)** $\forall p_0, p_1 \in \wp(\mathbf{Q})[\ \tilde{\partial}_C^{\mathrm{b}}(p_0 \hat{\cup} p_1) = \tilde{\partial}_C^{\mathrm{b}}(p_0) \hat{\cup} \tilde{\partial}_C^{\mathrm{b}}(p_1)\ ]$.

**(2)** $\forall p_0, p_1 \in \mathbf{P}_{\mathrm{cl}}[\ \tilde{\partial}_C(p_0 \hat{\cup} p_1) = \tilde{\partial}_C(p_0) \hat{\cup} \tilde{\partial}_C(p_1)\ ]$. ■

**Proof.** See § 6.A.3. ■

**Proof of Lemma 6.14** Here we will prove (6.81) for op $\equiv \|$ (for the other combinators, the claim can be established in a similar fashion). Let us define two mappings $F, G : (\mathcal{L}_{\tilde{1}}[\emptyset])^2 \to \mathbf{P}_{\mathrm{cl}}$ as follows: For $(s_0, s_1) \in (\mathcal{L}_{\tilde{1}}[\emptyset])^2$,

$$\text{(i) } F(s_0, s_1) = \mathcal{C}_1^{\mathrm{sr}}[\![s_0 \| s_1]\!], \quad \text{(ii) } G(s_0, s_1) = \tilde{\|}(\mathcal{C}_1^{\mathrm{sr}}[\![s_0]\!], \mathcal{C}_1^{\mathrm{sr}}[\![s_1]\!]). \tag{6.83}$$

Let us show that both $F$ and $G$ are the fixed-point of some higher-order mapping

$$\Phi_{\|} : ((\mathcal{L}_{\tilde{1}}[\emptyset])^2 \to \mathbf{P}_{\mathrm{cl}}) \to ((\mathcal{L}_{\tilde{1}}[\emptyset])^2 \to \mathbf{P}_{\mathrm{cl}}),$$

from which (6.81) for op $\equiv \|$ immediately follows.

Let $s_0, s_1$. First, we observe

$$\mathcal{B}(\mathcal{C}_1^{\mathrm{sr}}[\![s_0 \| s_1]\!]) = \mathcal{B}(\tilde{\|}(\mathcal{C}_1^{\mathrm{sr}}[\![s_0]\!], \mathcal{C}_1^{\mathrm{sr}}[\![s_0]\!])) = \tilde{\|}_{\mathrm{b}}(\mathcal{B}(\mathcal{C}_1^{\mathrm{sr}}[\![s_0]\!]), \mathcal{B}(\mathcal{C}_1^{\mathrm{sr}}[\![s_0]\!])). \tag{6.84}$$

(For the definition of $\mathcal{B}(\cdot)$, see Definition 6.1.) For every $c \in \mathbf{C}_1$,

$$\begin{aligned}
&F(s_0, s_1)[\langle c \rangle] \\
&= \bigcup \{\psi[\mathcal{C}_1^{\mathrm{sr}}[\![s']\!]] : (s_0 \| s_1) \xrightarrow{c}_1 s'\} \\
&= \bigcup \{\psi[\mathcal{C}_1^{\mathrm{sr}}[\![s_0' \| s_1]\!]] : s_0 \xrightarrow{c}_1 s_0'\} \cup \bigcup \{\psi[\mathcal{C}_1^{\mathrm{sr}}[\![s_0 \| s_1']\!]] : s_1 \xrightarrow{c}_1 s_1'\} \\
&= \psi[\bigcup \{F(s_0', s_1) : s_0 \xrightarrow{c}_1 s_0'\} \cup \bigcup \{F(s_0, s_1') : s_1 \xrightarrow{c}_1 s_1'\}].
\end{aligned}$$

Also,

$$\begin{aligned}
&F(s_0, s_1)[\tau] \\
&= \bigcup \{\psi[\mathcal{C}_1^{\mathrm{sr}}[\![s']\!]] : (s_0 \| s_1) \xrightarrow{\tau}_1 s'\} \\
&= \bigcup \{\psi[\mathcal{C}_1^{\mathrm{sr}}[\![s_0' \| s_1]\!]] : s_0 \xrightarrow{\tau}_1 s_0'\} \cup \bigcup \{\psi[\mathcal{C}_1^{\mathrm{sr}}[\![s_0 \| s_1']\!]] : s_1 \xrightarrow{\tau}_1 s_1'\} \\
&\quad \bigcup_{c \in \mathrm{cact}(s_0) \cap \overline{\mathrm{cact}(s_1)}} [\psi[\bigcup \{\mathcal{C}_1^{\mathrm{sr}}[\![s_0' \| s_1']\!] : s_0 \xrightarrow{c}_1 s_0' \wedge s_0 \xrightarrow{\bar{c}}_1 s_0'\}]] \\
&= \psi[\bigcup \{F(s_0', s_1) : s_0 \xrightarrow{\tau}_1 s_0'\} \cup \bigcup \{F(s_0, s_1') : s_1 \xrightarrow{\tau}_1 s_1'\} \\
&\qquad \cup \bigcup_{c \in \mathrm{cact}(s_0) \cap \overline{\mathrm{cact}(s_1)}} [\bigcup \{F(s_0', s_1') : s_0 \xrightarrow{c}_1 s_0' \wedge s_0 \xrightarrow{\bar{c}}_1 s_0'\}]].
\end{aligned}$$

Now we define $\Phi_{\|}$ so that $F = \Phi_{\|}(F)$ as follows: For $\tilde{F} \in ((\mathcal{L}_{\tilde{1}}[\emptyset])^2 \to \mathbf{P}_{\mathrm{cl}})$, and $(s_0, s_1) \in (\mathcal{L}_{\tilde{1}}[\emptyset])^2$,

$$\begin{aligned}
&\Phi_{\|}(\tilde{F})(s_0, s_1) \\
&= \tilde{\|}_{\mathrm{b}}(\mathcal{B}(\mathcal{C}_1^{\mathrm{sr}}[\![s_0]\!]), \mathcal{B}(\mathcal{C}_1^{\mathrm{sr}}[\![s_0]\!])) \\
&\quad \cup \bigcup_{c \in \mathbf{C}_1} [\langle c \rangle \cdot (\psi[\bigcup \{F(s_0', s_1) : s_0 \xrightarrow{c}_1 s_0'\} \\
&\qquad\qquad \cup \bigcup \{F(s_0, s_1') : s_1 \xrightarrow{c}_1 s_1'\}] \;)] \\
&\quad \cup \langle \tau \rangle \cdot \psi[\bigcup \{F(s_0', s_1) : s_0 \xrightarrow{\tau}_1 s_0'\} \\
&\qquad \cup \bigcup \{F(s_0, s_1') : s_1 \xrightarrow{\tau}_1 s_1'\} \\
&\qquad \cup \bigcup_{c \in \mathrm{cact}(s_0) \cap \overline{\mathrm{cact}(s_1)}} [ \\
&\qquad\quad \bigcup \{F(s_0', s_1') : s_0 \xrightarrow{c}_1 s_0' \wedge s_1 \xrightarrow{\bar{c}}_1 s_1'\}]].
\end{aligned} \tag{6.85}$$

Obviously $\Phi_{\|}$ is a contraction on $((\mathcal{L}_{\tilde{1}}[\emptyset])^2 \to \mathbf{P}_{\mathrm{cl}})$, and therefore, $F = \mathrm{fix}_{\mathrm{I}}(\Phi_{\|})$.

Using (6.83) and Proposition 6.22, we can easily check that $G = \Phi_{\|}(G)$. Thus, we have $F = \mathrm{fix}_{\mathrm{I}}(\Phi_{\|}) = G$, i.e., (6.81) for op $\equiv \|$ holds. ∎

### 6.5.2 Equivalence between $\mathcal{M}_1^{\mathrm{sr}}$ and $\mathcal{D}_1^{\mathrm{sr}}$

**Lemma 6.15** *For every $S \in \mathcal{L}_{\tilde{1}}$, one has*

$$\forall \zeta \in \mathrm{SeVal}[\ \mathcal{M}_1^{\mathrm{sr}}[\![S]\!](\zeta) = \mathcal{D}_1^{\mathrm{sr}}[\![S]\!](\zeta)\ ].\ \blacksquare \tag{6.86}$$

**Proof.** We will prove this by induction on the structure of $S$, or more precisely, we will prove, by induction, that the following holds for every $N \in \omega$:

$$\forall S \in \mathcal{L}_{\tilde{1}}(N), \forall \zeta \in \mathrm{SeVal}[\ \mathcal{M}_1^{\mathrm{sr}}[\![S]\!](\zeta) = \mathcal{D}_1^{\mathrm{sr}}[\![S]\!](\zeta)\ ]. \tag{6.87}$$

**Induction Base.** For $N = 0$, (6.87) holds vacuously, since $\mathcal{L}_{\tilde{1}}(0) = \emptyset$.
**Induction Step.** Let $n \in \omega$ and suppose that (6.87) holds for $N = n$. We will prove that (6.87) holds for $N = n + 1$. Let $S \in \mathcal{L}_{\tilde{1}}(n + 1)$. We will prove (6.86). By the definition of $\mathcal{L}_{\tilde{1}}(n + 1)$, one of the following three conditions holds:

$$S \in \mathcal{X}_{\mathcal{P}}; \tag{6.88}$$

$$S \equiv \mathsf{op}(\tilde{S}_0, \ldots, \tilde{S}_{r-1}) \text{ with op being an } r\text{-ary combinator } (r \in 3), \text{ and } \tilde{S}_i \in \mathcal{L}_{\tilde{1}}(n); \tag{6.89}$$

$$S \equiv (\mu X.\ \tilde{S}) \text{ with some } X \in \mathcal{X}_{\mathcal{P}} \text{ and } \tilde{S} \in \mathcal{L}_{\tilde{1}}(n). \tag{6.90}$$

In the case (6.88) holds, (6.86) immediately follows from the definitions of $\mathcal{M}_1^{\mathrm{sr}}$ and $\mathcal{D}_1^{\mathrm{sr}}$.

In the case (6.89) holds, (6.86) immediately follows from the induction hypothesis.

In the case (6.90) holds, one obtains (6.86) as follows: Let $\zeta \in \mathrm{SeVal}$ and put

$$\Phi = (\lambda p \in \mathbf{P}_{\mathrm{cl}}.\ \mathcal{M}_1^{\mathrm{sr}}[\![\tilde{S}]\!](\zeta[p/X])),$$

$$\Psi = (\lambda p \in \mathbf{P}_{\mathrm{cl}}.\ \mathcal{D}_1^{\mathrm{sr}}[\![\tilde{S}]\!](\zeta[p/X])).$$

Then $\Phi = \Psi$, by the induction hypothesis, and therefore,

$$\begin{aligned}
&\mathcal{M}_1^{\mathrm{sr}}[\![S]\!](\zeta) = \mathrm{fix}(\Phi) = \lim_{n\in\omega}[\,\Phi^n(\tilde{\bot})\,] \\
&= \lim_{n\in\omega}[\,\Psi^n(\tilde{\bot})\,] = \bigsqcup_{n\in\omega}[\,\Psi^n(\tilde{\bot})\,] \quad \text{(by Lemma 6.1)} \\
&= \mathrm{fix}_{\mathrm{l}}(\Psi) = \mathcal{D}_1^{\mathrm{sr}}[\![S]\!](\zeta).
\end{aligned}$$

Thus, one has (6.86). ■

### 6.5.3 The Relationship between $\mathcal{D}_1^{\mathrm{sr}}$ and $\mathcal{D}_1^{\mathrm{wf}}$

In this section, we will show that for every statement $s$, $\dot{\mathcal{D}}_1^{\mathrm{wf}}[\![s]\!]$ is obtained from $\dot{\mathcal{D}}_1^{\mathrm{sr}}[\![s]\!]$ by applying $H$ (which is the claim of Lemma 6.20 (2) below). In the proof of this fact, a key role is played by the notion of *neutrality* defined by:

**Definition 6.33** For $r \in \omega$ and $f \in (\mathbf{P}_{\mathrm{cl}})^r \to \mathbf{P}_{\mathrm{cl}}$, we say the hiding operation $H$ is *neutral* w.r.t. $f$ iff the following holds:

$$\forall p_0, \ldots, p_{r-1} \in \mathbf{P}_{\mathrm{cl}}[\ H(f(H(p_0), \ldots, H(p_{r-1}))) = H(f(p_0, \ldots, p_{r-1})). \blacksquare$$

The following four lemmas together states that the hiding operation $H$ is neutral w.r.t. all the semantic operations.

**Lemma 6.16** (Neutrality of Hiding w.r.t. $\tilde{+}$)

$$\forall p_0, p_1 \in \mathbf{P}_{\mathrm{cl}}[\ H(H(p_0)\tilde{+}H(p_1)) = H(p_0\tilde{+}p_1)\ ]. \blacksquare$$

**Proof.** See § 6.B.1. ∎

**Lemma 6.17** (Neutrality of Hiding w.r.t. $\tilde{\|}$)

$$\forall p, p' \in \mathbf{P}_{\mathrm{cl}}[\ H(H(p)\tilde{\|}H(p')) = H(p\tilde{\|}p')\ ]. \blacksquare$$

**Proof.** See § 6.B.2. ∎

**Lemma 6.18** (Neutrality of Hiding w.r.t. $\tilde{;}$)

$$\forall p, p' \in \mathbf{P}_{\mathrm{cl}}[\ H(H(p)\,\tilde{;}\,H(p')) = H(p\,\tilde{;}\,p')\ ]. \blacksquare$$

**Proof.** See § 6.B.3. ∎

**Lemma 6.19** (Neutrality of Hiding w.r.t. $\widetilde{\partial}_C$) *For every* $C \in \wp(\mathbf{C}_1)$,

$$\forall p \in \mathbf{P}_{\mathrm{cl}}[\ H(\widetilde{\partial}_C(H(p))) = H(\widetilde{\partial}_C(p))\ ]. \blacksquare$$

**Proof.** See § 6.B.4 ∎

Summarizing Lemmas 6.16–6.19, we have established that the hiding operations $H$ is neutral w.r.t. all the semantic operations $\tilde{+}$, $\tilde{\|}$, $\tilde{+}$, and $\widetilde{\partial}_C$ ($C \in \wp(\mathbf{C}_1)$). Using this fact, we can show the following connection between $\mathcal{D}_{\bar{1}}^{\mathrm{sr}}$ and $\mathcal{D}_{\bar{1}}^{\mathrm{wf}}$:

**Lemma 6.20 (1)** *For every* $S \in \mathcal{L}_{\bar{1}}$, *one has*

$$\forall \zeta \in \mathrm{SeVal}[\ H(\mathcal{D}_{\bar{1}}^{\mathrm{sr}}[\![S]\!](\zeta)) = \mathcal{D}_{\bar{1}}^{\mathrm{wf}}[\![S]\!](H \circ \zeta)\ ]. \tag{6.91}$$

**(2)** *For every* $s \in \mathcal{L}_{\bar{1}}[\emptyset]$, *one has*

$$H(\dot{\mathcal{D}}_{\bar{1}}^{\mathrm{sr}}[\![s]\!]) = \dot{\mathcal{D}}_{\bar{1}}^{\mathrm{wf}}[\![s]\!]. \blacksquare \tag{6.92}$$

**Proof. (1)** We will prove, by induction, that the following holds for every $N \in \omega$:

$$\forall S \in \mathcal{L}_{\bar{1}}(N), \forall \zeta \in \mathrm{SeVal}[\ H(\mathcal{D}_{\bar{1}}^{\mathrm{sr}}[\![S]\!](\zeta)) = \mathcal{D}_{\bar{1}}^{\mathrm{wf}}[\![S]\!](H \circ \zeta)\ ]. \tag{6.93}$$

**Induction Base.** For $N = 0$, (6.93) holds vacuously since $\mathcal{L}_{\bar{1}}(0) = \emptyset$.
**Induction Step.** Let $n \in \omega$ and suppose that (6.93) holds for $N = n$. We will prove (6.93) for $N = n + 1$. Let $S \in \mathcal{L}_{\bar{1}}(n + 1)$. Let us prove (6.91). By the definition of $\mathcal{L}_{\bar{1}}(n + 1)$, one of the following three conditions holds:

$$S \in \mathcal{X}_{\mathcal{P}}, \tag{6.94}$$

$$S \equiv \mathrm{op}(\tilde{S}_0, \ldots, \tilde{S}_{r-1}) \text{ with op being an } r\text{-ary combinator } (r \in 3), \text{ and } \tilde{S}_i \in \mathcal{L}_{\bar{1}}(n), \tag{6.95}$$

$$S \equiv (\mu X.\, \tilde{S}) \text{ with some } X \in \mathcal{X}_{\mathcal{P}} \text{ and } \tilde{S} \in \mathcal{L}_{\bar{1}}(n). \tag{6.96}$$

We distinguish three cases according to which of the above conditions is satisfied.

**Case 1.** Suppose (6.94) holds. Then, For every $\zeta \in \mathrm{SeVal}$, one has

$$H(\mathcal{D}_{\bar{1}}^{\mathrm{sr}}[\![S]\!](\zeta)) = H(\zeta(S)) = (H \circ \zeta)(S) = \mathcal{D}_{\bar{1}}^{\mathrm{wf}}[\![S]\!](H \circ \zeta).$$

Thus, one has (6.91).

**Case 2.** Suppose (6.95) holds. Then, for every $\zeta \in \mathrm{SeVal}$,

$$\begin{aligned}
&H(\mathcal{D}_{\bar{1}}^{\mathrm{sr}}[\![S]\!](\zeta)) = H(\mathcal{D}_{\bar{1}}^{\mathrm{sr}}[\![F(\tilde{S}_0, \ldots, \tilde{S}_{r-1})]\!](\zeta)) \\
&= H(\widetilde{\mathrm{op}}(\mathcal{D}_{\bar{1}}^{\mathrm{sr}}[\![\tilde{S}_0]\!](\zeta), \ldots, \mathcal{D}_{\bar{1}}^{\mathrm{sr}}[\![\tilde{S}_{r-1}]\!](\zeta))) \\
&= H(\widetilde{\mathrm{op}}(H(\mathcal{D}_{\bar{1}}^{\mathrm{sr}}[\![\tilde{S}_0]\!](\zeta)), \ldots, H(\mathcal{D}_{\bar{1}}^{\mathrm{sr}}[\![\tilde{S}_{r-1}]\!](\zeta)))) \\
&\qquad \text{(by the neutrality of } H \text{ w.r.t. } \widetilde{\mathrm{op}}) \\
&= \widehat{\mathrm{op}}(\mathcal{D}_{\bar{1}}^{\mathrm{wf}}[\![\tilde{S}_0]\!](H \circ \zeta), \ldots, \mathcal{D}_{\bar{1}}^{\mathrm{wf}}[\![\tilde{S}_{r-1}]\!](H \circ \zeta)) \\
&\qquad \text{(by the definition of } \widehat{\mathrm{op}} \text{ and the induction hypothesis)} \\
&= \mathcal{D}_{\bar{1}}^{\mathrm{wf}}[\![\mathrm{op}(\tilde{S}_0, \ldots, \tilde{S}_{r-1})]\!](H \circ \zeta) = \mathcal{D}_{\bar{1}}^{\mathrm{wf}}[\![S]\!](H \circ \zeta).
\end{aligned}$$

Thus, one has (6.91).

**Case 3.** Suppose (6.96) holds, and let $\zeta \in \mathrm{SeVal}$. Then, putting

$$\Phi = (\lambda p \in \mathbf{P}_{\mathrm{cl}}.\, \mathcal{D}_{\bar{1}}^{\mathrm{sr}}[\![\tilde{S}]\!](\zeta[p/X])),$$

one has

$$\begin{aligned}
&H(\mathcal{D}_{\bar{1}}^{\mathrm{sr}}[\![S]\!](\zeta)) = H(\mathrm{fix}_{\mathrm{l}}(\Phi)) = H(\textstyle\bigsqcup_{i \in \omega}[\,\Phi^i(\{\langle\bot\rangle\})\,]) \\
&= \textstyle\bigsqcup_{i \in \omega}[\,H(\Phi^i(\{\langle\bot\rangle\}))\,] \quad \text{(by the continuity of } H).
\end{aligned} \tag{6.97}$$

Also, putting

$$\Psi = (\lambda p \in \mathbf{P}_{\mathrm{cl}}.\, \mathcal{D}_{\bar{1}}^{\mathrm{wf}}[\![\tilde{S}]\!]((H \circ \zeta)[p/X])),$$

one has

$$\mathcal{D}_{\bar{1}}^{\mathrm{wf}}[\![S]\!](H \circ \zeta) = \mathrm{fix}_{\mathrm{l}}(\Psi) = \bigsqcup_{i \in \omega}[\Psi^i(\{\langle\bot\rangle\})]. \tag{6.98}$$

Thus, it suffices, for showing (6.91), to show that the following holds for every $i \in \omega$:

$$H(\Phi^i(\{\langle\bot\rangle\})) = \Psi^i(\{\langle\bot\rangle\}). \tag{6.99}$$

We will show this by induction on $i$.

Obviously (6.99) holds for $i = 0$. Let $j \in \omega$ and suppose that (6.99) holds for $i = j$. Then,

$$\Phi^{j+1}(\{\langle\bot\rangle\}) = \Phi(\Phi^j(\{\langle\bot\rangle\})) = \mathcal{D}_{\bar{1}}^{\mathrm{sr}}[\![\tilde{S}]\!](\zeta[\Phi^j(\{\langle\bot\rangle\})/X]),$$

and therefore,

$$H(\Phi^{j+1}(\{\langle\bot\rangle\})) = H(\mathcal{D}_1^{\mathrm{sr}}[\![\tilde{S}]\!](\zeta[\Phi^j(\{\langle\bot\rangle\})/X]))$$
$$= \mathcal{D}_1^{\mathrm{wf}}[\![\tilde{S}]\!](H \circ (\zeta[\Phi^j(\{\langle\bot\rangle\})/X])) \quad \text{(by the induction hypothesis)}$$
$$= \mathcal{D}_1^{\mathrm{wf}}[\![\tilde{S}]\!]((H \circ \zeta)[\Psi^j(\{\langle\bot\rangle\})/X])$$

(because

$$H \circ (\zeta[\Phi^i(\{\langle\bot\rangle\})/X]) = (H \circ \zeta)[H(\Phi^i(\{\langle\bot\rangle\}))/X]$$
$$= (H \circ \zeta)[\Psi^i(\{\langle\bot\rangle\})/X] \quad \text{(by (6.99) for } i = j) \text{ )}$$

$$= \Psi(\Psi^j(\{\langle\bot\rangle\})) = \Psi^{j+1}(\{\langle\bot\rangle\}).$$

Hence, one has (6.99) for $i = j+1$. Thus, for every $i \in \omega$, (6.99) holds, and therefore, one has (6.91).

**(2)** This part immediately follows from part (1). ■

### 6.5.4 Equivalence between $\mathcal{C}_1^{\mathrm{wf}}$ and $\mathcal{D}_1^{\mathrm{wf}}$

The following lemma immediately follows from Definition 6.12, Lemma 6.13, Lemma 6.15, and Lemma 6.20:

**Lemma 6.21** *For every $s \in \mathcal{L}_{\bar{1}}[\emptyset]$, one has*

$$\mathcal{C}_1^{\mathrm{wf}}[\![s]\!] = \dot{\mathcal{D}}_1^{\mathrm{wf}}[\![s]\!]. \blacksquare$$

**Proof.** Let $s \in \mathcal{L}_{\bar{1}}[\emptyset]$. Then

$$\mathcal{C}_1^{\mathrm{wf}}[\![s]\!] = H(\mathcal{C}_1^{\mathrm{sr}}[\![s]\!]) \quad \text{(by Definition 6.12)}$$
$$= H(\dot{\mathcal{M}}_1^{\mathrm{sr}}[\![s]\!]) \quad \text{(by Lemma 6.13)}$$
$$= H(\dot{\mathcal{D}}_1^{\mathrm{sr}}[\![s]\!]) \quad \text{(by Lemma 6.15)}$$
$$= \dot{\mathcal{D}}_1^{\mathrm{wf}}[\![s]\!] \quad \text{(by Lemma 6.20 (2))}. \blacksquare$$

### 6.5.5 Full Abstractness of $\mathcal{D}_1^{\mathrm{wf}}$ w.r.t. $\mathcal{O}_1^{\mathrm{mf}}$

The order-theoretic model $\mathcal{D}_1^{\mathrm{wf}}$ is *fully abstract* w.r.t. the flattened weak linear model $\mathcal{O}_1^{\mathrm{mf}}$, provided that the set $\mathbf{C}_1$ of communication actions is infinite:

**Theorem 6.1** *If $\mathbf{C}_1$ is infinite, then for every $s_0, s_1 \in \mathcal{L}_{\bar{1}}[\emptyset]$, one has*

$$\begin{array}{l} \dot{\mathcal{D}}_1^{\mathrm{wf}}[\![s_0]\!] = \dot{\mathcal{D}}_1^{\mathrm{wf}}[\![s_1]\!] \Leftrightarrow \\ \forall X, \forall S \in \mathcal{L}_{\bar{1}}[X][\ \mathcal{O}_1^{\mathrm{mf}}[\![S[s_0/X]]\!] = \mathcal{O}_1^{\mathrm{mf}}[\![S[s_1/X]]\!]\ ]. \blacksquare \end{array} \tag{6.100}$$

**Proof.** Suppose $\mathbf{C}_1$ is infinite, and let $s_0, s_1 \in \mathcal{L}_{\bar{1}}[\emptyset]$.

The $\Leftarrow$-part of (6.100) immediately follows from Lemma 6.4 and Lemma 6.21.

Also, we obtain the $\Rightarrow$-part of (6.100) from Proposition 6.4, Lemma 6.12, and Lemma 6.21 as follows: If

$$\dot{\mathcal{D}}_1^{\rm wf}[\![s_0]\!] = \dot{\mathcal{D}}_1^{\rm wf}[\![s_1]\!], \tag{6.101}$$

then for every $X \in \mathcal{X}_{\mathcal{P}}$, $S \in \mathcal{L}_{\hat{1}}[X]$, $\zeta \in$ SeVal, and $i \in 2$,

$$\begin{aligned}
&\mathcal{O}_1^{\rm mf}[\![S[s_i/X]]\!] = \mathcal{A}_1^{\rm wf}(\mathcal{C}_1^{\rm wf}[\![S[s_i/X]]\!]) \quad \text{(by Proposition 6.4)}\\
&= \mathcal{A}_1^{\rm wf}(\dot{\mathcal{D}}_1^{\rm wf}[\![S[s_i/X]]\!]) \quad \text{(by Lemma 6.21)}\\
&= \mathcal{A}_1^{\rm wf}(\mathcal{D}_1^{\rm wf}[\![S]\!](\zeta[\mathcal{D}_1^{\rm wf}[\![s_i]\!]/X])) \quad \text{(by Lemma 6.12).}
\end{aligned}$$

From this and (6.101), it follows that

$$\mathcal{O}_1^{\rm mf}[\![S[s_0/X]]\!] = \mathcal{O}_1^{\rm mf}[\![S[s_1/X]]\!]. \blacksquare$$

## 6.6 Comparison of a Variant of $\mathcal{D}_1^{\rm wf}$ and Other Models

We conclude this chapter with a comparison between a slightly simplified variant $\mathcal{D}_{\hat{1}}^{\rm wf}$ of the fully abstract model $\mathcal{D}_1^{\rm wf}$ given in § 6.4.5, and two well-known models for communicating processes: the *improved failures model* $\mathcal{N}$ of Brookes and Roscoe ([BR 84]) and the *strong Acceptance tree model* $\mathbf{AT}_{\rm s}$ of Hennessy ([Hen 88]); these three models $\mathcal{D}_{\hat{1}}^{\rm wf}$, $\mathcal{N}$, and $\mathbf{AT}_{\rm S}$ are shown to be isomorphic. We also give a comparison between these models and the two models $\mathcal{C}_1^{\rm w}$ and $\mathcal{C}_1^{\rm wi}$ introduced in Chapter 5.

To give the comparison, we replace the choice combinator '+' by the *internal choice operator* '$\oplus$' with the following two transition axioms:

$$\text{(i) } (s_0 \oplus s_1) \xrightarrow{\tau}_1 s_0, \quad \text{(ii) } (s_0 \oplus s_1) \xrightarrow{\tau}_1 s_1. \tag{6.102}$$

Thus the distinction between a *direct* and *indirect* deadlocks is no longer necessary (this distinction has been needed only for defining denotational interpretation of '+'); we obtain a slightly modified language $\mathcal{L}_{\hat{1}}$, for which denotational models are constructed more easily.

**Definition 6.34** Let $\mathbf{Sig}_{\hat{1}} = (\mathbf{Fun}_{\hat{1}}, \text{arity}_{\hat{1}})$ with

$$\mathbf{Fun}_{\hat{1}} = (\mathbf{Fun}_1 \setminus \{\text{'+'}\}) \cup \{\text{'}\oplus\text{'}\},$$

and for op $\in \mathbf{Fun}_{\hat{1}}$, let

$$\text{arity}_{\hat{1}}(\text{op}) = \begin{cases} 2 & \text{if op} = \oplus, \\ \text{arity}_1(\text{op}) & \text{otherwise.} \end{cases}$$

Also, we define a language $\mathcal{L}_{\hat{1}}$ on the basis of $\mathbf{Sig}_{\hat{1}}$, just as $\mathcal{L}_{\hat{1}}$ has been defined on the basis of $\mathbf{Sig}_1$. Let $\mathcal{L}_{\hat{1}}[\emptyset]$ be the set of closed elements $S$ of $\mathcal{L}_{\hat{1}}$. ∎

### 6.6.1 A Simplified Variant of $\mathcal{D}_{1}^{\mathrm{wf}}$

For the simplified language $\mathcal{L}_{\hat{1}}$, we construct a denotational $\mathcal{D}_{\hat{1}}^{\mathrm{wf}}$ by simplifying $\mathcal{D}_{1}^{\mathrm{wf}}$ accordingly.

In the construction of $\mathcal{D}_{\hat{1}}^{\mathrm{wf}}$, we take $\mathbf{C}_{1}^{\surd}$, instead of $\mathbf{A}_{1}^{\surd}$, as the set of actions; this is for the convenience of comparing $\mathcal{D}_{\hat{1}}^{\mathrm{wf}}$ and the other two models which are constructed based on $\mathbf{C}_{1}^{\surd}$.

We define the semantic domain $\mathbf{P}_{\hat{1}}$ for $\mathcal{D}_{\hat{1}}^{\mathrm{wf}}$ just as $\mathbf{P}_{\mathrm{cl}}$, but on the basis of $\mathbf{C}_{1}^{\surd}$ (instead of $\mathbf{A}_{1}^{\surd}$) by:

**Definition 6.35** (Semantic Domain $\mathbf{P}_{\hat{1}}$ for $\mathcal{D}_{\hat{1}}^{\mathrm{wf}}$)

**(1)** **(i)** $\mathbf{Q}_{\hat{1}}^{\mathrm{fin}} = (\mathbf{C}_1)^{<\omega} \cdot (\delta[\wp(\mathbf{C}_{1}^{\surd})] \cup \{\langle\bot\rangle, \langle\surd\rangle\})$.

**(ii)** $\mathbf{Q}_{\hat{1}} = \mathbf{Q}_{\hat{1}}^{\mathrm{fin}} \cup (\mathbf{C}_1)^{\omega}$.

**(2)** Let $\mathbf{P}_{\hat{1}}$ be the set of elements $p$ of $\wp_{+\mathrm{fcl}}(\mathbf{Q}_{\hat{1}})$ satisfying the following conditions (i)–(iii):

**(i)** $p$ is *bounded in the action part*, i.e., satisfies (6.6).

**(ii)** $p$ is *prefix-closed* in the sense that

$$\begin{aligned}&\forall w \in (\mathbf{C}_1)^{<\omega}, \forall q \in \mathbf{Q}_{\hat{1}}[\ w \cdot q \in p \Rightarrow \\ &\quad w \cdot \langle\bot\rangle \in p \vee w \cdot \langle\delta(\emptyset)\rangle \in p\ ].\end{aligned} \tag{6.103}$$

**(iii)** $p$ is *downward closed* in the sense that

$$\begin{aligned}&\forall w \in (\mathbf{C}_1)^{<\omega}, \forall \Gamma \in \wp(\mathbf{C}_{1}^{\surd})[\ w \cdot \langle\delta(\Gamma)\rangle \in p \Rightarrow \\ &\quad \forall \Gamma' \in \wp(\Gamma)[\ w \cdot \langle\delta(\Gamma')\rangle \in p\ ]].\end{aligned} \tag{6.104}$$

**(iv)** $p$ is *upward closed w.r.t. disabled actions* in the sense that

$$\begin{aligned}&\forall w \in (\mathbf{C}_1)^{<\omega}, \forall \Gamma \in \wp(\mathbf{C}_{1}^{\surd})[\ w \cdot \langle\delta(\Gamma)\rangle \in p \Rightarrow \\ &\quad w \cdot \langle\delta(\Gamma \cup (\mathbf{C}_{1}^{\surd} \setminus \mathrm{act}(p[w])))\rangle \in p\ ].\ \blacksquare\end{aligned} \tag{6.105}$$

Obviously $(\mathbf{P}_{\hat{1}}, \sqsubseteq_{\mathrm{s}})$ is a pos; further, it is easily shown that $(\mathbf{P}_{\hat{1}}, \sqsubseteq_{\mathrm{s}}, \{\langle\bot\rangle\})$ is a cpo.

We can define semantic operations $\widetilde{\mathrm{op}}_{\hat{1}}$ on $\mathbf{P}_{\hat{1}}$ for combinators $\mathrm{op} \in \mathbf{Fun}_{\hat{1}}$ as follows: First, we define a domain $\mathbf{P}'_{\hat{1}}$ just as $\mathbf{P}_{\hat{1}}$ but on the basis of $\mathbf{A}_{1}^{\surd}$ instead of $\mathbf{C}_{1}^{\surd}$, and semantic operations $\widetilde{\mathrm{op}}'_{\hat{1}}$ are defined on $\mathbf{P}'_{\hat{1}}$ for $\mathrm{op} \in \mathbf{Fun}_{\hat{1}}$ just as in § 6.4.1. (There semantic operations $\widetilde{\mathrm{op}}$ have been defined on $\mathbf{P}_{\mathrm{cl}}$ for combinators $\mathrm{op} \in \mathbf{Fun}_1$.) Obviously $\mathbf{P}_{\hat{1}}$ is a suset of $\mathbf{P}'_{\hat{1}}$, and $\mathbf{P}_{\hat{1}}$ is closed under the operations $\widetilde{\mathrm{op}}'_{\hat{1}}$, i.e.,

$$\forall r \in 3, \forall \mathrm{op} \in \mathbf{Fun}_{\hat{1}}[\ \widetilde{\mathrm{op}}'_{\hat{1}}[(\mathbf{P}_{\hat{1}})^r] \subseteq \mathbf{P}_{\hat{1}}\ ].$$

We define the semantic operation $\widetilde{\mathrm{op}}_{\hat{1}} : (\mathbf{P}_{\hat{1}})^r \to \mathbf{P}_{\hat{1}}$ by:

$$\widetilde{\mathrm{op}}_{\hat{1}} = \widetilde{\mathrm{op}}'_{\hat{1}} \quad (\mathbf{P}_{\hat{1}})^r.$$

The simplified variant $\mathcal{D}_{\hat{1}}^{\mathrm{wf}}$ is defined in terms of the semantic operations $\widetilde{\mathrm{op}}_{\hat{1}}$, just as $\mathcal{D}_{1}^{\mathrm{wf}}$ has been defined in terms of the semantic operations $\widetilde{\mathrm{op}}$.

## 6.6.2 Comparison with Improved Failures Model

In this subsection, we will show that a simplified variant $\mathcal{D}_{\mathrm{i}}^{\mathrm{wf}}$ of the weak order-theoretic model $\mathcal{D}_1^{\mathrm{wf}}$ is isomorphic to the *improved failures model* $\mathcal{N}$ proposed by Brookes and Roscoe in [BR 84]. Similar isomorphism has been established by de Bakker et al. in [BMO 87] for *linear semantics*, and they raised there an analogous problem for failures semantics as a topic for future research; here we study the problem along the lines of [BMO 87], and establish a result analogous to that of [BMO 87].

### Comparison of Semantic Domains

We define the semantic domain $\mathbf{P}_{\mathrm{br}}$ for the improved failures model $\mathcal{N}$ as in Definition 6.36 below. (This definition is seemingly rather different from the original definition of [BR 84]; however the domain $\mathbf{P}_{\mathrm{br}}$ is isomorphic to a sub-cpo of the original domain $\mathbf{N}$ of [BR 84], as explained in Remark 6.3 below.)

**Definition 6.36** (Semantic Domain for the Improved Failures Model)

**(1)** A subset $p$ of $\mathbf{Q}_{\mathrm{i}}^{\mathrm{fin}}$ is said to be *saturated in* $\mathbf{Q}_{\mathrm{i}}^{\mathrm{fin}}$ iff

$$\forall q, q' \in \mathbf{Q}_{\mathrm{i}}^{\mathrm{fin}}[\ q \in p \wedge q \sqsubseteq q' \Rightarrow q' \in p\ ]. \tag{6.106}$$

**(2)** Let $\mathbf{P}_{\mathrm{br}}$ be the set of elements $p$ of $\wp_+(\mathbf{Q}_{\mathrm{i}}^{\mathrm{fin}})$ satisfying the following conditions.

**(i)** $\mathrm{mini}(p)$ is bounded in the action part, i.e.,

$$\forall n \in \omega[\ \{\mathrm{strip}(q^{[n]}) : q \in \mathrm{mini}(p)\} \text{ is finite }]./a \tag{6.107}$$

**(ii)** $p$ is *prefix-closed* in the sense of (6.103).

**(iii)** $p$ is downward closed in the sense of (6.104).

**(iv)** $p$ is *upward closed w.r.t. disabled actions* in the sense of (6.105).

**(v)** $p$ is saturated in $\mathbf{Q}_{\mathrm{i}}^{\mathrm{fin}}$.

**(3)** We define the *nondeterminism ordering* $\sqsubseteq_{\mathrm{n}}$ on $\mathbf{P}_{\mathrm{br}}$ by:

$$\sqsubseteq_{\mathrm{n}} = \{(p_0, p_1) \in (\mathbf{P}_{\mathrm{br}})^2 : p_0 \supseteq p_1\}.$$

We sometimes write $p_0 \sqsubseteq_{\mathrm{n}} p_1$ to denote that $(p_0, p_1) \in \sqsubseteq_{\mathrm{n}}$, as usual. ∎

**Remark 6.3** **(1)** In [BR 84], processes (elements of $\mathbf{N}$) are defined as pairs $(F, D)$ with

$$F \subseteq (\mathbf{C}_1)^{<\omega} \times \wp(\mathbf{C}_1^{\checkmark}), \quad D \subseteq (\mathbf{C}_1)^{<\omega}.$$

Such a pair $(F, D)$ corresponds to the following set which is an element of $\mathbf{P}_{\mathrm{bf}}$:

$$\{w \cdot \langle \delta(\Gamma) \rangle : (w, \Gamma) \in F\} \cup \{w \cdot \langle \bot \rangle : w \in D\}.$$

We adopt the above definition of the semantic domain for $\mathcal{N}$, for the convenience in comparing $\mathcal{N}$ with $\mathcal{D}_{\hat{\imath}}^{\mathrm{wf}}$.

**(2)** In [BR 84], the model $\mathbf{N}$ was defined by seven conditions (N1)–(N5), (D1) and (D2) (cf. § 3 of [BR 84]). Condition (N1) corresponds to nonemptiness, which is satisfied by all elements of $\wp_{+}(\mathbf{Q}_{\hat{\imath}}^{\mathrm{fin}})$; conditions (N2), (N3), (N4) correspond to (ii), (iii), (iv) in Definition 6.36 (2), respectively; the conjunction of (D1) and (D2) corresponds to (v) in Definition 6.36 (2). Condition (i) in Definition 6.36 (2) is added for the convenience in comparing $\mathcal{N}$ with $\mathcal{D}_{\hat{\imath}}^{\mathrm{wf}}$. Condition (N5) follows from condition (iv) with the help of conditions (i), (iii), (v).
Summarizing, conditions (i)–(v) are equivalent to the seven conditions in [BR 84] plus condition (i). That is, $\mathbf{P}_{\mathrm{br}}$ is isomorphic to the sub-cpo consisting of those elements of $\mathbf{N}$ which are bounded in the action-part. ∎

Every element of $\mathbf{P}_{\mathrm{br}}$ is *strongly prefix-closed* in the following sense:

**Proposition 6.25** *Let* $p \in \mathbf{P}_{\mathrm{br}}$. *Then,*

$$\forall w \in (\mathbf{C}_1)^{<\omega}, \forall q \in \mathbf{Q}_{\hat{\imath}}[\ w \cdot q \in p \Rightarrow w \cdot \langle \delta(\emptyset) \rangle \in p\ ]. \blacksquare \tag{6.108}$$

**Proof.** Fix $p \in \mathbf{P}_{\mathrm{br}}$. Let $w \in (\mathbf{C}_1)^{<\omega}$, $q \in \mathbf{Q}_{\hat{\imath}}$ such that $w \cdot q \in p$. Then, either $w \cdot \langle \perp \rangle \in p$ or $w \cdot \langle \delta(\emptyset) \rangle \in p$, since $p$ satisfies (6.103. When $w \cdot \langle \perp \rangle \in p$, one has $w \cdot \langle \delta(\emptyset) \rangle \in p$, since $p$ is saturated. Thus, in both cases, one has $w \cdot \langle \delta(\emptyset) \rangle \in p$. Summarizing the above, one has (6.108). ∎

Obviously $(\mathbf{P}_{\mathrm{br}}, \sqsubseteq_{\mathrm{n}})$ is a pos; further, it is easily shown that $(\mathbf{P}_{\mathrm{br}}, \sqsubseteq_{\mathrm{n}})$ is a cpo.
For establishing the isomorphism between $\mathbf{P}_{\hat{\imath}}$ and $\mathbf{P}_{\mathrm{br}}$, we will employ a *saturation function* satr$(\cdot)$ defined by:

**Definition 6.37** **(1)** A subset $p$ of $\mathbf{Q}_{\hat{\imath}}$ is said to be *saturated in* $\mathbf{Q}_{\hat{\imath}}$ iff

$$\forall q, q' \in \mathbf{Q}_{\hat{\imath}}[\ q \in p \wedge q \sqsubseteq q' \Rightarrow q' \in p\ ]. \tag{6.109}$$

**(2)** We define a *saturation function* satr : $\wp(\mathbf{Q}_{\hat{\imath}}) \to \wp(\mathbf{Q}_{\hat{\imath}})$ as follows: For every $p \in \wp(\mathbf{Q}_{\hat{\imath}})$,

$$\mathrm{satr}(p) = \{q \in \mathbf{Q}_{\hat{\imath}} : \exists q' \in p[\ q' \sqsubseteq q\ ]\}.$$

We call satr$(p)$ the *saturation* of $p$. ∎

From the above definition, we immediately have the following proposition:

**Proposition 6.26** *Let* $p \in \wp(\mathbf{Q}_{\hat{\imath}})$.

**(1)** *Taking saturation of* $p$ *preserves the minimal part* mini$(p)$ *of* $p$ *in the sense that*

$$\mathrm{mini}(\mathrm{satr}(p)) = \mathrm{mini}(p).$$

**(2)** *Taking saturation of* $p$ *preserves closedness in the sense that if* $p$ *is closed, then* satr$(p)$ *is also closed.* ∎

Using the saturation function, we define an mapping $\phi_{\mathrm{br}}$ from $\mathbf{P}_{\hat{1}}$ to $\mathbf{P}_{\mathrm{br}}$ by:

**Definition 6.38** (Isomorphism between $\mathbf{P}_{\hat{1}}$ and $\mathbf{P}_{\mathrm{br}}$)

**(1)** Let $\phi_{\mathrm{br}} : \mathbf{P}_{\hat{1}} \to \wp(\mathbf{Q}_{\hat{1}}^{\mathrm{fin}})$ be defined as follows: For every $p \in \mathbf{P}_{\hat{1}}$,

$$\phi_{\mathrm{br}}(p) = \mathrm{satr}(p) \cap \mathbf{Q}_{\hat{1}}^{\mathrm{fin}}.$$

**(2)** Let $\psi_{\mathrm{br}} : \mathbf{P}_{\mathrm{br}} \to \wp(\mathbf{Q}_{\hat{1}})$ be defined as follows: For every $p \in \mathbf{P}_{\mathrm{br}}$,

$$\psi(p) = \mathrm{mini}(p^{\mathrm{cls}}). \blacksquare$$

The following proposition immediately follows from the definitions of $\mathbf{P}_{\hat{1}}$, $\mathrm{P}_{\mathrm{br}}$, $\phi_{\mathrm{br}}$, and $\psi_{\mathrm{rm}}$:

**Proposition 6.27** **(1)** $\phi_{\mathrm{br}}[\mathbf{P}_{\hat{1}}] \subseteq \mathbf{P}_{\mathrm{br}}$.

**(2)** $\psi_{\mathrm{br}}[\mathbf{P}_{\mathrm{br}}] \subseteq \mathbf{P}_{\hat{1}}$. $\blacksquare$

From the first and second parts of the this lemma, one has

$$\phi_{\mathrm{br}} : \mathbf{P}_{\hat{1}} \to \mathbf{P}_{\mathrm{br}}, \text{ and } \psi_{\mathrm{br}} : \mathbf{P}_{\mathrm{br}} \to \mathbf{P}_{\hat{1}},$$

respectively.

The two cpo's $(\mathbf{P}_{\hat{1}}, \sqsubseteq_{\mathrm{s}})$ and $(\mathbf{P}_{\mathrm{br}}, \sqsubseteq_{\mathrm{n}})$ are *isomorphic* as cpo's, with $\phi_{\mathrm{br}}$ and $\psi_{\mathrm{br}}$ being isomorphisms between them:

**Lemma 6.22** (Isomorphism between $\mathbf{P}_{\hat{1}}$ and $\mathbf{P}_{\mathrm{br}}$)

**(1)** $\psi_{\mathrm{br}} \circ \phi_{\mathrm{br}} = \mathrm{id}_{\mathbf{P}_{\hat{1}}}$.

**(2)** $\phi_{\mathrm{br}} \circ \psi_{\mathrm{br}} = \mathrm{id}_{\mathbf{P}_{\mathrm{br}}}$.

**(3)** $\phi_{\mathrm{br}}$ *is a continuous function from* $(\mathbf{P}_{\hat{1}}, \sqsubseteq_{\mathrm{s}})$ *to* $(\mathbf{P}_{\mathrm{br}}, \sqsubseteq_{\mathrm{n}})$.

**(4)** $\psi_{\mathrm{br}}$ *is a continuous function from* $(\mathbf{P}_{\mathrm{br}}, \sqsubseteq_{\mathrm{n}})$ *to* $(\mathbf{P}_{\hat{1}}, \sqsubseteq_{\mathrm{s}})$. $\blacksquare$

In order to prove this lemma, we need the following proposition:

**Proposition 6.28** **(1)** *Let* $p \in \wp(\mathbf{Q}_{\hat{1}})$. *If* $p$ *is closed and prefix-closed, then*

$$(p \cap \mathbf{Q}_{\hat{1}}^{\mathrm{fin}})^{\mathrm{cls}} = p.$$

**(2)** *For every* $p \in \mathbf{P}_{\hat{1}}$,

$$(\mathrm{satr}(p) \cap \mathbf{Q}_{\hat{1}}^{\mathrm{fin}})^{\mathrm{cls}} = \mathrm{satr}(p). \blacksquare$$

**Proof.** **(1)** Let $p \in \wp(\mathbf{Q}_{\hat{1}})$ and suppose $p$ is closed and prefix-closed. It suffices to prove two propositions (6.110) and (6.111) below.

$$(p \cap \mathbf{Q}_{\hat{1}}^{\mathrm{fin}})^{\mathrm{cls}} \cap \mathbf{Q}_{\hat{1}}^{\mathrm{fin}} = p \cap \mathbf{Q}_{\hat{1}}^{\mathrm{fin}}. \tag{6.110}$$

$$(p \cap \mathbf{Q}_{\hat{1}}^{\mathrm{fin}})^{\mathrm{cls}} \cap (\mathbf{C}_1)^{\omega} = p \cap (\mathbf{C}_1)^{\omega}. \tag{6.111}$$

First, let us show (6.110). Since all accumulation points of every subset of $\mathbf{Q}_{\hat{1}}$ are contained in $(\mathbf{C}_1)^\omega$, one has

$$(p \cap \mathbf{Q}_{\hat{1}}^{\text{fin}})^{\text{cls}} \setminus (p \cap \mathbf{Q}_{\hat{1}}^{\text{fin}}) \subseteq (\mathbf{C}_1)^\omega,$$

and therefore,

$$((p \cap \mathbf{Q}_{\hat{1}}^{\text{fin}})^{\text{cls}} \setminus (p \cap \mathbf{Q}_{\hat{1}}^{\text{fin}})) \cap \mathbf{Q}_{\hat{1}}^{\text{fin}} \subseteq (\mathbf{C}_1)^\omega \cap \mathbf{Q}_{\hat{1}}^{\text{fin}} = \emptyset. \tag{6.112}$$

Further

$$\begin{aligned}
&(p \cap \mathbf{Q}_{\hat{1}}^{\text{fin}})^{\text{cls}} \cap \mathbf{Q}_{\hat{1}}^{\text{fin}} \\
&= (((p \cap \mathbf{Q}_{\hat{1}}^{\text{fin}})^{\text{cls}} \setminus (p \cap \mathbf{Q}_{\hat{1}}^{\text{fin}})) \cup (p \cap \mathbf{Q}_{\hat{1}}^{\text{fin}})) \cap \mathbf{Q}_{\hat{1}}^{\text{fin}} \\
&= (((p \cap \mathbf{Q}_{\hat{1}}^{\text{fin}})^{\text{cls}} \setminus (p \cap \mathbf{Q}_{\hat{1}}^{\text{fin}})) \cap \mathbf{Q}_{\hat{1}}^{\text{fin}}) \cup (p \cap \mathbf{Q}_{\hat{1}}^{\text{fin}}) \\
&= p \cap \mathbf{Q}_{\hat{1}}^{\text{fin}} \quad (\text{by } (6.112)).
\end{aligned}$$

Thus one obtains (6.110).

Next, let us show (6.111). One has

$$(p \cap \mathbf{Q}_{\hat{1}}^{\text{fin}})^{\text{cls}} \subseteq p^{\text{cls}} = p, \quad (\text{since } p \text{ is closed}).$$

and therefore,

$$(p \cap \mathbf{Q}_{\hat{1}}^{\text{fin}})^{\text{cls}} \cap (\mathbf{C}_1)^\omega \subseteq p \cap (\mathbf{C}_1)^\omega.$$

We will show the reverse inclusion:

$$p \cap (\mathbf{C}_1)^\omega \subseteq (p \cap \mathbf{Q}_{\hat{1}}^{\text{fin}})^{\text{cls}} \cap (\mathbf{C}_1)^\omega. \tag{6.113}$$

Let $q \in p \cap (\mathbf{C}_1)^\omega$. Since $p$ is prefix-closed, there exists $q' \in \mathbf{Q}_{\hat{1}}^{\text{fin}}$, for each $n$, such that

$$\exists q' \in \mathbf{Q}_{\hat{1}}^{\text{fin}}[\ (q \quad n) \cdot q' \in p\ ].$$

Choose such an element$q'$ for each $n$ and denote it by $q'_n$. Then,

$$\forall n \in \omega[\ (q \quad n) \cdot q'_n \in p \cap \mathbf{Q}_{\hat{1}}^{\text{fin}}\ ].$$

Thus, $q \in (p \cap \mathbf{Q}_{\hat{1}}^{\text{fin}})^{\text{cls}}$. Since $q$ has been an arbitrary element of $p \cap (\mathbf{C}_1)^\omega$, one has (6.113).

**(2)** Let $p \in \mathbf{P}_{\hat{1}}$. Then, satr$(p)$ is prefix-closed by the definition of $\mathbf{P}_{\hat{1}}$, and is closed by Proposition 6.26 (2). Thus, applying part (1), one has

$$(\text{satr}(p) \cap \mathbf{Q}_{\hat{1}}^{\text{fin}})^{\text{cls}} = \text{satr}(p). \blacksquare$$

**Proof of Lemma 6.22**

**(1)** It suffices to show that

$$\forall p \in \mathbf{P}_{\hat{1}}[\ \psi_{\mathrm{br}}(\phi_{\mathrm{br}}(p)) = p\ ]. \tag{6.114}$$

Let $p \in \mathbf{P}_{\hat{1}}$. Then,

$$\begin{aligned}
&\psi_{\mathrm{br}}(\phi_{\mathrm{br}}(p)) \\
&= \mathrm{mini}((\mathrm{satr}(p) \cap \mathbf{Q}_{\hat{1}}^{\mathrm{fin}})^{\mathrm{cls}}) \quad \text{(by the definitions of } \phi_{\mathrm{br}} \text{ and } \psi_{\mathrm{br}}) \\
&= \mathrm{mini}(\mathrm{satr}(p)) \quad \text{(by Proposition 6.28 (2))} \\
&= \mathrm{mini}(p) \quad \text{(by Proposition 6.26 (1))} \\
&= p \quad \text{(since } p \text{ is flat).}
\end{aligned}$$

Thus one has (6.114).
**(2)** It suffices to show that

$$\forall p \in \mathbf{P}_{\mathrm{br}}[\ \phi_{\mathrm{br}}(\psi_{\mathrm{br}}(p)) = p\ ]. \tag{6.115}$$

Let $p \in \mathbf{P}_{\mathrm{br}}$. Then,

$$\begin{aligned}
&\phi_{\mathrm{br}}(\psi_{\mathrm{br}}(p)) \\
&= \mathrm{satr}(\mathrm{mini}(p^{\mathrm{cls}})) \cap \mathbf{Q}_{\hat{1}}^{\mathrm{fin}} \quad \text{(by the definitions of } \phi_{\mathrm{br}} \text{ and } \psi_{\mathrm{br}}) \qquad (6.116) \\
&= \mathrm{satr}(p^{\mathrm{cls}}) \cap \mathbf{Q}_{\hat{1}}^{\mathrm{fin}} \quad \text{(by Proposition 6.26 (1)).}
\end{aligned}$$

Further, one has

$$\mathrm{satr}(p^{\mathrm{cls}}) = p^{\mathrm{cls}}, \tag{6.117}$$

since $p^{\mathrm{cls}}$ is saturated in $\mathbf{Q}_{\hat{1}}$, which follows from the fact $p$ is saturated in $\mathbf{Q}_{\hat{1}}^{\mathrm{fin}}$.
By (6.116) and (6.117), one has

$$\phi_{\mathrm{br}}(\psi_{\mathrm{br}}(p)) = p^{\mathrm{cls}} \cap \mathbf{Q}_{\hat{1}}^{\mathrm{fin}} = p.$$

Thus, one has (6.115).
**(3)** It can be easily shown that $\phi_{\mathrm{br}}$ is monotonic, i.e.,

$$\forall p_0, p_1 \in \mathbf{P}_{\hat{1}}[\ p_0 \sqsubseteq_{\mathrm{s}} p_1 \ \Rightarrow\ \phi_{\mathrm{br}}(p_0) \supseteq \phi_{\mathrm{br}}(p_1)\ ]. \tag{6.118}$$

It suffices to show that the following holds for every $\langle p_1 \rangle_{n\in\omega} \in \mathrm{Chan}(\mathbf{P}_{\hat{1}}, \subset\subset)$:

$$\phi_{\mathrm{br}}(\bigsqcup_{n\in\omega}[\,p_n\,]) = \bigcap_{n\in\omega}[\,\phi_{\mathrm{br}}(p_n)\,]. \tag{6.119}$$

Let $\langle p_1 \rangle_{n\in\omega} \in \mathrm{Chan}(\mathbf{P}_{\hat{1}}, \subset\subset)$. We will show (6.119).
From (6.118), the monotonicity of $\phi_{\mathrm{br}}$, it immediately follows that

$$\phi_{\mathrm{br}}(\bigsqcup_{n\in\omega}[\,p_n\,]) \subseteq \bigcap_{n\in\omega}[\,\phi_{\mathrm{br}}(p_n)\,].$$

Thus, it suffices to show that

$$\phi_{\mathrm{br}}(\bigsqcup_{n\in\omega}[p_n]) \supseteq \bigcap_{n\in\omega}[\phi_{\mathrm{br}}(p_n)]. \tag{6.120}$$

Let $q \in \bigcap_{n\in\omega}[\phi_{\mathrm{br}}(p_n)]$. By the definition of $phi_{\mathrm{BR}}$, for every $n \in \omega$, there is $\hat{q}_n \sqsubseteq q$. Fix such an element $\hat{q}_n$ for every $n$. Since $\{q' \in \mathbf{Q}_{\hat{1}} : q' \sqsubseteq q\}$ is finite, there exists $q' \in \mathbf{Q}_{\hat{1}}$ such that $\{q' \in \mathbf{Q}_{\hat{1}} : \hat{q}_n = q'\}$ is infinite. Fix such $q'$. By this and the Interpolation Theorem there is $N \in \omega$ such that

$$\forall n \geq N[\ \hat{q}_n = q'\ ].$$

Thus, $q' \in \bigsqcup_{n\in\omega}[p_n]$, and therefore,

$$q' \in \mathrm{str}(\bigsqcup_{n\in\omega}[p_n,].$$

Thus, one obtains (6.120).

**(4)** First, let us prove that $\psi_{\mathrm{br}}$ is monotonic, i.e., that

$$\forall p_0, p_1 \in \mathbf{P}_{\mathrm{br}}[\ p_0 \supseteq p_1 \Rightarrow \psi_{\mathrm{br}}(p_0) \sqsubseteq_{\mathrm{br}} \psi_{\mathrm{br}}(p_1)\ ]. \tag{6.121}$$

Let $p_0, p_1 \in \mathbf{P}_{\mathrm{br}}$ such that $p_0 \supseteq p_1$. Then, obviously $(p_0)^{\mathrm{cls}} \supseteq (p_1)^{\mathrm{cls}}$, and therefore, $(p_0)^{\mathrm{cls}} \sqsubseteq_{\mathrm{s}} (p_1)^{\mathrm{cls}}$. By this and the fact that $\mathrm{mini}(\cdot)$ preserves the order $\sqsubseteq_{\mathrm{s}}$, one has $\mathrm{mini}((p_0)^{\mathrm{cls}}) \sqsubseteq_{\mathrm{s}} \mathrm{mini}((p_1)^{\mathrm{cls}})$, i.e., $\psi_{\mathrm{br}}(p_0) \sqsubseteq_{\mathrm{s}} \psi_{\mathrm{br}}(p_1)$. Summarizing the above, one obtains (6.121).

Next, let us prove that $\psi_{\mathrm{br}}$ is continuous, i.e., that the following holds for every $\langle p_n\rangle_{n\in\omega} \in \mathrm{Chain}(\mathbf{P}_{\mathrm{br}}, \supseteq)$:

$$\psi_{\mathrm{br}}(\bigcap_{n\in\omega}[p_n]) = \bigsqcup_{n\in\omega}[\psi_{\mathrm{br}}(p_n)]. \tag{6.122}$$

Fix $\langle p_n\rangle_{n\in\omega} \in \mathrm{Chain}(\mathbf{P}_{\mathrm{br}}, \supseteq)$. From (6.121), the monotonicity of $\psi_{\mathrm{br}}$, it immediately follows that

$$\psi_{\mathrm{br}}(\bigcap_{n\in\omega}[p_n]) \sqsupseteq_{\mathrm{s}} \bigsqcup_{n\in\omega}[\psi_{\mathrm{br}}(p_n)].$$

Let us show the other inequality:

$$\psi_{\mathrm{br}}(\bigcap_{n\in\omega}[p_n]) \sqsubseteq_{\mathrm{s}} \bigsqcup_{n\in\omega}[\psi_{\mathrm{br}}(p_n)]. \tag{6.123}$$

For this purpose, it suffices, by the definition $\phi_{\mathrm{br}}$, to show the following:

$$(\bigcap_{n\in\omega}[p_n])^{\mathrm{cls}} \supseteq \bigsqcup_{n\in\omega}[\mathrm{mini}((p_n)^{\mathrm{cls}})]. \tag{6.124}$$

Let $q \in \bigsqcup_{n\in\omega}[\mathrm{mini}((p_n)^{\mathrm{cls}})]$. We will show that

$$q \in (\bigcap_{n\in\omega}[p_n])^{\mathrm{cls}}. \tag{6.125}$$

We distinguish two cases according to whether $q$ is finite or infinite.
**Case 1.** Suppose $q$ is finite. Then, by the definition of $\bigsqcup_{n\in\omega}[\,\mathrm{mini}((p_n)^{\mathrm{cls}})\,]$, there is $N \in \omega$ such that

$$\forall n \geq N[\ q \in \mathrm{mini}((p_n)^{\mathrm{cls}})\ ].$$

Fix such a number $N$ and let $n \geq N$. Then, one has $q \in (p_n)^{\mathrm{cls}}$, and moreover, $q \in p_n$, since $q$ is finite. Thus $\forall n \geq N[\ q \in p_n\ ]$. Also, since $\langle p_n \rangle_{n\in\omega}$ is a $\supseteq$-chain, one has $\forall n \leq N[\ q \in p_n\ ]$. Thus, $\forall n \in \omega[\ q \in p_n\ ]$, and therefore, $q \in \bigcap_{n\in\omega}[\,p_n\,]$. Hence, one has (6.125).
**Case 2.** Suppose $q$ is infinite. Then, by the definition of $\bigsqcup_{n\in\omega}[\,\mathrm{mini}((p_n)^{\mathrm{cls}})\,]$, there exists $\langle q_n \rangle_{n\in\omega} \in \mathrm{Chain}(\mathbf{Q}_{\mathrm{br}}, \sqsubseteq)$ such that

$$\forall n \in \omega[\ q_n \in \mathrm{mini}((p_n)^{\mathrm{cls}})\ ] \wedge \bigsqcup_{n\in\omega}[\,q_n\,] = q.$$

From this and the definition of $\bigsqcup_{n\in\omega}[\,q_n\,]$, it follows that for every $n \in \omega$, there exits $N \in \omega$ such that

$$\forall n \geq N, \exists q \in \mathbf{Q}_{\hat{1}}[\ (q \quad n) \cdot q \in \mathrm{mini}((p_n)^{\mathrm{cls}})\ ].$$

For each $n$, we fix such a number $N$, and denote it by $N(n)$. Fix $n \in \omega$. Then, by Proposition 6.25, one has

$$\forall k \geq N(n)[\ (q \quad n) \cdot \langle \delta(\emptyset) \rangle \in \mathrm{mini}((p_n)^{\mathrm{cls}})\ ],$$

and therefore,

$$\forall k \geq N(n)[\ (q \quad n) \cdot \langle \delta(\emptyset) \rangle \in p_n\ ]. \tag{6.126}$$

Also, since $\langle p_n \rangle_{n\in\omega}$ is a $\supseteq$-chain, one has

$$\forall k \leq N(n)[\ (q \quad n) \cdot \langle \delta(\emptyset) \rangle \in p_n\ ]. \tag{6.127}$$

By (6.126) and (6.127), one has

$$\forall k \in \omega[\ (q \quad n) \cdot \langle \delta(\emptyset) \rangle \in p_n\ ],$$

and therefore,

$$(q \quad n) \cdot \langle \delta(\emptyset) \rangle \in \bigcap_{n\in\omega}[\,p_n\,].$$

Since $n$ has been an arbitrary element of $\omega$, one has

$$\forall n \in \omega[\ (q \quad n) \cdot \langle \delta(\emptyset) \rangle \in \bigcap_{n\in\omega}[\,p_n\,]\ ].$$

Thus, one has (6.125).

Summarizing the above, in both Case 1 and Case 2, one has (6.125). Since $q$ has been an arbitrary element of $\bigsqcup_{n\in\omega}[\,\mathrm{mini}((p_n)^{\mathrm{cls}})\,]$, one has (6.124). ■

### Comparison of Denotational Models

For each $r \in 3$ and $f \in \Sigma_{\hat{1}}^{(r)}$, a semantic operation $\tilde{f}_{\hat{1}} : (\mathbf{P}_{\hat{1}})^r \to \mathbf{P}_{\hat{1}}$ is defined just as $\tilde{f} : (\mathbf{P}_{\mathrm{cl}})^r \to \mathbf{P}_{\mathrm{cl}}$, with certain appropriate modification (or simplification). In term of these semantics operations $\tilde{f}_{\hat{1}}$ ($f \in \mathbf{Fun}_{\hat{1}}$), a cpo-based denotational model $\mathcal{D}_{\hat{1}}^{\mathrm{wf}} : \mathcal{L}_{\hat{1}} \to ((\mathcal{X}_{\mathcal{P}} \to \mathbf{P}_{\hat{1}}) \to \mathbf{P}_{\hat{1}})$ is defined, in the same way as $\mathcal{D}_{1}^{\mathrm{wf}}$ has been defined in terms of $\tilde{f}$ ($f \in \mathbf{Fun}_{\hat{1}}$).

Also let $\mathcal{N} : \mathcal{L}_{\hat{1}} \to ((\mathcal{X}_{\mathcal{P}} \to \mathbf{P}_{\mathrm{br}}) \to \mathbf{P}_{\mathrm{br}})$ be the *improved failures model* defined in [BR 84] on the basis of $\mathbf{P}_{\mathrm{br}}$.

We define $\dot{\mathcal{D}}_{\hat{1}}^{\mathrm{wf}} : \mathcal{L}_{\hat{1}} \to \mathbf{P}_{\hat{1}}$ and $\dot{\mathcal{N}} : \mathcal{L}_{\hat{1}} \to \mathbf{P}_{\mathrm{br}}$ from $\mathcal{D}_{\hat{1}}^{\mathrm{wf}}$ and $\mathcal{N}$, respectively, just as $\dot{\mathcal{D}}_{1}^{\mathrm{w}}$ has been defined from $\mathcal{D}_{1}^{\mathrm{w}}$

**Lemma 6.23** *Let $S \in \mathcal{L}_{\hat{1}}$. Then*

**(1)** $\forall \zeta \in (\mathcal{X}_{\mathcal{P}} \to \mathbf{P}_{\hat{1}})[\ \phi_{\mathrm{br}}(\mathcal{D}_{\hat{1}}^{\mathrm{wf}}[\![S]\!](\zeta)) = \mathcal{N}[\![S]\!](\phi_{\mathrm{br}} \circ \zeta)\ ].$

**(2)** $\forall \zeta \in (\mathcal{X}_{\mathcal{P}} \to \mathbf{P}_{\mathrm{br}})[\ \psi_{\mathrm{br}}(\mathcal{N}[\![S]\!](\zeta)) = \mathcal{D}_{\hat{1}}^{\mathrm{wf}}[\![S]\!](\psi_{\mathrm{br}} \circ \zeta)\ ].$ ∎

The next proposition immediately follows from this lemma:

**Proposition 6.29** *Let $s_0, s_1 \in \mathcal{L}_{\hat{1}}$. Then*

$$\dot{\mathcal{D}}_{\hat{1}}^{\mathrm{wf}}[\![s_0]\!] \sqsubseteq_{\mathrm{s}} \dot{\mathcal{D}}_{\hat{1}}^{\mathrm{wf}}[\![s_1]\!] \Leftrightarrow \dot{\mathcal{N}}[\![s_0]\!] \sqsubseteq_{\mathrm{n}} \dot{\mathcal{N}}[\![s_1]\!]. \blacksquare \tag{6.128}$$

**Proof.** Let $s_0, s_1 \in \mathcal{L}_{\hat{1}}$. We will prove the $\Rightarrow$-part of (6.128) (the $\Leftarrow$-part can be proved in a similar fashion).

Suppose

$$\dot{\mathcal{D}}_{\hat{1}}^{\mathrm{wf}}[\![s_0]\!] \sqsubseteq_{\mathrm{s}} \dot{\mathcal{D}}_{\hat{1}}^{\mathrm{wf}}[\![s_1]\!].$$

Then, one has

$$\phi_{\mathrm{br}}(\dot{\mathcal{D}}_{\hat{1}}^{\mathrm{wf}}[\![s_0]\!]) \sqsubseteq_{\mathrm{s}} \phi_{\mathrm{br}}(\dot{\mathcal{D}}_{\hat{1}}^{\mathrm{wf}}[\![s_1]\!]).$$

From this and Lemma 6.23 (1), it follows that

$$\dot{\mathcal{N}}[\![s_0]\!] \sqsubseteq_{\mathrm{n}} \dot{\mathcal{N}}[\![s_1]\!].$$

Thus, one has the $\Rightarrow$-part of (6.128). ∎

### Comparison of Operational Models

An operational model $\mathcal{C}_{\hat{1}}^{\mathrm{wf}} : \mathcal{L}_{\hat{1}}[\emptyset] \to \wp(\mathbf{Q}_{\hat{1}})$ is defined in terms of the transition relations $\xrightarrow{a}_1$, just as $\mathcal{C}_{1}^{\mathrm{wf}}$ by:

**Definition 6.39** For $s \in \mathcal{L}_{\hat{1}}[\emptyset]$, let

$$\mathcal{C}_{\hat{1}}^{\mathrm{wf}}[\![s]\!] = \mathrm{mini}(\mathcal{T}_{\omega}(s) \cup \mathcal{T}_{\surd}(s) \cup \mathcal{T}_{\perp}(s) \cup \mathcal{F}(s)), \tag{6.129}$$

where $\mathcal{T}_{\omega}(s)$, $\mathcal{T}_{\surd}(s)$, $\mathcal{T}_{\perp}(s)$ are the sets defined in Definition 5.6, and $\mathcal{F}(s)$ is the set defined in Definition 5.8. ∎

The equality between $\mathcal{C}_{\hat{\mathrm{i}}}^{\mathrm{wf}}$ and $\dot{\mathcal{D}}_{\hat{\mathrm{i}}}^{\mathrm{wf}}$ can be established just as that between $\mathcal{C}_{\mathrm{i}}^{\mathrm{wf}}$ and $\dot{\mathcal{D}}_{\mathrm{i}}^{\mathrm{wf}}$:

**Proposition 6.30** *For every $s \in \mathcal{L}_{\hat{\mathrm{i}}}[\emptyset]$, one has*

$$\mathcal{C}_{\hat{\mathrm{i}}}^{\mathrm{wf}}[\![s]\!] = \dot{\mathcal{D}}_{\hat{\mathrm{i}}}^{\mathrm{wf}}[\![s]\!]. \blacksquare$$

**Proof.** Similar to the proof of Lemma 6.21. ■

In [BR 84], the authors do not give an operational characterization of $\mathcal{N}$; such a characterization, however, is found in [OH 86]. In [OH 86], the authors define a model $\mathrm{Obs}_{\mathcal{F}}[\![\cdot]\!]$ (for $\mathcal{L}_{\hat{\mathrm{i}}}[\emptyset]$), which turns out to equal to $\dot{\mathcal{N}}$, operationally in terms of the transition relations $\xrightarrow{a}_1$ as follows (cf. § 13 of [OH 86]):

**Definition 6.40** (Operational Characterization of $\mathcal{N}$) Let $\mathrm{Obs}_{\mathcal{F}}[\![\cdot]\!] : \mathcal{L}_{\hat{\mathrm{i}}}[\emptyset] \to \wp(\mathbf{Q}_{\mathrm{br}})$ be defined as follows: For $s \in \mathcal{L}_{\hat{\mathrm{i}}}$, let

$$\begin{aligned}\mathrm{Obs}_{\mathcal{F}}[\![s]\!] = {} & \{q \in \mathbf{Q}_{\mathrm{br}} : \exists w \in (\mathbf{C}_1)^{<\omega}[\ s \xRightarrow{w\cdot\langle\bot\rangle}_1 \ \wedge w\cdot\langle\bot\rangle \sqsubseteq_{\mathrm{s}} q\ ]\} \\ & \cup\{w\cdot\langle\delta(\Gamma)\rangle : w \in (\mathbf{C}_1)^{<\omega} \wedge \Gamma \in \wp(\mathbf{C}_1^{\surd}) \wedge \\ & \quad \exists s'[\ s \xRightarrow{w} s' \wedge \tau \notin \mathrm{act}_1(s') \\ & \quad \wedge \Gamma \cap \mathrm{act}_1(s') = \emptyset\ ]\}. \blacksquare\end{aligned} \tag{6.130}$$

From the definitions of $\mathcal{C}_{\hat{\mathrm{i}}}^{\mathrm{wf}}$ and $\mathrm{Obs}_{\mathcal{F}}[\![\cdot]\!]$, the following proposition immediately follows:

**Proposition 6.31** *For every $s \in \mathcal{L}_{\hat{\mathrm{i}}}[\emptyset]$, one has*

$$\mathrm{Obs}_{\mathcal{F}}[\![s]\!] = \phi_{\mathrm{br}}(\mathcal{C}_{\hat{\mathrm{i}}}^{\mathrm{wf}}[\![s]\!]). \blacksquare$$

The equality between $\mathrm{Obs}_{\mathcal{F}}[\![\cdot]\!]$ and $\dot{\mathcal{N}}$ follows from that between $\mathcal{C}_{\hat{\mathrm{i}}}^{\mathrm{wf}}$ and $\dot{\mathcal{D}}_{\hat{\mathrm{i}}}^{\mathrm{wf}}$:

**Proposition 6.32** *For every $s \in \mathcal{L}_{\hat{\mathrm{i}}}[\emptyset]$, one has*

$$\mathrm{Obs}_{\mathcal{F}}[\![s]\!] = \dot{\mathcal{N}}[\![s]\!]. \blacksquare$$

**Proof.** Let $s \in \mathcal{L}_{\hat{\mathrm{i}}}[\emptyset]$. Then,

$$\begin{aligned}\mathrm{Obs}_{\mathcal{F}}[\![s]\!] &= \phi_{\mathrm{br}}(\mathcal{C}_{\hat{\mathrm{i}}}^{\mathrm{wf}}[\![s]\!]) \quad \text{(by Proposition 6.31)} \\ &= \phi_{\mathrm{br}}(\dot{\mathcal{D}}_{\hat{\mathrm{i}}}^{\mathrm{wf}}[\![s]\!]) \quad \text{(by Proposition 6.30)} \\ &= \dot{\mathcal{N}}[\![s]\!]. \quad \text{(by Lemma 6.23 (1))}. \blacksquare\end{aligned}$$

We remark that this result is established in [OH 86] without mentioning $\mathcal{C}_{\hat{\mathrm{i}}}^{\mathrm{wf}}$ or $\dot{\mathcal{D}}_{\hat{\mathrm{i}}}^{\mathrm{wf}}$, but the proof in [OH 86] seems less intuitive than the one given above.

### 6.6.3 Comparison with Acceptance Tree Model

It can be shown that the improved failures model $\mathcal{N}$ is isomorphic to the *strong Acceptance Tree model* $\mathbf{AT}_s$ which is fully abstract w.r.t. the operational equivalence $\eqsim_{\text{must}}$ induced by the *must tests* (cf. Proposition 7.1 of [DeN 87] and the Historical Note of [Hen 88]). Thus, we can also show that our model $\mathcal{D}_{\bar{1}}^{\text{wf}}$ is isomorphic to the *strong Acceptance Tree model* $\mathbf{AT}_S[\![\cdot]\!]$.

The model $\mathbf{AT}_s$ is originally defined for a language which involves *action-prefixing* instead of sequential composition and no successful termination (cf. [Hen 88]); we slightly extend the model $\mathbf{AT}_s$ so as to include sequential composition and successful termination.

This extension preserves all the results of [Hen 88], except that we have to slightly modify the notion of *computation* in accordance with the language extension:

**Definition 6.41** Let $s \in \mathcal{L}_{\bar{1}}[\emptyset]$ and $e$ be an *experimenter* in the sense of [Hen 88]. A *computation* starting from $(e, s)$ is a sequence $\langle (e_n, s_n) \rangle_{n\in\nu}$ (with $\nu \in \omega \cup \{\omega\}$) which satisfies either (6.131) or (6.132) below:

$$\nu = \omega \wedge \forall n \in \nu[\ (e_n \| s_n) \xrightarrow{\tau}_1 (e_{n+1} \| s_{n+1})\ ]. \tag{6.131}$$

$$\begin{aligned} &\nu \in \omega \wedge \forall n \in (\nu - 1)[\ (e_n \| s_n) \xrightarrow{\tau}_1 (e_{n+1} \| s_{n+1})\ ] \\ &\wedge (\text{act}_1(e_{\nu-1} \| s_{\nu-1}) = \emptyset \vee \surd \in \text{act}_1(e_{\nu-1} \| s_{\nu-1})). \blacksquare \end{aligned} \tag{6.132}$$

Here we do not give the definition of $\eqsim_{\text{must}}$ or $\mathbf{AT}_S[\![\cdot]\!]$; instead, we give an operational relation $\equiv_{\text{must}}$ which is shown to coincide with $\eqsim_{\text{must}}$ (cf. Theorem 4.4.6 of [Hen 88]).

For the definition of $\equiv_{\text{must}}$, we need some preliminaries. First, we rephrase several auxiliary notations given in § 2.5 of [Hen 88], with some modifications in accordance with the language extension:

**Definition 6.42 (1)** For $\mathcal{A}_0, \mathcal{A}_1 \in \wp(\wp(\mathbf{C}_1^{\surd}) \cup \{\surd\})$, we write $\mathcal{A}_1 \subset\subset \mathcal{A}_0$ to denote that

$$\begin{aligned} &\forall A_1 \in \mathcal{A}_1 \cap \wp(\mathbf{C}_1^{\surd}), \exists A_0 \in \mathcal{A}_0[\ A_0 \subseteq A_1\ ] \\ &\wedge (\surd \in \mathcal{A}_1 \Rightarrow \surd \in \mathcal{A}_0). \end{aligned}$$

**(2)** For $s \in \mathcal{L}_{\bar{1}}[\emptyset]$, let $\mathbf{S}(s) = \{\gamma \in \mathbf{C}_1^{\surd} : s \overset{\gamma}{\Longrightarrow}_1\}$.

**(3)** For $s \in \mathcal{L}_{\bar{1}}[\emptyset]$ and $w \in (\mathbf{C}_1)^{<\omega}$, let

$$\mathcal{A}(s, w) = \{\mathbf{S}(s') : s \overset{w}{\Longrightarrow}_1 s'\} \cup \{\surd : \exists s'[\ s \overset{w}{\Longrightarrow}_1 s' \xrightarrow{\surd}_1\ ]\}.$$

**(4)** For $s \in \mathcal{L}_{\bar{1}}[\emptyset]$ and $w \in (\mathbf{C}_1)^{<\omega}$, we write $s \downarrow w$ to denote that

$$\forall w' \leq_{\text{p}} w[\ \neg(s_0 \overset{w' \cdot \langle \perp \rangle}{\Longrightarrow}_1)\ ] \blacksquare$$

In terms of the notations given above, a preorder $\ll_{\text{must}}$ is defined in § 2.5 of [Hen 88] as in Definition 6.43 below; we define $\equiv_{\text{must}}$ to be the kernel of $\ll_{\text{must}}$:

**Definition 6.43 (1)** A binary relation $\ll_{\mathrm{must}}$ on $\mathcal{L}_{\hat{1}}[\emptyset]$ is defined as follows: For $s_0, s_1 \in \mathcal{L}_{\hat{1}}[\emptyset]$, let $s_0 \ll_{\mathrm{must}} s_1$ iff

$$\forall w \in (\mathbf{C}_1)^{<\omega}[\ s_0 \downarrow w \Rightarrow s_1 \downarrow w \wedge \mathcal{A}(s_1, w) \subset\subset \mathcal{A}(s_0, w)\ ].$$

**(2)** Let $\equiv_{\mathrm{must}}$ be the *kernel* of $\ll_{\mathrm{must}}$, i.e., let

$$\equiv_{\mathrm{must}} = \ll_{\mathrm{must}} \cap (\ll_{\mathrm{must}})^{-1}.$$

Namely, for $s_0, s_1 \in \mathcal{L}_{\hat{1}}[\emptyset]$, let $s_0 \equiv_{\mathrm{must}} s_1$ iff

$$s_0 \ll_{\mathrm{must}} s_1 \wedge s_1 \ll_{\mathrm{must}} s_0. \blacksquare$$

Then we have the following lemma stating that the congruence induced by $\mathcal{C}_{\hat{1}}^{\mathrm{wf}}$ coincides with $\equiv_{\mathrm{must}}$:

**Lemma 6.24** *Let* $s_0, s_1 \in \mathcal{L}_{\hat{1}}[\emptyset]$.

**(1)** $$s_0 \ll_{\mathrm{must}} s_1 \Leftrightarrow \mathcal{C}_{\hat{1}}^{\mathrm{wf}}[\![s_0]\!] \sqsubseteq_{\mathrm{s}} \mathcal{C}_{\hat{1}}^{\mathrm{wf}}[\![s_1]\!]. \tag{6.133}$$

**(2)** $$s_0 \equiv_{\mathrm{must}} s_1 \Leftrightarrow \mathcal{C}_{\hat{1}}^{\mathrm{wf}}[\![s_0]\!] = \mathcal{C}_{\hat{1}}^{\mathrm{wf}}[\![s_1]\!]. \blacksquare \tag{6.134}$$

**Proof.** Let $s_0, s_1 \in \mathcal{L}_{\hat{1}}[\emptyset]$.
**(1)** First, we will prove the $\Rightarrow$-part of (6.133). Suppose

$$s_0 \ll_{\mathrm{must}} s_1. \tag{6.135}$$

We will show that $\mathcal{C}_{\hat{1}}^{\mathrm{wf}}[\![s_0]\!] \sqsubseteq_{\mathrm{s}} \mathcal{C}_{\hat{1}}^{\mathrm{wf}}[\![s_0]\!]$. For this purpose, it suffices to show that $\mathrm{Obs}_{\mathcal{F}}[\![s_0]\!] \sqsubseteq_{\mathrm{n}} \mathrm{Obs}_{\mathcal{F}}[\![s_1]\!]$, i.e., that

$$\mathrm{Obs}_{\mathcal{F}}[\![s_0]\!] \supseteq \mathrm{Obs}_{\mathcal{F}}[\![s_1]\!]. \tag{6.136}$$

Let $q \in \mathrm{Obs}_{\mathcal{F}}[\![s_1]\!]$. We will show that $q \in \mathrm{Obs}_{\mathcal{F}}[\![s_0]\!]$, by distinguishing two cases according to whether or not

$$\exists w \in (\mathbf{C}_1)^{<\omega}[\ w \cdot \langle\bot\rangle \in \mathrm{Obs}_{\mathcal{F}}[\![s_1]\!] \wedge w \leq_{\mathrm{p}} \mathrm{strip}(q)\ ]. \tag{6.137}$$

**Case 1.** Suppose (6.137) holds, and fix such a sequence $w$. If it holds that

$$\neg\exists w' \leq_{\mathrm{p}} w[\ s_0 \xrightarrow{w'\cdot\langle\bot\rangle}\ ],$$

then, one has

$$\neg\exists w \in (\mathbf{C}_1)^{<\omega}[\ w \cdot \langle\bot\rangle \in \mathrm{Obs}_{\mathcal{F}}[\![s_1]\!]\ ]$$

(by the definition of $\ll_{\mathrm{must}}$), which contradicts (6.137). Thus one has

$$\exists w' \leq_{\mathrm{p}} w[\ s_0 \xrightarrow{w'\cdot\langle\bot\rangle}\ ].$$

From this and the definition of $\mathrm{Obs}_{\mathcal{F}}[\![\cdot]\!]$, it follows that $q \in \mathrm{Obs}_{\mathcal{F}}[\![s_0]\!]$.
**Case 2.** Suppose (6.137) does not holds. Then, either

$$q = \mathrm{strip}(q) \cdot \langle\delta(\Gamma_1)\rangle$$

for some $\Gamma_1 \in \wp(\mathbf{C}_1^{\checkmark})$, or

$$q = \text{strip}(q) \cdot \langle \surd \rangle.$$

We consider the first case (the same result can be obtained more easily in the second case).

Fix such a set $\Gamma_1$. By (6.135), it holds that

$$\neg \exists w \leq_{\mathrm{p}} \text{strip}(q)[\ s_0 \xrightarrow{w \cdot \langle \perp \rangle}\ ].$$

Also, by the definition of $\mathcal{A}(p_1, \text{strip}(q))$, one has

$$\exists A_1 \in \wp(\mathbf{C}_1^{\checkmark} \setminus \Gamma_1)[\ A_1 \in \mathcal{A}(p_1, \text{strip}(q))\ ].$$

Fix such a set $A_1$. From (6.135) and the definition of $\ll_{\text{must}}$, it follows that

$$\exists A_0 \in \mathcal{A}(s_0, \text{strip}(q))[\ A_0 \subseteq A_2\ ].$$

Fix such a set $A_0$. There exists a statement $s_0'$ such that

$$s_0 \xrightarrow{\text{strip}(q)} s_0' \wedge A_0 = \mathbf{S}(s_0').$$

Thus, there exits another statement $s_0''$ such that

$$s_0 \xrightarrow{\text{strip}(q)} s_0'' \wedge \tau \notin \text{act}_1(s_0'') \wedge A_0 = \mathbf{S}(s_0') \supseteq \text{act}_1(s_0'').$$

Fix such a statement $s_0''$. One has

$$\Gamma_1 \subseteq \mathbf{C}_1^{\checkmark} \setminus A_1 \subseteq \mathbf{C}_1^{\checkmark} \setminus A_0 \subseteq \mathbf{C}_1^{\checkmark} \setminus \text{act}_1(s_0'').$$

Thus,

$$q = \text{strip}(q) \cdot \langle \delta(\Gamma_1) \rangle \in \text{Obs}_{\mathcal{F}}[\![s_0]\!].$$

Summarizing the above, in both cases, one has $q \in \text{Obs}_{\mathcal{F}}[\![s_0]\!]$. Since $q$ has been an arbitrary element of $\text{Obs}_{\mathcal{F}}[\![s_0]\!]$, one has the desired conclusion (6.136).

Next, we will prove $\Leftarrow$-part of (6.133). Suppose

$$\mathcal{C}_1^{\text{wf}}[\![s_0]\!] \sqsubseteq_{\text{s}} \mathcal{C}_1^{\text{wf}}[\![s_1]\!]. \tag{6.138}$$

We will show that $s_0 \ll_{\text{must}} s_1$. For this purpose, it suffices to show that

$$\begin{aligned} &\forall w \in (\mathbf{C}_1)^{<\omega}[\ \forall w' \leq_{\mathrm{p}} w[\ \neg(s_0 \xrightarrow{w' \cdot \langle \perp \rangle}{}_1)\ ] \Rightarrow \\ &\quad \forall w' \leq_{\mathrm{p}} w[\ \neg(s_1 \xrightarrow{w' \cdot \langle \perp \rangle}{}_1)\ ] \wedge \mathcal{A}(s_1, w) \subset\subset \mathcal{A}(s_0, w)\ ]. \end{aligned} \tag{6.139}$$

Let $w \in (\mathbf{C}_1)^{<\omega}$ and suppose that

$$\forall w' \leq_{\mathrm{p}} w[\ \neg(s_0 \xrightarrow{w' \cdot \langle \perp \rangle}{}_1)\ ]. \tag{6.140}$$

First, let us prove, by contradiction, that

$$\forall w' \leq_{\text{p}} w[\ \neg(s_1 \xRightarrow{w' \cdot \langle \bot \rangle}_1)\ ]. \tag{6.141}$$

Assume this does not hold, then there exists $w' \leq_{\text{p}} w$ such that

$$s_1 \xRightarrow{w' \cdot \langle \bot \rangle}_1 .$$

Let $\hat{w}$ be the shortest sequence of such sequences $w'$. Then, by the definition of $\mathcal{C}_1^{\text{wf}}$, one has $\hat{w} \cdot \langle \bot \rangle \in \mathcal{C}_1^{\text{wf}}[\![s_1]\!]$. From this and (6.138), it follows that

$$\exists \hat{w}' \leq_{\text{p}} \hat{w}[\ \hat{w}' \cdot \langle \bot \rangle \in \mathcal{C}_1^{\text{wf}}[\![s_0]\!]\ ],$$

but this contradicts (6.140). Thus, (6.141) must hold.

Next, let us prove that $\mathcal{A}(s_1, w) \subset\subset \mathcal{A}(s_0, w)$, i.e., that

$$\begin{aligned} &\textbf{(i)}\ \forall A_1 \in \mathcal{A}(s_1, w) \cap \wp(\mathbf{C}_1^{\checkmark}), \exists A_0 \in \mathcal{A}(s_0, w)[\ A_0 \subseteq A_1\ ], \\ &\textbf{(ii)}\ (\checkmark \in \mathcal{A}(s_1, w) \Rightarrow \checkmark \in \mathcal{A}(s_0, w)). \end{aligned} \tag{6.142}$$

We will prove (6.142) (i) (the other part (6.142) (ii) can be shown more easily). Let $A_1 \in \mathcal{A}(s_1, w)$. Then, by the definition of $\mathcal{A}(s_1, w)$, there exists $s_1'$ such that

$$s_1 \xRightarrow{w}_1 s_1' \wedge A_1 = \mathbf{S}(s_1').$$

Further, by (6.141), there exists $s_1''$ such that

$$s_1' \xRightarrow{\epsilon}_1 s_1'' \wedge \tau \in \text{act}_1(s_1'') \wedge \text{act}_1(s_1'') \subseteq \mathbf{S}(s_1') = \mathbf{S}(s_1') = A_1. \tag{6.143}$$

Fix such a statement $s_1''$, and let $\Gamma_1 = \mathbf{C}_1^{\checkmark} \setminus \text{act}_1(s_1'')$. Then, by (6.143), one has

$$\Gamma_1 \supseteq \mathbf{C}_1^{\checkmark} \setminus A_1. \tag{6.144}$$

Further, $w \cdot \langle \delta(\Gamma_1) \rangle \in \mathcal{C}_1^{\text{wf}}[\![s_1]\!]$. From this and (6.138), it follows that

$$w \cdot \langle \delta(\Gamma_1) \rangle \in \mathcal{C}_1^{\text{wf}}[\![s_0]\!].$$

Thus, there exits $s_0'$ such that

$$s_0 \xRightarrow{w}_1 s_0' \wedge \tau \notin \text{act}_1(s_0') \wedge \Gamma \cap \text{act}_1(p_0') = \emptyset.$$

Since $\text{act}_1(s_0') = \mathbf{S}(s_0')$, one has $\Gamma_1 \cap \mathbf{S}(s_0') = \emptyset$, and therefore,

$$\mathbf{S}(s_0') \subseteq \mathbf{C}_1^{\checkmark} \setminus \Gamma_1 = \text{act}_1(s_1'') \subseteq A_1.$$

Observing $\mathbf{S}(s_0') \in \mathcal{A}(s_0, w)$, one has

$$\exists A_0 \in \mathcal{A}(s_0, w)[\ A_0 \subseteq A_1\ ]$$

Since $A_1$ has been arbitrary, one has (6.142), i.e., $\mathcal{A}(s_1, w) \subset\subset \mathcal{A}(s_0, w)$.

Summarizing the above, one has (6.139), i.e., it holds that $s_0 \ll_{\text{must}} s_1$.

**(2)** This part immediately follows from part (1). ■

### 6.6.4 Hierarchy of Weak Compositional Models for Communicating Processes

For comparing various models for $\mathcal{L}_{\hat{1}}$, recall the notations:

$$\mathcal{M} \equiv_{\text{abs}} \mathcal{M}', \quad \mathcal{M} \leq_{\text{abs}} \mathcal{M}', \quad \mathcal{M} <_{\text{abs}} \mathcal{M}',$$

given in Notation 3.1, where $\mathcal{M}$ and $\mathcal{M}'$ are models for $\mathcal{L}_{\emptyset}$.

In order to compare with the denotational models defined hitherto in this chapter, we define two models $\mathcal{C}_{\hat{1}}^{\text{w}}$ and $\mathcal{C}_{\hat{1}}^{\text{wi}}$ for $\mathcal{L}_{\hat{1}}[\emptyset]$ operationally; the models $\mathcal{C}_{\hat{1}}^{\text{w}}$ and $\mathcal{C}_{\hat{1}}^{\text{wi}}$ are those simplified variants of $\mathcal{C}_{\text{i}}^{\text{w}}$ and $\mathcal{C}_{\text{i}}^{\text{wi}}$ defined in Chapter 5, respectively, which ignore the refusal part.

**Definition 6.44** The two model $\mathcal{C}_{\hat{1}}^{\text{w}} : \mathcal{L}_{\hat{1}}[\emptyset] \to \wp(\mathbf{Q}_{\hat{1}})$ are defined as follows: For every $s \in \mathcal{L}_{\hat{1}}[\emptyset]$, let

$$\mathcal{C}_{\hat{1}}^{\text{w}}[\![s]\!] = \mathcal{T}_{\omega}(s) \cup \mathcal{T}_{\surd}(s) \cup \mathcal{T}_{\perp}(s) \cup \mathcal{F}(s),$$

$$\mathcal{C}_{\hat{1}}^{\text{w}}[\![s]\!] = \Lambda(\mathcal{C}_{\hat{1}}^{\text{w}}[\![s]\!]),$$

where $\mathcal{T}_{\omega}(s)$, $\mathcal{T}_{\surd}(s)$, $\mathcal{T}_{\perp}(s)$ are the sets defined in Definition 5.6, $\mathcal{F}(s)$ is the set defined in Definition 5.8 (1), and $\Lambda$ is the pruning function defined in Definition 5.8 (3). ∎

The compositionality of $\mathcal{C}_{\hat{1}}^{\text{w}}$ and $\mathcal{C}_{\hat{1}}^{\text{wi}}$ can be established in the same way as the proof of the compositionality of $\mathcal{C}_{\text{i}}^{\text{w}}$ and $\mathcal{C}_{\text{i}}^{\text{wi}}$ in Chapter 5, with the obvious modifications; we omit the proof.

The classical *trace model* $\mathbf{T} : \mathcal{L}_{\hat{1}}[\emptyset] \to \wp((\mathbf{C}_1)^{<\omega})$ is defined operationally as follows:

**Definition 6.45** For $s \in \mathcal{L}_{\hat{1}}[\emptyset]$, let

$$\mathcal{T}[\![s]\!] = \{w \in (\mathbf{C}_1)^{<\omega} : s \overset{w}{\Longrightarrow}_1\}. \blacksquare$$

As is shown in [Hoa 85], the model $\mathcal{T}$ can be constructed denotationally on the basis of the cpo $(\wp((\mathbf{C}_1)^{<\omega}), \subseteq, \emptyset)$, and thus, $\mathcal{T}$ is compositional. Further, $\mathcal{T}$ and the *weak Acceptance Tree model* $\mathbf{AT}_{\text{w}}$ of [Hen 88] are equally abstract (cf. Theorem 4.4.6 (a) of [Hen 88]).

In [Hen 88], the *Acceptance Tree model* $\mathbf{AT}$ is defined so that the following holds (cf. Theorem 4.4.6 (c) of [Hen 88]):

$$\begin{aligned} &\forall s_0, s_1 \in \mathcal{L}_0[\emptyset][\ \mathbf{AT}[\![s_0]\!] = \mathbf{AT}[\![s_1]\!] \Leftrightarrow \\ &\quad \mathbf{AT}_{\text{w}}[\![s_0]\!] = \mathbf{AT}_{\text{w}}[\![s_1]\!] \wedge \mathbf{AT}_{\text{s}}[\![s_0]\!] = \mathbf{AT}_{\text{s}}[\![s_1]\!]\ ]. \end{aligned} \tag{6.145}$$

The hierarchy of compositional models for the language $\mathcal{L}_{\hat{1}}$ defined hitherto is summarized in Figure 6.2.

We have shown in this section so far that

$$\mathcal{C}_{\hat{1}}^{\text{wf}} \equiv_{\text{abs}} \dot{\mathcal{D}}_{\hat{1}}^{\text{wf}} \equiv_{\text{abs}} \dot{\mathcal{N}} \equiv_{\text{abs}} \mathbf{AT}_{\text{s}}.$$

$$\mathcal{C}_{\hat{1}}^{\mathrm{w}} <_{\mathrm{abs}} \mathcal{C}_{\hat{1}}^{\mathrm{wi}} <_{\mathrm{abs}} \mathbf{AT} <_{\mathrm{abs}} \begin{cases} \mathcal{T} \equiv_{\mathrm{abs}} \mathbf{AT}_{\mathrm{w}} \\ \mathcal{C}_{\hat{1}}^{\mathrm{wf}} \equiv_{\mathrm{abs}} \dot{\mathcal{D}}_{\hat{1}}^{\mathrm{wf}} \equiv_{\mathrm{abs}} \dot{\mathcal{N}} \equiv_{\mathrm{abs}} \mathbf{AT}_{\mathrm{s}} \end{cases}$$

Figure 6.2: Hierarchy of Weak Compositional Models

Also, it easily follows from (6.145) that

$$\mathbf{AT} <_{\mathrm{abs}} \mathbf{AT}_{\mathrm{w}}.$$

Thus, for establishing the claim of Figure 6.2, it suffices to show that

$$\mathcal{C}_{\hat{1}}^{\mathrm{w}} <_{\mathrm{abs}} \mathcal{C}_{\hat{1}}^{\mathrm{wi}} <_{\mathrm{abs}} \mathbf{AT} <_{\mathrm{abs}} \mathcal{C}_{\hat{1}}^{\mathrm{wf}}. \tag{6.146}$$

We will establish this in the sequel of this section.

**Proposition 6.33** $\mathbf{AT} <_{\mathrm{abs}} \mathcal{C}_{\hat{1}}^{\mathrm{wf}}$. ∎

**Proof.** By (6.145) and the fact that $\mathbf{AT}_{\mathrm{s}} \equiv_{\mathrm{abs}} \mathcal{C}_{\hat{1}}^{\mathrm{wf}}$, it suffices to show that

$$\exists s_0, s_1 \in \mathcal{L}_{\hat{1}}[\emptyset][\ \mathcal{T}[\![s_0]\!] \neq \mathcal{T}[\![s_1]\!] \wedge \mathcal{C}_{\hat{1}}^{\mathrm{wf}}[\![s_0]\!] = \mathcal{C}_{\hat{1}}^{\mathrm{wf}}[\![s_1]\!]\ ]. \tag{6.147}$$

Fixing $c \in \mathbf{C}_1$, let us put

$$s_0 \equiv (\mu X.\ \tau; X), \quad s_1 \equiv s_0 \oplus c.$$

Then,

$$\mathcal{C}_{\hat{1}}^{\mathrm{wf}}[\![s_0]\!] = \mathcal{C}_{\hat{1}}^{\mathrm{wf}}[\![s_1]\!] = \{\langle\perp\rangle\},$$

but

$$\mathcal{T}[\![s_0]\!] \neq \mathcal{T}[\![s_1]\!],$$

since

$$\langle c\rangle \in \mathcal{T}[\![s_1]\!] \setminus \mathcal{T}[\![s_0]\!].$$

Thus, one has the desired result (6.147). ∎

**Proposition 6.34** $\mathcal{C}_{\hat{1}}^{\mathrm{wi}} <_{\mathrm{abs}} \mathbf{AT}$. ∎

**Proof.** First, we will show

$$\mathcal{C}_{\hat{1}}^{\mathrm{wi}} \leq_{\mathrm{abs}} \mathbf{AT}. \tag{6.148}$$

To show this, it suffices to show that $\mathcal{C}_{\hat{1}}^{\mathrm{wi}} \leq_{\mathrm{abs}} \mathcal{T}$, i.e. that

$$\forall s_0, s_1 \in \mathcal{L}_{\bar{1}}[\emptyset][\ \mathcal{C}_{\bar{1}}^{\mathrm{wi}}[\![s_0]\!] = \mathcal{C}_{\bar{1}}^{\mathrm{wi}}[\![s_1]\!] \Rightarrow \mathcal{T}[\![s_0]\!] = \mathcal{T}[\![s_1]\!]\ ], \tag{6.149}$$

since (6.145) holds and

$$\mathcal{C}_{\bar{1}}^{\mathrm{wi}} \leq_{\mathrm{abs}} \mathcal{C}_{\bar{1}}^{\mathrm{wf}} = \mathbf{AT}_{\mathrm{s}}.$$

We will show (6.149). Let $s_0, s_1 \in \mathcal{L}_{\bar{1}}[\emptyset]$ such that

$$\mathcal{C}_{\bar{1}}^{\mathrm{wi}}[\![s_0]\!] = \mathcal{C}_{\bar{1}}^{\mathrm{wi}}[\![s_1]\!]. \tag{6.150}$$

Let $w \in \mathcal{T}[\![s_0]\!]$. Then, there exists $s_0'$ such that $s_0 \stackrel{w}{\Longrightarrow}_1 s_0'$. Fix such $s_0'$. From the fact that the transition system $(\mathcal{L}_{\bar{1}}[\emptyset], \langle \stackrel{a}{\longrightarrow} \rangle_{a \in \mathbf{A}_1})$ is finitely branching, it follows that

$$\langle \perp \rangle \in \mathcal{C}_{\bar{1}}^{\mathrm{wi}}[\![s_0']\!] \vee \langle \delta(\emptyset) \rangle \in \mathcal{C}_{\bar{1}}^{\mathrm{wi}}[\![s_0']\!],$$

with the help of König's Lemma (cf., e.g., [Kun 80] Lemma 5.6). In both cases, one has $w \cdot q \in \mathcal{C}_{\bar{1}}^{\mathrm{wi}}[\![s_0]\!]$ with some $q \in \{\langle \perp \rangle, \langle \delta(\emptyset) \rangle\}$. From this and (6.150), it follows that $w \cdot q \in \mathcal{C}_{\bar{1}}^{\mathrm{wi}}[\![s_1]\!]$, and thus, $w \in \mathcal{T}[\![s_1]\!]$.

Summarizing the above, one obtains (6.149), and thus, (6.148).

Next, let us show that

$$\neg(\mathcal{C}_{\bar{1}}^{\mathrm{wi}} \equiv_{\mathrm{abs}} \mathbf{AT}). \tag{6.151}$$

To show this, it suffices to show that

$$\exists s_0, s_1 [\ \mathcal{C}_{\bar{1}}^{\mathrm{wi}}[\![s_0]\!] \neq \mathcal{C}_{\bar{1}}^{\mathrm{wi}}[\![s_1]\!] \wedge \mathbf{AT}[\![s_0]\!] = \mathbf{AT}[\![s_1]\!]\ ]. \tag{6.152}$$

Let

$$s_0 \equiv (\mu X.\ \tau; X) \oplus (c; (\mu X.\ \tau; X)), \quad s_1 \equiv (\mu X.\ \tau; X) \oplus c.$$

Then, one has

$$\mathcal{T}[\![s_0]\!] = \mathcal{T}[\![s_1]\!] = \{\langle c \rangle\}, \quad \mathcal{C}_{\bar{1}}^{\mathrm{wf}}[\![s_0]\!] = \mathcal{C}_{\bar{1}}^{\mathrm{wf}}[\![s_1]\!] = \{\langle \perp \rangle\},$$

and therefore, $\mathbf{AT}[\![s_0]\!] = \mathbf{AT}[\![s_1]\!]$. But,

$$\langle c, \perp \rangle \in \mathcal{C}_{\bar{1}}^{\mathrm{wi}}[\![s_0]\!] \setminus \mathcal{C}_{\bar{1}}^{\mathrm{wi}}[\![s_1]\!],$$

and thus, $\mathcal{C}_{\bar{1}}^{\mathrm{wi}}[\![s_0]\!] \neq \mathcal{C}_{\bar{1}}^{\mathrm{wi}}[\![s_1]\!]$. Thus one has (6.152), and therefore, (6.151).

Summarizing the above, one has the desired result that $\mathcal{C}_{\bar{1}}^{\mathrm{wi}} <_{\mathrm{abs}} \mathbf{AT}$. ■

**Proposition 6.35** $\mathcal{C}_{\bar{1}}^{\mathrm{w}} <_{\mathrm{abs}} \mathcal{C}_{\bar{1}}^{\mathrm{wi}}$. ■

**Proof.** By the definitions of $\mathcal{C}_{\bar{1}}^{\mathrm{w}}$ and $\mathcal{C}_{\bar{1}}^{\mathrm{wi}}$, it immediately follows that

$$\mathcal{C}_{\bar{1}}^{\mathrm{w}} \leq_{\mathrm{abs}} \mathcal{C}_{\bar{1}}^{\mathrm{wi}}.$$

Thus, it suffices to show that

$$\exists s_0, s_1 [\ \mathcal{C}_{\bar{1}}^{\mathrm{w}}[\![s_0]\!] \neq \mathcal{C}_{\bar{1}}^{\mathrm{w}}[\![s_1]\!] \wedge \mathcal{C}_{\bar{1}}^{\mathrm{wi}}[\![s_0]\!] = \mathcal{C}_{\bar{1}}^{\mathrm{wi}}[\![s_1]\!]\ ]. \tag{6.153}$$

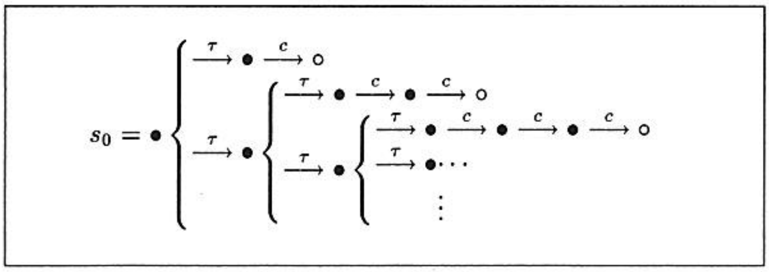

Figure 6.3: Synchronization Tree of a Statement $s_0$ in $\mathcal{L}_{\mathrm{i}}$

Let

$$s_0 \equiv (\mu X.\ c \oplus (X; c)), \quad s_1 \equiv (\mu X.\ (c; X)) \oplus s_0$$

(see Figure 6.3 for a pictorial representation of $s_0$). Then obviously

$$\mathcal{C}_{\mathrm{i}}^{\mathrm{wi}}[\![s_0]\!] = \mathcal{C}_{\mathrm{i}}^{\mathrm{wi}}[\![s_1]\!]$$

but

$$\mathcal{C}_{\mathrm{i}}^{\mathrm{w}}[\![s_0]\!] \neq \mathcal{C}_{\mathrm{i}}^{\mathrm{w}}[\![s_1]\!],$$

since

$$c^{\omega} \in \mathcal{C}_{\mathrm{i}}^{\mathrm{w}}[\![s_1]\!] \setminus \mathcal{C}_{\mathrm{i}}^{\mathrm{w}}[\![s_0]\!].$$

Thus, one has (6.153), and therefore, the claim of this proposition. ■

Summarizing Propositions 6.35–6.33, (6.146) is shown, and thus, the claim of Figure 6.2 is established.

**Remark 6.4** Let $\sim_i$ be an operationally defined equivalence over a language REC(**Sig**), the set of terms generated by a signature **Sig** with $\mu$-notation, and let $\mathcal{D}_i$ be a cpo-based model (for $\mathcal{L}$) which is induced from a **Sig**-*domain* $\mathbf{Dom}_i = (\mathbf{D}_i, \langle \tilde{f}_i : f \in \mathbf{Sig} \rangle)$ and is fully abstract w.r.t. $\sim_i$ ($i = 0, 1$). (Here by a **Sig**-domain, we refer to a pair $(\mathbf{D}, \langle \tilde{f} : f \in \mathbf{Sig} \rangle)$ such that $\mathbf{D}$ is a cpo, and $\tilde{f}$ is a continuous interpretation of $f$ on $\mathbf{D}$ ($f \in \mathbf{Sig}$).) Then, a fully abstract cpo-based model w.r.t. the conjunction of $\sim_0$ and $\sim_1$ can be constructed just as the model induced from the product **Sig**-domain $(\mathbf{D}_0 \times \mathbf{D}_1, \langle \tilde{f} : f \in \mathbf{Sig} \rangle)$ of $\mathbf{Dom}_0$ and $\mathbf{Dom}_1$, where for each $f \in \mathbf{Sig}$ of arity $r$, $\tilde{f}$ is defined by:

$$\tilde{f} = (\lambda((x_0, y_0), \ldots, (x_{r-1}, y_{r-1})) \in (\mathbf{D}_0 \times \mathbf{D}_1)^r.$$
$$(\tilde{f}_0(x_0, \ldots, x_{r-1}), \tilde{f}_1(y_0, \ldots, y_{r-1}))).$$

In this way, a cpo-based model which is isomorphic to the Acceptance tree model **AT** is induced from the product $\mathbf{Sig}_1$-domain of the two $\mathbf{Sig}_1$-domains underling $\mathcal{C}_{\mathrm{i}}^{\mathrm{wf}}$ and $\mathcal{T}$. ■

# 6.A Proof of Distributivity of Semantic Operations

## 6.A.1 Proof of Proposition 6.22

Let $\mathbf{M}_{\|} = ((\mathbf{P}_{\mathrm{cl}})^3 \to \mathbf{P}_{\mathrm{cl}})$, and let us define $F, G \in ((\mathbf{P}_{\mathrm{cl}})^3 \to \wp(\mathbf{Q}))$ by:

$$F = (\lambda(p, p_0, p_1) \in (\mathbf{P}_{\mathrm{cl}})^3.\ p \,\tilde{\|}\, (p_0 \,\hat{\cup}\, p_1)),$$

$$G = (\lambda(p, p_0, p_1) \in (\mathbf{P}_{\mathrm{cl}})^3.\ (p \,\tilde{\|}\, p_0) \,\hat{\cup}\, (p \,\tilde{\|}\, p_1)).$$

Then, it is easily checked that $F, G \in \mathbf{M}_{\|}$. We will establish that $F = G$ by showing both $F$ and $G$ are the fixed-point of a contractive higher-order mapping $\Psi_{\|}$ from $\mathbf{M}_{\|}$ to itself.

Let $p, p_0, p_1 \in \mathbf{P}_{\mathrm{cl}}$. By Proposition 6.9,

$$\begin{aligned} F(p, p_0, p_1) = {} & \tilde{\|}_{\mathrm{b}}(p, p_0 \,\hat{\cup}\, p_1) \\ & \hat{\cup}\, \tilde{\lfloor\!\lfloor}(p, p_0 \,\hat{\cup}\, p_1) \,\hat{\cup}\, \tilde{\lfloor\!\lfloor}(p_0 \,\hat{\cup}\, p_1, p) \,\hat{\cup}\, \tilde{|}(p, p_0 \,\hat{\cup}\, p_1). \end{aligned} \tag{6.154}$$

Also,

$$\begin{aligned} G(p, p_0, p_1) = {} & (\tilde{\|}_{\mathrm{b}}(p, p_0) \,\hat{\cup}\, \tilde{\|}_{\mathrm{b}}(p, p_1)) \\ & \hat{\cup}\, (\tilde{\lfloor\!\lfloor}(p, p_0) \,\hat{\cup}\, \tilde{\lfloor\!\lfloor}(p, p_1)) \,\hat{\cup}\, (\tilde{\lfloor\!\lfloor}(p_0, p) \,\hat{\cup}\, \tilde{\lfloor\!\lfloor}(p_1, p)) \\ & \hat{\cup}\, (\tilde{|}(p, p_0) \,\hat{\cup}\, \tilde{|}(p, p_1)). \end{aligned} \tag{6.155}$$

Below let us define the higher-order mapping $\Psi_{\|}$. By the definition of $\tilde{\lfloor\!\lfloor}$, one has

$$\begin{aligned} & \tilde{\lfloor\!\lfloor}(p, p_0 \,\hat{\cup}\, p_1) = \bigcup_{a \in \mathrm{act}(p)} [\, \langle a \rangle \cdot \hat{\bigcup}_{j \in 2} [p[\langle a \rangle] \,\tilde{\|}\, p_j\,]] \\ & = \bigcup_{a \in \mathrm{act}(p)} [\, \langle a \rangle \cdot F(p[\langle a \rangle], p_1, p_2)\,]. \end{aligned} \tag{6.156}$$

Also, one has

$$\begin{aligned} & \tilde{\lfloor\!\lfloor}(p_0 \,\hat{\cup}\, p_1, p) \\ & = \bigcup_{(j,k) \in \{(0,1),(1,0)\}} [\bigcup_{a \in \mathrm{act}(p_j) \setminus \mathrm{act}(p_k)} [\, \langle a \rangle \cdot (p_j[\langle a \rangle] \,\tilde{\|}\, p)\,]] \\ & \quad \cup \bigcup_{a \in \mathrm{act}(p_0) \cap \mathrm{act}(p_1)} [\, \langle a \rangle \cdot F(p, p_0[\langle a \rangle], p_1[\langle a \rangle])\,]. \end{aligned} \tag{6.157}$$

Further,

$$\begin{aligned} & \tilde{|}(p, p_0 \,\hat{\cup}\, p_1) \\ & = \hat{\bigcup} \{ \langle \tau \rangle \cdot F(p[\langle c \rangle], p_0[\langle \bar{c} \rangle], p_1[\langle \bar{c} \rangle]) : \\ & \qquad c \in \mathrm{act}(p) \,\wedge\, \textstyle\bigwedge_{j \in 2} [\bar{c} \in \mathrm{act}(p_j)]\} \\ & \quad \hat{\cup} \hat{\bigcup}_{(j,k) \in \{(0,1),(1,0)\}} [\hat{\bigcup} \{ \langle \tau \rangle \cdot (p[\langle c \rangle] \,\tilde{\|}\, p_j[\langle \bar{c} \rangle]) : c \in \mathrm{act}(p) \,\wedge \\ & \qquad\qquad \bar{c} \in \mathrm{act}(p_j) \,\wedge\, \bar{c} \notin \mathrm{act}(p_k)\}]. \end{aligned} \tag{6.158}$$

Now let us define $\Psi_{\|} \in (\mathbf{M}_{\|} \to ((\mathbf{P}_{\mathrm{cl}})^3 \to \wp(\mathbf{Q})))$ together with auxiliary mappings $\Psi_{\lfloor\!\lfloor}, \Psi'_{\lfloor\!\lfloor}, \Psi_{|} \in (\mathbf{M}_{\|} \to ((\mathbf{P}_{\mathrm{cl}})^3 \to \wp(\mathbf{Q})))$ as follows: For $F' \in \mathbf{M}_{\|}$ and $p', p'_0, p'_1 \in \mathbf{P}_{\mathrm{cl}}$,

$$\Psi_{\|}(F')(p', p_0', p_1')$$
$$= \tilde{\|}_{\mathrm{b}}(p', p_0', p_1') \,\hat{\cup}\, \Psi_{\|\!\!\_}(F')(p', p_0', p_1') \,\hat{\cup}\, \Psi'_{\|\!\!\_}(F')(p', p_0', p_1')$$
$$\hat{\cup}\, \Psi_{|}(F')(p', p_0', p_1'),$$

where,

$$\Psi_{\|\!\!\_}(F')(p', p_0', p_1') = \bigcup_{a \in \mathrm{act}(p)} [\langle a \rangle \cdot F(p[\langle a \rangle], p_1, p_2)],$$

$$\Psi'_{\|\!\!\_}(F')(p', p_0', p_1')$$
$$= \bigcup_{(j,k) \in \{(0,1),(1,0)\}} [\bigcup_{a \in \mathrm{act}(p_j) \setminus \mathrm{act}(p_k)} [\langle a \rangle \cdot (p_j[\langle a \rangle] \tilde{\|} p)]]$$
$$\cup \bigcup_{a \in \mathrm{act}(p_0) \cap \mathrm{act}(p_1)} [\langle a \rangle \cdot F(p, p_0[\langle a \rangle], p_1[\langle a \rangle])],$$

$$\Psi_{|}(F')(p', p_0', p_1')$$
$$= \hat{\bigcup} \{\langle \tau \rangle \cdot F'(p'[\langle c \rangle], p_0'[\langle \bar{c} \rangle], p_1'[\langle \bar{c} \rangle]) : c \in \mathrm{act}(p') \,\wedge\, \bigwedge_{j \in 2} [\bar{c} \in \mathrm{act}(p_j')]\}$$
$$\hat{\cup} \hat{\bigcup}_{(j,k) \in \{(0,1),(1,0)\}} [\hat{\bigcup} \{\langle \tau \rangle \cdot (p'[\langle c \rangle] \tilde{\|} p_j'[\langle \bar{c} \rangle]) : c \in \mathrm{act}(p') \,\wedge$$
$$\bar{c} \in \mathrm{act}(p_j') \,\wedge\, \bar{c} \notin \mathrm{act}(p_k')\}].$$

Then, it can be easily checked that

$$\forall F' \in \mathbf{M}_{\|}, \forall p, p_0, p_1 \in \mathbf{P}_{\mathrm{cl}}[\ \Psi_{\|}(F')(p, p_0, p_1) \in \mathbf{P}_{\mathrm{cl}}\ ],$$

i.e., that $\Psi_{\|} \in (\mathbf{M}_{\|} \to \mathbf{M}_{\|})$. Further, it follows easily from the definition of $\Psi_{\|}$ that $\Psi_{\|}$ is a contraction from $\mathbf{M}_{\|}$ to itself.

By (6.156), (6.157), (6.158), one has $\tilde{\|\!\!\_}(p, p_0 \,\hat{\cup}\, p_1) = \Psi_{\|\!\!\_}(F)(p, p_0 \,\hat{\cup}\, p_1)$, $\tilde{\|\!\!\_}(p_0 \,\hat{\cup}\, p_1, p) = \Psi'_{\|\!\!\_}(F)(p, p_0 \,\hat{\cup}\, p_1)$, and $\tilde{|}(p_0 \,\hat{\cup}\, p_1, p) = \Psi_{|}(F)(p, p_0 \,\hat{\cup}\, p_1)$. Thus

$$F(p, p_1, p_2) = \Psi_{\|}(F)(p, p_1, p_2).$$

Since $p, p_1, p_2$ have been chosen to be arbitrary elements of $\mathbf{P}_{\mathrm{cl}}$, one has $F = \Psi_{\|}(F)$. From this and the contractivity of $\Psi_{\|}$, it follows that $F = \mathrm{fix}(\Psi_{\|})$.

Next, let us show that $G$ is also the fixed-point of $\Psi_{\|}$. First, one can easily show the following by a case analysis on whether $\langle \bot \rangle \in p$ or not and on whether $\langle \bot \rangle \in p_0 \,\hat{\cup}\, p_1$ or not:

$$\hat{\bigcup}_{j \in 2} [\tilde{\|}_{\mathrm{b}}(p, p_j)] = \tilde{\|}_{\mathrm{b}}(p, p_0 \,\hat{\cup}\, p_1). \tag{6.159}$$

Second, one has

$$\hat{\bigcup}_{j \in 2} [\tilde{\|\!\!\_}(p, p_j)] = \Psi_{\|\!\!\_}(G)(p, p_0, p_1), \tag{6.160}$$

because

$$\hat{\bigcup}_{j \in 2} [\tilde{\|\!\!\_}(p, p_j)]$$
$$= \hat{\bigcup}_{j \in 2} [\bigcup_{a \in \mathrm{act}(p)} [\langle a \rangle \cdot (p[\langle a \rangle] \tilde{\|} p_j)]]$$

$$= \bigcup_{a \in \mathrm{act}(p)} [\langle a \rangle \cdot \hat{\bigcup}_{j \in 2} [p[\langle a \rangle] \tilde{\|} p_j]]$$

$$= \bigcup_{a \in \mathrm{act}(p)} [\langle a \rangle \cdot G(p[\langle a \rangle], p_0, p_1)]$$

$$= \Psi_{\Vert\!\lfloor}(G)(p, p_0, p_1).$$

Also, one has

$$\hat{\bigcup}_{j \in 2} [\tilde{\Vert\!\lfloor}(p_j, p)] = \Psi'_{\Vert\!\lfloor}(G)(p, p_0, p_1), \tag{6.161}$$

because for $(j, k) \in \{(0,1), (1,0)\}$,

$$\tilde{\Vert\!\lfloor}(p_j, p) = \bigcup_{a \in \mathrm{act}(p_j)} [\langle a \rangle \cdot \tilde{\|}(p, p_j[\langle a \rangle])]$$

$$= \bigcup_{a \in \mathrm{act}(p_j) \cap \mathrm{act}(p_k)} [\langle a \rangle \cdot \tilde{\|}(p, p_j[\langle a \rangle])]$$
$$\cup \bigcup_{a \in \mathrm{act}(p_j) \setminus \mathrm{act}(p_k)} [\langle a \rangle \cdot \tilde{\|}(p, p_j[\langle a \rangle])],$$

and therefore,

$$\hat{\bigcup}_{j \in 2} [\tilde{\Vert\!\lfloor}(p_j, p)]$$

$$= \hat{\bigcup}_{(j,k) \in \{(0,1),(1,0)\}} [\bigcup_{a \in \mathrm{act}(p_j) \cap \mathrm{act}(p_k)} [\langle a \rangle \cdot \tilde{\|}(p, p_j[\langle a \rangle])]$$
$$\cup \bigcup_{a \in \mathrm{act}(p_j) \setminus \mathrm{act}(p_k)} [\langle a \rangle \cdot \tilde{\|}(p, p_j[\langle a \rangle])]\ ]$$

$$= \bigcup_{a \in \mathrm{act}(p_0) \cap \mathrm{act}(p_1)} [\langle a \rangle \cdot \hat{\bigcup}_{j \in 2} [\tilde{\|}(p, p_j[\langle a \rangle])]]$$
$$\cup \bigcup_{(j,k) \in \{(0,1),(1,0)\}} [\bigcup_{a \in \mathrm{act}(p_j) \setminus \mathrm{act}(p_k)} [\langle a \rangle \cdot \tilde{\|}(p, p_j[\langle a \rangle])]]$$

$$= \bigcup_{a \in \mathrm{act}(p_0) \cap \mathrm{act}(p_1)} [\langle a \rangle \cdot G(p, p_0[\langle a \rangle], p_1[\langle a \rangle])]$$
$$\cup \bigcup_{(j,k) \in \{(0,1),(1,0)\}} [\bigcup_{a \in \mathrm{act}(p_j) \setminus \mathrm{act}(p_k)} [\langle a \rangle \cdot \tilde{\|}(p, p_j[\langle a \rangle])]]$$

$$= \Psi'_{\Vert\!\lfloor}(G)(p, p_0, p_1).$$

Finally, one has

$$\bigcup_{j \in 2} [\tilde{|}(p, p_j)] = \Psi_{|}(G)(p, p_0, p_1), \tag{6.162}$$

because for $(j, k) \in \{(0,1), (1,0)\}$,

$$\tilde{|}(p, p_j) = \hat{\bigcup} \{ \langle \tau \rangle \cdot (p[\langle c \rangle] \tilde{\|} p_j[\langle \bar{c} \rangle]) : c \in \mathrm{act}(p) \ \wedge\ \bar{c} \in \mathrm{act}(p_j) \}$$

$$= \hat{\bigcup} \{ \langle \tau \rangle \cdot (p[\langle c \rangle] \tilde{\|} p_j[\langle \bar{c} \rangle]) : c \in \mathrm{act}(p) \ \wedge\ \bar{c} \in \mathrm{act}(p_j) \cap \mathrm{act}(p_k) \}$$
$$\hat{\cup} \hat{\bigcup} \{ \langle \tau \rangle \cdot (p[\langle c \rangle] \tilde{\|} p_j[\langle \bar{c} \rangle]) : c \in \mathrm{act}(p) \ \wedge\ \bar{c} \in \mathrm{act}(p_j) \setminus \mathrm{act}(p_k) \},$$

and therefore,

$$\hat{\bigcup}_{j \in 2} [\tilde{|}(p, p_j)]$$

$$= \hat{\bigcup}\{\langle\tau\rangle \cdot (\hat{\bigcup}_{j\in 2}[p[\langle c\rangle]\tilde{\|}p_j[\langle\bar{c}\rangle]]) : c \in \mathrm{act}(p) \wedge \bar{c} \in \mathrm{act}(p_j) \cap \mathrm{act}(p_k)\}$$
$$\hat{\cup}\hat{\bigcup}_{(j,k)\in\{(0,1),(1,0)\}}[\hat{\bigcup}\{\langle\tau\rangle \cdot (p[\langle c\rangle]\tilde{\|}p_j[\langle\bar{c}\rangle]) : c \in \mathrm{act}(p) \wedge \bar{c} \in \mathrm{act}(p_j) \setminus \mathrm{act}(p_k)\}]$$
$$= \hat{\bigcup}\{\langle\tau\rangle \cdot G(p[\langle c\rangle], p_0[\langle\bar{c}\rangle], p_1[\langle\bar{c}\rangle]) : c \in \mathrm{act}(p) \wedge \bar{c} \in \mathrm{act}(p_j) \cap \mathrm{act}(p_k)\}$$
$$\hat{\cup}\hat{\bigcup}_{(j,k)\in\{(0,1),(1,0)\}}[\hat{\bigcup}\{\langle\tau\rangle \cdot (p[\langle c\rangle]\tilde{\|}p_j[\langle\bar{c}\rangle]) : c \in \mathrm{act}(p) \wedge \bar{c} \in \mathrm{act}(p_j) \setminus \mathrm{act}(p_k)\}]$$
$$= \Psi_{\|}(G)(p, p_0, p_1).$$

By (6.161), (6.160), (6.161) and (6.162), one has

$$G(p, p_0, p_1) = \Psi_{\|}(G)(p, p_0, p_1).$$

Thus, $G = \Psi_{\|}(G)$, and therefore, $G = \mathrm{fix}(\Psi_{\|})$.

Summarizing the above, one has the desired result that $F = G$. ■

### 6.A.2 Proof of Proposition 6.23

**(1)** This part follows immediately from the definition of $\tilde{;}_{\mathrm{b}}$ in Definition 6.19 (5).
**(2)** This part can be shown by a fixed-point argument similar to the proof of Proposition 6.22. ■

### 6.A.3 Proof of Proposition 6.24

**(1)** This part follows immediately from the definition of $\tilde{\partial}^{\mathrm{b}}_C$ in Definition 6.20 (1).
**(2)** This part can be shown by a fixed-point argument similar to the proof of Proposition 6.22. ■

## 6.B Proof of Neutrality of Hiding w.r.t. Semantic Operations

### 6.B.1 Proof of Lemma 6.16

We need a few preliminaries for the proof of this lemma.

**Proposition 6.36** *For every* $p_0, p_1 \in \mathbf{P}_{\mathrm{cl}}$,

$$H(H(p_0)\tilde{+}H(p_1)) = H(p_0)\tilde{+}H(p_1).$$ ■

**Proof.** This is shown as follows:

$$H(H(p_0)\tilde{+}H(p_1))$$
$$= \mathrm{mini}(h[H(p_0)\tilde{+}H(p_1)]) \quad \text{(by the definition of } H\text{)}$$
$$= \mathrm{mini}(H(p_0)\tilde{+}H(p_1))$$

(because $\forall q \in (H(p_0)\tilde{+}H(p_1))[\ h(q) = q\ ]$, and therefore, $h[H(p_0)\tilde{+}H(p_1)] = H(p_0)\tilde{+}H(p_1)$)

$$= H(p_0)\tilde{+}H(p_1)$$

(because $H(p_0)\tilde{+}H(p_1)$ is flat by the definition of $\tilde{+}$). ■

**Proof of Lemma 6.16** Let $p_0, p_1 \in \mathbf{P}_{\rm cl}$. We will show

$$H(H(p_0)\tilde{+}H(p_1)) = H(p_0\tilde{+}p_1). \tag{6.163}$$

For showing this, it suffices, by Proposition 6.36, to show

$$H(p_0)\tilde{+}H(p_1) = H(p_0\tilde{+}p_1). \tag{6.164}$$

We consider the case $\tau \in \mathrm{act}(p_0) \wedge \tau \in \mathrm{act}(p_1)$ (in the other case, the same result can be obtained more straightforwardly).

First, one has, for $j \in 2$,

$$\mathcal{B}(H(p_j)) = \mathcal{B}(p_j) \,\hat{\cup}\, \mathcal{B}(H(p_j[\tau])),$$

and that

$$\tilde{\Delta}(H(p_j)) = \tilde{\Delta}(p_j) \cup \tilde{\Delta}(H(p_j[\tau])) = \tilde{\Delta}(p_j).$$

Thus, by the definition of $\tilde{+}_{\rm b}$, one has

$$\begin{aligned}
&\tilde{+}_{\rm b}(H(p_0), H(p_1))\\
&= \hat{\bigcup}_{j\in 2}[\mathcal{B}(H(p_j)) \setminus \tilde{\Delta}(H(p_j))] \,\hat{\cup}\, (\tilde{\Delta}(H(p_0)) \cap \tilde{\Delta}(H(p_1)))\\
&= \hat{\bigcup}_{j\in 2}[(\mathcal{B}(p_j) \setminus \tilde{\Delta}(p_j)) \,\hat{\cup}\, \mathcal{B}(H(p_j[\tau]))] \,\hat{\cup}\, (\tilde{\Delta}(H(p_0)) \cap \tilde{\Delta}(H(p_1))).
\end{aligned}$$

Thus,

$$\begin{aligned}
&\tilde{+}_{\rm b}(H(p_0), H(p_1))\\
&= \tilde{+}_{\rm b}(\mathcal{B}(p_0), \mathcal{B}(p_1)) \,\hat{\cup}\, \mathcal{B}(H(p_0[\tau])) \,\hat{\cup}\, \mathcal{B}(H(p_1[\tau])).
\end{aligned} \tag{6.165}$$

Finally,

$$\begin{aligned}
&H(p_0)\tilde{+}H(p_0)\\
&= \tilde{+}_{\rm b}(\mathcal{B}(p_0), \mathcal{B}(p_1)) \,\hat{\cup}\, \mathcal{B}(H(p_0[\tau])) \,\hat{\cup}\, \mathcal{B}(H(p_1[\tau]))\\
&\quad \hat{\cup} \hat{\bigcup}_{j\in 2}[\oplus(H(p_j[\tau])) \,\hat{\cup}\, \hat{\bigcup}_{c\in \mathrm{cact}(p_j)}[\langle c\rangle \cdot H(p_j[\langle c\rangle])]]\\
&= \tilde{+}_{\rm b}(\mathcal{B}(p_0), \mathcal{B}(p_1))\\
&\quad \hat{\cup} \hat{\bigcup}_{j\in 2}[\mathcal{B}(H(p_j[\tau])) \,\hat{\cup} \oplus (H(p_j[\tau])) \,\hat{\cup}\, \hat{\bigcup}_{c\in \mathrm{cact}(p_j)}[\langle c\rangle \cdot H(p_j[\langle c\rangle])]]\\
&= \tilde{+}_{\rm b}(\mathcal{B}(p_0), \mathcal{B}(p_1)) \,\hat{\cup}\, H(\oplus(p_0)) \,\hat{\cup}\, H(\oplus(p_1))
\end{aligned}$$

(because, for $j \in 2$,

$$\begin{aligned}
&\mathcal{B}(H(p_j[\tau])) \,\hat{\cup} \oplus (H(p_j[\tau]))\\
&\hat{\cup} \hat{\bigcup}_{c\in \mathrm{cact}(p_j)}[\langle c\rangle \cdot H(p_j[\langle c\rangle])]\\
&= H(p_j[\tau]) \,\hat{\cup}\, \hat{\bigcup}_{c\in \mathrm{cact}(p_j)}[\langle c\rangle \cdot H(p_j[\langle c\rangle])]\\
&= H(\oplus(p_j))
\end{aligned}$$

)

$$\begin{aligned}
&= H(\tilde{+}_{\rm b}(\mathcal{B}(p_0), \mathcal{B}(p_1)) \,\hat{\cup} \oplus (p_0) \,\hat{\cup} \oplus (p_1))\\
&= H(p_0\tilde{+}p_1).
\end{aligned}$$

Thus, one has (6.164). ∎

## 6.B.2 Proof of Lemma 6.17

We need a few preliminaries for the proof of this lemma.

First, we have the distributivity of $H$ w.r.t. $\hat{\cup}$:

**Proposition 6.37** (Distributivity of $H$ w.r.t. $\hat{\cup}$)

$$\forall p_1, p_2 \in \wp(\mathbf{Q})[\ H(p_1 \hat{\cup} p_2) = H(p_1) \hat{\cup} H(p_2)\ ]. \blacksquare$$

The operation $\tilde{\|}$ is *commutative*:

**Proposition 6.38** $\forall p_1, p_2 \in \mathbf{P}[\ p_1 \tilde{\|} p_2 = p_2 \tilde{\|} p_1\ ]$. $\blacksquare$

**Definition 6.46 (1)** For $p \in \wp(\mathbf{Q})$, let $\mathrm{Lgt}(p) = \sup(\mathrm{lgt}[p])$, where the supremum is taken in $\omega \cup \{\omega\}$. We say $p$ is *bounded in length*, when $\mathrm{Lgt}(p) < \omega$.

**(2)** Let $\mathbf{P}_{\mathrm{cl}}^{\mathrm{bl}}$ be the set of those elements of $\mathbf{P}_{\mathrm{cl}}$ that are bounded in length. That is,

$$\mathbf{P}_{\mathrm{cl}}^{\mathrm{bl}} = \{p \in \mathbf{P}_{\mathrm{cl}} : \mathrm{Lgt}(p) < \omega\}. \blacksquare$$

**Proof of Lemma 6.17** By Proposition 6.38, it suffices, for establishing the lemma, to show that

$$\forall p, p' \in \mathbf{P}_{\mathrm{cl}}[\ H(H(p) \tilde{\|} p') = H(p \tilde{\|} p')\ ]. \tag{6.166}$$

Let $p, p' \in \mathbf{P}_{\mathrm{cl}}$. By the continuity of $H$ and $\tilde{\|}$, one has

$$H(H(p) \tilde{\|} p') = \bigsqcup_{n \in \omega} [H(H((p)^{[n]}) \tilde{\|} (p')^{[n]})].$$

Likewise,

$$H(p \tilde{\|} p') = \bigsqcup_{n \in \omega} [H((p)^{[n]} \tilde{\|} (p')^{[n]})].$$

Thus, for showing (6.166), it suffices to show that the following holds for every $p, p' \in \mathbf{P}_{\mathrm{cl}}^{\mathrm{bl}}$:

$$H(H(p) \tilde{\|} p') = H(p \tilde{\|} p'). \tag{6.167}$$

We will prove this by induction on $\mathrm{Lgt}(p) + \mathrm{Lgt}(p')$. Note that $\mathrm{Lgt}(p) + \mathrm{Lgt}(p') \geq 2$, since $\mathrm{Lgt}(p) \geq 1$ and $\mathrm{Lgt}(p') \geq 1$.

**Induction Base.** Suppose $\mathrm{Lgt}(p) + \mathrm{Lgt}(p') = 2$. Then $\mathrm{Lgt}(p) = 1$, and therefore, $H(p) = p$. Thus, one immediately obtains (6.167).

**Induction Step.** Let $n \geq 2$ and suppose that (6.167) holds for every $(p, p') \in (\mathbf{P}_{\mathrm{cl}}^{\mathrm{bl}})^2$ with $\mathrm{Lgt}(p) + \mathrm{Lgt}(p') \leq n$. Let $(p, p') \in (\mathbf{P}_{\mathrm{cl}}^{\mathrm{bl}})^2$ with $\mathrm{Lgt}(p) + \mathrm{Lgt}(p') = n+1$. We will prove (6.167). When $\mathrm{Lgt}(p) = 1$, we have (6.167) as in the induction base. Suppose $\mathrm{Lgt}(p) \geq 2$.

If $\langle \bot \rangle \in p'$, then both sides of (6.167) equal to $\{\langle \bot \rangle\}$, and therefore, (6.167) holds.

Let us consider the case $\langle \bot \rangle \notin p'$. We distinguish two cases according to whether $\tau \in \mathrm{act}(p)$ or not. We consider the case $\tau \in \mathrm{act}(p)$ (the same result can be obtained in the case $\tau \notin \mathrm{act}(p)$ in a similar fashion). In this case,

$$p = (p \cap \mathbf{B}) \cup \langle\tau\rangle \cdot p[\langle\tau\rangle] \cup \bigcup_{c \in \mathrm{act}(p) \cap \mathbf{C}_1} [\langle c\rangle \cdot p[\langle c\rangle]], \tag{6.168}$$

and therefore,

$$H(p) = (p \cap \mathbf{B}) \,\hat{\cup}\, H(p[\langle\tau\rangle]) \cup \hat{\bigcup}_{c \in \mathrm{act}(p) \cap \mathbf{C}_1} [\langle c\rangle \cdot H(p[\langle c\rangle])].$$

Hence,

$$\begin{aligned} H(p)\tilde{\|}p' = {} & \tilde{\|}_{\mathrm{b}}((p \cap \mathbf{B}) \cup (H(p[\langle\tau\rangle]) \cap \mathbf{B}), p' \cap \mathbf{B}) \\ & \hat{\cup}\, \tilde{\lfloor\!\lfloor}(p', H(p)) \\ & \hat{\cup}\, \tilde{\lfloor\!\lfloor}(H(p[\langle\tau\rangle]), p') \\ & \hat{\cup} \hat{\bigcup}_{c \in \mathrm{act}(p) \cap \mathbf{C}_1} [\langle c\rangle \cdot \tilde{\|}(H(p[\langle c\rangle]), p')] \\ & \hat{\cup}\, \tilde{|}(H(p[\langle\tau\rangle]), p') \\ & \hat{\cup} \hat{\bigcup}_{c \in \mathrm{act}(p) \cap \overline{\mathrm{act}(p') \cap \mathbf{C}_1}} [\langle\tau\rangle \cdot \tilde{\|}(H(p[\langle c\rangle]), p'[\langle\bar{c}\rangle])], \end{aligned}$$

and therefore,

$$\begin{aligned} & H(H(p)\tilde{\|}p') \\ & = \tilde{\|}_{\mathrm{b}}(p \cap \mathbf{B}, p' \cap \mathbf{B}) \,\hat{\cup}\, \tilde{\|}_{\mathrm{b}}(H(p[\langle\tau\rangle]) \cap \mathbf{B}, p' \cap \mathbf{B}) \\ & \quad \hat{\cup}\, H(\tilde{\lfloor\!\lfloor}(p', H(p))) \\ & \quad \hat{\cup}\, H(\tilde{\lfloor\!\lfloor}(H(p[\langle\tau\rangle]), p')) \\ & \quad \hat{\cup} \hat{\bigcup}_{c \in \mathrm{act}(p) \cap \mathbf{C}_1} [\langle c\rangle \cdot H(\tilde{\|}(H(p[\langle c\rangle]), p'))] \\ & \quad \hat{\cup}\, H(\tilde{|}(H(p[\langle\tau\rangle]), p')) \\ & \quad \hat{\cup} \hat{\bigcup}_{c \in \mathrm{act}(p) \cap \overline{\mathrm{act}(p') \cap \mathbf{C}_1}} [\langle\tau\rangle \cdot \tilde{\|}(H(p[\langle c\rangle]), p'[\langle\bar{c}\rangle])]. \end{aligned} \tag{6.169}$$

Further, since $H(p) \sqsubseteq_{\mathrm{s}} H(p[\langle\tau\rangle])$, one has

$$\tilde{\lfloor\!\lfloor}(p', H(p)) \sqsubseteq_{\mathrm{s}} \tilde{\lfloor\!\lfloor}(p', H(p[\langle\tau\rangle])),$$

and therefore,

$$\tilde{\lfloor\!\lfloor}(p', H(p)) = \tilde{\lfloor\!\lfloor}(p', H(p)) \,\hat{\cup}\, \tilde{\lfloor\!\lfloor}(p', H(p[\langle\tau\rangle])).$$

From this and Proposition 6.37, it follows that

$$H(\tilde{\lfloor\!\lfloor}(p', H(p))) = H(\tilde{\lfloor\!\lfloor}(p', H(p))) \,\hat{\cup}\, H(\tilde{\lfloor\!\lfloor}(p', H(p[\langle\tau\rangle]))). \tag{6.170}$$

By (6.169) and (6.170), one has

$$\begin{aligned} & H(H(p)\tilde{\|}p') \\ & = \tilde{\|}_{\mathrm{b}}(p \cap \mathbf{B}, p' \cap \mathbf{B}) \\ & \quad \hat{\cup}\, H(\tilde{\lfloor\!\lfloor}(p', H(p))) \\ & \quad \hat{\cup}\, \hat{\cup} \hat{\bigcup}_{c \in \mathrm{act}(p) \cap \mathbf{C}_1} [\langle c\rangle \cdot H(\tilde{\|}(H(p[\langle c\rangle]), p'))] \\ & \quad \hat{\cup} \hat{\bigcup}_{c \in \mathrm{act}(p) \cap \overline{\mathrm{act}(p') \cap \mathbf{C}_1}} [\langle\tau\rangle \cdot \tilde{\|}(H(p[\langle c\rangle]), p'[\langle\bar{c}\rangle])] \\ & \quad \hat{\cup}\, (\tilde{\|}_{\mathrm{b}}(H(p[\langle\tau\rangle]) \cap \mathbf{B}, p' \cap \mathbf{B}) \,\hat{\cup}\, H(\tilde{\lfloor\!\lfloor}(p', H(p[\langle\tau\rangle])))) \\ & \qquad \hat{\cup}\, H(\tilde{\lfloor\!\lfloor}(H(p[\langle\tau\rangle]), p')) \,\hat{\cup}\, H(\tilde{|}(H(p[\langle\tau\rangle]), p'))\ ). \end{aligned}$$

By this and the fact that

$$\begin{aligned}
&\tilde{\|}_{\mathrm{b}}(H(p[\langle\tau\rangle])\cap\mathbf{B},p'\cap\mathbf{B})\,\hat{\cup}\,H(\tilde{\lfloor\!\lfloor}(p',H(p[\langle\tau\rangle])))\\
&\hat{\cup}\,H(\tilde{\lfloor\!\lfloor}(H(p[\langle\tau\rangle]),p'))\,\hat{\cup}\,H(\tilde{|}(H(p[\langle\tau\rangle]),p'))\\
&=H(\tilde{\|}_{\mathrm{b}}(H(p[\langle\tau\rangle])\cap\mathbf{B},p'\cap\mathbf{B})\,\hat{\cup}\,\tilde{\lfloor\!\lfloor}(p',H(p[\langle\tau\rangle]))\\
&\qquad\hat{\cup}\,\tilde{\lfloor\!\lfloor}(H(p[\langle\tau\rangle]),p')\,\hat{\cup}\,\tilde{|}(H(p[\langle\tau\rangle]),p'))\\
&=H(H(p[\langle\tau\rangle])\tilde{\|}p'),
\end{aligned}$$

one has

$$\begin{aligned}
&H(H(p)\tilde{\|}p')\\
&=\tilde{\|}_{\mathrm{b}}(p\cap\mathbf{B},p'\cap\mathbf{B})\\
&\quad\hat{\cup}\,\hat{\bigcup}_{a\in\mathrm{act}(p')}[(\langle a\rangle\setminus\tau)\cdot\tilde{\|}(H(p),p'[\langle a\rangle])]\\
&\quad\hat{\cup}\,\hat{\bigcup}_{c\in\mathrm{act}(p)\cap\mathbf{C}_1}[\langle c\rangle\cdot H(\tilde{\|}(H(p[\langle c\rangle]),p'))]\\
&\quad\hat{\cup}\,\hat{\bigcup}_{c\in\mathrm{act}(p)\cap\overline{\mathrm{act}(p')\cap\mathbf{C}_1}}[\tilde{\|}(H(p[\langle c\rangle]),p'[\langle\bar{c}\rangle])]\\
&\quad\hat{\cup}\,H(H(p[\langle\tau\rangle])\tilde{\|}p').
\end{aligned}\tag{6.171}$$

Also, by (6.168), one has

$$\begin{aligned}
p\tilde{\|}p'=\;&\tilde{\|}_{\mathrm{b}}(p\cap\mathbf{B},p'\cap\mathbf{B})\\
&\hat{\cup}\,\hat{\bigcup}_{a\in\mathrm{act}(p')}[\langle a\rangle\cdot\tilde{\|}(p,p'[\langle a\rangle])]\\
&\hat{\cup}\,\langle\tau\rangle\cdot\tilde{\|}(p[\langle\tau\rangle],p')\\
&\hat{\cup}\,\hat{\bigcup}_{c\in\mathrm{act}(p)\cap\mathbf{C}_1}[\langle c\rangle\cdot\tilde{\|}(p[\langle c\rangle],p')]\\
&\hat{\cup}\,\hat{\bigcup}_{c\in\mathrm{act}(p)\cap\overline{\mathrm{act}(p')\cap\mathbf{C}_1}}[\langle\tau\rangle\cdot\tilde{\|}(p[\langle c\rangle],p'[\langle\bar{c}\rangle])],
\end{aligned}$$

and therefore,

$$\begin{aligned}
H(p\tilde{\|}p')=\;&\tilde{\|}_{\mathrm{b}}(p\cap\mathbf{B},p'\cap\mathbf{B})\\
&\hat{\cup}\,\hat{\bigcup}_{a\in\mathrm{act}(p')}[(\langle a\rangle\setminus\tau)\cdot H(\tilde{\|}(p,p'[\langle a\rangle]))]\\
&\hat{\cup}\,H(\tilde{\|}(p[\langle\tau\rangle],p'))\\
&\hat{\cup}\,\hat{\bigcup}_{c\in\mathrm{act}(p)\cap\mathbf{C}_1}[\langle c\rangle\cdot H(\tilde{\|}(p[\langle c\rangle],p'))]\\
&\hat{\cup}\,\hat{\bigcup}_{c\in\mathrm{act}(p)\cap\overline{\mathrm{act}(p')\cap\mathbf{C}_1}}[H(\tilde{\|}(p[\langle c\rangle],p'[\langle\bar{c}\rangle]))],
\end{aligned}\tag{6.172}$$

By the induction hypothesis, one can show that the right-hand sides of (6.169) and (6.172) are equal. Thus, one has (6.167). ■

## 6.B.3 Proof of Lemma 6.18

This lemma is proved by induction as Lemma 6.17. The following proposition is employed in the proof of the above lemma:

**Proposition 6.39**
$\forall p,p'\in\mathbf{P}_{\mathrm{cl}}[\;H(H(p)\tilde{;}H(p'))=H(p)\tilde{;}H(p')\;]$. ■

**Proof.** This can be proved in a similar fashion to the proof of Proposition 6.36. ■

**Proof of Lemma 6.18** By Proposition 6.39, it suffices to show

$$\forall p, p' \in \mathbf{P}^{\mathrm{bl}}_{\mathrm{cl}}[\ H(p) \,\tilde{;}\, H(p') = H(p \,\tilde{;}\, p')\ ].$$

For showing this, it suffices, by the continuity $H$ w.r.t. $\sqsubseteq_{\mathrm{s}}$, to show that the following holds for every $p, p' \in \mathbf{P}^{\mathrm{bl}}_{\mathrm{cl}}$:

$$H(p) \,\tilde{;}\, H(p') = H(p \,\tilde{;}\, p'). \tag{6.173}$$

We will prove this by induction on $\mathrm{Lgt}(p) + \mathrm{Lgt}(p')$ in a similar fashion to the proof of Lemma 6.17.

**Induction Base.** Suppose $\mathrm{Lgt}(p) + \mathrm{Lgt}(p') = 2$. Then $\mathrm{Lgt}(p) = \mathrm{Lgt}(p') = 1$. Hence, $H(p) = p$ and $H(p') = p'$, and therefore, one has (6.173) as follows:

$$H(p) \,\tilde{;}\, H(p') = p \,\tilde{;}\, p' = H(p \,\tilde{;}\, p').$$

**Induction Step.** Let $n \geq 2$ and suppose that (6.173) holds for every $(p, p') \in (\mathbf{P}^{\mathrm{bl}}_{\mathrm{cl}})^2$ with $\mathrm{Lgt}(p)+\mathrm{Lgt}(p') \leq n$. Let $(p, p') \in (\mathbf{P}^{\mathrm{bl}}_{\mathrm{cl}})^2$ with $\mathrm{Lgt}(p)+\mathrm{Lgt}(p') \leq n+1$. We will prove (6.173).

First, let us show

$$\mathcal{B}(H(p) \,\tilde{;}\, H(p')) = \mathcal{B}(H(p \,\tilde{;}\, p')). \tag{6.174}$$

We consider the case $\langle\surd\rangle \in p$ (the same result can be obtained more straightforwardly in the other case $\langle\surd\rangle \notin p$). One has

$$\begin{aligned}
&\mathcal{B}(H(p \,\tilde{;}\, p')) \\
&= \tilde{;}_{\mathrm{b}}(\mathcal{B}(p), \mathcal{B}(p')) \,\hat{\cup}\, \mathcal{B}(H(p[\langle\tau\rangle]) \,\tilde{;}\, p') \,\hat{\cup}\, \mathcal{B}(H(p'[\langle\tau\rangle])) \\
&= \tilde{;}_{\mathrm{b}}(\mathcal{B}(p), \mathcal{B}(p')) \,\hat{\cup}\, \mathcal{B}(H(p'[\langle\tau\rangle])) \,\hat{\cup}\, \tilde{;}_{\mathrm{b}}(\mathcal{B}(H(p[\langle\tau\rangle])), \mathcal{B}(H(p')))
\end{aligned}$$

(because, by the induction hypothesis, one has

$$H(p[\langle\tau\rangle] \,\tilde{;}\, p') = H(p[\langle\tau\rangle]) \,\tilde{;}\, H(p')$$

and therefore,

$$\begin{aligned}
&\mathcal{B}(H(p[\langle\tau\rangle] \,\tilde{;}\, p')) = \mathcal{B}(H(p[\langle\tau\rangle]) \,\tilde{;}\, H(p')) \\
&= \tilde{;}_{\mathrm{b}}(\mathcal{B}(H(p[\langle\tau\rangle])), \mathcal{B}(H(p'))).
\end{aligned}$$

)

$$= \tilde{;}_{\mathrm{b}}(\mathcal{B}(p), \mathcal{B}(p') \,\hat{\cup}\, \mathcal{B}(p'[\langle\tau\rangle])) \hat{\cup}\, \tilde{;}_{\mathrm{b}}(\mathcal{B}(H(p[\langle\tau\rangle])), \mathcal{B}(H(p')))$$

(because

$$\begin{aligned}
&\tilde{;}_{\mathrm{b}}(\mathcal{B}(p), \mathcal{B}(p')) \,\hat{\cup}\, \mathcal{B}(H(p'[\langle\tau\rangle])) \\
&= \tilde{;}_{\mathrm{b}}(\mathcal{B}(p), \mathcal{B}(p') \,\hat{\cup}\, \mathcal{B}(H(p'[\langle\tau\rangle]))),
\end{aligned}$$

which follows straightforwardly from the definition of $\tilde{;}_{\mathrm{b}}$ and the $\Delta$-condition)

$$= \tilde{;}_{\mathrm{b}}(\mathcal{B}(p), \mathcal{B}(H(p'))) \hat{\cup}\, \tilde{;}_{\mathrm{b}}(\mathcal{B}(H(p[\langle\tau\rangle])), \mathcal{B}(H(p')))$$

(because $\mathcal{B}(p') \,\hat{\cup}\, \mathcal{B}(p'[\langle\tau\rangle]) = \mathcal{B}(H(p'))$)

$$= \tilde{;}_{\mathrm{b}}(\mathcal{B}(p) \,\hat{\cup}\, \mathcal{B}(H(p[\langle\tau\rangle])), \mathcal{B}(H(p')))$$
(by Proposition 6.23 (1))
$$= \tilde{;}_{\mathrm{b}}(\mathcal{B}(H(p)), \mathcal{B}(H(p'))).$$

Thus, one has (6.174).

Next, let us show

$$\oplus\,(H(p) \,\tilde{;}\, H(p')) = \oplus(H(p \,\tilde{;}\, p')). \tag{6.175}$$

We consider the case that $\tau \in \mathrm{act}(p) \wedge \tau \in \mathrm{act}(p') \wedge \langle\surd\rangle \in p \wedge \langle\surd\rangle \in H(p[\langle\tau\rangle])$ (in the other case, the same result can be obtained more straightforwardly).

Since

$$H(p) = \mathcal{B}(p) \,\hat{\cup}\, p[\langle\tau\rangle] \,\hat{\cup}\, \hat{\bigcup}_{c\in\mathrm{cact}(p)}[\langle c\rangle \cdot H(p[\langle c\rangle]),$$

one has

$$\begin{aligned}\oplus\,(H(p) \,\tilde{;}\, H(p')) = {} & \hat{\bigcup}_{c\in\mathrm{cact}(H(p[\langle\tau\rangle]))}[\langle c\rangle \cdot (H(p[\langle\tau\rangle])[\langle c\rangle] \,\tilde{;}\, H(p'))] \\ & \hat{\cup}\, \hat{\bigcup}_{c\in\mathrm{cact}(p)}[\langle c\rangle \cdot (H(p[\langle c\rangle]) \cdot H(p'))] \\ & \hat{\cup} \oplus (H(p')).\end{aligned} \tag{6.176}$$

Also, by the definition $\tilde{;}$, one has

$$\begin{aligned}p \,\tilde{;}\, p' = {} & \tilde{;}_{\mathrm{b}}(p, p') \,\hat{\cup}\, \langle\tau\rangle \cdot (p[\langle\tau\rangle] \,\tilde{;}\, p') \,\hat{\cup}\, \hat{\bigcup}_{c\in\mathrm{cact}(p)}[\langle c\rangle \cdot (p[\langle c\rangle] \,\tilde{;}\, p')] \\ & \hat{\cup}\, \langle\tau\rangle \cdot p'[\langle\tau\rangle] \,\hat{\cup}\, \hat{\bigcup}_{c\in\mathrm{cact}(p')}[\langle c\rangle \cdot p'[\langle c\rangle]],\end{aligned}$$

and therefore,

$$\begin{aligned}\oplus\,(H(p \,\tilde{;}\, p')) = {} & \oplus(H(p[\langle\tau\rangle] \,\tilde{;}\, p')) \,\hat{\cup}\, \hat{\bigcup}_{c\in\mathrm{cact}(p)}[\langle c\rangle \cdot H(p[\langle c\rangle] \,\tilde{;}\, p')] \\ & \hat{\cup} \oplus (H(p'[\langle\tau\rangle])) \,\hat{\cup}\, \bigcup_{c\in\mathrm{cact}(p')}[\langle c\rangle \cdot H(p'[\langle c\rangle])].\end{aligned} \tag{6.177}$$

We will show that the right-hand sides of (6.176) and (6.177) are equal. First, one has

$$p' = \mathcal{B}(p') \,\hat{\cup}\, \langle\tau\rangle \cdot p'[\langle\tau\rangle] \,\hat{\cup}\, \hat{\bigcup}_{c\in\mathrm{cact}(p')}[\langle c\rangle \cdot p'[\langle c\rangle]],$$

and therefore,

$$\oplus\,(H(p')) = \oplus(H(p'[\langle\tau\rangle])) \,\hat{\cup}\, \hat{\bigcup}_{c\in\mathrm{cact}(p')}[\langle c\rangle \cdot H(p'[\langle c\rangle])]. \tag{6.178}$$

Also, by the induction hypothesis, one has

$$H(p[\langle\tau\rangle] \,\tilde{;}\, p') = H(p[\langle\tau\rangle]) \,\tilde{;}\, H(p'),$$

and therefore,

$$\begin{aligned}& \oplus(H(p[\langle\tau\rangle] \,\tilde{;}\, p')) = \oplus(H(p[\langle\tau\rangle]) \,\tilde{;}\, H(p')) \\ & = \hat{\bigcup}_{c\in\mathrm{cact}(H(p[\langle\tau\rangle]))}[\langle c\rangle \cdot (H(p[\langle\tau\rangle])[\langle c\rangle] \,\tilde{;}\, H(p'))] \,\hat{\cup} \oplus (H(p')),\end{aligned} \tag{6.179}$$

where $\oplus(H(p'))$ in the right-hand side is necessary under the assumption that $\langle\surd\rangle \in H(p[\langle\tau\rangle])$.

Further, by the induction hypothesis, one has

$$\begin{aligned}&\hat{\bigcup}_{c\in\text{cact}(p)}[\langle c\rangle \cdot H(p[\langle c\rangle] \mathbin{\bar{;}} p')]\\&= \hat{\bigcup}_{c\in\text{cact}(p)}[\langle c\rangle \cdot H(p[\langle c\rangle]) \mathbin{\bar{;}} H(p')].\end{aligned} \tag{6.180}$$

From (6.178) (6.179), and (6.180), it follows that the right-hand sides of (6.176) and (6.177) are equal. Thus, one has (6.175)

By (6.174) and (6.175), one has (6.173). ■

### 6.B.4 Proof of Lemma 6.19

This lemma is proved by induction as Lemmas 6.17 and 6.18. The following proposition is employed in the proof of the above lemma:

**Proposition 6.40** *For every* $C \in \wp(\mathbf{C}_1)$,

$$\forall p \in \mathbf{P}_{\text{cl}}[\ H(\widetilde{\partial}_C(H(p))) = \widetilde{\partial}_C(H(p))\ ].\ ■$$

**Proof.** This can be proved in a similar fashion to the proof of Proposition 6.36. ■

**Proof of Lemma 6.19** Let $C \in \wp(\mathbf{C}_1)$. By Proposition 6.40, it suffices to show

$$\forall p \in \mathbf{P}_{\text{cl}}[\ \widetilde{\partial}_C(H(p)) = H(\widetilde{\partial}_C(p))\ ].$$

For showing this, it suffices, by the continuity $H$ w.r.t. $\sqsubseteq_{\text{s}}$, to show that the following holds for every $p \in \mathbf{P}_{\text{cl}}^{bl}$:

$$\widetilde{\partial}_C(H(p)) = H(\widetilde{\partial}_C(p)). \tag{6.181}$$

We will prove this by induction on $\text{Lgt}(p)$ in a similar fashion to the proof of Lemma 6.17.

**Induction Base.** For $p$ with $\text{Lgt}(p) = 1$, one has (6.181), because

$$\widetilde{\partial}_C(H(p)) = \widetilde{\partial}_C(p) = H(\widetilde{\partial}_C(p)).$$

**Induction Step.** Let $n \leq 1$ and suppose that (6.181) holds for every $p \in \mathbf{P}_{\text{cl}}^{\text{bl}}$ with $\text{Lgt}(p) \leq n$. Let $p \in \mathbf{P}_{\text{cl}}^{\text{bl}}$ with $\text{Lgt}(p) = n + 1$. We will show (6.181) in the case $\tau \in \text{act}(p)$ (in the other case, the same result can be obtained more straightforwardly).

By the definition of $H$, one has

$$H(p) = H(\mathcal{B}(p) \mathbin{\hat{\cup}} H(p[\langle\tau\rangle]) \mathbin{\hat{\cup}} \tilde{\bigcup}_{c\in\text{cact}(p)}[\langle c\rangle \cdot H(p[\langle c\rangle])]).$$

Thus,

$$
\begin{aligned}
&\widetilde{\partial}_C(H(p)) \\
&= \widetilde{\partial}^{\mathrm{b}}_C(\mathcal{B}(p) \,\hat{\cup}\, \mathcal{B}(H(p[\langle\tau\rangle]))) \\
&\quad \hat{\cup} \hat{\bigcup}_{c\in \mathrm{act}(H(p[\langle\tau\rangle]))\setminus C}[\langle c\rangle \cdot \widetilde{\partial}_C(H(p[\langle\tau\rangle])[\langle c\rangle])] \\
&\quad \hat{\cup} \hat{\bigcup}_{c\in \mathrm{act}(p)\setminus C}[\langle c\rangle \cdot \widetilde{\partial}_C(H(p[\langle c\rangle]))] \\
&= \widetilde{\partial}^{\mathrm{b}}_C(\mathcal{B}(p)) \\
&\quad \hat{\cup} (\widetilde{\partial}^{\mathrm{b}}_C(\mathcal{B}(H(p[\langle\tau\rangle]))) \,\hat{\cup}\, \hat{\bigcup}_{c\in \mathrm{act}(H(p[\langle\tau\rangle]))\setminus C}[\langle c\rangle \cdot \widetilde{\partial}_C(H(p[\langle\tau\rangle])[\langle c\rangle])]) \\
&\quad \hat{\cup} \hat{\bigcup}_{c\in \mathrm{act}(p)\setminus C}[\langle c\rangle \cdot H(\widetilde{\partial}_C(p[\langle c\rangle]))]
\end{aligned}
$$

(because, for each $c \in \mathrm{act}(p) \setminus C$,

$$\widetilde{\partial}_C(H(p[\langle c\rangle])) = H(\widetilde{\partial}_C(p[\langle c\rangle]))$$

by the induction hypothesis, and because

$$
\begin{aligned}
&\widetilde{\partial}^{\mathrm{b}}_C(\mathcal{B}(p) \,\hat{\cup}\, \mathcal{B}(H(p[\langle\tau\rangle]))) \\
&= \widetilde{\partial}^{\mathrm{b}}_C(\mathcal{B}(p)) \,\hat{\cup}\, \widetilde{\partial}^{\mathrm{b}}_C(\mathcal{B}(H(p[\langle\tau\rangle])))
\end{aligned}
$$

by Proposition 6.24 )

$$
\begin{aligned}
&= \widetilde{\partial}^{\mathrm{b}}_C(\mathcal{B}(p)) \\
&\quad \hat{\cup}\, H(\widetilde{\partial}_C(p[\langle\tau\rangle])) \\
&\quad \hat{\cup} \hat{\bigcup}_{c\in \mathrm{act}(p)\setminus C}[\langle c\rangle \cdot H(\widetilde{\partial}_C(p[\langle c\rangle]))]
\end{aligned}
$$

(because

$$
\begin{aligned}
&\widetilde{\partial}_C(H(p[\langle\tau\rangle])) \\
&= \widetilde{\partial}^{\mathrm{b}}_C(\mathcal{B}(H(p[\langle\tau\rangle]))) \\
&\quad \hat{\cup} \hat{\bigcup}_{c\in \mathrm{act}(H(p[\langle\tau\rangle]))\setminus C}[\langle c\rangle \cdot \widetilde{\partial}_C(H(p[\langle\tau\rangle])[\langle c\rangle])] \\
&\qquad \text{(by the definition of } \widetilde{\partial}_C) \\
&= H(\widetilde{\partial}_C(p[\langle\tau\rangle])) \quad \text{(by the induction hypothesis)} \quad )
\end{aligned}
$$

$$
\begin{aligned}
&= H(\widetilde{\partial}^{\mathrm{b}}_C(\mathcal{B}(p)) \,\hat{\cup}\, \hat{\bigcup}_{a\in \mathrm{act}(p)\setminus C}[\langle a\rangle \cdot \widetilde{\partial}_C(p[\langle a\rangle])]) \\
&= H(\widetilde{\partial}_C(p)).
\end{aligned}
$$

Thus, one has (6.181). ■

# Part III

# Models for Nonuniform Languages

# Chapter 7

# Fully Abstract Denotational Models for Nonuniform Concurrent Languages

This chapter investigates *full abstractness* of denotational models w.r.t. operational ones for two concurrent languages. The languages are *nonuniform* in the sense that the meaning of atomic statements generally depends on the current state. The first language $\mathcal{L}_3$ has *parallel composition* but no communication, whereas the second one $\mathcal{L}_4$ has CSP-like *communications* in addition. For each language $\mathcal{L}_i$ $(i = 3, 4)$, an operational model $\mathcal{O}_i$ is introduced in terms of a Plotkin-style transition system, while a denotational model $\mathcal{M}_i$ for $\mathcal{L}_i$ is defined compositionally using interpreted operations of the language, with meanings of recursive programs as fixed-points in appropriate complete metric spaces. The full abstractness is shown by means of a context with parallel composition:

> *Given two statements $s_1$ and $s_2$ with different denotational meanings, a suitable statement $T$ is constructed such that the operational meanings of $s_1 \| T$ and $s_2 \| T$ are distinct.*

A combinatorial method for constructing such $T$ is proposed. Thereby the full abstractness of $\mathcal{M}_3$ (resp. $\mathcal{M}_4$) w.r.t. $\mathcal{O}_3$ (resp. $\mathcal{O}_4$) is established. That is, $\mathcal{M}_i$ is most abstract of those models $\mathcal{C}$ which are compositional and satisfy $\mathcal{O}_i = \mathcal{A} \circ \mathcal{C}$ for some abstraction function $\mathcal{A}$ $(i = 3, 4)$.

For $\mathcal{L}_4$, another operational model $\mathcal{O}_4^*$ is defined in terms of the same transition system in terms of which $\mathcal{O}_4$ is defined. The model $\mathcal{O}_4^*$ is more abstract than $\mathcal{O}_4$ in the sense that $\mathcal{O}_4^*$ ignores *states* whereas $\mathcal{O}_4$ involves them.

The following connection between $\mathcal{O}_4$ and $\mathcal{O}_4^*$ is established:

$$\forall s_1, s_2 \in \mathcal{L}_4[\ \forall C[\ C \text{ is a } \textit{context} \text{ of } \mathcal{L}_4 \Rightarrow \mathcal{O}_4^*[\![C[s_1]]\!] = \mathcal{O}_4^*[\![C[s_2]]\!]\ ] \\ \Rightarrow \mathcal{O}_4[\![s_1]\!] = \mathcal{O}_4[\![s_2]\!].$$

From this and the full abstractness of $\mathcal{M}_4$ w.r.t. $\mathcal{O}_4$, the full abstractness of $\mathcal{M}_4$ w.r.t. $\mathcal{O}_4^*$ immediately follows.

## 7.1 Introduction

This chapter investigates *full abstractness* of denotational models w.r.t. operational ones for two concurrent languages. The languages are *nonuniform* in the sense that the meaning of atomic statements generally depends on the current state. In particular, they have individual variables which store values, and the elementary actions are (mainly) value assignments to these variables. The first language $\mathcal{L}_3$ has *parallel composition* but no communication, whereas the second one $\mathcal{L}_4$ has CSP-like *communications* in addition. Both of the two languages have recursion. For each of $\mathcal{L}_i$ $(i = 3, 4)$, an operational model $\mathcal{O}_i$ for $\mathcal{L}_i$ is introduced in terms of a Plotkin-style transition system, while a denotational model $\mathcal{M}_i$ for $\mathcal{L}_i$ is defined compositionally using interpreted operations of the language and some fixed-point method for defining the meanings of recursive programs.

We show that, with the nonuniform languages, one needs to represent, in the meaning of a process, possible interactions between the process and its environment. Merely recording observations of initial and final states or possible computation sequences is not enough to obtain compositionality. One needs sequences in which there are *gaps* between steps to represent possible actions of the environment. This is essential in order to interpret parallel composition compositionally. Furthermore, the model one obtains by adding this information is in fact *fully abstract* w.r.t. the operational semantics, which is established by showing how to construct contexts that distinguish processes with different meanings.

The *full abstractness problem* for programming languages was first raised by Milner in [Mil 75]. In general, a model $\mathcal{M}$ for a language $\mathcal{L}$ is called *fully abstract* w.r.t. another model $\mathcal{O}$, if it makes *just enough* distinctions in order to be correct (and thus compositional) w.r.t. $\mathcal{O}$. In other words, it is fully abstract w.r.t. $\mathcal{O}$, if the following holds for every $s_1, s_2 \in \mathcal{L}$:

$$\begin{aligned}&\mathcal{M}[\![s_1]\!] = \mathcal{M}[\![s_2]\!] \Leftrightarrow \\ &\forall C[\ C \text{ is a } \textit{context} \text{ of } \mathcal{L} \Rightarrow \mathcal{O}[\![C[s_1]]\!] = \mathcal{O}[\![C[s_2]]\!]\ ],\end{aligned}$$

where a *context* is a statement consisting of the language constructs of $\mathcal{L}$ and a *place-holder* (or a *hole*) $X$, and $C[s]$ denotes the result of substituting $s$ for $X$ in $C$.[1] If $\mathcal{M}$ is fully abstract w.r.t. $\mathcal{O}$, then $\mathcal{M}$ is most abstract of those models $\mathcal{C}$ which are compositional and satisfy $\mathcal{O} = \mathcal{A} \circ \mathcal{C}$ for some abstraction function $\mathcal{A}$, i.e., for each of these $\mathcal{C}$'s, there is an *abstraction function* $\mathcal{A}^*$ such that $\mathcal{A}^* \circ \mathcal{C} = \mathcal{M}$. The models $\mathcal{M}_i$ $(i = 3, 4)$ will be *denotational* in the sense that apart from being compositional, they treat infinite behavior by means of some fixed point construction.

[1] For an operational or denotational model $\mathcal{M}$ for a language $\mathcal{L}$ and a statement $s \in \mathcal{L}$, the notation $\mathcal{M}[\![s]\!]$ is used to denote the value of $\mathcal{M}$ at $s$.

The mathematical domains we use are *complete metric spaces* ([Niv 79], [BZ 82]). In general, the metric approach may have, as a tool in programming language semantics, some advantages over the use of the more traditional complete partial orders: First, many definitions can be given as the (by Banach's theorem) unique fixed-point of some higher-order function. Second, a metric powerdomain can be easily defined (as the collection of closed or compact subsets of a given complete metric space). In comparison, ordered powerdomains are easily defined as well (by means of ideal completion), but often the characterization of their elements is rather technical. For some example of the application of metric spaces to semantics, see for instance [ABKR 89], [BM 88], [Bak 91].

The main body of this chapter consists of § 7.2 and § 7.3.

In § 7.2, the first language $\mathcal{L}_3$ is introduced; an operational model $\mathcal{O}_3$ is presented in terms of a Plotkin-style transition system; a denotational model $\mathcal{M}_3$ for $\mathcal{L}_3$ is defined on the basis of a complete metric space consisting of sets of streams of pairs of states with some additional information. First, the correctness of $\mathcal{M}_3$ w.r.t. $\mathcal{O}_3$ is established, as in [Rut 89] and [BR 91], by means of the fixed-point method introduced in [KR 90]. The full abstractness of $\mathcal{M}_3$ is shown by means of a context with parallel composition:

> *Given two statements $s_1, s_2 \in \mathcal{L}_3$ with different denotational meanings, a suitable statement $T$ called a* tester *is constructed such that the operational meanings of $s_1 \| T$ and $s_2 \| T$ are distinct.*[2] (7.1)

A combinatorial method called the *testing method*, which is the key idea of this chapter, is proposed for constructing such a tester (Lemma 7.10). This is in general applicable to denotational models with a domain consisting of sets of streams of pairs of states (possibly with some additional information). Thereby, we can construct testers having the following property:

> *Given a process $p$ and a finite sequence*
>
> $$r = \langle(\sigma_i, \sigma_i')\rangle_{i \in n},$$
>
> *we can construct a tester $T$ and an executable sequence*
>
> $$\tilde{r} = \langle(\tilde{\sigma}_i, \tilde{\sigma}')\langle_{i \in k},$$
>
> *with $k \geq n$ such that for every process $p'$, the parallel composition $p' \,\tilde{\|}\, \mathcal{M}_3[\![T]\!]$ can execute $\tilde{r}$ if there is some sequence $q$ such that $\langle(\sigma_i, \sigma_i')\rangle_{i \in n} \cdot q \in p'$, and the converse of this holds for $p' = p$. Intuitively, for such $T$ and $\tilde{r}$, the process $p$ is* forced *to execute the steps $\langle(\sigma_i, \sigma_i')\rangle_{i \in n}$ (maybe not consecutively but in this order), when $p \,\tilde{\|}\, \mathcal{M}_3[\![T]\!]$ executes the steps $\langle(\tilde{\sigma}_i, \tilde{\sigma}_i')\rangle_{i \in k}$ consecutively.*

By the above property, we can construct such testers $T$ as in (7.1):

[2] The variable $T$ is used to denote a statement when it is considered a tester, while the typical variable for the set of statements is $s$.

*If $s_1$ and $s_2$ are distinct in their denotational meanings, then, putting $p_i = \mathcal{M}_3[\![s_i]\!]$ ($i = 1, 2$), there exists some sequence $r$ such that $r \cdot q \in p_1$ for some $q$ but $r \cdot q \notin p_2$ for every $q$ (or vice versa). By constructing a tester $T$ and an executable sequence $\tilde{r}$ for $r$ and $p = p_2$ as above, one has $\tilde{r} \in \mathcal{M}_3[\![s_1]\!] \tilde{\|} \mathcal{M}_3[\![T]\!]$ and $\tilde{r} \notin \mathcal{M}_3[\![s_2]\!] \tilde{\|} \mathcal{M}_3[\![T]\!]$. Thus one has a difference between the operational meanings of the two statements $s_1 \| T$ and $s_2 \| T$.*

The full abstractness of $\mathcal{M}_3$ is established by means of the testing method as described above.

In § 7.3, the second language $\mathcal{L}_4$ is introduced; an operational model $\mathcal{O}_4$ for $\mathcal{L}_4$ is given as in § 7.2. The domain of a denotational model $\mathcal{M}_4$ for $\mathcal{L}_4$ is a kind of *failures model*, which was introduced in [BHR 85], and is adapted here to the nonuniform setting. Each element of the domain is a set consisting of such elements that are represented as $(\langle(\sigma_i, a_i, \sigma_i')\rangle_i, (\sigma'', \Gamma))$, where $\sigma_i$, $\sigma_i'$, and $\sigma''$ are states, $a_i$ is an action, and $\Gamma$ is a set of *communication sorts*. These elements are called *failures*; the parts $\langle(\sigma_i, a_i, \sigma_i')\rangle_i$ and $(\sigma'', \Gamma)$ are called a *trace* and a *refusal*, respectively. First, the correctness of $\mathcal{M}_4$ is established as in § 7.2. Then, the full abstractness of $\mathcal{M}_4$ is established by a combination of the testing method and the method proposed by Bergstra, Klop, and Olderog in [BKO 88] to establish the full abstractness of a *failures model* for a uniform language without recursion. This method was adapted by Rutten in [Rut 89] so as to employ it for a language with recursion in the framework of complete metric spaces, which suggests how to use it in the present setting. Given two statements $s_1$ and $s_2$ of $\mathcal{L}_4$ which are distinct in their denotational meanings, then the denotational meanings are distinct in the trace parts or in the refusal parts. When the distinction is in the trace parts, we can construct a tester by the method described above, otherwise we can construct a tester by the method of [BKO 88].

It might be argued that $\mathcal{O}_4$ involves too much information, because one might be interested only in actions and not in states. (For example, let us consider a calculator which reads expressions and writes their values. We are interested only in the input/output actions, not in the states of its registers.) From this viewpoint, another operational model $\mathcal{O}_4^*$ for $\mathcal{L}_4$ is presented; the model $\mathcal{O}_4^*$ is defined in terms of the same transition system in terms of which $\mathcal{O}_4$ is defined.

The model $\mathcal{O}_4^*$ is more abstract than $\mathcal{O}_4$ in the sense that the former ignores states whereas the latter does not: For each statement $s$, $\mathcal{O}_4[\![s]\!]$ is a set of sequences of actions, and an abstraction function $\mathcal{A}_4^*$ is defined such that

$$\mathcal{O}_4^* = \mathcal{A}_4^* \circ \mathcal{O}_4. \tag{7.2}$$

Furthermore we show that the following holds for every $s_1, s_2 \in \mathcal{L}_4$:

$$\begin{aligned}&\forall C[\ C \text{ is a } \textit{context} \text{ of } \mathcal{L}_4 \Rightarrow \mathcal{O}_4^*[\![C[s_1]]\!] = \mathcal{O}_4^*[\![C[s_2]]\!]\ ] \\ &\Rightarrow \mathcal{O}_4[\![s_1]\!] = \mathcal{O}_4[\![s_2]\!].\end{aligned} \tag{7.3}$$

This property is called the *completeness* of $\mathcal{O}_4$ w.r.t. $\mathcal{O}_4^*$. From this and the full abstractness of $\mathcal{M}_4$ w.r.t. $\mathcal{O}_4$, the full abstractness of $\mathcal{M}_4$ w.r.t. $\mathcal{O}_4^*$ is immediately obtained as follows:

The full abstractness of $\mathcal{M}_4$ w.r.t. $\mathcal{O}_4^*$ refers to the fact that the following holds for every $s_1, s_2 \in \mathcal{L}_4$:

$$\begin{array}{l} \mathcal{M}_4[\![s_1]\!] = \mathcal{M}_4[\![s_2]\!] \Leftrightarrow \\ \forall C[C \text{ is a } \textit{context} \text{ of } \mathcal{L}_4 \Rightarrow \mathcal{O}_4^*[\![C[s_1]]\!] = \mathcal{O}_4^*[\![C[s_2]]\!]\ ]. \end{array} \tag{7.4}$$

The $\Rightarrow$-part follows immediately from (7.2) and the full abstractness of $\mathcal{M}_4$ w.r.t. $\mathcal{O}_4$. The $\Leftarrow$-part is shown by showing its contrapositive: Suppose $\mathcal{M}_4[\![s_1]\!] \neq \mathcal{M}_4[\![s_2]\!]$. Then by the full abstractness of $\mathcal{M}_4$ w.r.t. $\mathcal{O}_4$, there is a context $C$ such that $\mathcal{O}_4[\![C[s_1]]\!] \neq \mathcal{O}_4[\![C[s_2]]\!]$; by this and (7.3), there is another context $C^*$ such that $\mathcal{O}_4^*[\![C^*[C[s_1]]]\!] \neq \mathcal{O}_4^*[\![C^*[C[s_2]]]\!]$. Thus we have the $\Leftarrow$-part of (7.4).

Finally, in § 7.4, some remarks on related works and future work are given.

Several mathematical proofs are deferred to the appendices.

Closely related to this chapter is the work of Hennessy and Plotkin [HP 79]. The language treated there, which we denote by $\mathcal{L}_{\text{co}}$, is very similar to our first language $\mathcal{L}_3$, except that it contains "co", a *coroutine* construct, as well as the usual interleaving. In [HP 79], a denotational model $\mathcal{V}$ for $\mathcal{L}_{\text{co}}$ is constructed and the full abstractness of $\mathcal{V}$ is established. Interestingly, we can construct a fully abstract model $\mathcal{M}_{\text{co}}$ for $\mathcal{L}_{\text{co}}$, by slightly modifying $\mathcal{M}_3$; thus the two models $\mathcal{V}$, $\mathcal{M}_{\text{co}}$ turn out to be isomorphic (see § 7.2.6 for more comparison with [HP 79]).

The work of Roscoe [Ros 84] is also related to this chapter. The language treated there, a large subset of occam, is similar to our second language $\mathcal{L}_4$ in many respects. However, unlike individual variables in $\mathcal{L}_4$, variables in occam are not shared by two or more parallel processes. Thus, the model proposed in [Ros 84] is different from $\mathcal{M}_4$ in the way for involving states into the meaning of a statement (see § 7.3.7 for more comparison with [Ros 84]).

## 7.2 A Nonuniform Language with Parallel Composition

The first language $\mathcal{L}_3$ is a *nonuniform* language with recursion and *parallel composition* but no communication.

First, an operational model $\mathcal{O}_3$ is introduced in terms of a Plotkin-style transition system.

Then a denotational model $\mathcal{M}_3$ is defined compositionally by means of interpreted operations of the language, with meanings of recursive programs as fixed-points of the denotational semantic domain, a complete metric space consisting of sets of streams of pairs of states.

The correctness of $\mathcal{M}_3$ w.r.t. $\mathcal{O}_3$ is established, as in [Rut 89] and [BR 91], by means of the fixed-point method introduced in [KR 90].

Finally, full abstractness of $\mathcal{M}_3$ is shown by means of a context with parallel composition:

*Given two statements $s_1$ and $s_2$ with different denotational meanings, a suitable statement $T$ is constructed such that the operational meanings of $s_1 \,\|\, T$ and $s_2 \,\|\, T$ are distinct.*

For constructing such $T$, a combinatorial method called the *testing method*, is introduced in Lemma 7.10 (*Testing Lemma*). By means of this, the full abstractness of $\mathcal{M}_3$ w.r.t. $\mathcal{O}_3$ is established.

### 7.2.1 The Language $\mathcal{L}_3$

The language $\mathcal{L}_3$ is the simplest nonuniform concurrent language with recursion: It has parallel composition but no communication, and its elementary actions consist only of value assignments to variables.

**Notation 7.1** **(1)** Let $(v \in)$ $\mathbf{V}$ denote some abstract domain of values.

**(2)** Let $(x \in)$ $\mathcal{IV}$ denote the set of *individual variables*. It is assumed there is a total order $\leq_{\text{iv}}$ on $\mathcal{IV}$. For $x_0, x_1 \in \mathcal{IV}$, let $x_0 <_{\text{iv}} x_1$ iff $x_0 \leq_{\text{iv}} x_1$ and $x_0 \neq x_1$.

**(3)** Let $(\sigma \in)$ $\Sigma$ denote the domain of *states*: $\Sigma = (\mathcal{IV} \to \mathbf{V})$.

**(4)** Let $(e \in)$ VExp denote the set of *value expressions*.

**(5)** Let $(b \in)$ BExp denote the set of *Boolean expressions*.

**(6)** The logical true and false values are denoted by tt and ff, respectively. ∎

We assume a simple syntax (not specified here) for $e$ and $b$. 'Simple' ensures at least that

$$\textit{no side-effects or nontermination occurs in their evaluation.} \tag{7.5}$$

The evaluations of $e$ and $b$ in state $\sigma$ are denoted by $[\![e]\!](\sigma)$ and $[\![b]\!](\sigma)$, respectively. For each $e \in$ VExp, it is assumed that

$$\exists \mathcal{X} \in \wp_{\text{fin}}(\mathcal{IV}), \forall \sigma_0, \sigma_1 \in \Sigma[\ \sigma_0 \quad \mathcal{X} = \sigma_0 \quad \mathcal{X} \Rightarrow [\![e]\!][\sigma_0] = [\![e]\!][\sigma_0]\ ]. \tag{7.6}$$

Likewise, for each $b \in$ BExp, it is assumed that

$$\exists \mathcal{X} \in \wp_{\text{fin}}(\mathcal{IV}), \forall \sigma_0, \sigma_1 \in \Sigma[\ \sigma_0 \quad \mathcal{X} = \sigma_0 \quad \mathcal{X} \Rightarrow [\![b]\!][\sigma_0] = [\![b]\!][\sigma_0]\ ]. \tag{7.7}$$

Property (7.6) (resp. (7.7)) follows from the syntactical fact that only a finite number of individual variables occur in $e$ (resp. in $b$). The full abstractness of a denotational model is established under assumptions (7.5), (7.6), and (7.7).

**Definition 7.1** (Language $\mathcal{L}_3$) Let $P$ range over $\mathcal{RV}$, the set of *recursion variables*, and let $X$ range over $\mathcal{X}_{\mathcal{P}}$, the set of *statement variables*. Note that recursion variables are used as names of statements defined by recursion, while statement variables are used as place holders for defining *contexts* of a language.

**(1)** For every $x \in \mathcal{IV}$ and $e \in$ VExp, the construct "$(x := e)$"is a nullary combinator on statements called an *assignment*;[3] let **Asg**, the set of *assignments*, be defined by:

[3] Constructs which represent statements and have no parameter are called *nullary combinators on statements*.

$$\mathbf{Asg} = \{(x := e) : x \in \mathcal{IV} \wedge e \in \mathrm{VExp}\}.$$

Also, for every $b \in \mathrm{BExp}$, the construct "$\mathbf{if}(b, \cdot, \cdot)$" is a binary combinator on statements called an *conditional*; let **Cond**, the set of *conditionals*, be defined by:

$$\mathbf{Cond} = \{\mathbf{if}(b, \cdot, \cdot) : b \in \mathrm{BExp}\}.$$

A single-sorted *signature* $\mathbf{Sig}_3 = (\mathbf{Fun}_3, \mathrm{arity}_3(\cdot))$ is defined as follows, with $(\mathsf{op} \in)$ $\mathbf{Fun}_3$ being a set of *combinators* and $\mathrm{airty}_3(\cdot)$ a function which maps each combinator to its *arity*:

$$\mathbf{Fun}_3 = \{\mathbf{0}, \mathbf{e}\} \cup \mathbf{Asg} \cup \mathcal{RV} \cup \{;, +, \|\} \cup \mathbf{Cond}.$$

$$\mathrm{arity}_3(\mathsf{op}) = \begin{cases} 0 & \text{if } \mathsf{op} \in \{\mathbf{0}, \mathbf{e}\} \cup \mathbf{Asg} \cup \mathcal{RV}, \\ 2 & \text{if } \mathsf{op} \in \{+, \|, ;\} \cup \mathbf{Cond}, \end{cases}$$

where the nullary combinator: "**0**" and "**e**" represet *inaction*, and *termination*, respectively; the combinators: ";", "+", and "$\|$" represent *sequential composition*, *alternative choice*, *parallel composition*, respectively. For $r \in 3$, we put

$$\mathbf{Fun}_3^{(r)} = \{\mathsf{op} \in \mathbf{Fun}_3 : \mathrm{arity}_3(\mathsf{op}) = r\}.$$

**(2)** The set of statements of the nonuniform concurrent language $(S \in)$ $\mathcal{L}_3$ is defined as the set of terms generated by the signature $\mathbf{Sig}_3$ and the variable set $\mathcal{X}_\mathcal{P}$. That is, $\mathcal{L}_3$ is defined by the following grammar: [4]

$$S ::= \mathbf{0} \mid \mathbf{e} \mid (x := e) \mid (S_0; S_1) \mid (S_0 + S_1) \mid (S_0 \parallel S_1) \\ \mid \mathbf{if}(b, S_0, S_1) \mid P \mid X.$$

Let $\mathrm{FV}(S)$ denote the set of statement variables contained in $S$.

**(3)** Let $(s \in)$ $\mathcal{L}_3[\emptyset]$ be the set of statements with no statement variable, i.e., let

$$\mathcal{L}_3[\emptyset] = \{S \in \mathcal{L}_3 : \mathrm{FV}(S) = \emptyset\}.$$

For $X \in \mathcal{X}_\mathcal{P}$, let

$$\mathcal{L}_3[X] = \{S \in \mathcal{L}_3 : \mathrm{FV}(S) \subseteq \{X\}\}.$$

**(4)** The set of *guarded statements* $(g \in)$ $\mathcal{G}_3$ is defined by the following grammar:

$$g ::= \mathbf{0} \mid (x := e) \mid (g; s) \mid \mid (g_0 + g_1) \mid (g_0 \parallel g_1) \mid \mathbf{if}(b, g_0, g_1).$$

**(5)** We assume that each recursion variable $P$ is associated with an element $g_P$ of $\mathcal{G}_3$ by a set of declarations $D = \{(P, g_P)\}_{P \in \mathcal{RV}}$. A *program* consists of a pair $(s, D)$. ∎

[4] In this language, the precedence of ';' , '+', and '$\|$' is higher than that of ',' occurring in the construct $\mathbf{if}(\cdot, \cdot, \cdot)$.

In the sequel of this section, we fix a declaration set $D = \{(P, g_P)\}_{P \in \mathcal{RV}}$.

We introduce the notion of a *context* and some uses of it as follows:

**Notation 7.2** Let $\mathcal{L}$ be a language formulated as the set of terms generated by a signature **Sig** and a set $\{X_i\}$ of variables.

**(1)** For $S \in \mathcal{L}$ and a sequence of distinct variables $\langle X_1, \ldots, X_n \rangle$, the pair

$$(S, \langle X_1, \ldots, X_n \rangle)$$

is called a *context* of $\mathcal{L}$. We sometimes write

$$S_{\langle X_1, \ldots, X_n \rangle} \text{ for } (S, \langle X_1, \ldots, X_n \rangle).$$

When the notation $S_{\langle X_1, \ldots, X_n \rangle}$ is used, it is always assumed that

$$\mathrm{FV}(S) \subseteq \{X_1, \ldots, X_n\}.$$

**(2)** For a context $S_{\langle X_1, \ldots, X_n \rangle}$ and $S_1, \ldots, S_n \in \mathcal{L}$, the notation

$$S[\langle S_1, \ldots, S_n \rangle / \langle X_1, \ldots, X_n \rangle]$$

denotes the result of simultaneously replacing $X_i$ in $S$ with $S_i$, $i \in \bar{n}$. More simply, we sometimes write

$$S_{\langle X_1, \ldots, X_n \rangle}[S_1, \ldots, S_n] \text{ for } S[\langle S_1, \ldots, S_n \rangle / \langle X_1, \ldots, X_n \rangle].$$

**(3)** Let $\mathcal{I}$ be an *interpretation*, i.e., a mapping which maps each combinator in **Sig** to a corresponding semantic operation on a semantic domain $\mathbf{P}$ (see [Rut 90] for a formal definition of an interpretation for a signature); let $S_{\langle X_1, \ldots, X_n \rangle}$ be a context. For $p_1, \ldots, p_n \in \mathbf{P}$, let $[\![S]\!]^{\mathcal{I}}[\langle X_1, \ldots, X_n \rangle / \langle p_1, \ldots, p_n \rangle]$ denote the interpretation of $S$ under $\mathcal{I}$ with the assignment of the value $p_i$ to the variable $X_i$, $i \in \bar{n}$. More simply, we sometimes write

$$[\![S_{\langle X_1, \ldots, X_n \rangle}]\!]^{\mathcal{I}}(p_1, \ldots, p_n)$$

for

$$[\![S]\!]^{\mathcal{I}}[\langle p_1, \ldots, p_n \rangle / \langle X_1, \ldots, X_n \rangle]. \blacksquare$$

### 7.2.2 Operational Model $\mathcal{O}_3$ for $\mathcal{L}_3$

Let $(\alpha \in)$ $\mathbf{A}_3^{\surd}$, the set of actions in $\mathcal{L}_3$, be defined by:

$$\mathbf{A}_3^{\surd} = \{\tau, \surd\},$$

where $\tau$ and $\surd$ are symbols representing a *internal move* and *successful termination*, respectively.

The operational model $\mathcal{O}_3$ rests on a transition system $\langle \xrightarrow{\alpha}_3 : \alpha \in \{\tau, \surd\} \rangle$, of the style of [Plo 81]. For $s_1, s_2 \in \mathcal{L}_3[\emptyset]$ and $\sigma_1, \sigma_2 \in \Sigma$, we write $(s_1, \sigma_1) \xrightarrow{\tau}_3 (s_2, \sigma_2)$ (resp. $(s_1, \sigma_1) \xrightarrow{\surd}_3$) for $((s_1, \sigma_1), (s_2, \sigma_2)) \in \xrightarrow{\tau}_3$ (resp. $(s_1, \sigma_1) \in \xrightarrow{\surd}_3$) for easier readability.

**Definition 7.2** (Transition Relations $\xrightarrow{\alpha}_3$) The transition relations $\xrightarrow{\tau}_3$ and $\xrightarrow{\surd}_3$ are defined as the smallest relations satisfying the following rules (1)–(7-2). For $\sigma \in \Sigma$, $x \in \mathcal{IV}$, and $v \in \mathbf{V}$, the notation $\sigma[v/x]$ is used to denote a state $\sigma'$ which is the same as $\sigma$ except that $\sigma'(x) = v$.

**(1)** $(\mathbf{e}, \sigma) \xrightarrow{\surd}_3$ .

**(2)** $((x := e), \sigma) \xrightarrow{\tau}_3 (\mathbf{e}, \sigma[[\![e]\!](\sigma)/x])$.

**(3-1)** $\dfrac{(s_1, \sigma) \xrightarrow{\tau}_3 (s', \sigma')}{((s_1; s_2), \sigma) \xrightarrow{\tau}_3 ((s'; \sigma_2), \sigma')}$.

**(3-2)** $\dfrac{(s_1, \sigma) \xrightarrow{\epsilon}_3, (s_2, \sigma) \xrightarrow{\tau}_3 (s', \sigma')}{((s_1; s_2), \sigma) \xrightarrow{\tau}_3 (s', \sigma')}$.

**(3-3)** $\dfrac{(s_1, \sigma) \xrightarrow{\epsilon}_3, (s_2, \sigma) \xrightarrow{\epsilon}_3}{((s_1; s_2), \sigma) \xrightarrow{\epsilon}_3}$.

**(4-1)** $\dfrac{(s_1, \sigma) \xrightarrow{\tau}_3 (s, \sigma')}{\begin{array}{l}(s_1 + s_2, \sigma) \xrightarrow{\tau}_3 (s, \sigma') \\ (s_2 + s_1, \sigma) \xrightarrow{\tau}_3 (s, \sigma')\end{array}}$.

**(4-2)** $\dfrac{(s_1, \sigma) \xrightarrow{\surd}_3}{\begin{array}{l}(s_1 + s_2, \sigma) \xrightarrow{\surd}_3 \\ (s_2 + s_1, \sigma) \xrightarrow{\surd}_3\end{array}}$.

**(5-1)** $\dfrac{(s_1, \sigma) \xrightarrow{\tau}_3 (s, \sigma')}{\begin{array}{l}(s_1 \parallel s_2, \sigma) \xrightarrow{\tau}_3 (s \parallel s_2, \sigma') \\ (s_2 \parallel s_1, \sigma) \xrightarrow{\tau}_3 (s_2 \parallel s, \sigma')\end{array}}$.

**(5-2)** $\dfrac{(s_1, \sigma) \xrightarrow{\surd}_3, (s_2, \sigma) \xrightarrow{\surd}_3}{(s_1 \parallel s_2, \sigma) \xrightarrow{\surd}_3}$.

**(6-1)** $\dfrac{(s_1, \sigma) \xrightarrow{\tau}_3 (s, \sigma')}{(\mathbf{if}(b, s_1, s_2), \sigma) \xrightarrow{\tau}_3 (s, \sigma')} \quad ([\![e]\!](\sigma) = \mathrm{tt})$.

**(6-2)** $\dfrac{(s_1, \sigma) \xrightarrow{\surd}_3}{(\mathbf{if}(b, s_1, s_2), \sigma) \xrightarrow{\surd}_3} \quad ([\![e]\!](\sigma) = \mathrm{tt})$.

**(6-3)** $\dfrac{(s_2, \sigma) \xrightarrow{\tau}_3 (s, \sigma')}{(\mathbf{if}(b, s_1, s_2), \sigma) \xrightarrow{\tau}_3 (s, \sigma')} \quad ([\![e]\!](\sigma) = \mathrm{ff})$.

$$\textbf{(6-4)}\ \frac{(s_2,\sigma)\xrightarrow{\surd}_3}{(\mathbf{if}(b,s_1,s_2),\sigma)\xrightarrow{\surd}_3}\quad(\llbracket e\rrbracket(\sigma)=\mathrm{ff}).$$

$$\textbf{(7-1)}\ \frac{(g_P,\sigma)\xrightarrow{\tau}_3(s',\sigma')}{(P,\sigma)\xrightarrow{\tau}_3(s',\sigma')}\quad((P,g_P)\in D).$$

$$\textbf{(7-2)}\ \frac{(g_P,\sigma)\xrightarrow{\surd}_3}{(P,\sigma)\xrightarrow{\surd}_3}\quad((P,g_P)\in D).$$

The last rule, called the *recursion rule*, stipulates that for each declaration $(P, g_P) \in D$, transitions of the recursion variable $P$ are derived from those of its body $g_P$, as usual. ∎

A few auxiliary functions are introduced as preliminaries to the definition of the operational model.

**Definition 7.3** Let $s \in \mathcal{L}_3[\emptyset]$ and $\sigma \in \Sigma$. We define $\mathrm{IDrv}_3(s,\sigma)$, the set of immediate derivatives of $(s,\sigma)$, be defined by:

$$\mathrm{IDrv}_3(s,\sigma)=\{(s',\sigma'):(s,\sigma)\xrightarrow{\tau}_3(s',\sigma')\}.$$

Also, we define $\mathrm{act}_3(s,\sigma)$, the set of actions of $(s,\sigma)$, by:

$$\begin{aligned}\mathrm{act}_3(s,\sigma)=\{\alpha\in\{\tau,\surd\}:(&\alpha=\tau\wedge\mathrm{IDrv}(s,\sigma)\neq\emptyset)\\ &\vee(\alpha=\surd\wedge(s,\sigma)\xrightarrow{\surd}_3)\}.\end{aligned}$$ ∎

Let us call a statement $s \in \mathcal{L}_3[\emptyset]$ *finitely branching* iff for every $\sigma \in \Sigma$, $\mathrm{IDrv}_3(s,\sigma)$ is finite. Then, the transition relation $\xrightarrow{\tau}_3$ is *finitely branching* in the following sense:

**Lemma 7.1** *Every $s \in \mathcal{L}_3[\emptyset]$ is finitely branching, i.e., for every $\sigma$, the set $\{(s',\sigma'):(s,\sigma)\xrightarrow{\tau}_3(s',\sigma')\}$ is finite.* ∎

**Proof.** Let us say $s$ is a *guard statement*, if $s$ is finitely branching and

$$\forall\sigma[\ \surd\notin\mathrm{act}_3(s,\sigma)\ ].$$

First, it is shown that

$$\forall g\in\mathcal{G}_3[\ g\text{ is a guard statement }],\tag{7.8}$$

by induction on the structure $g$ using the following facts (7.9)–(7.12):

$$\mathbf{0}\text{ is a guard statement.}\tag{7.9}$$

$$\forall x\in\mathcal{IV},\forall e\in\mathrm{VExp},\forall\sigma\in\Sigma[\ (x:=e)\text{ is a guard statement }].\tag{7.10}$$

$$\forall s_1,s_2\in\mathcal{L}_3[\emptyset][\text{ if }s_1\text{ is a guard statement, then }(s_1;s_2)\text{ is also. a guard statement }]\tag{7.11}$$

$$\forall s_1, s_2 \in \mathcal{L}_3[\emptyset][\text{ if both } s_1 \text{ and } s_2 \text{ are guard statements, then } (s_1 + s_2), (s_1 \parallel s_2), \mathbf{if}(b, s_1 + s_2) \text{ are also guard statements }]. \tag{7.12}$$

Also, one has

$$\forall s_1, s_2 \in \mathcal{L}_3[\emptyset][\text{ if both } s_1 \text{ and } s_2 \text{ are finitely branching, then. } (s_1 + s_2), (s_1 \parallel s_2), \text{ and } \mathbf{if}(b, s_1 + s_2) \text{ are also finitely branching }] \tag{7.13}$$

Using (7.8) and (7.13), one has

$$\forall s \in \mathcal{L}_3[\emptyset][\ s \text{ is finitely branching }]$$

by induction on the structure of $s$. ■

As a preliminary to the definition of the operational model, we define an auxiliary semantic domain $\mathbf{R}_3$ by:

**Definition 7.4** Let

$$\mathbf{R}_3 = (\Sigma^{<\omega} \cdot \{\langle \delta \rangle, \langle \surd \rangle\}) \cup \Sigma^{\omega},$$

where $\delta$ is a symbol representing deadlock. We use the variables $\rho$ ranging over $\mathbf{R}_3$ and $\chi$ ranging over $\{\langle \delta \rangle, \langle \surd \rangle\}$. ■

An operational model $\mathcal{O}_3$ is defined by means of $\xrightarrow{\alpha}_3$ as the fixed-point of a higher-order mapping $\Psi_3^{\mathcal{O}}$.

**Definition 7.5** (Operational Model $\mathcal{O}_3$ for $\mathcal{L}_3$)

**(1)** Let $\mathbf{M}_3^{\mathcal{O}} = (\mathcal{L}_3[\emptyset] \to (\Sigma \to \wp_{+\text{cl}}(\mathbf{R}_3)))$, which is equipped with a metric $d$ defined as in § 2.2. Then, let $\Psi_3^{\mathcal{O}} : \mathbf{M}_3^{\mathcal{O}} \to \mathbf{M}_3^{\mathcal{O}}$ be defined as follows: For $f \in \mathbf{M}_3^{\mathcal{O}}$, $s \in \mathcal{L}_3[\emptyset]$, and $\sigma \in \Sigma$,

$$\begin{aligned}
&\Psi_3^{\mathcal{O}}(f)(s)(\sigma) \\
&= \bigcup\{\langle \sigma' \rangle \cdot f(s')(\sigma') : (s', \sigma') \in \text{IDrv}_3(s, \sigma)\} \\
&\quad \cup \text{if}(\surd \in \text{act}(s, \sigma), \{\langle \surd \rangle\}, \emptyset) \\
&\quad \cup \text{if}(\text{act}(s, \sigma) = \emptyset, \{\langle \delta \rangle\}, \emptyset).
\end{aligned}$$

The right-hand side of the above equation is closed by Lemma 7.1, and therefore, indeed $\Psi_3^{\mathcal{O}}(f) \in \mathbf{M}_3^{\mathcal{O}}$. Moreover, it immediately follows from the above definition that for $f, f' \in \mathbf{M}_3^{\mathcal{O}}$, $d(\Psi_3^{\mathcal{O}}(f), \Psi_3^{\mathcal{O}}(f)) \leq \kappa \cdot d(f, f')$, where $\kappa$ ($< 1$) is the fixed positive real number introduced in § 2.2. Thus, $\Psi_3^{\mathcal{O}}$ is a contraction from $\mathbf{M}_3^{\mathcal{O}}$ to $\mathbf{M}_3^{\mathcal{O}}$.

**(2)** Let the operational model $\mathcal{O}_3$ be the unique fixed-point of $\Psi_3^{\mathcal{O}}$. By the definition, one has $\mathcal{O}_3 : \mathcal{L}_3[\emptyset] \to (\Sigma \to \wp_{+\text{cl}}(\mathbf{R}_3))$, and for each $s \in \mathcal{L}_3[\emptyset]$ and $\sigma \in \Sigma$,

$$\begin{aligned}&\mathcal{O}_3[\![s]\!](\sigma)\\ =\;& \bigcup\{\langle\sigma'\rangle\cdot\mathcal{O}_3[\![s']\!](\sigma') : (s',\sigma')\in \mathrm{IDrv}_3(s,\sigma)\}\\ &\cup \mathrm{if}(\surd\in \mathrm{act}(s,\sigma), \{\langle\surd\rangle\}, \emptyset)\\ &\cup \mathrm{if}(\mathrm{act}(s,\sigma)=\emptyset, \{\langle\,\delta\,\rangle\},\emptyset). \blacksquare\end{aligned}$$

Note that $\mathcal{O}_3$ is not compositional as the following example shows.

**Example 7.1** Let $x \in \mathcal{IV}$. Then

$$\begin{aligned}\mathcal{O}_3[\![(x := 0); (x := x+1)]\!] &= \mathcal{O}_3[\![(x := 0); (x := 1)]\!]\\ &= (\lambda\sigma : \{\langle(\sigma[0/x], \sigma[1/x]), \surd\rangle\}),\end{aligned}$$

but

$$\begin{aligned}&\mathcal{O}_3[\![((x := 0); (x := x+1)) \parallel ((x := 2); \mathbf{0})]\!]\\ \neq\;& \mathcal{O}_3[\![((x := 0); (x := 1)) \parallel ((x := 2); \mathbf{0})]\!],\end{aligned}$$

since for every $\sigma$,

$$\begin{aligned}&\langle\sigma[0/x], \sigma[2/x], \sigma[3/x], \surd\rangle\\ \in\;& \mathcal{O}_3[\![((x := 0); (x := x+1)) \parallel ((x := 2); \mathbf{0})]\!](\sigma),\end{aligned}$$

but

$$\begin{aligned}&\langle\sigma[0/x], \sigma[2/x], \sigma[3/x], \surd\rangle\\ \notin\;& \mathcal{O}_3[\![((x := 0); (x := x+1)) \parallel ((x := 2); \mathbf{0})]\!](\sigma). \blacksquare\end{aligned}$$

### 7.2.3 Denotational Model $\mathcal{M}_3$ for $\mathcal{L}_3$

The denotational model $\mathcal{M}_3$ is defined compositionally by means of interpreted operations of the language.

The denotational semantic domain $\mathbf{P}_3$ is a complete metric space consisting of sets of streams of pairs of states. The meaning of a recursion variable $P$ with the declaration $(P, g_P)$ is defined as the fixed-point of the contraction which maps each process $p \in \mathbf{P}_3$ to the interpretation of $g_P$ under the interpreted operations with the assignment of $p$ to $P$. It turns out that the fixed-point is the unique solution of the equation $P = g_P$ under the interpretation consisting of the interpreted operations.

Recall that a real number $\kappa$ with $0 < \kappa < 1$ is fixed in this thesis. The domain $\mathbf{P}_3$ is defined by:

**Definition 7.6** (Denotational Semantic Domain $\mathbf{P}_3$ for $\mathcal{L}_3$)

**(1)** Let us use the variable $\Gamma$ ranging over $\wp(\{\surd\})$. For $\Gamma \in \wp(\{\surd\})$, let $\delta(\Gamma) = (\,\delta\,, \Gamma)$. Let

$$(\Upsilon \in)\ \mathbf{B}_3 = \{(\sigma, \surd) : \sigma \in \Sigma\} \cup \{(\sigma, \delta(\Gamma)) : \sigma \in \Sigma \wedge \Gamma \in \wp(\{\surd\})\}.$$

Further, let

$$\hat{\mathbf{B}}_3 = \{(\sigma, \surd) : \sigma \in \Sigma\} \cup \{(\sigma, \delta(\{\surd\})) : \sigma \in \Sigma\}.$$

**(2)** Let $\mathbf{Q}_3$ be the unique solution of

$$\mathbf{Q}_3 \cong \mathbf{B}_3 \uplus ((\Sigma \times \Sigma) \times \mathrm{id}_\kappa(\mathbf{Q}_3)).$$

One has

$$\mathbf{Q}_3 = ((\Sigma \times \Sigma)^{<\omega} \cdot \mathbf{B}_3) \cup (\Sigma \times \Sigma)^\omega.$$

**(3)** For $p \in \wp_{+\mathrm{cl}}(\mathbf{Q}_3)$, and $r \in (\Sigma \times \Sigma)^{<\omega}$, the *remainder* of $p$ with prefix $r$, denoted by $p[r]$, is defined by $p[r] = \{q \in \mathbf{Q}_3 : r \cdot q \in p\}$.

**(4)** The *initial state* of a sequence $q \in \mathbf{Q}_3 \cup (\Sigma \times \Sigma)^+$, denoted by $\mathrm{istate}_3(q)$, is defined as follows: Let $\mathrm{istate}_3(q) = \sigma$ if $q = \langle(\sigma, \sigma')\rangle \cdot q'$, and let $\mathrm{istate}_3(q) = \sigma''$ if $q = \langle(\sigma'', \delta(\Gamma))\rangle$.

**(5)** Let $p \in \wp_{+\mathrm{cl}}(\mathbf{Q}_3)$ and $\sigma \in \Sigma$. Let $p\langle\sigma\rangle$ be the set of those elements of $p$ whose initial state is $\sigma$, i.e., let

$$p\langle\sigma\rangle = \{q \in p : \mathrm{istate}_3(q) = \sigma\}.$$

Also, let

$$\begin{aligned}\mathrm{act}_3(p, \sigma) = {} & \mathrm{if}(\exists\sigma'[\ p[\langle(\sigma, \sigma')\rangle] \neq \emptyset\ ], \{\tau\}, \emptyset) \\ & \cup \mathrm{if}(\langle(\sigma, \surd)\rangle \in p, \{\surd\}, \emptyset).\end{aligned}$$

**(6)** Let $p \in \wp_{+\mathrm{cl}}(\mathbf{Q}_3)$ and $n \in \omega$. We say $p$ satisfies *the disabled-$\tau$ condition at level $n$* iff

$$\begin{aligned}&\forall r \in (\Sigma \times \Sigma)^n, \forall\sigma \in \Sigma[\ p[r] \neq \emptyset \wedge \tau \notin \mathrm{act}_3(p[r], \sigma) \Rightarrow \\ &\quad \exists\Gamma \in \wp(\{\surd\})[\ r \cdot \langle(\sigma, \delta(\Gamma))\rangle \in p\ ]].\end{aligned}$$

We say $p$ satisfies *the disabled $\tau$ condition* iff it satisfies the disabled $\tau$ condition at every level $n \in \omega$.

**(7)** Let $p \in \wp_{+\mathrm{cl}}(\mathbf{Q}_3)$ and $n \in \omega$. We say $p$ is *downward closed at level $n$* iff

$$\forall r \in (\Sigma \times \Sigma)^n, \forall\sigma[\ r \cdot \langle(\sigma, \delta(\{\surd\}))\rangle \in p \Rightarrow r \cdot \langle(\sigma, \delta(\emptyset))\rangle \in p\ ].$$

We say $p$ is *downward closed* iff it is downward closed at every level $n \in \omega$.

**(8)** Let $p \in \wp_{+\mathrm{cl}}(\mathbf{Q}_3)$ and $n \in \omega$. We say $p$ is *upward closed w.r.t. disabled actions at level $n$* iff

$$\begin{aligned}&\forall r \in (\Sigma \times \Sigma)^n, \forall\sigma[\ r \cdot \langle(\sigma, \delta(\emptyset))\rangle \in p \wedge \surd \notin \mathrm{act}_3(p[r], \sigma) \Rightarrow \\ &\quad r \cdot \langle(\sigma, \delta(\emptyset))\rangle \in p\ ].\end{aligned}$$

We say $p$ is *upward closed w.r.t. disabled actions* iff it is upward closed w.r.t. disabled actions at every level $n \in \omega$.

**(9)** Let $\mathbf{P}_3$, the domain of processes for $\mathcal{L}_3$, be the set of elements $p$ of $\wp_{+\mathrm{cl}}(\mathbf{Q}_3)$ satisfying the following conditions (i)–(iii):

**(i)** $p$ satisfies the disabled-$\tau$ condition

**(ii)** $p$ is downward closed.

**(iii)** $p$ is upward closed w.r.t. disabled actions. ∎

**Remark 7.1** A subset $\mathbf{P}$ of $\wp_{+\mathrm{cl}}(\mathbf{Q}_3)$ is said to be *closed under taking remainders* iff

$$\forall p \in \mathbf{P}, \forall r \in (\Sigma \times \Sigma)^{<\omega}[\ p[r] \neq \emptyset \Rightarrow p[r] \in \mathbf{P}\ ].$$

Given an arbitrary subset $\mathbf{P}_0$ of $\wp_{+\mathrm{cl}}(\mathbf{Q}_3)$, it is routine to check that the largest subset $\mathbf{P}_0'$ of $\wp_{+\mathrm{cl}}(\mathbf{Q}_3)$ which is included in $\mathbf{P}_0$ and closed under taking remainders is given by: $\mathbf{P}_0' = \{p \in \wp_{+\mathrm{cl}}(\mathbf{Q}_3) : \forall r \in (\Sigma \times \Sigma)^{<\omega}[\ p[r] \neq \emptyset \Rightarrow p[r] \in \mathbf{P}_0\ ]\}$. Thus, $\mathbf{P}_3$ is the largest subset of $\wp_{+\mathrm{cl}}(\mathbf{Q}_3)$ which is included in $\mathbf{P}_3^{(0)}$ and closed under taking remainders, where $\mathbf{P}_3^{(0)}$ is the set of elements of $\wp_{+\mathrm{cl}}(\mathbf{Q})$ satisfying the following conditions (i)–(iii):

**(i)** $p$ satisfies the disabled-$\tau$ condition at level 0.

**(ii)** $p$ is downward closed at level 0.

**(iii)** $p$ is upward closed w.r.t. disabled actions at level 0. ∎

The conditions (i)–(iii) in Definition 7.6 (9) are necessary for defining the interpretations of combinators in $\mathbf{Fun}_3^{(2)} \parallel$ as operations on $\mathbf{P}_3$ in the sequel.

**Lemma 7.2** *The set $\mathbf{P}_3$ is closed in $\wp_{+\mathrm{cl}}(\mathbf{Q}_3)$, and therefore, $\mathbf{P}_3$ is a complete metric space with the original metric of $\wp_{+\mathrm{cl}}(\mathbf{Q}_3)$ restricted to it.* ∎

**Proof.** The closedness can be established using Lemma 2.2 as follows: For $n \in \omega$, let $\mathbf{P}_3^{(n)}$ be the set of elements $p$ of $\wp_{+\mathrm{cl}}(\mathbf{Q}_3)$ satisfying the following conditions (i)–(iii):

**(i)** $p$ satisfies the disabled-$\tau$ condition at level $n$.

**(ii)** $p$ is downward closed at level $n$.

**(iii)** $p$ is upward closed w.r.t. disabled actions at level $n$.

Then, it immediately follows that

$$\forall p \in \wp_{+\mathrm{cl}}(\mathbf{Q}_3)[\ p \in \mathbf{P}_3^{(n)} \Leftrightarrow \tilde{\pi}_{n+1}(p) \in \mathbf{P}_3^{(n)}\ ].$$

Thus $\mathbf{P}_3^{(n)}$ is finitely characterized. Obviously, $\mathbf{P}_3 = \bigcap_{n\in\omega}[\mathbf{P}_3^{(n)}]$. Hence, by Lemma 2.2, $\mathbf{P}_3$ is closed. ∎

The interpretation $\mathcal{I}_3$ for the signature $\mathbf{Sig}_3$ is defined as follows:

**Definition 7.7** (Interpretation $\mathcal{I}_3$ for $\mathbf{Sig}_3$)

**(1)** $\tilde{\mathbf{0}}_3 = \{(\sigma, \delta(\Gamma)) : \sigma \in \Sigma \wedge \Gamma \in \wp(\{\surd\})\}$.

**(2)** $\tilde{\mathbf{e}}_3 = \{(\sigma, \surd) : \sigma \in \Sigma\} \cup \{(\sigma, \delta(\emptyset)) : \sigma \in \Sigma\}$.

**(3)** For $x \in \mathcal{IV}$ and $e \in \mathrm{VExp}$, the process $\mathrm{asg}_3(x, e) \in \mathbf{P}_3$, which is the interpretation of the assignment statement "$(x := e)$", is defined by:

$$\text{asg}_3(x,e) = \{\langle(\sigma, \sigma[[\![e]\!](\sigma)/x])\rangle : \sigma \in \Sigma\} \cdot \tilde{\mathbf{e}}_3,$$

where

$$\langle(\sigma, \sigma[[\![e]\!](\sigma)/x])\rangle \cdot \tilde{\mathbf{e}}_3$$

denotes the concatenation of $\langle(\sigma, \sigma[[\![e]\!](\sigma)/x])\rangle$ and $\tilde{\mathbf{e}}_3$.

**(4)** The semantic operation $\tilde{;}_3 : (\mathbf{P}_3)^{\underline{2}} \to \mathbf{P}_3$ corresponding to the combinator ';' is defined recursively as follows: For every $p_1, p_2 \in \mathbf{P}_3$,

$$\begin{aligned}\tilde{;}_3(p_1,p_2) = &\ \tilde{;}_3^{\delta}(p_1,p_2)\\ &\cup \textstyle\bigcup_{\sigma\in\Sigma}[\text{if}(\langle(\sigma,\surd)\rangle \in p_1, (p_2)^+, \emptyset)]\\ &\cup \textstyle\bigcup\{\tilde{;}_3(p_1[\langle(\sigma,\sigma')\rangle], p_2) : p_1[\langle(\sigma,\sigma')\rangle] \neq \emptyset\},\end{aligned}$$

where

$$\begin{aligned}&\tilde{;}_3^{\delta}(p_1,p_2)\\ &= \textstyle\bigcup_{\sigma\in\Sigma}[\{\langle(\sigma,\delta(\Gamma))\rangle : \Gamma \in \wp(\{\surd\}) \wedge \langle(\sigma,\delta(\Gamma\cup\{\surd\}))\rangle \in p_1\\ &\qquad \vee (\langle(\sigma,\delta(\Gamma\setminus\{\surd\}))\rangle \in p_1 \wedge \langle(\sigma,\delta(\Gamma))\rangle \in p_2)\}].\end{aligned}$$

Formally the operation $\tilde{;}_3$ is defined as a suitably defined contraction as the operation $\tilde{\|}_3$ is defined in Part (6) below.

**(5)** For $p \in \mathbf{P}_3$, $p \setminus \tilde{\mathbf{0}}_3$ is called the *action part* of $p$ and denoted by $p^+$, and the set $p \cap \tilde{\mathbf{0}}_3$ is called the *inaction part* of $p$. The action part of the alternative composition of two processes is the union of the action parts of those processes, and its inaction part is the intersection of the inaction parts of them. That is, for $p_1, p_2 \in \mathbf{P}_3$,

$$p_1 \tilde{+} p_2 = (p_1)^+ \cup (p_2)^+ \cup (p_1 \cap p_2 \cap \tilde{\mathbf{0}}_3).$$

**(6)** The parallel composition $\tilde{\|}_3 : (\mathbf{P}_3)^{\underline{2}} \to \mathbf{P}_3$ is defined recursively as follows: For every $p_1, p_2 \in \mathbf{P}_3$,

$$\tilde{\|}_3(p_1,p_2) = \tilde{\lfloor\!\lfloor}_3(p_1,p_2) \cup \tilde{\lfloor\!\lfloor}_3(p_2,p_1) \cup \tilde{\|}_3^{\surd}(p_1,p_2) \cup \tilde{\|}_3^{\delta}(p_1,p_2), \qquad (7.14)$$

where

$$\tilde{\lfloor\!\lfloor}_3(p_1,p_2) = \bigcup\{\langle(\sigma,\sigma')\rangle \cdot \tilde{\|}_3(p_1[\langle(\sigma,\sigma')\rangle], p_2) : p_1[\langle(\sigma,\sigma')\rangle] \neq \emptyset\},$$

$$\tilde{\|}_3^{\surd}(p_1,p_2) = \{\langle(\sigma,\surd)\rangle : \sigma \in \Sigma\} \cap p_1 \cap p_2,$$

$$\begin{aligned}&\tilde{\|}_3^{\delta}(p_1,p_2)\\ &= \{\langle(\sigma,\delta(\Gamma))\rangle : (\langle(\sigma,\delta(\Gamma))\rangle \in p_1 \wedge \langle(\sigma,\delta(\Gamma\setminus\{\surd\}))\rangle \in p_2)\\ &\qquad \vee (\langle(\sigma,\delta(\Gamma\setminus\{\surd\}))\rangle \in p_1 \wedge \langle(\sigma,\delta(\Gamma))\rangle \in p_2)\}.\end{aligned}$$

Formally the operation $\tilde{\|}_3$ is defined as the fixed-point of a suitably defined contraction: Let $\mathbf{M}_3^{\|} = ((\mathbf{P}_3)^{\underline{2}} \to \mathbf{P}_3)$, $\Omega_3^{\|} : \mathbf{M}_3^{\|} \to \mathbf{M}_3^{\|}$ be defined as follows: For $F \in \mathbf{M}_3^{\|}$, and $p_1, p_2 \in \mathbf{P}_3$,

$$\Omega_3^{\|}(F)(p_1, p_2)$$
$$= \Omega_3^{\|\!\!\_}(F)(p_1, p_2) \cup \Omega_3^{\|\!\!\_}(F)(p_2, p_1) \cup \tilde{\|}_3^{\checkmark}(p_1, p_2) \cup \tilde{\|}_3^{\delta}(p_1, p_2),$$

where

$$\Omega_3^{\|\!\!\_}(F)(p_1, p_2) = \bigcup \{\langle(\sigma, \sigma')\rangle \cdot F(p_1[\langle(\sigma, \sigma')\rangle], p_2) : p_1[\langle(\sigma, \sigma')\rangle] \neq \emptyset\}.$$

It is shown that $\Omega_3^{\|}(F)(p_1, p_2)$ satisfies the disabled $\tau$-condition at level 0 as follows: Let $\sigma \in \Sigma$, and suppose

$$\tau \notin \mathrm{act}_3(\Omega_3^{\|}(F)(p_1, p_2))$$

Then, by the definition of $\Omega_3^{\|}$, one has

$$\tau \notin \mathrm{act}_3(p_1, \sigma) \text{ and } \tau \notin \mathrm{act}_3(p_2, \sigma)$$

Thus, by the fact that $p_1$ and $p_2$ satisfy the disabled-$\tau$ condition and are downward closed, one has

$$\langle(\sigma, \delta(\emptyset))\rangle \in \tilde{\|}_3^{\delta}(p_1, p_2).$$

Likewise, it can be shown that $\Omega_3^{\|}(F)(p_1, p_2)$ is downward closed and upward closed w.r.t. disabled actions both at level 0.
Moreover $\Omega_3^{\|}(F)(p_1, p_2)$ satisfies the disabled $\tau$ at level $n$, and is downward closed and upward closed w.r.t. disabled actions, at level $n$ $(n \geq 1)$, since $\Omega_3^{\|\!\!\_}(F)(s_1, s_2)$ and $\Omega_3^{\|\!\!\_}(F)(s_2, s_1)$ satisfy these conditions at level $n$. Hence $\Omega_3^{\|}(F)(p_1, p_2) \in \mathbf{P}_3$. It immediately follows that $\Omega_3^{\|}$ is a contraction. Let $\tilde{\|}_3 = \mathrm{fix}(\Omega_3^{\|})$, and $\tilde{\|\!\!\_}_3 = \Omega_3^{\|\!\!\_}(\tilde{\|}_3)$.

**(7)** For $b \in \mathrm{BExp}$, the function $\mathrm{if}_3(b) : \mathbf{P}_3 \times \mathbf{P}_3 \to \mathbf{P}_3$, which is the interpretation of the binary combinator "$\mathbf{if}(b, \cdot, \cdot)$" on statements, is defined as follows: For every $p_1, p_2 \in \mathbf{P}_3$,

$$\mathrm{if}_3(b)(p_1, p_2) = \bigcup_{\sigma \in \Sigma} [\mathrm{if}([\![b]\!](\sigma) = \mathrm{tt}, p_1\langle\sigma\rangle, p_2\langle\sigma\rangle)].$$

**(8)** Let

$$\begin{aligned} \mathcal{I}_3 = \{&(\mathbf{0}, \tilde{\mathbf{0}}_3), (\mathbf{e}, \tilde{\mathbf{e}}_3)\} \\ \cup \{&((x := e), \mathrm{asg}_3(x, e)) : x \in \mathcal{IV} \wedge e \in \mathrm{VExp}\} \\ \cup \{&(\mathbf{if}(b, \cdot, \cdot), \mathrm{if}_3(b)) : b \in \mathrm{BExp}\} \\ \cup \{&(;, \tilde{;}_3), (+, \tilde{+}_3), (\|, \tilde{\|}_3)\}. \blacksquare \end{aligned}$$

The next lemma follows immediately from Definition 7.7 (5). We shall use it for establishing the full abstractness of the denotational model $\mathcal{M}_3$ defined below.

**Lemma 7.3** *Let* $p, p_1, p_2 \in \mathbf{P}_3$.

**(1)** $\langle(\sigma, \sigma')\rangle \cdot q \in \tilde{\|}_3(p_1, p_2)$ *iff* $q \in \tilde{\|}_3(p_1[\langle(\sigma, \sigma')\rangle], p_2)$ *or* $q \in \tilde{\|}_3(p_1, p_2[\langle(\sigma, \sigma')\rangle])$.

**(2)** $\tilde{\|}_3(p_1, p_2) = \tilde{\|}_3(p_2, p_1)$.

**(3)** $\forall p \in \mathbf{P}_3, \forall r \in (\Sigma \times \Sigma)^{<\omega}[$
$r \cdot \langle(\sigma, \delta(\emptyset))\rangle \in p \Leftrightarrow r \cdot \langle(\sigma, \delta(\{\surd\}))\rangle \in \tilde{\|}_3(p, \tilde{\mathbf{0}}_3)\ ].$ ∎

In terms of the interpretation $\mathcal{I}_3$, the denotational model $\mathcal{M}_3$ is defined as follows:

**Definition 7.8** (Denotational Model $\mathcal{M}_3$ for $\mathcal{L}_3$) The model $\mathcal{M}_3 : \mathcal{L}_3[\emptyset] \to \mathbf{P}_3$ is defined by induction on the structure of $s \in \mathcal{L}_3[\emptyset]$.

**(1)** First, for each recursion variable $P$, $\mathcal{M}_3[\![P]\!]$ is defined as the fixed-point of a contraction defined in terms of the declarations. Let $D = \{(P, g_P)\}_{P \in \mathcal{RV}}$ be the set of declarations. Let $\mathbf{M}_3^{\mathcal{M}} = (\mathcal{RV} \to \mathbf{P}_3)$, and let $\Pi_3 : \mathbf{M}_3^{\mathcal{M}} \to \mathbf{M}_3^{\mathcal{M}}$ be defined as follows: For $\mathbf{p} \in \mathbf{M}_3^{\mathcal{M}}$, $P \in \mathcal{RV}$,

$$\Pi_3(\mathbf{p})(P) = [\![g_P]\!]^{\mathcal{I}_3}[\langle \mathbf{p}(Y_1^P), \ldots, \mathbf{p}(Y_{l(P)}^P)\rangle / \langle Y_1^P, \ldots, Y_{l(P)}^P\rangle],$$

where $\{Y_1^P, \ldots, Y_{l(P)}^P\}$ is the set of recursion variables contained in $g_P$. (See Notation 7.2 for the notation $[\![g_P]\!]^{\mathcal{I}_3}(\cdots)$.) The mapping $\Pi_3$ is a contraction from $\mathbf{M}_3^{\mathcal{M}}$ to $\mathbf{M}_3^{\mathcal{M}}$. Let $\mathbf{p}_0 = \text{fix}(\Pi_3)$. For $P \in \mathcal{RV}$, let us define $\mathcal{M}_3[\![P]\!]$, the denotational meaning of $P$ in $\mathcal{M}$, by:

$$\mathcal{M}_3[\![P]\!] = \mathbf{p}_0(P).$$

**(2)** Next, for a composite statement $s \in \mathcal{L}_3[\emptyset]$, $\mathcal{M}_3[\![s]\!]$ is defined as follows: For each $r \in \omega$ and combinator $F \in \mathbf{Fun}_3^{(r)}$, and $s_0, \ldots, s_{r-1} \in \mathcal{L}_3[\emptyset]$, let

$$\mathcal{M}_3[\![F(s_1, \ldots, s_r)]\!] = \mathcal{I}_3(F)(\mathcal{M}_3[\![s_1]\!], \ldots, \mathcal{M}_3[\![s_r]\!]),$$

where $\mathcal{I}_3(F)$ is the interpreted operation corresponding to $F$ in $\mathcal{I}_3$. ∎

Several properties including the so-called *image-finiteness* for elements of $\mathbf{P}_3$ are introduced. It is shown that the denotational meaning of each statement in $\mathcal{L}_3[\emptyset]$ has these properties; this fact is used for establishing the full abstractness of $\mathcal{M}_3$.

**Definition 7.9** (Image Finiteness for Elements of $\mathbf{P}_3$) Let $p \in \mathbf{P}_3$ and $n \in \omega$.

**(1)** The process $p$ is *image-finite at level* $n$, written $\text{IFin}_3^{(n)}(p)$, iff

$$\forall r \in (\Sigma \times \Sigma)^n, \forall \sigma[\ \{\sigma' \in \Sigma : r \cdot \langle(\sigma, \sigma')\rangle \in p_{[n+1]}\} \text{ is finite } ].$$

The process $p$ is *image-finite*, written $\text{IFin}_3(p)$, iff $\forall n \in \omega[\ \text{IFin}_3^{(n)}(p)\ ]$.

**(2)** We say that *only a finite number of individual variables are relevant to the non-terminating part of* $p$ *at level* $n$, written $\text{FIRN}_3^{(n)}(p)$, iff there exists $\mathcal{X} \in \wp_{\text{fin}}(\mathcal{IV})$ such that the following holds:

$$\begin{array}{l}\forall r \in (\Sigma \times \Sigma)^n, \forall \vec{\sigma} \in ((\mathcal{IV} \setminus \mathcal{X}) \to \mathbf{V})^n[\ r \in p_{[n]} \Leftrightarrow \\ \quad \forall i \in n[\ \pi_0^2(r(i)) \quad (\mathcal{IV} \setminus \mathcal{X}) = \pi_1^2(r(i)) \quad (\mathcal{IV} \setminus \mathcal{X})\ ] \\ \quad \wedge \langle((\pi_j^2(r(i)) \quad \mathcal{X}) \cup \vec{\sigma}(i))_{j \in 2}\rangle_{i \in n} \in p_{[n]}\ ].\end{array} \tag{7.15}$$

That is, for each $r \in (\Sigma \times \Sigma)^n$, if $r \in p_{[n]}$, then, in every step

$$r(i) = (\pi_0^2(r(i)), \pi_1^2(r(i)))$$

of $r$ $(i \in n)$, the value for $x \in (\mathcal{IV} \setminus \mathcal{X})$ is not changed, i.e.,

$$\pi_0^2(r(i)) \quad (\mathcal{IV} \setminus \mathcal{X}) = \pi_1^2(r(i)) \quad (\mathcal{IV} \setminus \mathcal{X}), \tag{7.16}$$

and one may change the value $\pi_j^2(r(i))(x)$ $(j \in 2)$ arbitrarily, i.e.,

$$\langle((\pi_j^2(r(i)) \quad \mathcal{X}) \cup \vec{\sigma}(i))_{j\in 2}\rangle_{i\in n} \in p_{[n]} \tag{7.17}$$

for arbitrary $\vec{\sigma} \in ((\mathcal{IV} \setminus \mathcal{X}) \to \mathbf{V})^n$. Conversely, for arbitrary

$$\vec{\sigma} \in ((\mathcal{IV} \setminus \mathcal{X}) \to \mathbf{V})^n,$$

if one has (7.16) and (7.17), then $r \in p_{[n]}$. (See Remark 7.3 below for a motivation of this definition.)

**(3)** Similarly, we say that *only a finite number of individual variables are relevant to the terminating part of $p$ at level $n$*, written $\mathrm{FIRT}_3^{(n)}(p)$, iff there exists $\mathcal{X} \in \wp_{\mathrm{fin}}(\mathcal{IV})$ such that

$$\begin{aligned}
&\forall q \in (\Sigma \times \Sigma)^n \cdot \mathbf{B}_3, \forall \vec{\sigma} \in ((\mathcal{IV} \setminus \mathcal{X}) \to \mathbf{V})^{n+1}[\ q \in p \Leftrightarrow \\
&\quad \forall i \in n[\ \pi_0^2(q(i)) \quad (\mathcal{IV} \setminus \mathcal{X}) = \pi_1^2(q(i)) \quad (\mathcal{IV} \setminus \mathcal{X})\ ] \wedge \\
&\qquad \langle((\pi_j^2(q(i)) \quad \mathcal{X}) \cup \vec{\sigma}(i))_{j\in 2}\rangle_{i\in n} \\
&\qquad \cdot\langle((\pi_0^2(q(n)) \quad \mathcal{X}) \cup \vec{\sigma}(n), \pi_1^2(q(n)))\rangle \in p\ ].
\end{aligned} \tag{7.18}$$

**(4)** We say that *only a finite number of individual variable are relevant to $p$*, written $\mathrm{FIR}_3(p)$, iff $\mathrm{FIRN}_3(p)$ and $\mathrm{FIRT}_3(p)$, where

$$\mathrm{FIRN}_3(p) \Leftrightarrow \forall n \in \omega[\ \mathrm{FIRN}_3^{(n)}(p)\ ],$$

$$\mathrm{FIRT}_3(p) \Leftrightarrow \forall n \in \omega[\ \mathrm{FIRT}_3^{(n)}(p)\ ].$$

**(5)** $\mathbf{P}_3^* = \{p \in \mathbf{P}_3 : \mathrm{IFin}_3(p) \wedge \mathrm{FIR}_3(p)\}$. ∎

**Remark 7.2** It immediately follows that $\{p \in \mathbf{P}_3 : \mathrm{IFin}_3(p)\}$ is the largest subset of $\mathbf{P}_3$ which is included in $\{p \in \mathbf{P}_3 : \mathrm{IFin}_3^{(0)}(p)\}$ and closed under taking remainders. ∎

**Remark 7.3** **(1)** Note that for some set $D$ of declarations and some statement $s$, we cannot take *one* $\mathcal{X} \in \wp_{\mathrm{fin}}(\mathcal{IV})$ such that (7.15) holds for *every* $n \in \omega$ and $p = \mathcal{M}[\![s]\!]$. For example, suppose $\mathcal{IV} = \{x_n : n \in \omega\}$ and $\mathcal{RV} = \{P_n : n \in \omega\}$, and let $D = \{(P_n, (x_n := 1); P_{n+1}) : n \in \omega\}$, $p = \mathcal{M}_3[\![P_0]\!]$. Then, the greater $n \in \omega$ is given, the greater $\mathcal{X} \in \wp_{\mathrm{fin}}(\mathcal{IV})$ should be taken so that one has (7.15).

**(2)** It is easy to check that for $\mathcal{X}_1, \mathcal{X}_2 \in \wp_{\mathrm{fin}}(\mathcal{IV})$ with $\mathcal{X}_1 \subseteq \mathcal{X}_2$, the property (7.15) (resp. (7.18)) for $\mathcal{X} = \mathcal{X}_1$ implies (7.15) (resp. (7.18)) for $\mathcal{X} = \mathcal{X}_2$. ∎

It turns out that the denotational meaning of each statement is a member of $\mathbf{P}_3^*$, which is used for establishing the full abstractness of $\mathcal{M}_3$.

**Lemma 7.4 (1)** *The set* $\mathbf{P}_3^*$ *is closed in* $\mathbf{P}_3$.

**(2)** $\forall p \in \mathbf{P}_3^*, \forall r \in (\Sigma \times \Sigma)^{<\omega}[\, p[r] \neq \emptyset \Rightarrow p[r] \in \mathbf{P}_3^* \,]$. *That is,* $\mathbf{P}_3^*$ *is closed under taking remainders.*

**(3)** *The set* $\mathbf{P}_3^*$ *is closed under all interpreted operations of* $\mathcal{L}_3$.

**(4)** $\mathcal{M}_3[\mathcal{L}_3[\emptyset]] \subseteq \mathbf{P}_3^*$.

**(5)** $\forall p \in \mathcal{M}_3[\mathcal{L}_3[\emptyset]], \forall r \in (\Sigma \times \Sigma)^{<\omega}[\, p[r] \neq \emptyset \Rightarrow p[r] \in \mathbf{P}_3^* \,]$. ∎

**Proof. (1)** This part is shown as Lemma 7.2.
**(2)** This part follows immediately from the definition of $\mathbf{P}_3^*$.
**(3)** This part follows immediately from the definitions of the operations in Definition 7.7. Note that for the two operations $\mathrm{asg}_3(x, e)$ and $\mathrm{if}_3(b)$, we need properties (7.6) and (7.7), respectively, for establishing the closedness.
**(4)** Let $\mathbf{inact}_3 = (\lambda P \in \mathcal{RV} : \tilde{\mathbf{0}}_3)$. By the definition of $\tilde{\mathbf{0}}_3$, one has

$$\mathbf{inact}_3 \in (\mathcal{RV} \to \mathbf{P}_3^*).$$

By **(3)**, for every $n \in \omega$, one has

$$(\Pi_3)^n(\mathbf{inact}_3) \in (\mathcal{RV} \to \mathbf{P}_3^*),$$

where $(\Pi_3)^n$ is the $n$-times iteration of $\Pi_3$.

The set $\mathbf{P}_3^*$ is a complete metric space by **(1)** , and therefore, by Banach's Theorem one has

$$\mathbf{p}_0 = \lim_n[(\Pi_3)^n(\mathbf{inact}_3)] \in (\mathcal{RV} \to \mathbf{P}_3^*).$$

Hence, for each $P \in \mathcal{RV}$, one has

$$\mathcal{M}_1[\![P]\!] = \mathbf{p}_0(P) \in \mathbf{P}_3^*.$$

From this and **(3)**, it follows that

$$\forall s \in \mathcal{L}_1[\mathcal{M}_1[\![s]\!] \in \mathbf{P}_3^*].$$

**(5)** This part follows immediately from **(2)** and **(4)**. ∎

### 7.2.4 Correctness of $\mathcal{M}_3$ with respect to $\mathcal{O}_3$

The correctness of the denotational model is shown as in [Rut 89]: For the denotational model $\mathcal{M}_3$, an alternative formulation, called an *intermediate model*, is given, in terms of the same transition system which was used for the definition of $\mathcal{O}_3$. Let $\mathcal{C}_3$ be the intermediate model. Then the correctness is proved by showing that, for an appropriate abstraction function $\mathcal{A}_3$, both $\mathcal{A}_3 \circ \mathcal{C}_3$ and $\mathcal{O}_3$ are a fixed-point of the same contraction, which by Banach's Theorem has a unique fixed-point.

### Intermediate Model for $\mathcal{L}_3$ and Semantic Equivalence

First, the intermediate model $\mathcal{C}_3$, which is an alternative formulation of $\mathcal{M}_3$, is defined in terms of the transition relation $\xrightarrow{\tau}_3$.

**Definition 7.10** (Intermediate Model $\mathcal{C}_3$ for $\mathcal{L}_3$)

**(1)** Let $\mathbf{M}_3 = (\mathcal{L}_3[\emptyset] \to \mathbf{P}_3)$, and let $\Psi_3 : \mathbf{M}_3 \to \mathbf{M}_3$ be defined as follows: For $F \in \mathbf{M}_3$, $s \in \mathcal{L}_3[\emptyset]$,

$$\begin{aligned}\Psi_3(F)(s) = & \bigcup\{\langle(\sigma,\sigma')\rangle \cdot F(s') : \sigma \in \Sigma \wedge (s,\sigma) \xrightarrow{\tau}_3 (s',\sigma')\} \\ & \cup \{\langle(\sigma,\surd)\rangle : \surd \in \mathrm{act}_3(s,\sigma)\} \\ & \cup\{\langle(\sigma,\delta(\Gamma))\rangle : \sigma \in \Sigma \wedge \Gamma \in \wp(\{\surd\}) \\ & \quad \wedge \tau \notin \mathrm{act}_3(s,\sigma) \wedge \mathrm{act}_3(s,\sigma) \cap \Gamma = \emptyset\}.\end{aligned}$$

The right-hand side of the above equation is closed by Lemma 7.1; $\Psi_3$ is a contraction from $\mathbf{M}_3$ to $\mathbf{M}_3$.

**(2)** Let $\mathcal{C}_3 = \mathrm{fix}(\Psi_3)$. By definition, it holds for every $s \in \mathcal{L}_3[\emptyset]$ that

$$\begin{aligned}\mathcal{C}_3[\![s]\!] = & \bigcup\{\langle(\sigma,\sigma')\rangle \cdot \mathcal{C}_3[\![s']\!] : \sigma \in \Sigma \wedge (s,\sigma) \xrightarrow{\tau}_3 (s',\sigma')\} \\ & \cup \{\langle(\sigma,\surd)\rangle : \surd \in \mathrm{act}_3(s,\sigma)\} \\ & \cup\{\langle(\sigma,\delta(\Gamma))\rangle : \sigma \in \Sigma \wedge \Gamma \in \wp(\{\surd\}) \\ & \quad \wedge \tau \notin \mathrm{act}_3(s,\sigma) \wedge \mathrm{act}_3(s,\sigma) \cap \Gamma = \emptyset\}. \blacksquare\end{aligned}$$

It turns out that $\mathcal{C}_3$ is identical to $\mathcal{M}_3$.

**Lemma 7.5** (Semantic Equivalence for $\mathcal{L}_3$)

**(1)** *Let $F$ be an combinator of $\mathcal{L}_3$ with arity $r$, and let $s_1, \ldots, s_r \in \mathcal{L}_3[\emptyset]$. Then one has*

$$\mathcal{C}_3[\![F(s_1,\ldots,s_r)]\!] = \mathcal{I}_3(F)(\mathcal{C}_3[\![s_1]\!],\ldots,\mathcal{C}_3[\![s_r]\!]).$$

**(2)** *For $s \in \mathcal{L}_3[\emptyset]$, one has $\mathcal{C}_3[\![s]\!] = \mathcal{M}_3[\![s]\!]$.* $\blacksquare$

As a preliminary to the proof of Lemma 7.5, we give the next lemma stating that the operation $\tilde{\|}_3$ is *distributive* w.r.t. set-theoretical union.

**Lemma 7.6** (Distributivity of $\tilde{\|}_3$ in $\mathbf{P}_3$) *Let $k, l \geq 1$. Then*

$$\begin{aligned}&\forall p_0,\ldots,p_{k-1},p'_0,\ldots,p'_{l-1} \in \mathbf{P}_3[\\ &\quad \tilde{\|}_3(\textstyle\bigcup_{i\in k}[p_i], \bigcup_{j\in l}[p'_j]) = \bigcup_{(i,j)\in k\times l}[\tilde{\|}_3(p_i,p'_j)]\ ]. \blacksquare\end{aligned}$$

**Proof.** See § 7.A. $\blacksquare$

**Proof of Lemma 7.5** (1) Here we prove the claim for the combinator $\|$. For the other combinators this is proved (more straightforwardly) in a similar fashion. Let $\mathbf{H}_3 = (\mathcal{L}_3[\emptyset] \times \mathcal{L}_3[\emptyset] \to \mathbf{P}_3)$, and let $F, G \in \mathbf{H}_3$ be defined as follows: For $s_1, s_2 \in \mathcal{L}_3[\emptyset]$,

$$F(s_1, s_2) = \mathcal{C}_3[\![s_1 \parallel s_2]\!], \quad G(s_1, s_2) = \tilde{\|}_3(\mathcal{C}_3[\![s_1]\!], \mathcal{C}_3[\![s_2]\!]).$$

Moreover, let $\mathcal{F}_3^{\|} : \mathbf{H}_3 \to \mathbf{H}_3$ be defined as follows: For $f \in \mathbf{H}_3$ and $s_1, s_2 \in \mathcal{L}_3[\emptyset]$,

$$\begin{aligned}&\mathcal{F}_3^{\|}(f)(s_1, s_2)\\ &= \mathcal{F}_3^{\llcorner\!\llcorner}(f)(s_1, s_2) \cup \mathcal{F}_3^{\llcorner\!\llcorner}(f)(s_2, s_1) \cup \mathcal{F}_3^{\surd}(f)(s_1, s_2) \cup \mathcal{F}_3^{\delta}(f)(s_1, s_2),\end{aligned}$$

where

$$\mathcal{F}_3^{\llcorner\!\llcorner}(f)(s_1, s_2) = \bigcup\{\langle(\sigma, \sigma')\rangle \cdot f(s_1', s_2) : (s_1, \sigma) \xrightarrow{\tau}_3 (s_1', \sigma')\},$$

$$\mathcal{F}_3^{\surd}(f)(s_1, s_2) = \{\langle(\sigma, \surd)\rangle : \surd \in \mathrm{act}_3(s_1, \sigma) \cap \mathrm{act}_3(s_2, \sigma)\},$$

$$\begin{aligned}\mathcal{F}_3^{\delta}(s_1, s_2) = \{\langle(\sigma, \delta(\Gamma))\rangle : {}&\tau \notin \mathrm{act}_3(s_1) \wedge \tau \notin \mathrm{act}_3(s_2) \wedge \\ &(\ ((\Gamma \setminus \{\surd\}) \cap \mathrm{act}_3(s_1) = \emptyset \wedge \Gamma \cap \mathrm{act}_3(s_2) = \emptyset)\\ &\vee (\Gamma \cap \mathrm{act}_3(s_1) = \emptyset \wedge (\Gamma \setminus \{\surd\}) \cap \mathrm{act}_3(s_2) = \emptyset)\ )\}.\end{aligned}$$

Then $\mathcal{F}_3^{\|}$ is a contraction. Let $s_1, s_2 \in \mathcal{L}_3[\emptyset]$. By the definitions of $\mathcal{C}_3$ and $\langle \xrightarrow{\alpha}_3 : \alpha \in \{\tau, \surd\}\rangle$, and Lemma 7.1, one has $F(s_1, s_2) = \mathcal{F}_3^{\|}(F)(s_1, s_2)$. That is, $F = \mathrm{fix}(\mathcal{F}_3^{\|})$.

Next, let us show that $G = \mathrm{fix}(\mathcal{F}_3^{\|})$. By the definition of $\tilde{\|}_3$, one has

$$\begin{aligned}G(s_1, s_2) = {}&\tilde{\llcorner\!\llcorner}_3(\mathcal{C}_3[\![s_1]\!], \mathcal{C}_3[\![s_2]\!]) \cup \tilde{\llcorner\!\llcorner}_3(\mathcal{C}_3[\![s_2]\!], \mathcal{C}_3[\![s_1]\!])\\ &\tilde{\|}_3^{\surd}(\mathcal{C}_3[\![s_1]\!], \mathcal{C}_3[\![s_2]\!]) \cup \tilde{\|}_3^{\delta}(\mathcal{C}_3[\![s_1]\!], \mathcal{C}_3[\![s_2]\!]).\end{aligned}$$

Thus, for showing $G = \mathrm{fix}(\mathcal{F}_3^{\|})$, it suffices to show

$$\tilde{\llcorner\!\llcorner}_3(\mathcal{C}_3[\![s_1]\!], \mathcal{C}_3[\![s_2]\!]) = \mathcal{F}_3^{\llcorner\!\llcorner}(G)(s_1, s_2), \tag{7.19}$$

$$\tilde{\|}_3^{\surd}(\mathcal{C}_3[\![s_1]\!], \mathcal{C}_3[\![s_2]\!]) = \mathcal{F}_3^{\surd}(s_1, s_2), \tag{7.20}$$

$$\tilde{\|}_3^{\delta}(\mathcal{C}_3[\![s_1]\!], \mathcal{C}_3[\![s_2]\!]) = \mathcal{F}_3^{\delta}(s_1, s_2). \tag{7.21}$$

The fact (7.19) is shown as follows:

$$\begin{aligned}&\tilde{\llcorner\!\llcorner}_3(\mathcal{C}_3[\![s_1]\!], \mathcal{C}_3[\![s_2]\!])\\ &= \bigcup\{\langle(\sigma, \sigma')\rangle \cdot \tilde{\|}_3(\mathcal{C}_3[\![s_1]\!][\langle(\sigma, \sigma')\rangle], \mathcal{C}_3[\![s_2]\!]) : \mathcal{C}_3[\![s_1]\!][\langle(\sigma, \sigma')\rangle] \neq \emptyset\}\\ &= \bigcup\{\langle(\sigma, \sigma')\rangle \cdot \tilde{\|}_3(\bigcup\{\mathcal{C}_3[\![s_1']\!] : (s_1, \sigma) \xrightarrow{\tau}_3 (s_1', \sigma')\}, \mathcal{C}_3[\![s_2]\!]) :\\ &\qquad \exists s_1' [\ (s_1, \sigma) \xrightarrow{\tau}_3 (s_1', \sigma')\ ]\}\\ &= \bigcup\{\langle(\sigma, \sigma')\rangle \cdot (\bigcup\{\tilde{\|}_3(\mathcal{C}_3[\![s_1']\!], \mathcal{C}_3[\![s_2]\!]) : (s_1, \sigma) \xrightarrow{\tau}_3 (s_1', \sigma')\}) :\\ &\qquad \exists s_1' [\ (s_1, \sigma) \xrightarrow{\tau}_3 (s_1', \sigma')\ ]\}\\ &\qquad \text{(by Lemma 7.6)}\\ &= \bigcup\{\langle(\sigma, \sigma')\rangle \cdot \tilde{\|}_3(\mathcal{C}_3[\![s_1']\!], \mathcal{C}_3[\![s_2]\!]) : (s_1, \sigma) \xrightarrow{\tau}_3 (s_1', \sigma')\}\\ &= \mathcal{F}_3^{\llcorner\!\llcorner}(G)(s_1, s_2).\end{aligned}$$

The other facts (7.20) and (7.21) follow immediately from the definition of $\tilde{\|}_3^{\delta}$. Thus one has $G(s_1, s_2) = \mathcal{F}_3^{\|}(G)(s_1, s_2)$, i.e., $G = \text{fix}(\mathcal{F}_3^{\|})$. Thus, by Banach's Theorem, one has $F = G$, i.e.,

$$\forall s_1, s_2 \in \mathcal{L}_3[\emptyset][\ \mathcal{C}_3[\![s_1 \parallel s_2]\!] = \tilde{\|}_3(\mathcal{C}_3[\![s_1]\!], \mathcal{C}_3[\![s_2]\!])\ ].$$

**(2)** First, let us show, for $P \in \mathcal{RV}$, that

$$\mathcal{C}_3[\![P]\!] = \mathcal{M}_3[\![P]\!]. \tag{7.22}$$

Let $(P, g_P) \in D$. Then,

$$\begin{aligned}\mathcal{C}_3[\![P]\!] &= \mathcal{C}_3[\![g_P]\!] \quad \text{(by the definition of } \mathcal{C}_3\text{)}\\ &= [\![g_P]\!]^{\mathcal{I}_3}[\langle \mathcal{C}_3[\![Y_1^P]\!], \ldots, \mathcal{C}_3[\![Y_{l(P)}^P]\!]\rangle / \langle Y_1^P, \ldots, Y_{l(P)}^P\rangle]\\ &\quad \text{(by (1))},\end{aligned} \tag{7.23}$$

where $\{Y_1^P, \ldots, Y_{l(P)}^P\}$ is the set of recursion variables contained in $g_P$. Hence $(\lambda P \in \mathcal{RV} : \mathcal{C}_3[\![P]\!])$ is the fixed-point of $\Pi_3$ defined in Definition 7.8. Therefore by the definition of $\mathcal{M}_3[\![P]\!]$, one has (7.22). Using this and part (1), one can show, by induction on the structure of $s \in \mathcal{L}_3[\emptyset]$, that

$$\forall s \in \mathcal{L}_3[\emptyset][\ \mathcal{C}_3[\![s]\!] = \mathcal{M}_3[\![s]\!]\ ]. \blacksquare$$

### Correctness of $\mathcal{M}_3$ with respect to $\mathcal{O}_3$

An *abstraction function* $\mathcal{A}_3 : \mathbf{P}_3 \to (\Sigma \to \wp_{+\text{cl}}(\mathbf{R}_3))$ is defined as follows. First, it is defined as the fixed-point of a higher-order contraction. Next, it is shown that for a process $p$, $\mathcal{A}(p)$ is characterized as the set of *histories* of *executable* elements of $p$, where the notions of *history* and *executability* are to be formally defined below.

**Definition 7.11** (Abstraction Function $\mathcal{A}_3$ for $\mathcal{L}_3$)

**(1)** Let $\mathbf{M}_3^{\mathcal{A}} = (\mathbf{P}_3^* \to (\Sigma \to \wp_{+\text{cl}}(\mathbf{R}_3)))$, and let $\Delta_3 : \mathbf{M}_3^{\mathcal{A}} \to \mathbf{M}_3^{\mathcal{A}}$ be defined as follows: For $F \in \mathbf{M}_3^{\mathcal{A}}$, $p \in \mathbf{P}_3^*$, and $\sigma \in \Sigma$,

$$\begin{aligned}\Delta_3(F)(p)(\sigma) = &\bigcup\{\langle\sigma'\rangle \cdot F(p[\langle(\sigma, \sigma')\rangle])(\sigma') : p[\langle(\sigma, \sigma')\rangle] \neq \emptyset\}\\ &\cup \text{if}(\langle(\sigma, \surd)\rangle \in p, \{\langle\surd\rangle\}, \emptyset)\\ &\cup \text{if}(\langle(\sigma, \delta(\{\surd\}))\rangle \in p, \{\langle\,\delta\,\rangle\}, \emptyset).\end{aligned}$$

Note that the right-hand side of the above equation is nonempty, since if

$$\forall \sigma'[\ p[\langle(\sigma, \sigma')\rangle] = \emptyset\ ] \wedge \langle(\sigma, \surd)\rangle \notin p,$$

then, $\langle(\sigma, \delta(\emptyset))\rangle \in p$ by the fact that $p$ satisfies the disabled-$\tau$ condition, and further, $\langle(\sigma, \delta(\{\surd\}))\rangle \in p$ by the fact that $p$ is upward closed w.r.t. disabled actions. Thus the mapping $\Delta_3$ is a contraction from $\mathbf{M}_3^{\mathcal{A}}$ to $\mathbf{M}_3^{\mathcal{A}}$.

**(2)** Let $\mathcal{A}_3 = \text{fix}(\Delta_3)$. By this definition, it holds for $p \in \mathbf{P}_3^*$ and $\sigma \in \Sigma$, that

$$\begin{aligned}\mathcal{A}_3(p)(\sigma) = &\bigcup\{\langle\sigma'\rangle \cdot \mathcal{A}_3(p[\langle(\sigma, \sigma')\rangle])(\sigma') : p[\langle(\sigma, \sigma')\rangle] \neq \emptyset\}\\ &\cup \text{if}(\langle(\sigma, \surd)\rangle \in p, \{\langle\surd\rangle\}, \emptyset)\\ &\cup \text{if}(\langle(\sigma, \delta(\{\surd\}))\rangle \in p, \{\langle\,\delta\,\rangle\}, \emptyset). \blacksquare\end{aligned}$$

The abstraction function is to be characterized in another way. First, we need some preliminary definitions.

Intuitively, a sequence $\langle(\sigma_i, \sigma_i')\rangle_i$ in a process represents a possibility of *executing* the step $(\sigma_i, \sigma_i')$ if the process is in the state $\sigma_i$. After this execution, the process is in the state $\sigma_i'$. Thus a sequence $\langle(\sigma_i, \sigma_i')\rangle_i$ such that the second component of each element $(\sigma_i, \sigma_i')$ is the same as the first component of the next element $(\sigma_{i+1}, \sigma_{i+1}')$ represents a possibility of executing the steps $(\sigma_0, \sigma_0'), (\sigma_1, \sigma_1'), \ldots$, and therefore is called *executable*. In other words, a sequence is executable when it has no gaps.

**Definition 7.12** (Histories of Elements of $\mathbf{Q}_3$) Let $q \in \mathbf{Q}_3 \cup (\Sigma \times \Sigma)^{<\omega}$.

**(1)** The sequence $q$ is *executable*, written $\mathrm{Exec}_3(q)$, iff either

$$\exists \nu \in \omega \cup \{\omega\}, \exists \langle(\sigma_i, \sigma_i')\rangle_{i\in\nu}[\ q = \langle(\sigma_i, \sigma_i')\rangle_{i\in\nu} \\ \wedge \forall i \in \nu[\ i+1 \in \nu \Rightarrow \sigma_i' = \sigma_{i+1}\ ]$$

or

$$\exists k \in \omega, \exists \langle(\sigma_i, \sigma_i')\rangle_{i\in k}, \exists \sigma_k, \exists \Upsilon \in \hat{\mathbf{B}}_3[\ q = \langle(\sigma_i, \sigma_i')\rangle_{i\in k} \cdot \langle(\sigma_k, \Upsilon)\rangle \\ \wedge \forall i \in k[\ \sigma_i' = \sigma_{i+1}\ ]\ ].$$

Let

$$\mathbf{E}_3 = \{q \in \mathbf{Q}_3 \cup (\Sigma \times \Sigma)^{<\omega} : \mathrm{Exec}_3(q)\}.$$

For $\sigma \in \Sigma$, let

$$\mathbf{E}_3\langle\sigma\rangle = \{q \in \mathbf{E}_3 \setminus \{\epsilon\} : \mathrm{istate}_3(q) = \sigma\}.$$

**(2)** Let $q$ be executable. The *history* of $q$, denoted by $\mathrm{hist}_3(q)$, is defined by

$$\mathrm{hist}_3(q) = \begin{cases} \langle\sigma_i'\rangle_{i\in\nu} & \text{if } q = \langle(\sigma_i, \sigma_i')\rangle_{i\in\nu}, \\ \langle\sigma_i'\rangle_{i\in k} \cdot \langle\surd\rangle & \text{if } q = \langle(\sigma_i, \sigma_i')\rangle_{i\in k} \cdot \langle(\sigma_k, \surd)\rangle, \\ \langle\sigma_i'\rangle_{i\in k} \cdot \langle\,\delta\,\rangle & \text{if } q = \langle(\sigma_i, \sigma_i')\rangle_{i\in k} \cdot \langle(\sigma_k, \delta(\{\surd\}))\rangle. \end{cases} \blacksquare$$

Now we can give another formulation of $\mathcal{A}_3$ as follows:

**Lemma 7.7** (Another Formulation of Abstraction Function $\mathcal{A}_3$)

**(1)** *For* $p \in \mathbf{P}_3^*, \sigma \in \Sigma$, *one has*

$$\mathcal{A}_3(p)(\sigma) = \{\mathrm{hist}_3(q) : q \in p \cap \mathbf{E}_3\langle\sigma\rangle\}.$$

**(2)** $\forall k \geq 1, \forall p_1, \ldots, p_k \in \mathbf{P}_3^*, \forall \sigma[\ \mathcal{A}_3(\bigcup_{i\in\bar{k}}[p_i])(\sigma) = \bigcup_{i\in\bar{k}}[\mathcal{A}_3(p_i)(\sigma)]\ ]$. ∎

**Proof.** See § 7.B. ∎

By means of this lemma, one has the correctness of $\mathcal{M}_3$.

**Lemma 7.8** (Correctness of $\mathcal{M}_3$)

**(1)** $\mathcal{A}_3 \circ \mathcal{C}_3 = \mathcal{O}_3$.

**(2)** $\mathcal{A}_3 \circ \mathcal{M}_3 = \mathcal{O}_3$.

**(3)** $\forall s_1, s_2 \in \mathcal{L}_3[\emptyset][\ \mathcal{M}_3[\![s_1]\!] = \mathcal{M}_3[\![s_2]\!] \Rightarrow$
$\forall X \in \mathcal{X}_{\mathcal{P}}, \forall S \in \mathcal{L}_3[X][\ \mathcal{O}_3[\![S[s_1/X]]\!] = \mathcal{O}_3[\![S[s_2/X]]\!]\ ]]$. ∎

**Proof.** **(1)** By showing that $\mathcal{A}_3 \circ \mathcal{C}_3$ is the fixed-point of $\Psi_3^{\mathcal{O}}$ defined in Definition 7.5.
**(2)** Immediate from part (1) and Lemma 7.5 (2).
**(3)** Immediate from part (2) and the compositional definition of $\mathcal{M}_3$. ∎

### 7.2.5 Full Abstractness of $\mathcal{M}_3$ with respect to $\mathcal{O}_3$

The full abstractness of $\mathcal{M}_3$ is shown by means of a context with parallel composition:

*Given two statements $s_1, s_2 \in \mathcal{L}_3[\emptyset]$ with different denotational meanings, a suitable statement $T$ called a* tester *is constructed such that the operational meanings of $s_1 \,\|\, T$ and $s_2 \,\|\, T$ are distinct.* (7.24)

A combinatorial method for constructing such a tester is proposed in Lemma 7.10 (*Testing Lemma*). Thereby, we can construct testers having the following property:

*Given a process $p$ and a finite sequence:*

$$r = \langle(\sigma_1, \sigma_1'), \ldots, (\sigma_n, \sigma_n')\rangle,$$

*we can construct a tester $T$ and an executable sequence*

$$\tilde{r} = \langle(\tilde{\sigma}_1, \tilde{\sigma}_1'), \ldots, (\tilde{\sigma}_n, \tilde{\sigma}_k')\rangle$$

*with $k \geq n$ such that for every process $p'$, the parallel composition $p' \,\tilde{\|}_3\, \mathcal{M}_3[\![T]\!]$ can execute $\tilde{r}$ if there is some sequence $q$ such that $\langle(\sigma_1, \sigma_1'), \ldots, (\sigma_n, \sigma_n')\rangle \cdot q \in p'$, i.e.,*

$$p'[\langle(\sigma_1, \sigma_1'), \ldots, (\sigma_n, \sigma_n')\rangle] \neq \emptyset,$$

*and the converse of this holds for $p' = p$. Intuitively, for such $T$ and $\tilde{r}$, the process $p$ is* forced *to execute the steps $(\sigma_1, \sigma_1'), \ldots, (\sigma_n, \sigma_n')$ (maybe not consecutively but in this order), when $p \,\tilde{\|}_3\, \mathcal{M}_3[\![T]\!]$ executes the steps $\langle(\tilde{\sigma}_1, \tilde{\sigma}_1'), \ldots, (\tilde{\sigma}_k, \tilde{\sigma}_k')\rangle$ consecutively.*

By the above property, we can construct such testers $T$ as in (7.24) as follows:

*If $s_1$ and $s_2$ are distinct in their denotational meanings, then, putting $p_i = \mathcal{M}_3[\![s_i]\!]$ $(i = 1, 2)$, there exists some sequence $r$ such that $p_1[r] \neq \emptyset$ but $p_2[r] = \emptyset$ (or vice versa). By constructing a tester $T$ and an executable sequence $\tilde{r}$ for $r$ and $p = p_2$ as above, one has $\tilde{r} \in \mathcal{M}_3[\![s_1]\!] \,\tilde{\|}_3\, \mathcal{M}_3[\![T]\!]$ and $\tilde{r} \notin \mathcal{M}_3[\![s_2]\!] \,\tilde{\|}_3\, \mathcal{M}_3[\![T]\!]$. Thus one has a difference between the operational meanings of the two statements $s_1 \,\|\, T$ and $s_2 \,\|\, T$.*

For $\mathcal{L}_3$, the set $\mathbf{Cont}_3$ of contexts is defined as usual in terms of substitution (see Definition 3.7). Then, following Definition 3.3, the full abstractness of models for $\mathcal{L}_3$ is defined by:

**Definition 7.13** (Full Abstractness) Let $\mathcal{O}$ and $\mathcal{M}$ be models for $\mathcal{L}_3$. We say $\mathcal{M}$ is *fully abstract* w.r.t. $\mathcal{O}$ iff the following holds for every $s_1, s_2 \in \mathcal{L}_3[\emptyset]$:

$$\forall X \in \mathcal{X}_{\mathcal{P}}, \forall S \in \mathcal{L}_3[X][\ \mathcal{O}[\![S_{\langle X \rangle}[s_1]]\!] = \mathcal{O}[\![S_{\langle X \rangle}[s_2]]\!]\ ]$$
$$\Leftrightarrow \mathcal{M}[\![s_1]\!] = \mathcal{M}[\![s_2]\!]. \blacksquare$$

Let us proceed to establish the full abstractness of $\mathcal{M}_3$ w.r.t. $\mathcal{O}_3$, stated by the following theorem, under the assumption that $\mathbf{V}$ *is infinite*. The reader might expect that the same result can be obtained without this assumption, but it is necessary. In fact, if $\mathbf{V}$ is *finite*, then $\mathcal{M}_3$ is *not* fully abstract w.r.t. $\mathcal{O}_3$ (see Example 7.2 in § 7.2.6).

**Theorem 7.1** (Full Abstractness of $\mathcal{M}_3$) *If* $\mathbf{V}$ *be infinite, then for every* $s_1, s_2 \in \mathcal{L}_3[\emptyset]$, *one has*

$$\begin{aligned}&\mathcal{M}_3[\![s_1]\!] = \mathcal{M}_3[\![s_2]\!] \Leftrightarrow \\ &\forall X \in \mathcal{X}_{\mathcal{P}}, \forall S \in \mathcal{L}_3[X][\ \mathcal{O}_3[\![S_{\langle X \rangle}[s_1]]\!] = \mathcal{O}_3[\![S_{\langle X \rangle}[s_2]]\!]\ ]. \blacksquare\end{aligned} \tag{7.25}$$

For establishing Theorem 7.1, we present the following lemma, from which Theorem 7.1 follows easily.

In the sequel of this chapter, we fix an elment $\bar{v}$ of $\mathbf{V}$, and for $\mathcal{X} \in \wp_{\text{fin}}(\mathcal{IV})$, set

$$\Sigma_{\mathcal{X}} = \{\sigma \in \Sigma : \forall x \in (\mathcal{IV} \setminus \mathcal{X})[\ \sigma(x) = \bar{v}\ ]\}.$$

**Lemma 7.9** (Uniform Distinction Lemma for $\mathcal{L}_3$) *If* $\mathbf{V}$ *is infinite, then the following hold for every* $\mathcal{X} \in \wp_{+\text{fin}}(\mathcal{IV})$.

**(1)** *For every* $r \in (\Sigma_{\mathcal{X}} \times \Sigma_{\mathcal{X}})^{<\omega}$,

$$\begin{aligned}&\forall p_1, p_2 \in \mathbf{P}_3^*[\ p_1[r] \neq \emptyset \wedge p_2[r] = \emptyset \Rightarrow \\ &\quad \forall \sigma_0 \in \Sigma_{\mathcal{X}}, \exists T \in \mathcal{L}_3[\emptyset][ \\ &\qquad \mathcal{A}_3(p_1 \tilde{\|}_3 \mathcal{M}_3[\![T]\!])(\sigma_0) \setminus \mathcal{A}_3(p_2 \tilde{\|}_3 \mathcal{M}_3[\![T]\!])(\sigma_0) \neq \emptyset\ ]].\end{aligned} \tag{7.26}$$

**(2)** *For every* $q \in (\Sigma_{\mathcal{X}} \times \Sigma_{\mathcal{X}})^{<\omega} \cdot \mathbf{B}_3$,

$$\begin{aligned}&\forall p_1, p_2 \in \mathbf{P}_3^*[\ q \in p_1 \setminus p_2 \Rightarrow \\ &\quad \forall \sigma_0 \in \Sigma_{\mathcal{X}}, \exists T \in \mathcal{L}_3[\emptyset][ \\ &\qquad \mathcal{A}_3(p_1 \tilde{\|}_3 \mathcal{M}_3[\![T]\!])(\sigma_0) \setminus \mathcal{A}_3(p_2 \tilde{\|}_3 \mathcal{M}_3[\![T]\!])(\sigma_0) \neq \emptyset\ ]]. \blacksquare\end{aligned} \tag{7.27}$$

The proof of this lemma is given later. First, Theorem 7.1 is proved using this lemma.

**Proof of Theorem 7.1.** Let $\mathbf{V}$ be infinite, and $s_1, s_2 \in \mathcal{L}_3[\emptyset]$. We will prove (7.25). The $\Rightarrow$-part of (7.25) is the claim of Lemma 7.8 (3). Let us prove the $\Leftarrow$-part. For this purpose, it suffices to show the following:

$$\begin{aligned}&\mathcal{M}_3[\![s_1]\!] \neq \mathcal{M}_3[\![s_2]\!] \Rightarrow \\ &\exists T \in \mathcal{L}_3[\emptyset][\; \mathcal{A}_3(\mathcal{M}_3[\![s_1]\!] \,\tilde{\|}_3\, \mathcal{M}_3[\![T]\!]) \neq \mathcal{A}_3(\mathcal{M}_3[\![s_2]\!] \,\tilde{\|}_3\, \mathcal{M}_3[\![T]\!]) \;].\end{aligned} \tag{7.28}$$

Let $p_1 = \mathcal{M}_3[\![s_1]\!]$, $p_2 = \mathcal{M}_3[\![s_2]\!]$, and suppose $p_1 \neq p_2$. We can assume, without loss of generality, that there exists $q$ such that $q \in p_1$ and $q \notin p_2$. The proof is given by distinguishing two cases according to whether $q$ is infinite or finite.

**Case 1.** Suppose $q$ is infinite. First, let us show by contradiction that there is $n \in \omega$ such that $q_{[n]} \notin (p_2)_{[n]}$. Assume, to the contrary, that $\forall n \in \omega[\; p_2[q_{[n]}] \neq \emptyset \;]$. Then, by the closedness of $p_2$, one has $q \in p_2$, which contradicts the fact $q \notin p_2$. Hence, there is $n \in \omega$ such that $p_2[q_{[n]}] = \emptyset$. From the fact $\mathrm{FIR}_3(p_i)$ $(i = 1, 2)$ and Remark 7.3 (2), it follows that there is $\mathcal{X} \in \wp_{+\mathrm{fin}}(\mathcal{IV})$ such that (7.15) holds for $p = p_i$ $(i = 1, 2)$. Fix such $\mathcal{X}$, and let $\bar{\sigma} = (\lambda x \in (\mathcal{IV} \setminus \mathcal{X}) : \bar{v})$, and $r = \langle ((\pi_j^2(q(i)) \;\; \mathcal{X}) \cup \bar{\sigma})_{j \in 2} \rangle_{i \in n}$. Then, $r \in (\Sigma_{\mathcal{X}} \times \Sigma_{\mathcal{X}})^n$. It follows from the fact that $q_{[n]} \in ((p_1)_{[n]} \setminus (p_2)_{[n]})$ and (7.15) for $p = p_i$ $(i = 1, 2)$, that $r \in ((p_1)_{[n]} \setminus (p_2)_{[n]})$. Thus applying Lemma 7.9 (1), one has $\exists T \in \mathcal{L}_3[\emptyset][\; \mathcal{A}_3(\mathcal{M}_3[\![s_1]\!] \,\tilde{\|}_3\, \mathcal{M}_3[\![T]\!]) \setminus \mathcal{A}_3(\mathcal{M}_3[\![s_2]\!] \,\tilde{\|}_3\, \mathcal{M}_3[\![T]\!]) \neq \emptyset \;]$.

**Case 2.** Suppose $q$ is finite. Then, one obtain the same result in a similar fashion to Case 1, but using Lemma 7.9 (2) instead of Lemma 7.9 (1) used in Case 1. ∎

**Proof of Lemma 7.9**

Testers for proving Lemma 7.9 (1) (resp. Lemma 7.9 (2)) are constructed by induction on the length $r \in (\Sigma_{\mathcal{X}} \times \Sigma_{\mathcal{X}})^{<\omega}$ (resp. $q \in (\Sigma_{\mathcal{X}} \times \Sigma_{\mathcal{X}})^{<\omega} \cdot \mathbf{B}_3$). The following lemma is used to construct testers for $r$ (or $q$) with length $n+1$ by means of testers for $r$ (or $q$) with length $n$. The assumption that $\mathbf{V}$ *is infinite* will be essentially used in the proof of Lemma 7.10.

**Lemma 7.10** (Testing Lemma for $\mathcal{L}_3$) *If $\mathbf{V}$ is infinite, then for $\mathcal{X} \in \wp_{+\mathrm{fin}}(\mathcal{IV})$, $p \in \mathbf{P}_3^*$, and $\sigma', \sigma'', \sigma_0 \in \Sigma_{\mathcal{X}}$, there are two finite sequences $r_1, r_2 \in (\Sigma_{\mathcal{X}} \times \Sigma_{\mathcal{X}})^{<\omega}$ such that the following hold:*

**(1)** $r_1 \cdot \langle (\sigma', \sigma'') \rangle \cdot r_2 \in \mathbf{E}_3\langle \sigma_0 \rangle$.

**(2)** *For every tester $T' \in \mathcal{L}_3[\emptyset]$, there exists another tester $T \in \mathcal{L}_3[\emptyset]$ such that the following hold:*

  **(i)** $\mathcal{M}_3[\![T]\!][r_1 \cdot r_2] = \mathcal{M}_3[\![T']\!]$,

  **(ii)** *The process $p$ is* forced *to execute the step $(\sigma', \sigma'')$ and forbidden to execute any other steps, when the parallel composition $p \,\tilde{\|}_3\, \mathcal{M}_3[\![T]\!]$ executes the sequence: $r_1 \cdot \langle (\sigma', \sigma'') \rangle \cdot r_2$. That is, the following holds for every $q' \in \mathbf{Q}_3$:*

$$\begin{aligned}&r_1 \cdot \langle (\sigma', \sigma'') \rangle \cdot r_2 \cdot q' \in p \,\tilde{\|}_3\, \mathcal{M}_3[\![T]\!] \\ &\Rightarrow\; p[\langle (\sigma', \sigma'') \rangle] \neq \emptyset \;\wedge\; q' \in p[\langle (\sigma', \sigma'') \rangle] \,\tilde{\|}_3\, \mathcal{M}_3[\![T']\!]. \;\blacksquare\end{aligned} \tag{7.29}$$

The proof of this lemma will be given later. First, we will prove the following corollary, and thereby, Lemma 7.9.

**Corollary 7.1** *Let* $\mathbf{V}$ *be inifite, and let* $\mathcal{X} \in \wp_{+\mathrm{fin}}(\mathcal{IV})$, $p \in \mathbf{P}_3^*$, $(\sigma', \sigma'') \in \Sigma_{\mathcal{X}} \times \Sigma_{\mathcal{X}}$, *and* $\sigma_0 \in \Sigma_{\mathcal{X}}$. *Then, there are two finite sequences* $\rho_1, \rho_2 \in (\Sigma_{\mathcal{X}})^{<\omega}$ *such that for every tester* $T' \in \mathcal{L}_3[\emptyset]$ *there exists another tester* $T \in \mathcal{L}_3[\emptyset]$ *such that, putting* $\sigma_1 = \mathrm{last}(\rho_1 \cdot \sigma'' \cdot \rho_2)$, *one has the following:*

**(1)** *For every* $p' \in \mathbf{P}_3^*$, *one has*

$$\begin{aligned} &\forall \rho' \in \mathbf{R}_3[\ p'[\langle(\sigma', \sigma'')\rangle] \neq \emptyset \\ &\quad \wedge \rho' \in \mathcal{A}_3(p'[\langle(\sigma', \sigma'')\rangle] \tilde{\|}_3 \mathcal{M}_3[\![T']\!])(\sigma_1) \\ &\quad \Rightarrow \rho_1 \cdot \sigma'' \cdot \rho_2 \cdot \rho' \in \mathcal{A}_3(p' \tilde{\|}_3 \mathcal{M}_3[\![T]\!])(\sigma_0)\ ]. \end{aligned} \tag{7.30}$$

**(2)** *For* $p' = p$, *the converse of (7.30) holds. That is,*

$$\begin{aligned} &\forall \rho' \in \mathbf{R}_3[\ \rho_1 \cdot \sigma'' \cdot \rho_2 \cdot \rho' \in \mathcal{A}_3(p \tilde{\|}_3 \mathcal{M}_3[\![T]\!])(\sigma_0) \Rightarrow \\ &\quad p[\langle(\sigma', \sigma'')\rangle] \neq \emptyset \wedge \rho' \in \mathcal{A}_3(p[\langle(\sigma', \sigma'')\rangle] \tilde{\|}_3 \mathcal{M}_3[\![T']\!])(\sigma_1)\ ]. \blacksquare \end{aligned} \tag{7.31}$$

**Proof.** Take $r_1$, $r_2$ as in Lemma 7.10, and put $\rho_1 = \mathrm{hist}_3(r_1)$, $\rho_2 = \mathrm{hist}_3(r_2)$, and let $\sigma_1 = \mathrm{last}(\rho_1 \cdot \sigma'' \cdot \rho_2)$. Also, for $T'$, take $T$ as in Lemma 7.10.
**Part (1).** Let $p' \in \mathbf{P}_3^*$, and $\rho' \in \mathbf{R}_3$. Suppose $p'[\langle(\sigma', \sigma'')\rangle] \neq \emptyset$ and $\rho' \in \mathcal{A}_3(p'[\langle(\sigma', \sigma'')\rangle] \tilde{\|}_3 \mathcal{M}_3[\![T']\!])(\sigma_1)$. Then, by Lemma 7.7 (1), there exists

$$q' \in (p'[\langle(\sigma', \sigma'')\rangle] \tilde{\|}_3 \mathcal{M}_3[\![T']\!])$$

such that

$$q' \in \mathbf{E}_3\langle\sigma_1\rangle \wedge \mathrm{hist}_3(q') = \rho'.$$

Fix such $q'$. By Lemma 7.10 **(1)**, one has $r_1 \cdot \langle(\sigma', \sigma'')\rangle \cdot r_2 \cdot q' \in \mathbf{E}_3\langle\sigma_0\rangle$. By Lemma 7.10 **(2)** **(i)**, $q' \in (p'[\langle(\sigma', \sigma'')\rangle] \tilde{\|}_3 \mathcal{M}_3[\![T]\!][r_1 \cdot r_2])$. Thus, applying $\Leftarrow$-part of Lemma 7.3 (1) successively, one has $r_2 \cdot q' \in (p'[\langle(\sigma', \sigma'')\rangle] \tilde{\|}_3 \mathcal{M}_3[\![T]\!][r_1])$, $\langle(\sigma', \sigma'')\rangle \cdot r_2 \cdot q' \in (p' \tilde{\|}_3 \mathcal{M}_3[\![T]\!][r_1])$, and $r_1 \cdot \langle(\sigma', \sigma'')\rangle \cdot r_2 \cdot q' \in (p' \tilde{\|}_3 \mathcal{M}_3[\![T]\!])$. Hence, $\rho_1 \cdot \sigma'' \cdot \rho_2 \cdot \rho' = \mathrm{hist}_3(r_1 \cdot \langle(\sigma', \sigma'')\rangle \cdot r_2 \cdot q') \in \mathcal{A}_3(p' \tilde{\|}_3 \mathcal{M}_3[\![T]\!])(\sigma_0)$.
**Part (2).** Let $\rho' \in \mathbf{R}_3$, and suppose $\rho_1 \cdot \sigma'' \cdot \rho_2 \cdot \rho' \in \mathcal{A}_3(p \tilde{\|}_3 \mathcal{M}_3[\![T]\!])(\sigma_0)$. Then, by Lemma 7.7 (1), there exists $q'$ such that $(*)$: $q' \in \mathbf{E}_3\langle\sigma_1\rangle \wedge \mathrm{hist}_3(q') = \rho'$. Fix such $q'$. By (7.29), one has $p[\langle(\sigma', \sigma'')\rangle] \neq \emptyset$ and $q' \in p[\langle(\sigma', \sigma'')\rangle] \tilde{\|}_3 \mathcal{M}_3[\![T']\!]$. Thus, by $(*)$, one has $\rho' = \mathrm{hist}_3(q') \in \mathcal{A}_3(p[\langle(\sigma', \sigma'')\rangle] \tilde{\|}_3 \mathcal{M}_3[\![T']\!])(\sigma_1)$. ■

**Proof of Lemma 7.9.** Let $\mathcal{X} \in \wp_{+\mathrm{fin}}(\mathcal{IV})$.
**Part (1).** We will prove (7.26) holds for every $r \in (\Sigma_{\mathcal{X}} \times \Sigma_{\mathcal{X}})^{<\omega}$ by induction on the length of $r$.
**Induction Base:** Let $\mathrm{lgt}(r) = 0$, i.e., let $r = \epsilon$, and let $p_1, p_2 \in \mathbf{P}_3^*$. Then, one has (7.26) vacuously, since $\forall p \in \mathbf{P}_3^*[\ p[\epsilon] = p \neq \emptyset\ ]$, and therefore, it does not hold that $p_1[r] \neq \emptyset \wedge p_2[r] = \emptyset$.

**Induction Step:** Let $k \in \omega$, and assume that the claim holds for every $r$ such that $\mathrm{lgt}(r) \leq k$. Fix an arbitrary sequence $r$ of length $k+1$, say $r = \langle(\sigma', \sigma'')\rangle \cdot \tilde{r}$. Let $p_1, p_2 \in \mathbf{P}_3^*$ such that $(*)$: **(i)** $p_1[r] \neq \emptyset$, **(ii)** $p_2[r] = \emptyset$. Finally let $\sigma_0 \in \Sigma_{\mathcal{X}}$. We distinguish two cases according to whether $p_2[\langle(\sigma', \sigma'')\rangle] = \emptyset$ or not.

**Case 1.** Suppose $p_2[\langle(\sigma', \sigma'')\rangle] = \emptyset$. Then, applying Corollary 7.1 with $p = p_2$ and $T' \equiv \mathbf{0}$, there are $\rho_1$, $\rho_2$, $T$ such that:

$$\begin{cases} \textbf{(i)} & \forall p \in \mathbf{P}_3^*, \forall \rho' \in \mathbf{R}_3[\ p[\langle(\sigma', \sigma'')\rangle] \neq \emptyset \\ & \quad \wedge \rho' \in \mathcal{A}_3(p[\langle(\sigma', \sigma'')\rangle] \tilde{\|}_3 \tilde{\mathbf{0}}_3)(\sigma_1) \\ & \quad \Rightarrow \rho_1 \cdot \sigma'' \cdot \rho_2 \cdot \rho' \in \mathcal{A}_3(p \,\tilde{\|}_3\, \mathcal{M}_3[\![T]\!])(\sigma_0)\ ], \\ \textbf{(ii)} & \forall \rho' \in \mathbf{R}_3[\ \rho_1 \cdot \sigma'' \cdot \rho_2 \cdot \rho' \in \mathcal{A}_3(p_2 \,\tilde{\|}_3\, \mathcal{M}_3[\![T]\!])(\sigma_0) \Rightarrow \\ & \quad p_2[\langle(\sigma', \sigma'')\rangle] \neq \emptyset \wedge \rho' \in \mathcal{A}_3(p_2[\langle(\sigma', \sigma'')\rangle] \tilde{\|}_3 \tilde{\mathbf{0}}_3)(\sigma_1)\ ], \end{cases} \tag{7.32}$$

where $\sigma_1 = \mathrm{last}(\rho_1 \cdot \sigma'' \cdot \rho_2)$. By $(*)$ (i), there exists $\rho' \in \mathcal{A}_3(p_1[r] \tilde{\|}_3 \tilde{\mathbf{0}}_3)(\sigma_1)$. Let us fix such $\rho'$. By (7.32) (i) for $p = p_1$, one has $\rho_1 \cdot \sigma'' \cdot \rho_2 \cdot \rho' \in \mathcal{A}_3(p_1 \,\tilde{\|}_3\, \mathcal{M}_3[\![T]\!])(\sigma_0)$. Next, assume (for the sake of contradiction) that $\rho_1 \cdot \sigma'' \cdot \rho_2 \cdot \rho' \in \mathcal{A}_3(p_2 \,\tilde{\|}_3\, \mathcal{M}_3[\![T]\!])(\sigma_0)$. Then, by (7.32) (ii), one has $p_2[\langle(\sigma', \sigma'')\rangle] \neq \emptyset$, which contradicts the fact

$$p_2[\langle(\sigma', \sigma'')\rangle] = \emptyset.$$

Hence, $\rho_1 \cdot \sigma'' \cdot \rho_2 \cdot \rho' \notin \mathcal{A}_3(p_2 \,\tilde{\|}_3\, \mathcal{M}_3[\![T]\!])(\sigma_0)$.

**Case 2.** Suppose $p_2[\langle(\sigma', \sigma'')\rangle] \neq \emptyset$, and let us denote $p_1[\langle(\sigma', \sigma'')\rangle]$ and $p_2[\langle(\sigma', \sigma'')\rangle]$ by $p_1'$ and $p_2'$, respectively. Then, one has, by $(*)$, that $(\dagger)$: $p_1'[\tilde{r}] \neq \emptyset \wedge p_2'[\tilde{r}] = \emptyset$. Applying Corollary 7.1 with $p = p_2$, there are $\rho_1, \rho_2$ such that for every $T' \in \mathcal{L}_3[\emptyset]$ there exists $T$ satisfying the following:

$$\begin{cases} \textbf{(i)} & \forall p \in \mathbf{P}_3^*, \forall \rho' \in \mathbf{R}_3[\ p[\langle(\sigma', \sigma'')\rangle] \neq \emptyset \\ & \quad \wedge \rho' \in \mathcal{A}_3(p[\langle(\sigma', \sigma'')\rangle] \tilde{\|}_3 \mathcal{M}_3[\![T']\!])(\sigma_1) \\ & \quad \Rightarrow \rho_1 \cdot \sigma'' \cdot \rho_2 \cdot \rho' \in \mathcal{A}_3(p \,\tilde{\|}_3\, \mathcal{M}_3[\![T]\!])(\sigma_0)\ ], \\ \textbf{(ii)} & \forall \rho' \in \mathbf{R}_3[\ \rho_1 \cdot \sigma'' \cdot \rho_2 \cdot \rho' \in \mathcal{A}_3(p_2 \,\tilde{\|}_3\, \mathcal{M}_3[\![T]\!])(\sigma_0) \Rightarrow \\ & \quad p_2[\langle(\sigma', \sigma'')\rangle] \neq \emptyset \wedge \\ & \quad \rho' \in \mathcal{A}_3(p_2[\langle(\sigma', \sigma'')\rangle] \tilde{\|}_3 \mathcal{M}_3[\![T']\!])(\sigma_1)\ ], \end{cases} \tag{7.33}$$

where $\sigma_1 = \mathrm{last}(\rho_1 \cdot \sigma'' \cdot \rho_2)$. By the induction hypothesis and $(\dagger)$, there are $T_0$ and $\rho'$ such that

$$\rho' \in \mathcal{A}_3(p_1' \,\tilde{\|}_3\, \mathcal{M}_3[\![T_0]\!])(\sigma_1) \setminus \mathcal{A}_3(p_2' \,\tilde{\|}_3\, \mathcal{M}_3[\![T_0]\!])(\sigma_1). \tag{7.34}$$

Let $\rho = \rho_1 \cdot \sigma'' \cdot \rho_2 \cdot \rho'$, and take $T$ such that (7.33) holds for $T' = T_0$. By (7.33) (i) for $p = p_1$ and (7.34), one has $\rho \in \mathcal{A}_3(p_1 \,\tilde{\|}_3\, \mathcal{M}_3[\![T]\!])(\sigma_0)$. Next, assume (for the sake of contradiction) that $\rho_1 \cdot \sigma'' \cdot \rho_2 \cdot \rho' \in \mathcal{A}_3(p_2 \,\tilde{\|}_3\, \mathcal{M}_3[\![T]\!])(\sigma_0)$. Then, it follows from (7.33) (ii) that $\rho' \in \mathcal{A}_3(p_2' \,\tilde{\|}_3\, \mathcal{M}_3[\![T_0]\!])(\sigma_1)$, which contradicts (7.34). Thus, $\rho_1 \cdot \sigma'' \cdot \rho_2 \cdot \rho' \notin \mathcal{A}_3(p_2 \,\tilde{\|}_3\, \mathcal{M}_3[\![T]\!])(\sigma_0)$. Summarizing, in this case too, there is $\rho$ such that $\rho \in \mathcal{A}_3(p_1 \,\tilde{\|}_3\, \mathcal{M}_3[\![T]\!])(\sigma_0) \setminus \mathcal{A}_3(p_2 \,\tilde{\|}_3\, \mathcal{M}_3[\![T]\!])(\sigma_0)$.

**Part (2).** In order to establish part (2), we will prove (7.27) holds for every $q \in (\Sigma_{\mathcal{X}} \times \Sigma_{\mathcal{X}})^{<\omega} \cdot \mathbf{B}_3$, by induction on the length of $q$.
**Induction Base:** Let $\mathrm{lgt}(q) = 1$, say $q = (\sigma', \Upsilon)$. Let $p_1, p_2 \in \mathbf{P}_3^*$ such that $q \in p_1 \setminus p_2$, and let $\sigma_0 \in \Sigma_{\mathcal{X}}$. Since $\mathcal{X}$ is finite and nonempty, we can put $\mathcal{X} = \{x_1, \ldots, x_r\}$. We distinguish three cases according to the value of $\Upsilon$.
**Case 1.** Suppose $\Upsilon = \surd$. Then, let us set

$$T \equiv (x_1 := \sigma'(x_1)); \cdots; (x_r := \sigma'(x_r)); \mathbf{e},$$

and $t = \mathcal{M}_3[\![T]\!]$. By the definition of $\tilde{\|}_3$, one has

$$\langle(\sigma_0', \sigma_1'), \ldots, (\sigma_{r-1}', \sigma_r'), (\sigma', \surd)\rangle \in p_1 \tilde{\|}_3 t,$$

i.e.,

$$\langle\sigma_1', \ldots, \sigma_r', \surd\rangle \in \mathcal{A}_3(p_1 \tilde{\|}_3 t)(\sigma_0),$$

where

$$\sigma_i' = \sigma_0[\langle\sigma'(x_1), \ldots, \sigma'(x_i)\rangle / \langle x_1, \ldots, x_i\rangle] \quad (i \in r+1).$$

Let us prove, by contradiction, that

$$\langle(\sigma_0', \sigma_1'), \ldots, (\sigma_{r-1}', \sigma_r'), (\sigma', \surd)\rangle \notin p_2 \tilde{\|}_3 t.$$

Indeed, if

$$\langle(\sigma_0', \sigma_1'), \ldots, (\sigma_{r-1}', \sigma_r'), \sigma'\rangle \in p_2 \tilde{\|}_3 t,$$

then the first $r$-steps

$$(\sigma_0', \sigma_1'), \ldots, (\sigma_{r-1}', \sigma_r')$$

must stem from the right-hand side $t$. Thus, it must hold that

$$(\sigma', \surd) \in p_2 \tilde{\|}_3 t[\langle(\sigma_0', \sigma_1'), \ldots, (\sigma_{r-1}', \sigma_r')\rangle] = p_2 \tilde{\|}_3 \tilde{\mathbf{e}}_3.$$

However, this is impossible since $(\sigma', \surd) \notin p_2$. Summarizing, one has

$$\langle(\sigma_0', \sigma_1'), \ldots, (\sigma_{r-1}', \sigma_r'), (\sigma', \surd)\rangle \notin p_2 \tilde{\|}_3 t,$$

i.e.,

$$\langle\sigma_1', \ldots, \sigma_r', \surd\rangle \notin \mathcal{A}_3(p_2 \tilde{\|}_3 t)(\sigma_0).$$

**Case 2.** Suppose $\Upsilon = \delta(\{\surd\})$. Then, let us set

$$T \equiv (x_1 := \sigma'(x_1)); \cdots; (x_r := \sigma'(x_r)); \mathbf{e},$$

and $t = \mathcal{M}_3[\![T]\!]$. We can show, as in Case 1, that

$$\langle(\sigma_0', \sigma_1'), \ldots, (\sigma_{r-1}', \sigma_r'), (\sigma', \delta(\{\surd\}))\rangle \in (p_1 \tilde{\|}_3 t) \setminus (p_2 \tilde{\|}_3 t),$$

i.e.,

$$\langle \sigma_1', \ldots, \sigma_r', \delta \rangle \in \mathcal{A}_3(p_1 \tilde{\|}_3 t)(\sigma_0 \setminus \mathcal{A}_3(p_2 \tilde{\|}_3 t)(\sigma_0).$$

**Case 3.** Suppose $\Upsilon = \delta(\emptyset)$. Then, by Lemma 7.3 (3), one has

$$\langle (\sigma', \delta(\{\surd\})) \rangle \in (p_1 \tilde{\|}_3 \tilde{\mathbf{0}}_3) \setminus (p_2 \tilde{\|}_3 \tilde{\mathbf{0}}_3).$$

Thus this case is reduced to Case 2.
**Induction Step:** Similar to the induction step of part **(1)**. ■

Finally let us prove Lemma 7.10. A crucial ingredient of the proof is the fact that the value of a variable can be changed from any value to any other value in *one atomic step*, by means of an assignment statement.

**Proof of Lemma 7.10.** The proof is formulated by supposing that $\mathcal{X}$ is reduced to one variable: $\mathcal{X} = \{x\}$, which simplifies the proof allowing us to identify a state $\sigma \in \Sigma_{\mathcal{X}}$ with its value $\sigma(x) \in \mathbf{V}$. However, the lemma still holds when $\mathcal{X}$ is composed of more than one variable, as established in § 7.C. For $v \in \mathbf{V}$, let $\bar{\sigma}(v) = (\lambda y \in \mathcal{IV} : \text{if}(y = x, v, \bar{v}))$.

Trying to construct a desired tester $T$, we first observe that the composition $p \tilde{\|}_3 \mathcal{M}_3[\![T]\!]$ must be in the state $\sigma'$ when $p$ executes the step $(\sigma', \sigma'')$. Therefore, if $\sigma_0(x) \neq \sigma'(x)$, then $\mathcal{M}[\![T]\!]$ must execute the step $(\sigma, \sigma')$ for some $\sigma$, and therefore, $T$ must have an assignment "$x := \sigma'(x)$" in it. Moreover, we need a trick for forbidding $p$ to execute the step $(\sigma, \sigma')$ instead of $\mathcal{M}[\![T]\!]$ and forbidding $\mathcal{M}[\![T]\!]$ to execute the step $(\sigma', \sigma'')$ instead of $p$. The proof of Lemma 7.10 is given by distinguishing two cases according to whether $\sigma_0(x) = \sigma'(x)$.
**Case 1.** When $\sigma_0(x) = \sigma'(x)$, we can easily construct two sequences $r_1$, $r_2$ satisfying **(1)** and **(2)** of Lemma 7.10 as follows: Let

$$r_1 = \epsilon, \;\; r_2 = \langle (\sigma'', \bar{\sigma}(v_1)) \rangle,$$

where $v_1$ is chosen such that

$$\textbf{(i)} \;\; v_1 \neq \sigma''(x), \quad \textbf{(ii)} \;\; v_1 \notin \{v \in \mathbf{V} : \langle (\sigma', \sigma''), (\sigma'', \bar{\sigma}(v)) \rangle \in p_{[2]}\}. \tag{7.35}$$

Note that the right-hand side of (7.35) (ii) is finite since $p$ is image-finite by Definition 7.9, and therefore, there is $v_1$ satisfying (7.35). It immediately follows that Lemma 7.10 **(1)** holds. Let us show Lemma 7.10 **(2)**. For every $T' \in \mathcal{L}_3[\emptyset]$, let $T \equiv (x := v_1); T'$. It immediately follows that **(2) (i)** holds. Let us show **(2) (ii)**, i.e., that (7.29) holds for every $q' \in \mathbf{Q}_3$.

Suppose that

$$\langle (\sigma', \sigma''), (\sigma'', \bar{\sigma}(v_1)) \rangle \cdot q' \in p \tilde{\|}_3 \mathcal{M}_3[\![T]\!].$$

Let us show that the first two steps $(\sigma', \sigma'')$ and $(\sigma'', \bar{\sigma}(v_1))$ must stem from $p$ and $\mathcal{M}_3[\![T]\!]$, respectively. The first step cannot stem from $\mathcal{M}_3[\![T]\!]$ by (7.35) (i). Also, the second step cannot stem from $p$ by (7.35) (ii). Thus one has the desired result.
**Case 2.** When $\sigma_0(x) \neq \sigma'(x)$, we can construct two sequences $r_1$, $r_2$ satisfying **(1)** and **(2)** of Lemma 7.10 as follows: Let $r_1 = \langle (\sigma_0, \sigma') \rangle$, and $r_2 = \langle (\sigma'', \bar{\sigma}(v_1)) \rangle$, where $v_1$ is chosen such that

$$\begin{cases} \textbf{(i)}\ v_1 \notin \{v \in \mathbf{V} : \langle(\sigma_0,\sigma''),(\sigma',\sigma''),(\sigma'',\bar{\sigma}(v))\rangle \in p_{[3]}\}, \\ \textbf{(ii)}\ v_1 \neq \sigma'(x), \quad \textbf{(iii)}\ v_1 \neq \sigma''(x), \\ \textbf{(iv)}\ v_1 \notin \{v \in \mathbf{V} : \langle(\sigma',\sigma''),(\sigma'',\bar{\sigma}(v))\rangle \in p_{[2]}\}. \end{cases} \tag{7.36}$$

Note that the right-hand sides of (7.36) (i) and (iv) are finite, since $p$ is image-finite by Definition 7.9, and therefore, there is $v_1$ satisfying (7.36). It immediately follows that **(1)** holds. Let us show **(2)**, i.e., that for every $T' \in \mathcal{L}_3[\emptyset]$, there exists $T \in \mathcal{L}_3[\emptyset]$ satisfying **(2) (i)**, **(2) (ii)**. A tester $T$ with these properties can be constructed in the following format: For $\bar{v}_0, \bar{v}', \bar{v}_1, \bar{v}_2 \in \mathbf{V}$, and $s \in \mathcal{L}_3[\emptyset]$, let

$$\begin{aligned} &F(\bar{v}_0, \bar{v}', \bar{v}_1, \bar{v}_2, s) \\ &\equiv \mathbf{if}(x = \bar{v}_0,\ \ (x := \bar{v}'); (x := \bar{v}_1); s,\ \ (x := \bar{v}_2); \mathbf{0}). \end{aligned} \tag{7.37}$$

We set $T \equiv F(\sigma_0(x), \sigma'(x), v_1, v_2, T')$, where $v_2$ is chosen such that $(*)$: **(i)** $v_2 \neq \sigma''(x)$, **(ii)** $v_2 \neq v_1$. In this case also, it immediately follows that **(2) (i)** holds. Let us show **(2) (ii)**, i.e., that (7.29) holds for every $q' \in \mathbf{Q}_3$. First, put $t' = \mathcal{M}_3[\![T']\!]$, $t = \mathcal{M}_3[\![T]\!]$.

Suppose that

$$\langle(\sigma_0,\sigma'),(\sigma',\sigma''),(\sigma'',\bar{\sigma}(v_1))\rangle \cdot q' \in p \,\tilde{\|}_3\, t.$$

Let us show that the first three steps $(\sigma_0,\sigma')$, $(\sigma',\sigma'')$, $(\sigma'',\bar{\sigma}(v_1))$ must stem from $t$, $p$, $t$, respectively.

First, let us show by contradiction that the first step $(\sigma_0, \sigma')$ cannot stem from $p$. Assume that the first step stems from $p$, i.e., that

$$\langle(\sigma',\sigma''),(\sigma'',\bar{\sigma}(v_1))\rangle \cdot q' \in p[\langle(\sigma_0,\sigma')\rangle] \,\tilde{\|}_3\, t.$$

Then the second step $(\sigma', \sigma'')$ must stem from either of $p[\langle(\sigma_0,\sigma')\rangle]$ or $t$; let us show that it can stem from neither of them. Suppose that the second step stems from $t$, i.e., that

$$\langle(\sigma'',\bar{\sigma}(v_1))\rangle \cdot q' \in p[\langle(\sigma_0,\sigma')\rangle] \tilde{\|}_3 t[\langle(\sigma',\sigma'')\rangle].$$

Then $t[\langle(\sigma',\sigma'')\rangle] \neq \emptyset$, and therefore, under the assumption $\sigma_0(x) \neq \sigma'(x)$, the assignment "$x := v_2$" must be executed in the second step, which yields $\sigma''(x) = v_2$. However this contradicts $(*)$ (i). Thus

$$\langle(\sigma'',\bar{\sigma}(v_1))\rangle \cdot q' \in p[\langle(\sigma_0,\sigma'),(\sigma',\sigma'')\rangle] \,\tilde{\|}_3\, t.$$

The third step $(\sigma'', \bar{\sigma}(v_1))$ cannot stem from $p[\langle(\sigma_0,\sigma'),(\sigma',\sigma'')\rangle]$, since, by (7.36) (i),

$$p[\langle(\sigma_0,\sigma'),(\sigma',\sigma''),(\sigma'',\bar{\sigma}(v_1))\rangle] = \emptyset.$$

Thus the third step must stem from $t$, which implies $v_1 = \sigma'(x)$ or $v_1 = v_2$. However both are impossible by (7.36) (ii) and $(*)$ (ii), respectively. Summarizing, the first step cannot stem from $p$, and therefore, it must stem from $t$. Thus one has

$$\langle(\sigma',\sigma''),(\sigma'',\bar{\sigma}(v_1))\rangle \cdot q' \in p \,\tilde{\|}_3\, t[\langle(\sigma_0,\sigma')\rangle].$$

Next let us show the second step $(\sigma', \sigma'')$ cannot stem from $t[\langle(\sigma_0, \sigma')\rangle]$. If it stems from $t[\langle(\sigma_0, \sigma')\rangle]$, then

$$t[\langle(\sigma_0, \sigma'), (\sigma', \sigma'')\rangle] \neq \emptyset,$$

which implies, by the form of $T$, that $\sigma''(x) = v_1$. This contradicts (7.36) (iii). Thus the second step must stem from $p$, and therefore,

$$(\sigma'', \bar{\sigma}(v_1)) \cdot q' \in p[\langle(\sigma', \sigma'')\rangle]\tilde{\|}_3 t[\langle(\sigma_0, \sigma')\rangle].$$

Finally, the third step $(\sigma'', \bar{\sigma}(v_1))$ cannot stem from $p[\langle(\sigma', \sigma'')\rangle]$, since

$$p[\langle(\sigma', \sigma''), (\sigma'', \bar{\sigma}(v_1))\rangle] = \emptyset.$$

by (7.36) (iv). Thus the third step must stem from $t[\langle(\sigma_0, \sigma')\rangle]$, and therefore,

$$q' \in p[\langle(\sigma', \sigma'')\rangle]\tilde{\|}_3 t[\langle(\sigma_0, \sigma'), (\sigma'', \bar{\sigma}(v_1))\rangle],$$

i.e.,

$$q' \in p[\langle(\sigma', \sigma'')\rangle]\tilde{\|}_3 \mathcal{M}_3[\![T']\!]. \blacksquare$$

**Remark 7.4** Note that if $\sigma_0(x) \neq \sigma'(x)$ and $\sigma'(x) \neq \sigma''(x)$, then a simpler tester $T \equiv (x := \sigma'(x)); (x := v_1); T'$, with $v_1$ satisfying (7.36), is sufficient to establish the above lemma. However if $\sigma_0(x) \neq \sigma'(x)$ and $\sigma'(x) = \sigma''(x)$, then we need a tester defined in the format (7.37), for excluding the possibility that the first three steps of the parallel composition may stem from $p$, $t$, and $t$, respectively. ∎

### 7.2.6 Comparison of $\mathcal{M}_3$ and Other Models

#### Comparison with a More Abstract Model than $\mathcal{M}_3$ for $\mathcal{L}_3$ with V Finite

As stated earlier, the assumption **V** *is infinite* is necessary for the full abstractness of $\mathcal{M}_3$. In fact, if **V** is *finite*, then we can construct another compositional model $\widehat{\mathcal{M}}_3$ which is correct w.r.t. $\mathcal{O}_3$ and *more* abstract than $\mathcal{M}_3$. Thus $\mathcal{M}_3$ cannot be fully abstract w.r.t. $\mathcal{O}_3$. The model $\widehat{\mathcal{M}}_3$ is constructed from $\mathcal{M}_3$ by abstracting from certain redundant information present in $\mathcal{M}_3$ as follows:

**Definition 7.14** Let $p \in \wp(\mathbf{Q}_3)$.

**(1)** Let $q \in p$, and $(n, i) \in \omega \times 2$. Let us say $q$ *is pruned away from* $p$ *at place* $(n, i)$ iff $q$ is infinite and

$$q \notin q_{[n+i]} \cdot \langle \mathbf{E}_3 \langle \pi_i^2(q(n)) \rangle \cap (\Sigma \times \Sigma)^\omega \rangle \subseteq p.$$

**(2)** A *pruning function* $\Lambda : \wp(\mathbf{Q}_3) \to \wp(\mathbf{Q}_3)$ is defined as follows:

$$\Lambda(p) = \{q \in p : \neg\exists(n, i) \in \omega \times 2[\ q \text{ is pruned away from } p \text{ at place } (n, i)\ ]\}.$$

**(3)** For $s \in \mathcal{L}_3[\emptyset]$, let $\widehat{\mathcal{M}}_3[\![s]\!] = \Lambda(\mathcal{M}_3[\![s]\!])$. ∎

Since *executable passes* in $\widehat{\mathcal{M}}_3[\![s]\!]$ are the same as those in $\mathcal{M}_3[\![s]\!]$ ($s \in \mathcal{L}_3[\emptyset]$) by the definition of $\Lambda$, one has the *correctness* of $\widehat{\mathcal{M}}_3$ w.r.t. $\mathcal{O}_3$:

**Lemma 7.11** $\mathcal{A}_3 \circ \widehat{\mathcal{M}}_3 = \mathcal{A}_3 \circ \mathcal{M}_3 = \mathcal{O}_3$. ∎

Moreover, we can show that $\widehat{\mathcal{M}}_3$ is compositional w.r.t. all the combinators of $\mathcal{L}_3$. For this purpose, we define another set of semantic operations from that defined in Definition 7.7. For each syntactical combinator $F$ with arity $r$ of $\mathcal{L}_3$, a semantic operation $\tilde{F}$ with domain $(\mathbf{P}_3)^r$ has been defined in Definition 7.7; we can extend the domain of $\tilde{F}$ from $(\mathbf{P}_3)^r$ to $(\wp(\mathbf{Q}_3))^r$ straightforwardly except for $F \equiv \|$. As to $\|$, we can extend the domain of $\tilde{\|}_3$ to $(\wp(\mathbf{Q}_3))^2$ by means of a merge operation on elements of $\mathbf{Q}_3$; this operation can be defined as in [Hor 92a], where merge operation on infinite sequences (taking communication into account) is defined.

**Definition 7.15** **(1)** Let $r \in \omega$. For a meaning function $\mathcal{M}$ with $\mathrm{dom}(\mathcal{M}) = \mathcal{L}_3[\emptyset]$, and $\vec{s} \in (\mathcal{L}_3[\emptyset])^r$, let $\mathcal{M}[\![\vec{s}]\!] = \langle \mathcal{M}[\![\vec{s}(i)]\!]\rangle_{i \in r}$. Also, for a function $f$ with $\mathrm{dom}(f) = \wp(\mathbf{Q}_3)$, and $\vec{p} \in (\wp(\mathbf{Q}_3))^r$, let $f(\vec{p}) = \langle f(\vec{p}(i))\rangle_{i \in r}$.

**(2)** Let $r \in 3$, and let $F \in \mathbf{Sig}_3^{(r)}$ and $\widetilde{F}$ be the semantic operation corresponding to $F$ in the interpretation structure $\mathcal{I}_3$, i.e., let $\widetilde{F} = \mathcal{I}_3(F)$. From $\widetilde{F}$, let us define another semantic operation $\widehat{F}$ as follows: For every $\vec{p} \in (\wp(\mathbf{Q}_3))^r$, let

$$\widehat{F}(\vec{p}) = \Lambda(\widetilde{F}(\vec{p})). \blacksquare$$

From the semantic operations $\widehat{F}$, one obtains the compositionality of $\widehat{D}_1$ w.r.t. all the combinators of $\mathcal{L}_3$:

**Lemma 7.12** *For every* $r \in \omega$ *and* $F \in \mathbf{Sig}_3^{(r)}$, *one has*

$$\forall \vec{s} \in (\mathcal{L}_3[\emptyset])^r[\ \widehat{\mathcal{M}}_3[\![F(\vec{s})]\!] = \widehat{F}(\widehat{\mathcal{M}}_3[\![\vec{s}]\!])\ ]. \blacksquare$$

**Proof.** Let $r \in \omega$ and $F \in \mathbf{Sig}_3^{(r)}$. It can be shown that

$$\forall \vec{p} \in (\wp(\mathbf{Q}_3))^r[\ \Lambda(\widetilde{F}(\vec{p})) = \Lambda(\widetilde{F}(\Lambda(\vec{p})))\ ]. \tag{7.38}$$

From this one obtains the desired result as follows: Let

$$\vec{s} \in (\mathcal{L}_3[\emptyset])^r, \quad \vec{p} = \mathcal{M}_3[\![\vec{s}]\!].$$

Then,

$$\begin{aligned}
&\widehat{\mathcal{M}}_3[\![F(\vec{s})]\!] = \Lambda(\mathcal{M}_3[\![F(\vec{s})]\!]) \quad (\text{by the definition of } \widehat{D}_1)\\
&= \Lambda(\widetilde{F}(\vec{p})) \quad (\text{by the compositionality of } \mathcal{M}_3)\\
&= \Lambda(\widetilde{F}(\Lambda(\vec{p}))) \quad (\text{by (7.38)})\\
&= \widehat{F}(\widehat{\mathcal{M}}_3[\![\vec{s}]\!]) \quad (\text{by the definitions of } \widehat{D}_1 \text{ and } \widehat{F}). \blacksquare
\end{aligned}$$

When $\mathbf{V}$ is finite, the model $\widehat{\mathcal{M}}_3$ is strictly more abstract than $\mathcal{M}_3$, as can seen from the following example. Thus $\mathcal{M}_3$ is not fully abstract in this case.

**Example 7.2** Assume $\mathbf{V} = \{0, 1\}$. Moreover, let us assume, for simplicity, that $\mathcal{IV} = \{x\}$. Then, $\Sigma$ is identified with $\mathbf{V}$. Let $g \equiv ((x := 0); P_0) + ((x := 1); P_0)$, and suppose $(P_0, g) \in D$. Then, setting

$$s_1 \equiv P_0 + \mathbf{if}(x = 0,\ (x := 0); \mathbf{if}(x = 0,\ P_0,\ \mathbf{0}),\ P_0),$$

and

$$s_2 \equiv \mathbf{if}(x = 0, ((x := 0); \mathbf{if}(x = 0,\ P_0,\ \mathbf{0})) + ((x := 1); P_0),\ P_0),$$

one has

$$\mathcal{M}_3[\![s_1]\!][(0,0) \cdot (1,1)] \neq \emptyset,$$

but

$$\mathcal{M}_3[\![s_2]\!][(0,0) \cdot (1,1)] = \emptyset.$$

Thus,

$$\mathcal{M}_3[\![s_1]\!] \neq \mathcal{M}_3[\![s_2]\!]. \tag{7.39}$$

However, by the definitions of $\widehat{\mathcal{M}}_3$ and $\Lambda$, one has

$$\begin{aligned} &\widehat{\mathcal{M}}_3[\![s_1]\!] = \Lambda(\mathcal{M}_3[\![s_1]\!]) \\ &= \Lambda(\mathcal{M}_3[\![s_2]\!]) = \widehat{\mathcal{M}}_3[\![s_2]\!] \\ &= \{q \in \mathcal{M}_3[\![s_2]\!] : q \text{ is finite} \vee q \text{ is infinite and executable}\}, \end{aligned} \tag{7.40}$$

since if $q$ $(\in \mathbf{Q}_3)$ is infinite and executable, then $q \in \mathcal{M}_3[\![s_i]\!]$ $(i = 1, 2)$. Thus, for every context $S_{\langle X \rangle} \in \mathcal{L}_3$, one has

$$\begin{aligned} &\mathcal{O}_3[\![S_{\langle X \rangle}[s_1]]\!] = \mathcal{A}_3(\widehat{\mathcal{M}}_3[\![S_{\langle X \rangle}[s_1]]\!]) \\ &= \mathcal{A}_3(\widehat{\mathcal{M}}_3[\![S_{\langle X \rangle}[s_2]]\!]) = \mathcal{O}_3[\![S_{\langle X \rangle}[s_2]]\!]. \end{aligned}$$

From this and (7.39), it follows that $\mathcal{M}_3$ is not fully abstract w.r.t. $\mathcal{O}_3$. ∎

Note that, when $\mathbf{V}$ is infinite, we cannot construct a statement yielding all infinite paths, such as $P_0$ in the above lemma; thus (†) in the above example does not hold when $\mathbf{V}$ is infinite. Moreover, for every $s \in \mathcal{L}_3[\emptyset]$, it is shown that

$$\mathcal{M}_3[\![s]\!] = \Lambda(\mathcal{M}_3[\![s]\!]) = \widehat{\mathcal{M}}_3[\![s]\!] \tag{7.41}$$

as follows: First, for every $q \in \mathcal{M}_3[\![s]\!]$, $(n, i) \in \omega \times 2$, it does not hold that $q_{[n+i]} \cdot \langle \mathbf{E}_3 \langle \pi_i^2(q(n)) \rangle \cap (\Sigma \times \Sigma)^\omega \rangle \subseteq p$, since $\mathcal{M}_3[\![s]\!]$ is image-finite by Lemma 7.4 (4). Hence, $q \in \mathcal{M}_3[\![s]\!]$ is not pruned away from $\mathcal{M}_3[\![s]\!]$ at place $(n, i)$. Thus, one has (7.41).

### Comparison with a Less Abstract Model than $\mathcal{M}_3$ for $\mathcal{L}_3$

In [BR 91], another denotational model $\mathcal{M}'_3$ for a language, which is the same as $\mathcal{L}_3$ except that it does not include **0** or **e**, was proposed. The model $\mathcal{M}'_1$ was presented on the basis of the domain: $\mathbf{P}'_3 = \wp_{+\mathrm{cl}}(\mathbf{Q}'_3)$, where $\mathbf{Q}'_3 \cong \{\epsilon\} \cup (\Sigma \to (\Sigma \times \mathbf{Q}'_3))$. The outline of $\mathcal{M}'_1$ is as follows; the interpretation of the parallel composition is omitted, since this is not necessary for the present purpose.

**(1)** $\mathcal{M}'_1[\![(x := e); s]\!] = \{(\lambda\sigma : (\sigma[[\![e]\!](\sigma)/x], q)) : q \in \mathcal{M}'_1[\![s]\!]\}$.

**(2)** The operation $\tilde{+}' : \mathbf{P}'_3 \times \mathbf{P}'_3 \to \mathbf{P}'_3$ is defined by: $\{\epsilon\} + p = p + \{\epsilon\} = p$, and, for $p_1, p_2 \neq \{\epsilon\}$, $p_1 + p_2$ is the set-theoretic union of $p_1$ and $p_2$.

**(3)** $\mathcal{M}'_1[\![\mathbf{if}(b, s_1, s_2])\!] = \{(\lambda\sigma : \mathrm{if}([\![b]\!](\sigma) = \mathrm{tt}, q_1(\sigma), q_2(\sigma))); q_1 \in \mathcal{M}'_1[\![s_1]\!] \wedge q_2 \in \mathcal{M}'_1[\![s_2]\!]\}$.

It turns out that $\mathcal{M}'_3$ is not fully abstract w.r.t. $\mathcal{O}_3$ as the next example shows. Thus, $\mathcal{M}'_3$ is less abstract than $\mathcal{M}_3$.

**Example 7.3** Let us assume, for simplicity, that $\mathcal{IV} = \{x\}$. Then, $\Sigma$ is identified with $\mathbf{V}$. Let

$$s_1 \equiv (x := 0) + (x := 1),$$

and

$$s_2 \equiv \mathbf{if}(x = 0, (x := 0), (x := 1)) + \mathbf{if}(x = 0, (x := 1), (x := 0)).$$

Then,

$$\mathcal{M}_3[\![s_1]\!] = \mathcal{M}_3[\![s_2]\!] = \{\langle(v, v')\rangle : v \in \mathbf{V} \wedge (v' = 0 \vee v' = 1)\} \cdot \tilde{\mathbf{e}}_3. \tag{7.42}$$

On the other hand $\mathcal{M}'_3[\![s_1]\!] = \{q_1, q_2\}$, where

$$q_1 = (\lambda v \in \mathbf{V} : (0, \epsilon)),$$

$$q_2 = (\lambda v \in \mathbf{V} : (1, \epsilon)).$$

Also, $\mathcal{M}'_3[\![s_2]\!] = \{q'_1, q'_2\}$, where

$$q'_1 = (\lambda v \in \mathbf{V} : \mathrm{if}(v = 0, (0, \epsilon), (1, \epsilon))),$$

$$q'_2 = (\lambda v \in \mathbf{V} : \mathrm{if}(v = 0, (1, \epsilon), (0, \epsilon))).$$

Hence

$$\mathcal{M}'_3[\![s_1]\!] \neq \mathcal{M}'_3[\![s_2]\!]. \tag{7.43}$$

If $\mathcal{M}'_3$ is also fully abstract, then one has

$$\forall s_1, s_2 \in \mathcal{L}_3[\emptyset][\ \mathcal{M}_3[\![s_1]\!] = \mathcal{M}_3[\![s_2]\!] \Leftrightarrow \mathcal{M}'_3[\![s_1]\!] = \mathcal{M}'_3[\![s_2]\!]\ ], \tag{7.44}$$

which contradicts (7.3) and (7.43). Hence $\mathcal{M}'_3$ cannot be fully abstract. ∎

## Comparison with Hennessy-Plotkin's Resumptions Model

The language treated in [HP 79], which we denote by $\mathcal{L}_{co}$, is very similar to $\mathcal{L}_3$, except that it contains "co", a *coroutine* construct, as well as the usual interleaving. On the basis of a set $(a \in)$ Act of *primitive actions*, $(s \in)\ \mathcal{L}_{co}$ is given by:

$$s ::= a \mid (s_1; s_2) \mid (s_1 + s_2) \mid (s_1 \parallel s_2) \mid (s_1 \ \mathbf{co}\ s_2)$$
$$\mid \mathbf{if}(b, s_1, s_2) \mid \mathbf{While}(b, s).$$

A transition relation $\rightarrow \subseteq \mathcal{L}_{co} \times \mathbf{Str}$ with $\mathbf{Str} = \Sigma \cup (\mathcal{L}_{co} \times \Sigma)$ is defined, as $\xrightarrow{\tau}_3$, with the help of a given interpretation $\mathcal{A} : \mathrm{Act} \rightarrow (\Sigma \rightarrow \Sigma)$ (see §2 of [HP 79]). The expression $(s, \sigma) \rightarrow \sigma'$ means that the configuration $(s, \sigma)$ can terminates with state $\sigma'$. The operational semantics $\mathcal{B}$ treated in [HP 79] is defined as follows: For every statement $s$ and state $\sigma$,

$$\mathcal{B}[\![s]\!](\sigma) = \{\sigma' : (s,\sigma) \rightarrow^* \sigma'\}$$
$$\cup \{\bot : \exists \langle (s_n, \sigma_n) \rangle_{n \in \omega}[\ (s_0, \sigma_0) = (s, \sigma)$$
$$\wedge \forall n \in \omega[\ (s_n, \sigma_n) \rightarrow (s_{n+1}, \sigma_{n+1})]]\}.$$

Obviously $\mathcal{B}$ is more abstract than another operational semantics

$$\mathcal{O}_{\mathbf{co}} : \mathcal{L}_{co} \rightarrow (\Sigma \rightarrow \wp_{+cl}(\Sigma^{\leq \omega} \setminus \{\epsilon\}))$$

which is constructed by slightly modifying $\mathcal{O}_3$ in the obvious way. Then, a denotational model $\mathcal{V}$ for $\mathcal{L}_{co}$ is defined on the basis of a domain $\mathbf{R}$ which is the solution of a domain equation in the category of *non-deterministic domains.* Furthermore, the full abstractness of $\mathcal{V}$ w.r.t. $\mathcal{B}$ is established under the following three assumptions (see the paragraph immediately preceding Lemma 5.6 of [HP 79]):

$$\begin{array}{l} \text{(i) The set } \Sigma \text{ of states is infinite.} \\ \text{(ii) For each } \sigma \in \Sigma, \text{ there exists a statement } K(\sigma) \in \mathrm{Act} \text{ such} \\ \quad \text{that } \forall \sigma'[\ \mathcal{A}[\![K(\sigma)]\!](\sigma') = \sigma\ ]. \\ \text{(iii) For each } \sigma \in \Sigma, \text{ there exists an expression is}(\sigma) \in \mathrm{BExp}, \\ \quad \text{such that } \forall \sigma'[\ [\![\mathrm{is}(\sigma)]\!](\sigma') = \mathrm{tt} \Leftrightarrow \sigma' = \sigma\ ]. \end{array} \tag{7.45}$$

We can construct a denotational model $\mathcal{M}_{\mathbf{co}}$ for $\mathcal{L}_{co}$ by slightly modifying $\mathcal{M}_3$. First, the underlying domain $\mathbf{P_{co}}$ is defined by slightly modifying $\mathbf{P}_3$ as follows: $\mathbf{P_{co}} = \wp_{+cl}(\mathbf{Q}_{co})$, where $\mathbf{Q}_{co}$ is the solution of domain equation:

$$\mathbf{Q}_{co} \cong (\Sigma \times \{(\surd, \sigma) : \sigma \in \Sigma\}) \uplus (\Sigma \times \Sigma) \times \mathrm{id}_{\kappa}(\mathbf{Q}_{co})$$

with '$\surd$' being some symbol standing for *termination.* Writing $\surd(\sigma)$ for $(\surd, \sigma)$ for the sake of easier readability, one has

$$\mathbf{Q}_{co} \cong (\Sigma \times \Sigma)^{<\omega} \cdot \{\langle (\sigma, \surd(\sigma')) \rangle : \sigma, \sigma' \in \Sigma\} \uplus (\Sigma \times \Sigma)^{\omega},$$

as with $\mathbf{Q}_3$. Then, the model $\mathcal{M}_{\mathbf{co}} : \mathcal{L}_{co} \rightarrow (\Sigma \rightarrow \mathbf{P_{co}})$ is defined by

$$\mathcal{M}_{\mathbf{co}}[\![s]\!](\sigma) = \mathcal{M}^{\mathrm{t}}_{\mathbf{co}}[\![s]\!](\sigma) \cup \mathcal{M}^{\mathrm{n}}_{\mathbf{co}}[\![s]\!](\sigma),$$

where $\mathcal{M}^{\mathrm{t}}_{\mathbf{co}}[\![s]\!](\sigma)$ and $\mathcal{M}^{\mathrm{n}}_{\mathbf{co}}[\![s]\!](\sigma)$ are the *terminating* and *non-terminating* parts of $\mathcal{M}_{\mathbf{co}}[\![s]\!](\sigma)$, respectively; these parts are defined as follows: First,

$$\begin{aligned}
&\mathcal{M}^{\mathrm{t}}_{\mathbf{co}}[\![s]\!](\sigma)\\
&= \{\langle(\sigma_i,\sigma_i')\rangle_{i\in n}\cdot\langle(\sigma_n,\surd(\sigma_n'))\rangle :\\
&\quad n\in\omega \wedge \sigma_0=\sigma\\
&\quad \wedge \exists\langle s_i\rangle_{i\in(n+1)}[\ s_0\equiv s \wedge \forall i\in n[\ (s_i,\sigma_i)\to(s_{i+1},\sigma_i')\ ]\\
&\qquad\qquad\qquad \wedge (s_n,\sigma_n)\to\sigma_n'\ ]\}.
\end{aligned}$$

Next,

$$\begin{aligned}
&\mathcal{M}^{\mathrm{n}}_{\mathbf{co}}[\![s]\!](\sigma)\\
&= \{\langle(\sigma_i,\sigma_i')\rangle_{i\in\omega} : \sigma_0=\sigma \wedge\\
&\quad \exists\langle s_i\rangle_{i\in\omega}[\ s_0\equiv s \wedge \forall i\in\omega[\ (s_i,\sigma_i)\to(s_{i+1},\sigma_i')\ ]]\}.
\end{aligned}$$

The model $\mathcal{M}_{\mathbf{CO}}$ can also be formulated by means of appropriate semantic operations and Banach's Theorem, as $\mathcal{M}_3$.

Interestingly, the full abstractness of $\mathcal{M}_{\mathbf{CO}}$ can also be established under the assumptions (7.45). Thus, the two models $\mathcal{V}$ and $\mathcal{M}_{\mathbf{CO}}$ are isomorphic in the sense of Lemma 3.1, whereas the two models are constructed rather differently. The proof of its full abstractness is outlined below.

**Proof of Full Abstractness of $\mathcal{M}_{\mathbf{CO}}$.** Let $s_1, s_2 \in \mathcal{L}_{\mathrm{co}}$ such that

$$\mathcal{M}_{\mathbf{co}}[\![s_1]\!] \neq \mathcal{M}_{\mathbf{co}}[\![s_2]\!].$$

Then, either

$$\mathcal{M}^{\mathrm{n}}_{\mathbf{co}}[\![s_1]\!] \neq \mathcal{M}^{\mathrm{n}}_{\mathbf{co}}[\![s_2]\!], \text{ or } \mathcal{M}^{\mathrm{t}}_{\mathbf{co}}[\![s_1]\!] \neq \mathcal{M}^{\mathrm{t}}_{\mathbf{co}}[\![s_2]\!].$$

Let us set $p_i = \mathcal{M}_{\mathbf{co}}[\![s_i]\!]$ $(i=1,2)$.

**Case 1.** Suppose $\mathcal{M}^{\mathrm{n}}_{\mathbf{co}}[\![s_1]\!] \neq \mathcal{M}^{\mathrm{n}}_{\mathbf{co}}[\![s_2]\!]$. Then, we can assume, without loss of generality, that there exists $q$ such that

$$q \in \mathcal{M}^{\mathrm{n}}_{\mathbf{co}}[\![s_1]\!] \setminus \mathcal{M}^{\mathrm{n}}_{\mathbf{co}}[\![s_2]\!].$$

Thus, by the closedness of $p_2$, there exists $\langle(\sigma_i,\sigma_i')\rangle_{i\in(m+1)}$ such that

$$\langle(\sigma_i,\sigma_i')\rangle_{i\in(m+1)} \in (p_1)_{[m+1]} \setminus (p_2)_{[m+1]} \tag{7.46}$$

for some $m\in\omega$. As in [HP 79], we can construct an appropriate tester $T_0$ for distinguishing $s_1$ and $s_2$ as follows: First, let

$$T_m \equiv \mathbf{if}(\mathrm{is}(\sigma_m'), K(\bar\sigma), K(\bar\sigma')),$$

where $\bar\sigma$, $\bar\sigma'$ will be chosen below. Then, $T_i$ $(i\in m,\ )$ are defined by:

$$T_i \equiv \mathbf{if}(\mathrm{is}(\sigma_i'), (K(\sigma_{i+1}); T_{i+1}), K(\bar\sigma')).$$

We choose $\bar\sigma$ and $\bar\sigma'$ so that

$$\bar{\sigma} \notin \bigcup_{k \in (m+1)} (\{\sigma : \langle(\sigma_i, \sigma'_i)\rangle_{i \in k} \cdot \langle(\sigma_k, \sigma)\rangle \in (p_2)_{[k+1]}\}), \tag{7.47}$$

$$\bar{\sigma}' \neq \bar{\sigma}. \tag{7.48}$$

Note that the right-hand side of (7.47) is finite since the transition relation $\rightarrow$ is finitely branching, and thus, by the assumption (7.45) (i), we can choose such states. Then, obviously one has

$$\langle(\sigma_0, \sigma'_0), (\sigma'_0, \sigma_1), \ldots, (\sigma_m, \sigma'_m), (\sigma_{m'}, \surd(\bar{\sigma}))\rangle \in \mathcal{M}_{\mathbf{co}}[\![s_1 \ \mathbf{co} \ T_0]\!],$$

and therefore,

$$\bar{\sigma} \in \mathcal{B}[\![s_1 \ \mathbf{co} \ T_0]\!](\sigma_0). \tag{7.49}$$

On the other hand, by the conditions (7.47) and (7.48), one can show that

$$\bar{\sigma} \in \mathcal{B}[\![s_2 \ \mathbf{co} \ T_0]\!](\sigma_0) \Rightarrow \langle(\sigma_i, \sigma'_i)\rangle_{i \in (m+1)} \in (p_2)_{[m+1]}.$$

Thus, since $\langle(\sigma_i, \sigma'_i)\rangle_{i \in (m+1)} \notin (p_2)_{[m+1]}$, one has

$$\bar{\sigma} \notin \mathcal{B}[\![s_2 \ \mathbf{co} \ T_0]\!](\sigma_0). \tag{7.50}$$

By (7.49) and (7.50), one has

$$\mathcal{B}[\![s_1 \ \mathbf{co} \ T_0]\!] \neq \mathcal{B}[\![s_2 \ \mathbf{co} \ T_0]\!].$$

**Case 2.** Suppose $\mathcal{M}^{\mathrm{t}}_{\mathbf{co}}[\![s_1]\!] \neq \mathcal{M}^{\mathrm{t}}_{\mathbf{co}}[\![s_2]\!]$. Then, we can assume, without loss of generality, that

$$\exists m \in \omega, \exists \langle(\sigma_i, \sigma'_i)\rangle_{i \in m} \cdot \langle(\sigma_m, \surd(\sigma'_m))\rangle \in p_1 \setminus p_2.$$

Fix such a sequence $\langle(\sigma_i, \sigma'_i)\rangle_{i \in m} \cdot \langle(\sigma_m, \surd(\sigma'_m))\rangle$. Further, let us choose $\bar{\sigma}$ so that

$$\bar{\sigma} \notin \{\sigma'_i : i \in (m+1)\} \cup \{\sigma : \langle(\sigma_i, \sigma'_i)\rangle_{i \in (m+1)} \cdot \langle(\sigma'_m, \sigma)\rangle \in (p_2)_{[m+2]}\}, \tag{7.51}$$

and let $T \equiv (K(\bar{\sigma})); T'$ with $T'$ being an arbitrary statement. Then, obviously one has

$$\langle(\sigma_i, \sigma'_i)\rangle_{i \in (m+1)} \cdot \langle(\sigma'_m, \bar{\sigma})\rangle \in (p_1; T)_{[m+2]}.$$

On the other hand, by the condition (7.51) it is impossible that

$$\langle(\sigma_i, \sigma'_i)\rangle_{i \in (m+1)} \cdot \langle(\sigma'_m, \bar{\sigma})\rangle \in (p_2; T)_{[m+2]}.$$

Hence, one has

$$\langle(\sigma_i, \sigma'_i)\rangle_{i \in (m+1)} \cdot \langle(\sigma'_m, \bar{\sigma})\rangle \in (p_1; T)_{[m+2]} \setminus (p_2; T)_{[m+2]}.$$

Thus, one obtains the same proposition as (7.46) in Case 1, replacing $\langle(\sigma_i, \sigma'_i)\rangle_{i \in (m+1)}$ by $\langle(\sigma_i, \sigma'_i)\rangle_{i \in (m+1)} \cdot \langle(\sigma'_m, \bar{\sigma})\rangle$, and $p_i$ by $(p_i; T)$ for $i = 1, 2$. Hence, one can construct $T_0$ such that

$$\mathcal{B}[\![(s_1; T) \ \mathbf{co} \ T_0]\!] \neq \mathcal{B}[\![(s_2; T) \ \mathbf{co} \ T_0]\!],$$

as in Case 1. ∎

The full abstractness result for $\mathcal{V}$ and $\mathcal{M}_{\mathbf{co}}$ essentially depends on the "co" construct; without this, the two models would not be fully abstract w.r.t. $\mathcal{B}$, which is also conjectured by Hennessy and Plotkin for $\mathcal{V}$ (see [HP 79] § 6).

## 7.3 A Nonuniform Language with Communication

The second language $\mathcal{L}_4$ is a nonuniform language which has CSP-like *communications* in addition to all constructs of the first language. An operational model $\mathcal{O}_4$ for $\mathcal{L}_4$ is given as in § 7.2.

The domain of a denotational model $\mathcal{M}_4$ for $\mathcal{L}_4$ is a kind of *failures model*, which was introduced in [BHR 85], adapted to the nonuniform setting. Each element of the domain is a set consisting of such elements as $(\langle(\sigma_i, a_i, \sigma_i')\rangle_i, (\sigma'', \Gamma))$, where $\sigma_i$, $\sigma_i'$, and $\sigma''$ are states, $a_i$ is an action and $\Gamma$ is a set of *communication sorts*. These elements are called *failures*; the parts $\langle(\sigma_i, a_i, \sigma_i')\rangle_i$ and $(\sigma'', \Gamma)$ are called a *trace* and a *refusal*, respectively.

First, the correctness of $\mathcal{M}_4$ is established as in § 7.2. Then, the full abstractness of $\mathcal{M}_4$ is established by a combination of the testing method introduced in § 7.2 and the method proposed by Bergstra, Klop, and Olderog in [BKO 88] to establish the full abstractness of a *failures model* for a uniform language without recursion. This method was adapted by Rutten in [Rut 89] so as to employ it for a language with recursion in the framework of complete metric spaces, which suggests how to use it in the present setting.

The full abstractness of the denotational model for $\mathcal{L}_4$ is established as follows: Given two statements $s_1$ and $s_2$ of $\mathcal{L}_4$ which are distinct in their denotational meanings, then the denotational meanings are distinct in the trace parts or in the refusal parts. When the distinction is in the trace parts, we can construct a tester by the testing method, otherwise we can construct a tester by the method of Bergstra, Klop, and Olderog.

### 7.3.1 The Language $\mathcal{L}_4$

In addition to all constructs of $\mathcal{L}_3$, the language $\mathcal{L}_4$ has CSP-like *communications*, i.e., it has *inputs* "$(c?\ x)$" and *outputs* "$(c!\ e)$" for all channels $c$, individual variables $x$, and value expressions $e$.

**Definition 7.16** (Language $\mathcal{L}_4$)

**(1)** Let $(c \in)$ **Chan** be the set of channels. For every $c \in$ **Chan** and $x \in \mathcal{IV}$, the construct $(c?\ x)$ is a nullary combinator representing an *input* through the channel $c$ with the input value assigned to the variable $x$; let **In**, the set of *inputs*, be defined by:

$$\mathbf{In} = \{(c?\ x) : c \in \mathbf{Chan} \wedge x \in \mathcal{IV}\}.$$

Also, for every $c \in$ **Chan** and $e \in$ VExp, the construct $(c!\ e)$ is a nullary combinator representing an *output* of the value $e$ through the channel $c$; let **Out**, the set of *outputs*, be defined by:

$$\mathbf{Out} = \{(c!\ e) : c \in \mathbf{Chan} \wedge e \in \mathrm{VExp}\}.$$

A single-sorted *signature* $\mathbf{Sig}_4 = (\mathbf{Fun}_4, \mathrm{arity}_4(\cdot))$ is defined as follows, with $(\mathrm{op} \in)$ $\mathbf{Fun}_4$ being a set of *combinators* and $\mathrm{arity}(\cdot)$ a function which maps each combinator to its *arity*:

$$\mathbf{Fun}_4 = \{\mathbf{0}, \mathbf{e}\} \cup \mathbf{Asg} \cup \mathbf{In} \cup \mathbf{Out} \cup \mathcal{RV} \\ \cup \{;, +, \|\} \cup \mathbf{Cond}.$$

$$\mathrm{arity}_4(\mathrm{op}) = \begin{cases} 0 & \text{if op} \in \{\mathbf{0}, \mathbf{e}\} \cup \mathbf{Asg} \cup \mathbf{In} \cup \mathbf{Out} \cup \mathcal{RV}, \\ 2 & \text{if op} \in \{+, \|, ;\} \cup \mathbf{Cond}, \end{cases}$$

**(2)** The set of statements of the nonuniform concurrent language $(S \in)$ $\mathcal{L}_4$ is defined as the set of terms generated by the signature $\mathbf{Sig}_4$ and the variable set $\mathcal{X}_\mathcal{P}$. That is, $\mathcal{L}_4$ is defined by the following grammar:

$$S ::= \mathbf{0} \mid \mathbf{e} \mid (x := e) \mid (c?\ x) \mid (c!\ e) \mid (S_0; S_1) \\ \mid (S_0 + S_1) \mid (S_0 \parallel S_1) \mid \mathbf{if}(b, S_0, S_1) \mid P \mid X,$$

where $P$ ranges over $\mathcal{RV}$, the set of recursion variables; $X$ range over $\mathcal{X}_\mathcal{P}$, as in Definition 7.1, and in addition, $c$ ranges over **Chan**, the set of *communication channels*.

**(3)** Let

$$(s \in)\ \mathcal{L}_4[\emptyset] = \{S \in \mathcal{L}_4 : \mathrm{FV}(S) = \emptyset\}.$$

Also, for $X \in \mathcal{X}_\mathcal{P}$, let

$$\mathcal{L}_4[X] = \{S \in \mathcal{L}_4 : \mathrm{FV}(S) \subseteq \{X\}\}.$$

**(4)** Then the set of *guarded statements* $(g \in)$ $\mathcal{G}_4$ is defined by the following grammar:

$$g ::= \mathbf{0} \mid (x := e) \mid (c!\ e) \mid (c?\ x) \mid \\ (g; s) \mid (g_1 + g_2) \mid g_1 \parallel g_2 \mid \mathbf{if}(b, g_1, g_2).$$

We assume that each recursion variable $P$ is associated with an element $g_P$ of $\mathcal{G}_4$ by a set of declarations $D = \{(P, g_P)\}_{P \in \mathcal{RV}}$. ∎

In the sequel of this section, we fix a declaration set $D = \{(P, g_P)\}_{P \in \mathcal{RV}}$.

### 7.3.2 Operational Model $\mathcal{O}_4$ for $\mathcal{L}_4$

An operational model $\mathcal{O}_4$ for $\mathcal{L}_4$ is defined in terms of a transition system $\langle \xrightarrow{\alpha}_4 : \alpha \rangle$. The following definition is given as a preliminary to the definition of $\langle \xrightarrow{\alpha}_4 : \alpha \rangle$.

**Definition 7.17** (Actions)

**(1)** The set of *communication sorts*, $(\gamma \in)$ $\mathbf{CS}_4$, is given by:

$$\mathbf{CS}_4 = \{c! : c \in \mathbf{Chan}\} \cup \{c? : c \in \mathbf{Chan}\}.$$

Let $\mathbf{CS}_4^{\checkmark} = \mathbf{CS}_4 \cup \{\checkmark\}$.

**(2)** Let $(a \in)\ \mathbf{A}_4$, the set of *actions* of $\mathcal{L}_4$, be defined by:

$$\mathbf{A}_4 = (\mathbf{CS}_4 \times \mathbf{V}) \cup \{\tau\}.$$

Let

$$(\alpha \in)\ \mathbf{A}_4^{\checkmark} = \mathbf{A}_4^{\checkmark} \cup \{\checkmark\}.$$

We sometimes write $c!v$ and $c?v$ for $(c!, v)$ and $(c?, v)$, respectively $(c, c \in$ **Chan** and $v \in \mathbf{V})$.

**(3)** The set of *action sorts*, $(A \in)$ **ASort**, is given by:

$$\mathbf{ASort} = \mathbf{CS}_4 \cup \{\tau, \checkmark\}.$$

**(4)** A function sort : $\mathbf{A}_4^{\checkmark} \to \mathbf{ASort}$ is defined as follows: For $\alpha \in \mathbf{A}_4^{\checkmark}$,

$$\text{sort}(\alpha) = \begin{cases} \gamma & \text{if } \alpha = (\gamma, v) \in \mathbf{CS}_4 \times \mathbf{V}, \\ \alpha & \text{otherwise.} \end{cases} \blacksquare$$

A transition system $\langle \xrightarrow{\alpha}_4 : \alpha \in \mathbf{A}_4^{\checkmark} \rangle$, with $\xrightarrow{\alpha}_4 \subseteq (\mathcal{L}_4[\emptyset])^2$ for $\alpha \neq \checkmark$ and $\xrightarrow{\checkmark}_4 \subseteq \mathcal{L}_4[\emptyset]$, is defined as follows:

**Definition 7.18** (Transition Relations $\xrightarrow{\alpha}_4$) The transition relations $\xrightarrow{\alpha}_4$ are defined as the smallest relations satisfying the following rules (1)–(10), where we write $(s_1, \sigma_1) \xrightarrow{a}_4 (s_2, \sigma_2)$ (resp. $(s_1, \sigma_1) \xrightarrow{\checkmark}_4$) for $((s_1, \sigma_1), (s_2, \sigma_2)) \in \xrightarrow{a}_4$ $(s_1, s_2 \in \mathcal{L}_4[\emptyset], \sigma_1, \sigma_2 \in \Sigma$, and $a \in \mathbf{A}_4)$, (resp. for $(s_1, \sigma_1) \in \xrightarrow{\checkmark}_4$).

**(1)** $(\mathbf{e}, \sigma) \xrightarrow{\checkmark}_4$ .

**(2)** $((x := e), \sigma) \xrightarrow{\tau}_4 (\mathbf{e}, \sigma[[\![e]\!](\sigma)/x])$.

**(3)** $((c!\ e), \sigma) \xrightarrow{(c!, [\![e]\!](\sigma))}_4 (\mathbf{e}, \sigma)$.

**(4)** $((c?\ x), \sigma) \xrightarrow{c?v}_4 (\mathbf{e}, \sigma[v/x])\ \ (v \in \mathbf{V})$.

**(5-1)** $$\frac{(s_1, \sigma) \xrightarrow{a}_4 (s', \sigma')}{((s_1; s_2), \sigma) \xrightarrow{a}_4 ((s'; \sigma_2), \sigma')}\ \ (a \in \mathbf{A}_4).$$

**(5-2)** $$\frac{(s_1, \sigma) \xrightarrow{\epsilon}_4, (s_2, \sigma) \xrightarrow{a}_4 (s', \sigma')}{((s_1; s_2), \sigma) \xrightarrow{a}_4 (s', \sigma')}(a \in \mathbf{A}_4).$$

**(5-3)** $$\frac{(s_1, \sigma) \xrightarrow{\epsilon}_4, (s_2, \sigma) \xrightarrow{\epsilon}_4}{((s_1; s_2), \sigma) \xrightarrow{\epsilon}_4}.$$

$$\textbf{(6-1)}\quad \frac{(s_1,\sigma) \xrightarrow{a}_4 (s,\sigma')}{\begin{array}{c}((s_1+s_2),\sigma) \xrightarrow{a}_4 (s,\sigma')\\ ((s_2+s_1),\sigma) \xrightarrow{a}_4 (s,\sigma')\end{array}}.$$

$$\textbf{(6-2)}\quad \frac{(s_1,\sigma) \xrightarrow{\surd}_4}{\begin{array}{c}((s_1+s_2),\sigma) \xrightarrow{\surd}_4\\ ((s_2+s_1),\sigma) \xrightarrow{\surd}_4\end{array}}.$$

$$\textbf{(7-1)}\quad \frac{(s_1,\sigma) \xrightarrow{a}_4 (s,\sigma')}{\begin{array}{c}((s_1 \parallel s_2),\sigma) \xrightarrow{a}_4 ((s \parallel s_2),\sigma')\\ ((s_2 \parallel s_1),\sigma) \xrightarrow{a}_4 ((s_2 \parallel s),\sigma')\end{array}} \quad (a \in \mathbf{A}_4).$$

$$\textbf{(7-2)}\quad \frac{(s_1,\sigma) \xrightarrow{c!v}_4 (s_1',\sigma),\ \ (s_2,\sigma) \xrightarrow{c?v}_4 (s_2',\sigma')}{\begin{array}{c}((s_1 \parallel s_2),\sigma) \xrightarrow{\tau}_4 ((s_1' \parallel s_2'),\sigma')\\ ((s_2 \parallel s_1),\sigma) \xrightarrow{\tau}_4 ((s_2' \parallel s_1'),\sigma')\end{array}} \quad (c \in \mathbf{Chan}, v \in \mathbf{V}).$$

$$\textbf{(7-3)}\quad \frac{(s_1,\sigma) \xrightarrow{\surd}_4,\ \ (s_2,\sigma) \xrightarrow{\surd}_4}{((s_1 \parallel s_2),\sigma) \xrightarrow{\surd}_4}.$$

$$\textbf{(8-1)}\quad \frac{(s_1,\sigma) \xrightarrow{a}_4 (s,\sigma')}{(\mathbf{if}(b,s_1,s_2),\sigma) \xrightarrow{a}_4 (s,\sigma')} \quad ([\![b]\!](\sigma) = \mathrm{tt}).$$

$$\textbf{(8-2)}\quad \frac{(s_1,\sigma) \xrightarrow{\surd}_4}{(\mathbf{if}(b,s_1,s_2),\sigma) \xrightarrow{\surd}_4} \quad ([\![b]\!](\sigma) = \mathrm{tt}).$$

$$\textbf{(8-3)}\quad \frac{(s_2,\sigma) \xrightarrow{a}_4 (s,\sigma')}{(\mathbf{if}(b,s_1,s_2),\sigma) \xrightarrow{a}_4 (s,\sigma')} \quad ([\![b]\!](\sigma) = \mathrm{ff}).$$

$$\textbf{(8-4)}\quad \frac{(s_2,\sigma) \xrightarrow{\surd}_4}{(\mathbf{if}(b,s_1,s_2),\sigma) \xrightarrow{\surd}_4} \quad ([\![b]\!](\sigma) = \mathrm{ff}).$$

$$\textbf{(9-1)}\quad \frac{(g_P,\sigma) \xrightarrow{a}_4 (s',\sigma')}{(P,\sigma) \xrightarrow{a}_4 (s',\sigma')} \quad ((P,g_P) \in D).$$

$$\textbf{(9-2)}\quad \frac{(g_P,\sigma) \xrightarrow{\surd}_4}{(P,\sigma) \xrightarrow{\surd}_4} \quad ((P,g_P) \in D).$$

For $(s,\sigma) \in \mathcal{L}_4[\emptyset] \times \Sigma$, let

$$\mathrm{act}_4(s,\sigma) = \{a \in \mathbf{A}_4 : \exists (s',\sigma') \in \mathcal{L}_4[\emptyset] \times \Sigma[\ (s,\sigma) \xrightarrow{a}_4 (s',\sigma')\ ]\}.$$

Moreover, let $\mathrm{sact}_4(s,\sigma) = \mathrm{sort}[\mathrm{act}_4(s,\sigma)]$. ∎

The family $\langle \xrightarrow{\alpha}_4 : \alpha \in \mathbf{A}_4^{\checkmark} \rangle$ of transition relation is *image finite* in the sense of part (1) of the following lemma:

**Lemma 7.13** *For every $s \in \mathcal{L}_4[\emptyset]$, $\sigma \in \Sigma$, the following hold:*

**(1)** *For every $a \in \mathbf{A}_4$, the set $\{(s',\sigma') \in \mathcal{L}_4[\emptyset] \times \Sigma : (s,\sigma) \xrightarrow{a} (s',\sigma')\}$ is finite.*

**(2)** $\mathrm{asort}(s,\sigma)$ *is finite.*

**(3)** *For every $c \in \mathbf{Chan}$, the set $\{v \in \mathbf{V} : (c!, v) \in \mathrm{act}_4(s,\sigma)\}$ is finite.* ∎

**Proof.** These are shown in a similar fashion to the proof of Lemma 7.1. ∎

As a preliminary to the definition of the operational model, we define an auxiliary semantic domain $\mathbf{R}_4$ by:

**Definition 7.19** Let

$$\mathbf{R}_4 = ((\mathbf{A}_4 \times \Sigma)^{<\omega} \cdot \{\langle \delta \rangle, \langle \surd \rangle\}) \cup (\mathbf{A}_4 \times \Sigma)^{\omega}.$$

We use the variable $\rho$ ranging over $\mathbf{R}_4$ and $o$ ranging over $(\Sigma \to \mathbf{R}_4)$. ∎

In terms of the transition relations $\xrightarrow{\alpha}_4$, the operational model $\mathcal{O}_4$ is defined as follows:

**Definition 7.20** (Operational Model $\mathcal{O}_4$ for $\mathcal{L}_4$)

**(1)** Let $\mathbf{M}_2^{\mathcal{O}} = (\mathcal{L}_4[\emptyset] \to (\Sigma \to \wp_{+\mathrm{cl}}(\mathbf{R}_4)))$, and let $\Psi_4^{\mathcal{O}} : \mathbf{M}_2^{\mathcal{O}} \to \mathbf{M}_2^{\mathcal{O}}$ be defined as follows: For $f \in \mathbf{M}_2^{\mathcal{O}}$, $s \in \mathcal{L}_4[\emptyset]$, and $\sigma \in \Sigma$,

$$\begin{aligned}\Psi_4^{\mathcal{O}}(f)(s)(\sigma) = & \bigcup\{(a,\sigma') \cdot f(s')(\sigma') : (s,\sigma) \xrightarrow{a}_4 (s',\sigma')\} \\ & \cup \mathrm{if}(\surd \in \mathrm{act}_4(s,\sigma), \{\langle \surd \rangle\}, \emptyset) \\ & \cup \mathrm{if}(\mathrm{act}_4(s,\sigma) = \emptyset, \{\langle \delta \rangle\} : \emptyset).\end{aligned}$$

It follows that $\Psi_4^{\mathcal{O}}$ is a contraction from $\mathbf{M}_2^{\mathcal{O}}$ to $\mathbf{M}_2^{\mathcal{O}}$, as in Definition 7.5.

**(2)** Let the operational model $\mathcal{O}_4$ be the unique fixed-point of $\Psi_4^{\mathcal{O}}$. By the definition, one has $\mathcal{O}_4 : \mathcal{L}_4[\emptyset] \to (\Sigma \to \wp_{+\mathrm{cl}}(\mathbf{R}_4))$, and for each $s \in \mathcal{L}_4[\emptyset]$ and $\sigma \in \Sigma$,

$$\begin{aligned}\mathcal{O}_4[\![s]\!](\sigma) = & \bigcup\{(a,\sigma') \cdot \mathcal{O}_4[\![s']\!](\sigma') : (s,\sigma) \xrightarrow{a}_4 (s',\sigma')\} \\ & \cup \mathrm{if}(\surd \in \mathrm{act}_4(s,\sigma), \{\langle \surd \rangle\}, \emptyset) \\ & \cup \mathrm{if}(\mathrm{act}_4(s,\sigma) = \emptyset, \{\langle \delta \rangle\} : \emptyset).\end{aligned} \tag{7.52}$$ ∎

The model $\mathcal{O}_4$, defined recursively above, can be characterized as follows:

**Lemma 7.14** *Let $s \in \mathcal{L}_4$ and $\sigma \in \Sigma$. For $\rho \in \mathbf{R}_4$, $\rho \in \mathcal{O}_4[\![s]\!](\sigma)$ iff one of the following conditions (7.53)–(7.55) is satisfied:*

$$\begin{aligned}\exists \langle (s_i,\sigma_i) \rangle_{i\in\omega}, \exists \langle a_i \rangle_{i\geq 1} [\ & \rho = \langle (a_{i+1},\sigma_{i+1}) \rangle_{i\in\omega} \wedge (s_0,\sigma_0) = (s,\sigma) \\ & \wedge \forall i \in \omega [\ (s_i,\sigma_i) \xrightarrow{a_{i+1}}_4 (s_{i+1},\sigma_{i+1})\ ]\ ].\end{aligned} \tag{7.53}$$

$$\exists n \in \omega, \exists \langle (s_i, \sigma_i) \rangle_{i \in (n+1)}, \exists \langle a_i \rangle_{i \in \bar{n}} [\ \rho = \langle (a_{i+1}, \sigma_{i+1}) \rangle_{i \in n} \cdot \langle \surd \rangle \quad (7.54)$$
$$\wedge (s_0, \sigma_0) = (s, \sigma) \wedge \forall i \in n [\ (s_i, \sigma_i) \xrightarrow{a_{i+1}}_4 (s_{i+1}, \sigma_{i+1})\ ]$$
$$\wedge \surd \in \mathrm{act}_4(s_n, \sigma_n)\ ].$$

$$\exists n \in \omega, \exists \langle (s_i, \sigma_i) \rangle_{i \in (n+1)}, \exists \langle a_i \rangle_{i \in \bar{n}} [\ \rho = \langle (a_{i+1}, \sigma_{i+1}) \rangle_{i \in n} \cdot \langle \delta \rangle \quad (7.55)$$
$$\wedge (s_0, \sigma_0) = (s, \sigma) \wedge \forall i \in n [\ (s_i, \sigma_i) \xrightarrow{a_{i+1}}_4 (s_{i+1}, \sigma_{i+1})\ ]$$
$$\wedge \mathrm{act}_4(s_n, \sigma_n) \subseteq \mathbf{CS}_4\ ].\ \blacksquare$$

**Proof.** Let $\mathcal{O}_4' : \mathcal{L}_4 \to (\Sigma \to \wp(\mathbf{R}_4))$ be defined so that for every $s$, $\sigma$, and $\rho$, $\rho \in \mathcal{O}_4'[\![s]\!](\sigma)$ iff one of the conditions (7.53)–(7.55) holds. Then, one has

$$\forall s \in \mathcal{L}_4, \forall \sigma \in \Sigma[\ \mathcal{O}_4'[\![s]\!](\sigma) \in \wp_{+\mathrm{cl}}(\mathbf{R}_4)\ ],$$

and $\mathbf{M}_2^{\mathcal{O}}(\mathcal{O}_4') = \mathcal{O}_4'$. Thus, $\mathcal{O}_4' = \mathrm{fix}(\mathbf{M}_2^{\mathcal{O}}) = \mathcal{O}_4$. $\blacksquare$

The second operational model $\mathcal{O}_4^*$ is defined also in terms of the transition relation $\to$ as in Lemma 7.14 but ignoring states.

**Definition 7.21** (Operational Model $\mathcal{O}_4^*$ for $\mathcal{L}_4$)

**(1)** Let $\mathbf{R}_4^* = ((\mathbf{A}_4)^{<\omega} \cdot \{\langle \surd \rangle, \langle \delta \rangle\}) \cup (\mathbf{A}_4)^{\omega}$.

**(2)** The function

$$\mathcal{O}_4^* : \mathcal{L}_4 \to (\Sigma \to \wp_{+\mathrm{cl}}(\mathbf{R}_4^*))$$

is defined as follows: Let $s \in \mathcal{L}_4[\emptyset][\emptyset]$ and $\sigma \in \Sigma$. For $\eta \in \mathbf{R}_4^*$, we put $\eta \in \mathcal{O}_4^*[\![s]\!](\sigma)$ iff one of the following conditions (7.56)–(7.58) is satisfied:

$$\exists \langle (s_i, \sigma_i) \rangle_{i \in \omega}, \exists \langle a_i \rangle_{i \geq 1} [\ \eta = \langle a_{i+1} \rangle_{i \in \omega} \wedge (s_0, \sigma_0) = (s, \sigma) \quad (7.56)$$
$$\wedge \forall i \in \omega [(s_i, \sigma_i) \xrightarrow{a_{i+1}}_4 (s_{i+1}, \sigma_{i+1})]\ ],$$

$$\exists n \in \omega, \exists \langle (s_i, \sigma_i) \rangle_{i \in (n+1)}, \exists \langle a_i \rangle_{i \in \bar{n}} [\ \eta = \langle a_{i+1} \rangle_{i \in n} \cdot \langle \surd \rangle \quad (7.57)$$
$$\wedge (s_0, \sigma_0) = (s, \sigma) \wedge \forall i \in n [\ (s_i, \sigma_i) \xrightarrow{a_{i+1}}_4 (s_{i+1}, \sigma_{i+1})\ ]$$
$$\wedge \surd \in \mathrm{act}_4(s_n, \sigma_n)\ ].$$

$$\exists n \in \omega, \exists \langle (s_i, \sigma_i) \rangle_{i \in (n+1)}, \exists \langle a_i \rangle_{i \in \bar{n}} [\ \eta = \langle a_{i+1} \rangle_{i \in n} \cdot \langle \delta \rangle \quad (7.58)$$
$$\wedge (s_0, \sigma_0) = (s, \sigma) \wedge \forall i \in n [\ (s_i, \sigma_i) \xrightarrow{a_{i+1}}_4 (s_{i+1}, \sigma_{i+1})\ ]$$
$$\wedge \mathrm{act}_4(s_n, \sigma_n) \subseteq \mathbf{CS}_4\ ].\ \blacksquare$$

### 7.3.3 Denotational Model $\mathcal{M}_4$ for $\mathcal{L}_4$

The domain of a denotational semantic domain $\mathbf{P}_4$ for $\mathcal{L}_4$ is a kind of *failures model*, which was introduced in [BHR 85], adapted to the nonuniform setting. Each element of the domain is a set consisting of such elements as $(\langle (\sigma_i, a_i, \sigma_i') \rangle_i, (\sigma'', \Gamma))$, where $\sigma_i$, $\sigma_i'$, and $\sigma''$ are states, $a_i$ is an action and $\Gamma$ is a set of *communication sorts*. These elements are called *failures*. Formally $\mathbf{P}_4$ is defined by:

**Definition 7.22** (Denotational Semantic Domain $\mathbf{P}_4$ for $\mathcal{L}_4$)

**(1)** Let us use the variable $\Gamma$ ranging over $\wp(\mathbf{CS}_4^{\checkmark})$. For $\Gamma \in \wp(\mathbf{CS}_4^{\checkmark})$, let $\delta(\Gamma) = (\,\delta\,, \Gamma)$. Let

$$(\Upsilon \in)\ \mathbf{B}_4 = \{(\sigma, \surd) : \sigma \in \Sigma\} \cup \{(\sigma, \delta(\Gamma)) : \sigma \in \Sigma \wedge \Gamma \in \wp(\mathbf{CS}_4^{\checkmark})\}.$$

Further, let

$$\begin{aligned}\hat{\mathbf{B}}_4 = &\{(\sigma, \surd) : \sigma \in \Sigma\}\\ &\cup \{(\sigma, \delta(\Gamma)) : \sigma \in \Sigma \wedge \Gamma \in \wp(\mathbf{CS}_4^{\checkmark}) \wedge \surd \in \Gamma\}.\end{aligned}$$

**(2)** Let $\mathbf{Q}_4$ be the unique solution of

$$\mathbf{Q}_4 \cong \mathbf{B}_4 \uplus ((\Sigma \times \mathbf{A}_4 \times \Sigma) \times \mathrm{id}_\kappa(\mathbf{Q}_4)),$$

where $\delta(\Gamma) = (\,\delta\,, \Gamma)$ as in Definition 7.6. One has

$$\mathbf{Q}_4 \cong ((\Sigma \times \mathbf{A}_4 \times \Sigma)^{<\omega} \cdot \mathbf{B}_4) \cup (\Sigma \times \mathbf{A}_4 \times \Sigma)^{\omega}.$$

**(3)** For $p \in \wp_{+\mathrm{cl}}(\mathbf{Q}_4)$, and $r \in (\Sigma \times \mathbf{A}_4 \times \Sigma)^{<\omega}$, the *remainder* of $p$ with prefix $r$, denoted by $p[r]$, is defined by: $p[r] = \{q' \in \mathbf{Q}_4 : r \cdot q' \in p\}$.

**(4)** For $q \in \mathbf{Q}_4 \cup (\Sigma \times \mathbf{A}_4 \times \Sigma)^+$, let $\mathrm{istate}_4(q) = \sigma$ if $q = \langle(\sigma, a, \sigma')\rangle \cdot q'$, and let $\mathrm{istate}_4(q) = \sigma''$ if $\exists\Gamma[\ q = \langle(\sigma'', \Gamma)\rangle\ ]$.

**(5)** For $p \in \wp_{+\mathrm{cl}}(\mathbf{Q}_4)$ and $\sigma \in \Sigma$, let

$$p\langle\sigma\rangle = \{q \in p : \mathrm{istate}_4(q) = \sigma\},$$

and

$$\begin{aligned}\mathrm{act}_4(p, \sigma) = &\{a \in \mathbf{A}_4 : \exists\sigma'[\ p[\langle(\sigma, a, \sigma')\rangle] \neq \emptyset\ ]\}\\ &\cup \mathrm{if}(\langle(\sigma, \surd)\rangle \in p, \{\surd\}, \emptyset).\end{aligned}$$

**(6)** Let $p \in \wp_{+\mathrm{cl}}(\mathbf{Q}_4)$ and $n \in \omega$. We say $p$ satisfies *the disabled $\tau$-dondition at leverl $n$* iff

$$\begin{aligned}&\forall r \in (\Sigma \times \mathbf{A}_4 \times \Sigma)^n, \forall\sigma \in \Sigma[\ p[r] \neq \emptyset \wedge \tau \notin \mathrm{act}_4(p[r], \sigma)\\ &\quad \Rightarrow \exists\Gamma \in \wp(\mathbf{CS}_4^{\checkmark})[\ r \cdot \langle(\sigma, \delta(\Gamma))\rangle \in p\ ]].\end{aligned}$$

We say $p$ satisfies *the disabled-$\tau$ condition* iff it satisfies the disabled-$\tau$ condition at every level $n \in \omega$.

**(7)** Let $p \in \wp_{+\mathrm{cl}}(\mathbf{Q}_4)$ and $n \in \omega$. We say $p$ is *downward closed at level $n$* iff

$$\begin{aligned}&\forall r \in (\Sigma \times \mathbf{A}_4 \times \Sigma)^n, \forall\sigma \in \Sigma, \forall\Gamma \in \wp(\mathbf{CS}_4^{\checkmark})[\ r \cdot \langle(\sigma, \delta(\Gamma))\rangle \in p\\ &\quad \Rightarrow \forall\Gamma' \in \wp(\Gamma)[\ r \cdot \langle(\sigma, \delta(\Gamma'))\rangle \in p\ ]].\end{aligned}$$

We say $p$ is *downward closed* iff it is downward closed at every level $n \in \omega$.

**(8)** Let $p \in \wp_{+\mathrm{cl}}(\mathbf{Q}_4)$ and $n \in \omega$. We say $p$ is *upward closed w.r.t. disabled actions at level $n$* iff

$$\begin{aligned}&\forall r \in (\Sigma \times \times \mathbf{A}_4 \times \Sigma)^n, \forall\sigma \in \Sigma, \forall\Gamma \in \wp(\mathbf{CS}_4^{\checkmark})[\\ &\quad r \cdot \langle(\sigma, \delta(\emptyset))\rangle \in p \Rightarrow\\ &\quad \forall\Gamma' \in \wp(\mathbf{CS}_4^{\checkmark} \setminus \mathrm{act}_4(p[r], \sigma))[\ r \cdot \langle(\sigma, \delta(\Gamma \cup \Gamma'))\rangle \in p\ ]].\end{aligned}$$

We say $p$ is *upward closed w.r.t. disabled actions* iff it is upward closed w.r.t. disabled actions at every level $n \in \omega$.

**(9)** Let $\mathbf{P}_4$, the domain of processes for $\mathcal{L}_3$, be the set of elements $p$ of $\wp_{+\mathrm{cl}}(\mathbf{Q}_4)$ satisfying the following conditions (i)–(iii):

**(i)** $p$ satisfies the disabled-$\tau$ condition

**(ii)** $p$ is downward closed.

**(iii)** $p$ is upward closed w.r.t. disabled actions.

**(10)** For $\gamma \in \mathbf{CS}_4$, let $\overline{\gamma} = c?$ if $\gamma = c!$; otherwise $\gamma = c?$ and let $\overline{\gamma} = c!$. Moreover, for $\Gamma \in \wp(\mathbf{CS}_4)$, let $\overline{\Gamma} = \{\overline{\gamma} : \gamma \in \Gamma\}$. ∎

We have the following lemma for $\mathbf{P}_4$, which is similar to Lemma 7.2 for $\mathbf{P}_3$.

**Lemma 7.15** *The set* $\mathbf{P}_4$ *is closed in* $\wp_{+\mathrm{cl}}(\mathbf{Q}_4)$, *and therefore,* $\mathbf{P}_4$ *is a complete metric space with the original metric of* $\wp_{+\mathrm{cl}}(\mathbf{Q}_4)$. ∎

**Proof.** This is proved in a similar fashion to the proof of Lemma 7.2. ∎

The interpretation $\mathcal{I}_4$ for the signature $\mathbf{Sig}_4$ is defined as follows:

**Definition 7.23** (Interpretation $\mathcal{I}_4$ for $\mathbf{Sig}_4$)

**(1)** $\tilde{\mathbf{0}}_4 = \{\langle(\sigma, \delta(\Gamma))\rangle : \sigma \in \Sigma \wedge \Gamma \in \wp(\mathbf{CS}_4^{\checkmark})\}$.

**(2)** $\tilde{\mathbf{e}}_4 = \{(\sigma, \surd) : \sigma \in \Sigma\} \cup \{(\sigma, \delta(\Gamma)) : \sigma \in \Sigma \wedge \Gamma \in \wp(\mathbf{CS}_4)\}$.

**(3)** For $x \in \mathcal{IV}$ and $e \in \mathrm{VExp}$, the process $\mathrm{asg}_4(x, e) \in \mathbf{P}_4$, which is the interpretation of the assignment statement "$(x := e)$", is defined by:

$$\mathrm{asg}_4(x, e) = \{\langle(\sigma, \tau, \sigma[[\![e]\!](\sigma)/x])\rangle : \sigma \in \Sigma\} \cdot \tilde{\mathbf{e}}_4.$$

**(4)** For $c \in \mathbf{Chan}$ and $e \in \mathrm{VExp}$, $\mathrm{out}_4(c, e) \in \mathbf{P}_4$, which is the interpretation of the output statement "$(c!\ e)$", is by:

$$\begin{aligned}\mathrm{out}_4(c, e) = &\{\langle(\sigma, (c!, [\![e]\!](\sigma)), \sigma)\rangle : \sigma \in \Sigma\} \cdot \tilde{\mathbf{e}}_4\\ &\cup\{\langle(\sigma, \delta(\Gamma))\rangle : \sigma \in \Sigma \wedge \Gamma \in \wp(\mathbf{CS}_4^{\checkmark} \setminus \{c!\})\}.\end{aligned}$$

**(5)** For $c \in \mathbf{Chan}$ and $x \in \mathcal{IV}$, $\mathrm{in}_4(c, x) \in \mathbf{P}_4$, which is the interpretation of the input statement "$(c?\ x)$", is defined by:

$$\begin{aligned}\mathrm{in}_4(c, x) = &\{\langle(\sigma, c?v, \sigma[v/x])\rangle : \sigma \in \Sigma \wedge v \in \mathbf{V}\} \cdot \tilde{\mathbf{e}}_4\\ &\cup \{\langle(\sigma, \delta(\Gamma))\rangle : \sigma \in \Sigma \wedge \Gamma \in \wp(\mathbf{CS}_4^{\checkmark} \setminus \{c?\})\}.\end{aligned}$$

**(6)** The semantic operation $\tilde{;}_4 : (\mathbf{P}_4)^{\underline{2}} \to \mathbf{P}_4$ corresponding to the combinator ';' is defined recursively as follows: For every $p_1, p_2 \in \mathbf{P}_4$,

$$\begin{aligned}\tilde{;}_4(p_1, p_2) = &\tilde{;}_4^{\delta}(p_1, p_2)\\ &\cup \textstyle\bigcup_{\sigma \in \Sigma}[\mathrm{if}(\langle(\sigma, \surd)\rangle \in p_1, (p_2)^+, \emptyset)]\\ &\cup \textstyle\bigcup\{\tilde{;}_4(p_1[\langle(\sigma, a, \sigma')\rangle], p_2) : p_1[\langle(\sigma, a, \sigma')\rangle] \neq \emptyset\},\end{aligned}$$

where

$$\tilde{;}^{\delta}_4(p_1,p_2)$$
$$= \bigcup_{\sigma\in\Sigma}[\{\langle(\sigma,\delta(\Gamma))\rangle : \Gamma \in \wp(\mathbf{CS}^{\checkmark}_4) \wedge \langle(\sigma,\delta(\Gamma \cup \{\surd\}))\rangle \in p_1$$
$$\vee (\langle(\sigma,\delta(\Gamma \setminus \{\surd\}))\rangle \in p_1 \wedge \langle(\sigma,\delta(\Gamma))\rangle \in p_2)\}].$$

Formally the operation $\tilde{;}_4$ is defined as a suitably defined contraction as the operation $\tilde{\|}_4$ is defined in Part (6) of Definition 7.7.

**(7)** For $p \in \mathbf{P}_4$, $p \setminus \tilde{\mathbf{0}}_4$ is called the *action part* of $p$ and denoted by $p^+$. For $p_1, p_2 \in \mathbf{P}_4$, $p_1 \tilde{+} p_2$ is defined as in Definition 7.7 by:

$$p_1 \tilde{+} p_2 = (p_1)^+ \cup (p_2)^+ \cup (p_1 \cap p_2 \cap \tilde{\mathbf{0}}_4).$$

**(8)** We have a unique operation $\tilde{\|}_4 : \mathbf{P}_4 \times \mathbf{P}_4 \to \mathbf{P}_4$ satisfying the following equation; the existence and uniqueness of such an operation are obtained as in Definition 7.7 (6). For $p_1, p_2 \in \mathbf{P}_4$,

$$\tilde{\|}_4(p_1,p_2) = \tilde{\|\!\!\lfloor}_4(p_1,p_2) \cup \tilde{\|\!\!\lfloor}_4(p_2,p_1) \cup \tilde{\lfloor}_4(p_1,p_2) \cup \tilde{\lfloor}_4(p_2,p_1) \cup \tilde{\|}^{\checkmark}_4(p_1,p_2) \cup \tilde{\|}^{\delta}_4(p_1,p_2), \tag{7.59}$$

where

$$\tilde{\|\!\!\lfloor}_4(p_1,p_2)$$
$$= \bigcup\{(\sigma,a,\sigma') \cdot \tilde{\|}_4(p_1[\langle(\sigma,a,\sigma')\rangle],p_2) : p_1[\langle(\sigma,a,\sigma')\rangle] \neq \emptyset\},$$

$$\tilde{\lfloor}_4(p_1,p_2)$$
$$= (\bigcup\{\langle(\sigma,\tau,\sigma')\rangle \cdot \tilde{\|}_4(p_1[\langle(\sigma,c!v,\sigma)\rangle],p_2[\langle(\sigma,c?v,\sigma')\rangle]) : \tag{7.60}$$
$$p_1[\langle(\sigma,c!v,\sigma)\rangle] \neq \emptyset \wedge p_2[\langle(\sigma,c?v,\sigma')\rangle] \neq \emptyset\})^{\mathrm{cls}},$$

$$\tilde{\|}^{\checkmark}_4(p_1,p_2) = \{\langle(\sigma,\surd)\rangle : \sigma \in \Sigma\} \cap p_1 \cap p_2,$$

$$\tilde{\|}^{\delta}_4(p_1,p_2)$$
$$= \{\langle(\sigma,\delta(\Gamma))\rangle : \exists\langle(\sigma,\delta(\Gamma_1))\rangle \in p_1, \exists\langle(\sigma,\delta(\Gamma_2))\rangle \in p_2[$$
$$(\mathbf{CS}_4 \setminus \Gamma_1) \cap \overline{(\mathbf{CS}_4 \setminus \Gamma_2)} = \emptyset$$
$$\wedge ((\Gamma \setminus \{\surd\} \subseteq \Gamma_1 \wedge \Gamma \subseteq \Gamma_2)$$
$$\vee (\Gamma \subseteq \Gamma_1 \wedge \Gamma \setminus \{\surd\} \subseteq \Gamma_2))]\}.$$

Note that taking closure in the right-hand side of (7.60) is necessary as Example 7.4 shows below.

**(9)** For $b \in \mathrm{BExp}$, $\mathrm{if}_4(b) : \mathbf{P}_4 \times \mathbf{P}_4 \to \mathbf{P}_4$ is defined as follows: For $p_1, p_2 \in \mathbf{P}_4$,

$$\mathrm{if}_4(b)(p_1,p_2) = \bigcup_{\sigma\in\Sigma}[\mathrm{if}([\![b]\!](\sigma) = \mathrm{tt}, p_1\langle\sigma\rangle, p_2\langle\sigma\rangle)].$$

**(10)** Let

$$\mathcal{I}_4 = \{(\mathbf{0},\tilde{\mathbf{0}}_4),(\mathbf{e},\tilde{\mathbf{e}}_4)\}$$
$$\cup\{((x := e),\mathrm{asg}_4(x,e)) : x \in \mathcal{IV} \wedge e \in \mathrm{VExp}\}$$
$$\cup\{((c!e),\mathrm{out}_4(c,e)) : c \in \mathbf{Chan} \wedge e \in \mathrm{VExp}\}$$
$$\cup\{((c?x),\mathrm{in}_4(c,x)) : c \in \mathbf{Chan} \wedge x \in \mathcal{IV}\}$$
$$\cup\{(\mathbf{if}(b,\cdot,\cdot),\mathrm{if}_4(b)) : b \in \mathrm{BExp}\}$$
$$\cup\{(;,\tilde{;}_4),(+,\tilde{+}_4),(\|,\tilde{\|}_4)\}. \blacksquare$$

**Example 7.4** Let us assume, for simplicity, that $\mathcal{IV} = \{x\}$ and $\mathbf{V} = \{v\}$. Then the set of states consists only of one state denoted by $v$. Moreover assume that $\mathbf{Chan} = \{c_i : i \in \omega\}$ and $c_i \neq c_j$ for $i \neq j$. Let $p_1$ and $p_2$ be defined by:

$$p_1 = \{r_n : n \in \omega\} \cdot \tilde{\mathbf{0}}_4, \;\; p_2 = \{\langle (v, c_n?v, v)\rangle : n \in \omega\} \cdot \tilde{\mathbf{0}}_4,$$

where

$$r_n = \langle (v, c_n!v, v)\rangle \cdot \underbrace{\langle (v, c_0!v, v), \ldots, (v, c_0!v, v)\rangle}_{n}.$$

Then $p_1$ and $p_2$ belong to $\mathbf{P}_4$, and moreover they are *image finite*, which notion is to be defined in Definition 7.25. Nevertheless, it is shown that the right-hand side of (7.60) without taking closure is not closed as follows. This set is $\{r'_n : n \in \omega\} \cdot \tilde{\mathbf{0}}_4$, where

$$r'_n = \langle (v, \tau, v)\rangle \cdot \underbrace{\langle (v, c_0!v, v) \cdots (v, c_0!v, v)\rangle}_{n}.$$

This is not closed, since the infinite sequence

$$\langle (v, \tau, v), (v, c_0!v, v), (v, c_0!v, v), \ldots\rangle$$

is a member of its closure but is not a member of it. ∎

The next lemma follows immediately from Definition 7.23 (7).

**Lemma 7.16 (1)** $\forall p_1, p_2 \in \mathbf{P}_4[\; \tilde{\|}_4(p_1, p_2) = \tilde{\|}_4(p_2, p_1)\; ]$.

**(2)** $\forall p \in \mathbf{P}_4, \forall r \in (\Sigma \times \mathbf{A}_4 \times \Sigma)^{<\omega}, \forall \Gamma \in \wp(\mathbf{CS}_4)[$
$r \cdot \langle (\sigma, \delta(\Gamma)) \in p \Leftrightarrow r \cdot \langle (\sigma, \delta(\Gamma \cup \{\surd\}))\rangle \in \tilde{\|}_4(p, \tilde{\mathbf{0}}_3)\; ]$. ∎

In terms of the interpretation $\mathcal{I}_4$, the denotational model $\mathcal{M}_4$ is defined by induction on the structure of $s \in \mathcal{L}_4[\emptyset]$, as in Definition 7.8.

**Definition 7.24** (Denotational Model $\mathcal{M}_4$ for $\mathcal{L}_4$) First, a contraction $\Pi_4$ from $\mathbf{M}_4^{\mathcal{M}} = (\mathcal{RV} \to \mathbf{P}_4)$ to itself is defined as in Definition 7.8 (1), using $\mathcal{I}_4$ instead of $\mathcal{I}_3$. Let $\mathbf{p}_0 = \text{fix}(\Pi_4)$, and for $P \in \mathcal{RV}$, let us define $\mathcal{M}_4[\![P]\!]$, the denotational meaning of $P$ in $\mathcal{M}_4$, by:

$$\mathcal{M}_4[\![P]\!] = \mathbf{p}_0(P).$$

Next, for each $r \in \omega$ and combinator $F \in \mathbf{Fun}_4^{(r)}$, and $s_0, \ldots, s_{r-1} \in \mathcal{L}_4[\emptyset]$, let

$$\mathcal{M}_4[\![F(s_1, \ldots, s_r)]\!] = \mathcal{I}_4(F)(\mathcal{M}_4[\![s_1]\!], \ldots, \mathcal{M}_4[\![s_r]\!]),$$

where $\mathcal{I}_4(F)$ is the interpreted operation corresponding to $F$ in $\mathcal{I}_4$. ∎

Several properties including the so-called *image finiteness* for elements of $\mathbf{P}_4$ are introduced. It will be shown that the denotational meaning of each statement in $\mathcal{L}_4[\emptyset]$ has these properties; this fact is used for establishing the full abstractness of $\mathcal{M}_4$.

**Definition 7.25** (Image Finiteness for Elements of $\mathbf{P}_4$) Let $p \in \mathbf{P}_4$ and $n \in \omega$.

**(1)** The process $p$ is *image finite at level n*, written $\mathrm{IFin}_4^{(n)}(p)$, iff

$$\begin{array}{l} \forall r \in (\Sigma \times \mathbf{A}_4 \times \Sigma)^n[\ p[r] \neq \emptyset \Rightarrow \\ \quad \forall \sigma \in \Sigma, \forall a \in \mathbf{A}_4[\ \{\sigma' \in \Sigma : p[r][\langle(\sigma, a, \sigma')\rangle] \neq \emptyset\} \text{ is finite }]]. \end{array}$$

The process $p$ is *image finite*, written $\mathrm{IFin}_4(p)$, iff

$$\forall n \in \omega[\ \mathrm{IFin}_4^{(n)}(p)\ ].$$

**(2)** The process $p$ is *finite w.r.t. action sorts at level n*, written $\mathrm{ASFin}^{(n)}(p)$, iff

$$\begin{array}{l} \forall r \in (\Sigma \times \mathbf{A}_4 \times \Sigma)^n[\ p[r] \neq \emptyset \Rightarrow \\ \quad \forall \sigma \in \Sigma[\ \mathrm{sact}_4(p[r], \sigma) \text{ is finite }]]. \end{array}$$

We write $\mathrm{ASFin}(p)$ to denote that $p$ is *finite w.r.t. action sorts at every level*, i.e., that

$$\forall n \in \omega[\ \mathrm{ASFin}^{(n)}(p)\ ].$$

**(3)** The process $p$ is *finite w.r.t. output values at level n*, written $\mathrm{OVFin}^{(n)}(p)$, iff

$$\begin{array}{l} \forall r \in (\Sigma \times \mathbf{A}_4 \times \Sigma)^{<n}[\ p[r] \neq \emptyset \Rightarrow \\ \quad \forall \sigma \in \Sigma, \forall c \in \mathbf{Chan}[ \\ \qquad \{v \in \mathbf{V} : \exists \sigma'[\ p[r][\langle(\sigma, c!v, \sigma')\rangle] \neq \emptyset\ ]\} \text{ is finite }]]. \end{array}$$

We write $\mathrm{OVFin}(p)$ to denote that $p$ is *finite w.r.t. output values at every level*, i.e., that

$$\forall n \in \omega[\ \mathrm{OVFin}^{(n)}(p)\ ].$$

**(4)** The process $p$ satisfies the *disjointness inaction condition at level n*, written $\mathrm{DIC}^{(n)}(p)$, iff

$$\begin{array}{l} \forall r \in (\Sigma \times \mathbf{A}_4 \times \Sigma)^n[\ p[r] \neq \emptyset \Rightarrow \\ \quad \forall \sigma \in \Sigma, \exists \mathcal{R} \subseteq \wp(\mathrm{sact}_4(p[r], \sigma) \cap \mathbf{CS}_4^{\checkmark})[ \\ \qquad \forall \Gamma \in \wp(\mathbf{CS}_4)[\ \langle(\sigma, \delta(\Gamma))\rangle \in p[r] \Leftrightarrow \exists R \in \mathcal{R}[\ \Gamma \cap R = \emptyset\ ]]]]. \end{array}$$

The process $p$ satisfies the *disjointness inaction condition*, written $\mathrm{DIC}(p)$, iff

$$\forall n \in \omega[\ \mathrm{DIC}^{(n)}(p)\ ].$$

(See Example 7.5, for a motivation of this definition.)

**(5)** The Property $\mathrm{FIRN}_4^{(n)}(p)$ is defined as $\mathrm{FIRN}_3^{(n)}(p)$ in Definition 7.9 (2). Formally, this is defined as follows: First, $\mathrm{FIRN}_4^{(n)}(p)$ iff there exists $\mathcal{X} \in \wp_{\mathrm{fin}}(\mathcal{IV})$ such that the following holds:

$$\begin{array}{l}\forall r \in (\Sigma \times \mathbf{A}_4 \times \Sigma)^n, \forall \vec{\sigma} \in ((\mathcal{IV} \setminus \mathcal{X}) \to \mathbf{V})^n[\ r \in p_{[n]} \Leftrightarrow \\ \quad \forall i \in n[\ \pi_0^3(r(i)) \quad (\mathcal{IV} \setminus \mathcal{X}) = \pi_2^3(r(i)) \quad (\mathcal{IV} \setminus \mathcal{X})\ ] \wedge \\ \quad \langle(((\pi_0^3(r(i)) \quad \mathcal{X}) \cup \vec{\sigma}(i), \pi_1^3(r(i)), (\pi_2^3(r(i)) \quad \mathcal{X}) \cup \vec{\sigma}(i))\rangle_{i \in n} \\ \quad \in p_{[n]}\ ].\end{array}$$

That is, for each $r \in (\Sigma \times \mathbf{A}_4 \times \Sigma)^n$, if $r \in p_{[n]}$, then, in every step

$$r(i) = (\pi_0^3(r(i)), \pi_1^3(r(i)), \pi_2^3(r(i)))$$

of $r$ $(i \in n)$, the value for $x \in \mathcal{IV} \setminus \mathcal{X}$ is not changed, i.e.,

$$\pi_0^3(r(i)) \quad (\mathcal{IV} \setminus \mathcal{X}) = \pi_2^3(r(i)) \quad (\mathcal{IV} \setminus \mathcal{X}), \tag{7.61}$$

and one may change the value $\pi_j^3(r(i))(x)$ $(j = 0, 2)$ arbitrarily, i.e.,

$$\begin{array}{l}\langle(((\pi_0^3(r(i)) \quad \mathcal{X}) \cup \vec{\sigma}(i), \pi_1^3(r(1)), (\pi_0^3(r(i)) \quad \mathcal{X}) \cup \vec{\sigma}(i))\rangle_{i \in n} \\ \in p_{[n]}\end{array} \tag{7.62}$$

for arbitrary $\vec{\sigma} \in ((\mathcal{IV}\setminus\mathcal{X}) \to \mathbf{V})^n$. Conversely, for arbitrary $\vec{\sigma} \in ((\mathcal{IV}\setminus\mathcal{X}) \to \mathbf{V})^n$, if one has (7.61) and (7.62), then $r \in p_{[n]}$.

**(6)** The property $\mathrm{FIRT}_4^{(n)}(p)$ is defined as $\mathrm{FIRT}_3^{(n)}(p)$ in Definition 7.9 (3) by:

$$\begin{array}{l}\mathrm{FIRT}_4^{(n)}(p) \Leftrightarrow \\ \exists \mathcal{X} \in \wp_{\mathrm{fin}}(\mathcal{IV}), \forall q \in (\Sigma \times \mathbf{A}_4 \times \Sigma)^n \cdot \mathbf{B}_4, \\ \forall \vec{\sigma} \in ((\mathcal{IV} \setminus \mathcal{X}) \to \mathbf{V})^{n+1}[\ q \in p \Leftrightarrow \\ \quad \forall i \in n[\ \pi_0^3(q(i)) \quad (\mathcal{IV} \setminus \mathcal{X}) = \pi_2^3(q(i)) \quad (\mathcal{IV} \setminus \mathcal{X})\ ] \wedge \\ \qquad \langle(((\pi_0^3(q(i)) \quad \mathcal{X}) \cup \vec{\sigma}(i), \pi_1^3(q(i)), (\pi_2^3(q(i)) \quad \mathcal{X}) \cup \vec{\sigma}(i))\rangle_{i \in n} \\ \qquad \cdot\langle(((\pi_0^2(q(n)) \quad \mathcal{X}) \cup \vec{\sigma}(n), \pi_1^2(q(n))))\rangle \\ \qquad \in p\ ].\end{array}$$

**(7)** $\mathrm{FIR}_4(p) \Leftrightarrow \mathrm{FIRN}_4(p) \wedge \mathrm{FIRT}_4(p)$

where

$$\mathrm{FIRN}_4(p) \Leftrightarrow \forall n \in \omega[\ \mathrm{FIRN}_4^{(n)}(p)\ ],$$

$$\mathrm{FIRT}_4(p) \Leftrightarrow \forall n \in \omega[\ \mathrm{FIRT}_4^{(n)}(p)\ ].$$

**(8)** $\mathbf{P}_4^* = \{p \in \mathbf{P}_4 : \mathrm{IFin}_4(p) \wedge \mathrm{ASFin}(p) \wedge \mathrm{OVFin}(p) \wedge \mathrm{DIC}(p) \wedge \mathrm{FIR}_4(p)\}$. ∎

**Remark 7.5** Though the condition $\mathrm{DIC}^{(0)}(\cdot)$ might seem too complicated, it is characterized in terms of a simpler condition $D(\cdot)$ defined as follows: For $p \in \mathbf{P}_4$,

$$\begin{array}{l}D(p) \Leftrightarrow \\ \forall \sigma[\ \exists \Gamma[\ (\sigma, \delta(\Gamma)) \in p\ ] \Rightarrow \\ \quad \exists R \subseteq \mathrm{sact}_4(s, \sigma) \cap \mathbf{CS}_4^{\checkmark}, \forall \Gamma[\ (\sigma, \delta(\Gamma)) \in p \Leftrightarrow \Gamma \cap R = \emptyset\ ]].\end{array} \tag{7.63}$$

Let $\mathbf{P}'$ be the smallest subset of $\mathbf{P}_4$ which includes $\{p \in \mathbf{P}_4 : D(p)\}$ and is closed under set-theoretical union, i.e., let

$$\mathbf{P}' = \{\bigcup \mathbf{P}'' : \mathbf{P}'' \subseteq \mathbf{P}_4 \land \bigcup \mathbf{P}'' \in \mathbf{P}_4 \land \forall p' \in \mathbf{P}''[\ D(p')\ ]\}.$$

Then, one has $\mathbf{P}' = \{p \in \mathbf{P}_4 : \mathrm{DIC}^{(0)}(p)\}$. The part $\mathbf{P}' \supseteq \{p \in \mathbf{P}_4 : \mathrm{DIC}^{(0)}(p)\}$ is shown as follows (the other part is shown more straightforwardly). Let $p \in \mathbf{P}_4$ with $\mathrm{DIC}^{(0)}(p)$, and $\Sigma' = \{\sigma : \exists\Gamma[\ (\sigma, \delta(\Gamma)) \in p\ ]\}$. Then for each $\sigma \in \Sigma'$, there exists $\mathcal{R}_\sigma$ such that

$$\forall\Gamma[\ (\sigma, \delta(\Gamma)) \in p \Leftrightarrow \exists R \in \mathcal{R}_\sigma[\ \Gamma \cap R = \emptyset\ ]].$$

Fix such $\mathcal{R}_\sigma$, and for each $\vec{R} \in \Pi_{\sigma\in\Sigma'}(\mathcal{R}_\sigma)$, put

$$p(\vec{R}) = \{q \in p : \mathrm{lgt}(q) \geq 2\} \cup \{(\sigma, \delta(\Gamma)) : \sigma \in \Sigma' \land \Gamma \cap \vec{R}(\sigma) = \emptyset\}.$$

Then, one has $D(p(\vec{R}))$ and $p = \bigcup\{p(\vec{R}) : \vec{R} \in \Pi_{\sigma\in\Sigma'}(\mathcal{R}_\sigma)\}$, and therefore, $p \in \mathbf{P}'$.

Also, as is obvious from Remark 7.1, the set $\{p \in \mathbf{P}_4 : \mathrm{DIC}(p)\}$ is defined as the largest subset of $\mathbf{P}_4$ which is included in $\{p \in \mathbf{P}_4 : \mathrm{DIC}^{(0)}(p)\}$ and *closed under taking remainders*, where closedness under taking remainders for subsets of $\mathbf{P}_4$ is defined as in Remark 7.1. It is easy to check that the downward closedness of $p \in \mathbf{P}_4$ follows from the fact that $\mathrm{DIC}(p)$. ∎

It turns out that the denotational meaning of each statement is a member of $\mathbf{P}_3^*$, which is used for establishing the full abstractness of $\mathcal{M}_4$.

**Lemma 7.17 (1)** *The set* $\mathbf{P}_4^*$ *is closed in* $\mathbf{P}_4$.

**(2)** $\forall p \in \mathbf{P}_4^*, \forall r \in (\Sigma \times \mathbf{A}_4 \times \Sigma)^{<\omega}[\ p[r] \neq \emptyset \Rightarrow p[r] \in \mathbf{P}_4^*\ ]$.

**(3)** *The set* $\mathbf{P}_4^*$ *is closed under all interpreted operations of* $\mathcal{L}_4$.

**(4)** $\mathcal{M}_4[\mathcal{L}_4[\emptyset]] \subseteq \mathbf{P}_4^*$.

**(5)** $\forall p \in \mathcal{M}_4[\mathcal{L}_4[\emptyset]], \forall r \in (\Sigma \times \Sigma)^{<\omega}[\ p[r] \neq \emptyset \Rightarrow p[r] \in \mathbf{P}_3^*\ ]$. ∎

**Proof.** These propositions are proved in a similar fashion to the proof of Lemma 7.4. Here we prove the essential part of **(3)**, i.e., that $\forall p_1, p_2 \in \mathbf{P}_4[\ \mathrm{DIC}(p_1) \land \mathrm{DIC}(p_2) \Rightarrow \mathrm{DIC}(p_1 \tilde{\|}_4 p_2)\ ]$. Let us show by induction on $n \in \omega$ that the following holds for every $n \in \omega$:

$$\forall p_1, p_2 \in \mathbf{P}_4[\ \mathrm{DIC}^{(n)}(p_1) \land \mathrm{DIC}^{(n)}(p_2) \Rightarrow \mathrm{DIC}^{(n)}(p_1 \tilde{\|}_4 p_2)\ ]. \tag{7.64}$$

**Induction Base:** Let $p_1, p_2 \in \mathbf{P}_4$ such that $\mathrm{DIC}^{(0)}(p_1)$ and $\mathrm{DIC}^{(0)}(p_2)$, and fix $\sigma \in \Sigma$. By the definition of $\mathrm{DIC}^{(0)}(\cdot)$, there exists $\mathcal{R}_i \subseteq \wp(\mathrm{sact}_4(p_i, \sigma) \cap \mathbf{CS}_4)$ such that

$$\forall\Gamma[\ (\sigma, \delta(\Gamma)) \in p_i \Leftrightarrow \exists R \in \mathcal{R}_i[\ \Gamma \cap R = \emptyset\ ]] \quad (i = 1, 2).$$

Let $\mathcal{R} = \{R_1 \cup R_2 : R_1 \in \mathcal{R}_1 \land R_2 \in \mathcal{R}_2 \land R_1 \cap \overline{R_2} = \emptyset\}$. Then one has, by the definitions of $\tilde{\|}_4$ and $\tilde{\|}_4^\delta$, that

$$\forall\Gamma[\ (\sigma, \delta(\Gamma)) \in p_1 \tilde{\|}_4 p_2 \Leftrightarrow \exists R \in \mathcal{R}[\ \Gamma \cap R = \emptyset\ ]],$$

which implies that $\mathrm{DIC}^{(0)}(p_1 \tilde{\|}_4 p_2)$.

**Induction Step:** For every $k \in \omega$, it immediately follows from the definition of $\tilde{\|}_4$, that (7.64) with $n = k + 1$ follows from (7.64) with $n = k$. ∎

### 7.3.4 Correctness of $\mathcal{M}_4$ with respect to $\mathcal{O}_4$

The correctness of $\mathcal{M}_4$ w.r.t. $\mathcal{O}_4$ is established as that of $\mathcal{M}_3$ w.r.t. $\mathcal{O}_3$, by means of an intermediate model $\mathcal{C}_4$.

#### Intermediate Model for $\mathcal{L}_4$ and Semantic Equivalence

First, the intermediate model $\mathcal{C}_4$, which is an alternative formulation of $\mathcal{M}_4$, is defined in terms of the transition relation $\longrightarrow_4$.

**Definition 7.26** (Intermediate Model $\mathcal{C}_4$ for $\mathcal{L}_4$) We have a unique mapping $\mathcal{C}_4 : \mathcal{L}_4[\emptyset] \to \mathbf{P}_4$ satisfying the following condition (the existence and uniqueness of such a mapping are obtained as in Definition 7.10): For $s \in \mathcal{L}_4[\emptyset]$,

$$\begin{aligned}\mathcal{C}_4[\![s]\!] = & \bigcup\{\langle(\sigma, a, \sigma')\rangle \cdot \mathcal{C}_4[\![s']\!] : (\sigma, a, \sigma') \in \Sigma \times \mathbf{A}_4 \times \Sigma \\ & \quad \wedge (s, \sigma) \xrightarrow{a}_4 (s', \sigma')\} \\ & \cup\{\langle(\sigma, \surd)\rangle : \sigma \in \Sigma \wedge \surd \in \mathrm{act}_4(s, \sigma)\} \\ & \cup\{\langle(\sigma, \delta(\Gamma))\rangle : \sigma \in \Sigma \wedge \Gamma \in \wp(\mathbf{CS}_4^{\surd}) \\ & \quad \wedge \tau \notin \mathrm{act}_4(s, \sigma) \wedge \Gamma \cap \mathrm{sact}_4(s, \sigma) = \emptyset\}.\ \blacksquare\end{aligned}$$

We have the distributivity of $\tilde{\|}_4$ in $\mathbf{P}_4$ as we had that in $\mathbf{P}_3$ (cf. Lemma 7.6).

**Lemma 7.18** (Distributivity of $\tilde{\|}_4$ in $\mathbf{P}_4$) *Let $k, l \geq 1$. Then*

$$\begin{aligned}&\forall p_0, \ldots, p_{k-1}, p'_0, \ldots, p'_{l-1} \in \mathbf{P}_4^*[\\ &\quad \tilde{\|}_4(\textstyle\bigcup_{i\in k}[p_i], \bigcup_{j\in l}[p'_j]) = \bigcup_{(i,j)\in k\times l}[\tilde{\|}_4(p_i, p'_j)]\,].\ \blacksquare\end{aligned}$$

**Proof.** See § 7.D. ■

By means of the above lemma, we will establish the equivalence between $\mathcal{M}_4$ and $\mathcal{C}_4$ as we have established Lemma 7.5.

**Lemma 7.19** (Semantic Equivalence for $\mathcal{L}_4$)

**(1)** *Let $F$ be an combinator of $\mathcal{L}_4$ with arity $r$, and let $s_1, \ldots, s_r \in \mathcal{L}_4[\emptyset]$. Then one has*

$$\mathcal{C}_4[\![F(s_1, \ldots, s_r)]\!] = \mathcal{I}_4(F)(\mathcal{C}_4[\![s_1]\!], \ldots, \mathcal{C}_4[\![s_r]\!]).$$

**(2)** *For $s \in \mathcal{L}_4[\emptyset]$, one has $\mathcal{C}_4[\![s]\!] = \mathcal{M}_4[\![s]\!]$.* ■

**Proof. (1)** The proof is similar to that of Lemma 7.5. Here we prove the claim for the combinator $\|$. For the other combinators this is proved (more straightforwardly) in a similar fashion. Let $\mathbf{H}_4 = (\mathcal{L}_4[\emptyset] \times \mathcal{L}_4[\emptyset] \to \mathbf{P}_4)$, and let $F, G \in \mathbf{H}_4$ be defined as follows: For $s_1, s_2 \in \mathcal{L}_4[\emptyset]$,

$$F(s_1, s_2) = \mathcal{C}_4[\![s_1 \parallel s_2]\!], \quad G(s_1, s_2) = \tilde{\|}_4(\mathcal{C}_4[\![s_1]\!], \mathcal{C}_4[\![s_2]\!]).$$

Moreover, let $\mathcal{F}_4^{\|} : \mathbf{H}_4 \to \mathbf{H}_4$ be defined as follows: For $f \in \mathbf{H}_4$ and $s_1, s_2 \in \mathcal{L}_4[\emptyset]$,

$$\begin{aligned}\mathcal{F}_4^{\parallel}(f)(s_1,s_2) = {} & \mathcal{F}_4^{\llcorner\!\llcorner}(f)(s_1,s_2) \cup \mathcal{F}_4^{\llcorner\!\llcorner}(f)(s_2,s_1)\\ & \cup \mathcal{F}_4^{\lfloor}(f)(s_1,s_2) \cup \mathcal{F}_4^{\lfloor}(f)(s_2,s_1)\\ & \cup \mathcal{F}_4^{\surd}(s_1,s_2) \cup \mathcal{F}_4^{\delta}(s_1,s_2),\end{aligned}$$

where

$$\mathcal{F}_4^{\llcorner\!\llcorner}(f)(s_1,s_2) = \bigcup\{\langle(\sigma,a,\sigma')\rangle \cdot f(s_1',s_2) : (s_1,\sigma) \xrightarrow{a}_4 (s_1',\sigma')\},$$

$$\begin{aligned}\mathcal{F}_4^{\lfloor}(f)(s_1,s_2) = \bigcup\{\langle(\sigma,\tau,\sigma')\rangle \cdot f(s_1',s_2') : & \\ \exists c, \exists v[\ (s_1,\sigma) \xrightarrow{c!v}_4 (s_1',\sigma) & \\ \wedge (s_2,\sigma) \xrightarrow{c?v}_4 (s_2',\sigma')\ ]\}, &\end{aligned}$$

$$\mathcal{F}_4^{\surd}(s_1,s_2) = \{\langle(\sigma,\surd)\rangle : \sigma \in \Sigma \wedge \surd \in \mathrm{act}_4(s_1,\sigma) \cap \mathrm{act}_4(s_2,\sigma)\},$$

and

$$\begin{aligned}\mathcal{F}_4^{\delta}(s_1,s_2) = \{\langle(\sigma,\delta(\Gamma))\rangle : {} & \tau \notin \mathrm{act}_4(s_1,\sigma) \wedge \tau \notin \mathrm{act}_4(s_1,\sigma)\\ & \wedge (\mathbf{CS}_4 \cap \mathrm{sact}_4(s_1,\sigma)) \cap \overline{(\mathbf{CS}_4 \cap \mathrm{sact}_4(s_2,\sigma))} = \emptyset\\ & \wedge \bigvee_{(i,j)=(1,2),(2,1)}[(\Gamma \setminus \{\surd\}) \cap \mathrm{sact}_4(s_i,\sigma) = \emptyset\\ & \qquad \wedge \Gamma \cap \mathrm{sact}_4(s_j,\sigma) = \emptyset]\ \}.\end{aligned}$$

Then, $\mathcal{F}_4^{\parallel}$ is a contraction.

Let $s_1, s_2 \in \mathcal{L}_4[\emptyset]$. By the definitions of $\mathcal{C}_4$ and $\rightarrow_4$, one has

$$F(s_1,s_2) = \mathcal{F}_4^{\parallel}(F)(s_1,s_2),$$

i.e., $F = \mathrm{fix}(\mathcal{F})$. Thus, for obtaining the desired result, it suffices to show $G = \mathcal{F}_4^{\parallel}(G)$. By the definition of $\tilde{\|}_4$, one has

$$\begin{aligned}G(s_1,s_2) = {} & \bigcup_{(i,j)=(1,2),(2,1)}[\ \tilde{\llcorner\!\llcorner}_4(\mathcal{C}_4[\![s_i]\!],\mathcal{C}_4[\![s_j]\!]) \cup \tilde{\lfloor}_4(\mathcal{C}_4[\![s_i]\!],\mathcal{C}_4[\![s_j]\!])\ ]\\ & \cup \tilde{\|}_4^{\surd}(\mathcal{C}_4[\![s_1]\!],\mathcal{C}_4[\![s_2]\!]) \cup \tilde{\|}_4^{\delta}(\mathcal{C}_4[\![s_1]\!],\mathcal{C}_4[\![s_2]\!]).\end{aligned}$$

Thus, for showing $G = \mathcal{F}_4^{\parallel}(G)$, it suffices to show

$$\tilde{\llcorner\!\llcorner}_4(\mathcal{C}_4[\![s_i]\!],\mathcal{C}_4[\![s_j]\!]) = \mathcal{F}_4^{\llcorner\!\llcorner}(G)(s_i,s_j) \quad ((i,j) = (1,2),(2,1)), \tag{7.65}$$

$$\tilde{\lfloor}_4(\mathcal{C}_4[\![s_i]\!],\mathcal{C}_4[\![s_j]\!]) = \mathcal{F}_4^{\lfloor}(G)(s_1,s_2) \quad ((i,j) = (1,2),(2,1)), \tag{7.66}$$

$$\tilde{\|}_4^{\surd}(\mathcal{C}_4[\![s_1]\!],\mathcal{C}_4[\![s_2]\!]) = \mathcal{F}_4^{\surd}(s_1,s_2), \tag{7.67}$$

and

$$\tilde{\|}_4^{\delta}(\mathcal{C}_4[\![s_1]\!],\mathcal{C}_4[\![s_2]\!]) = \mathcal{F}_4^{\delta}(s_1,s_2). \tag{7.68}$$

The fact (7.65) can be shown as (7.19) in the proof of Lemma 7.5 (1); (7.66) is shown as follows:

$$\begin{aligned}
&\tilde{\lfloor}_4(\mathcal{C}_4[\![s_i]\!], \mathcal{C}_4[\![s_j]\!]) \\
&= \bigcup\{\langle(\sigma,\tau,\sigma')\rangle \cdot \tilde{|\!|}_4((\mathcal{C}_4[\![s_i]\!][\langle(\sigma, c!v, \sigma)\rangle], \mathcal{C}_4[\![s_j]\!][\langle(\sigma, c?v, \sigma')\rangle])) : \\
&\qquad \mathcal{C}_4[\![s_i]\!][\langle(\sigma, c!v, \sigma)\rangle] \neq \emptyset \wedge \mathcal{C}_4[\![s_j]\!][\langle(\sigma, c?v, \sigma')\rangle] \neq \emptyset\}
\end{aligned}$$

(taking closure is omitted, since $\mathrm{ASFin}^{(0)}(\mathcal{O}_4[\![s_k]\!])$ and $\mathrm{OVFin}^{(0)}(\mathcal{O}_4[\![s_k]\!])$ $(k = 1, 2)$ by Lemma 7.13 (2) and (3), and therefore, the above set $\bigcup\{\langle(\sigma,\tau,\sigma')\rangle \cdots\}$ is closed)

$$\begin{aligned}
&= \bigcup\{\langle(\sigma,\tau,\sigma')\rangle \cdot \\
&\qquad \tilde{|\!|}_4(\bigcup\{\mathcal{C}_4[\![s_i']\!] : (s_i,\sigma) \xrightarrow{c!v}_4 (s_i',\sigma)\}, \\
&\qquad\quad \bigcup\{\mathcal{C}_4[\![s_j']\!] : (s_j,\sigma) \xrightarrow{c?v}_4 (s_j',\sigma')\}) : \\
&\qquad \exists s_i'[\ (s_i,\sigma) \xrightarrow{c!v}_4 (s_i',\sigma)\ ] \wedge \exists s_j'[\ (s_j,\sigma) \xrightarrow{c?v}_4 (s_2',\sigma')\ ]\} \\
&= \bigcup\{\langle(\sigma,\tau,\sigma')\rangle \cdot \\
&\qquad (\bigcup\{\tilde{|\!|}_4(\mathcal{C}_4[\![s_i']\!], \mathcal{C}_4[\![s_j']\!]) : \\
&\qquad\quad (s_i,\sigma) \xrightarrow{c!v}_4 (s_i',\sigma) \wedge (s_j,\sigma) \xrightarrow{c?v}_4 (s_j',\sigma')\}) : \\
&\qquad \exists s_i'[(s_i,\sigma) \xrightarrow{c!v}_4 (s_i',\sigma)] \wedge \exists s_j'[(s_j,\sigma) \xrightarrow{c?v}_4 (s_j',\sigma')]\}
\end{aligned}$$

(by Lemma 7.18)

$$= \mathcal{F}_4^{\lfloor}(G)(s_i, s_j).$$

The fact (7.67) immediately follows from the definitions of $\tilde{|\!|}_4^{\checkmark}$, $\mathcal{C}_4$, and $\mathcal{F}_4^{\checkmark}$.

For showing (7.68), it suffices, by the definition of $\tilde{|\!|}_4^{\delta}$, to show the following for every $(\sigma,\Gamma) \in \Sigma \times \wp(\mathbf{CS}_4^{\checkmark})$:

$$\begin{aligned}
&\exists\langle(\sigma,\delta(\Gamma_1))\rangle \in \mathcal{C}_4[\![s_1]\!], \exists\langle(\sigma,\delta(\Gamma_2))\rangle \in \mathcal{C}_4[\![s_2]\!][ \\
&\quad (\mathbf{CS}_4 \setminus \Gamma_1) \cap \overline{(\mathbf{CS}_4 \setminus \Gamma_2)} = \emptyset \\
&\quad \wedge \textstyle\bigvee_{(i,j)=(1,2),(2,1)}[(\Gamma \setminus \{\surd\}) \subseteq \Gamma_i \wedge \Gamma \subseteq \Gamma_j]\ ] \\
&\Leftrightarrow \tau \notin \mathrm{act}_4(s_1,\sigma) \wedge \tau \notin \mathrm{act}_4(s_2,\sigma) \\
&\quad \wedge (\mathbf{CS}_4 \cap \mathrm{sact}_4(s_1,\sigma)) \cap \overline{(\mathbf{CS}_4 \cap \mathrm{sact}_4(s_2,\sigma))} = \emptyset \\
&\quad \wedge \textstyle\bigvee_{(i,j)=(1,2),(2,1)}[(\Gamma \setminus \{\surd\}) \cap \mathrm{sact}_4(s_i,\sigma) = \emptyset \\
&\qquad\qquad \wedge \Gamma \cap \mathrm{sact}_4(s_j,\sigma) = \emptyset].
\end{aligned} \tag{7.69}$$

The $\Leftarrow$-part of (7.69) is obtained by putting

$$\Gamma_1 = \mathbf{CS}_4^{\checkmark} \setminus \mathrm{sact}_4(s_1,\sigma), \quad \Gamma_2 = \mathbf{CS}_4^{\checkmark} \setminus \mathrm{sact}_4(s_2,\sigma).$$

Let us show the $\Rightarrow$-part. Suppose the left-hand side of (7.69) holds, and fix such $\Gamma_1$, $\Gamma_2$. By the definition of $\mathcal{C}_4$,

$$\tau \notin \mathrm{act}_4(s_1,\sigma). \tag{7.70}$$

Moreover, $\Gamma_1 \cap \mathrm{sact}_4(s_1,\sigma) = \emptyset$, and therefore,

$$\mathbf{CS}_4 \cap \mathrm{sact}_4(s_1,\sigma) \subseteq \mathbf{CS}_4 \setminus \Gamma_1. \tag{7.71}$$

Similarly

$$\tau \notin \mathrm{act}_4(s_2, \sigma), \tag{7.72}$$

and $\mathrm{sact}_4(s_2, \sigma) \subseteq \mathbf{CS}_4 \setminus \Gamma_2$, i.e.,

$$\overline{(\mathbf{CS}_4 \cap \mathrm{sact}_4(s_2, \sigma))} \subseteq \overline{(\mathbf{CS}_4 \setminus \Gamma_2)}. \tag{7.73}$$

By (7.71) and (7.70), one has

$$(\mathbf{CS}_4 \cap \mathrm{sact}_4(s_1, \sigma)) \cap \overline{(\mathbf{CS}_4 \cap \mathrm{sact}_4(s_2, \sigma))} \subseteq (\mathbf{CS}_4 \setminus \Gamma_1) \cap \overline{(\mathbf{CS}_4 \setminus \Gamma_2)},$$

where the right-hand side is empty by the left-hand side of (7.69). Thus,

$$(\mathbf{CS}_4 \cap \mathrm{sact}_4(s_1, \sigma)) \cap \overline{(\mathbf{CS}_4 \cap \mathrm{sact}_4(s_2, \sigma))} = \emptyset. \tag{7.74}$$

Finally, by the left-hand side of (7.69), for some $(i, j) \in \{(1, 2), (2, 1)\}$, one has

$$(\Gamma \setminus \{\surd\}) \subseteq \Gamma_i \wedge \Gamma \subseteq \Gamma_j.$$

Fix such $(i, j)$. Then,

$$(\Gamma \setminus \{\surd\}) \subseteq \Gamma_i \subseteq \mathbf{CS}_4^{\surd} \setminus \mathrm{sact}_4(s_i, \sigma),$$

and therefore,

$$(\Gamma \setminus \{\surd\}) \cap \mathrm{sact}_4(s_i, \sigma) = \emptyset. \tag{7.75}$$

Similarly

$$\Gamma \cap \mathrm{sact}_4(s_j, \sigma) = \emptyset. \tag{7.76}$$

By (7.70), (7.72), (7.74), (7.75), (7.76), one has the right-hand side of (7.69). Thus one has the $\Rightarrow$-part of (7.69).

Summarizing, one has (7.69).

**(2)** Similar to the proof of part (2) of Lemma 7.5. ■

### Correctness of $\mathcal{M}_4$ with respect to $\mathcal{O}_4$

As a preliminary to the proof of the correctness, we will define an *abstraction function* $\mathcal{A}_4 : \mathbf{P}_4 \to (\Sigma \to \wp_{+\mathrm{cl}}(\mathbf{R}_4))$. As $\mathcal{A}_3$, this function is formulated in two ways, i.e., firstly as the fixed-point of a higher-order mapping, and secondly as the set of histories.

**Definition 7.27** (Abstraction Function $\mathcal{A}_4$ for $\mathcal{L}_4$) We have a unique mapping $\mathcal{A}_4 : \mathbf{P}_4 \to (\Sigma \to \wp_{+\mathrm{cl}}(\mathbf{R}_4))$ satisfying the following (the existence and uniqueness of such a mapping are obtained as in Definition 7.11): For every $p \in \mathbf{P}_4$, $\sigma \in \Sigma$,

$$\begin{aligned}\mathcal{A}_4(p)(\sigma) = & \bigcup\{\langle(a, \sigma')\rangle \cdot \mathcal{A}_4(p[\langle(\sigma, a, \sigma')\rangle])(\sigma') : p[\langle(\sigma, a, \sigma')\rangle] \neq \emptyset\} \\ & \cup \mathrm{if}(\langle(\sigma, \surd)\rangle \in p, \{\langle\surd\rangle\}, \emptyset) \\ & \cup \mathrm{if}(\exists \Gamma \in \wp(\mathbf{CS}_4^{\surd})[\ \surd \in \Gamma \wedge \langle(\sigma, \delta(\Gamma))\rangle \in p], \{\langle\,\delta\,\rangle\}, \emptyset).\end{aligned}$$

Note that the nonemptyness of $\mathcal{A}_4(p)(\sigma)$ follows from the fact that $p$ satisfies the disabled-$\tau$ condition and is upward closed w.r.t. disabled actions. ▮

The abstraction function is characterized in another way. First, we need some preliminary definitions.

**Definition 7.28** (Histories of Elements of $\mathbf{Q}_4$) Let

$$q \in \mathbf{Q}_4 \cup (\Sigma \times \mathbf{A}_4 \times \Sigma)^{<\omega}.$$

**(1)** The sequence $q$ is *executable*, written $\mathrm{Exec}_4(q)$, iff either

$$\exists \nu \in \omega \cup \{\omega\}, \exists \langle (\sigma_i, a_i, \sigma_i') \rangle_{i \in \nu} [\ q = \langle (\sigma_i, a_i, \sigma_i') \rangle_{i \in \nu} \\ \wedge \forall i \in \nu [\ i+1 \in \nu \Rightarrow \sigma_i' = \sigma_{i+1}\ ]]$$

or

$$\exists k \in \omega, \exists \langle (\sigma_i, a_i, \sigma_i') \rangle_{i \in k}, \exists \sigma_k, \exists \Upsilon \in \hat{\mathbf{B}}_4 [ \\ q = \langle (\sigma_i, a_i, \sigma_i') \rangle_{i \in k} \cdot \langle (\sigma_k, \Upsilon) \rangle \\ \wedge \forall i \in k [\ \sigma_i' = \sigma_{i+1}\ ]].$$

Let

$$\mathbf{E}_4 = \{q \in \mathbf{Q}_4 \cup (\Sigma \times \mathbf{A}_4 \times \Sigma)^{<\omega} : \mathrm{Exec}_4(q)\}.$$

For $\sigma \in \Sigma$, let

$$\mathbf{E}_4\langle\sigma\rangle = \{q \in \mathbf{E}_4 \setminus \{\epsilon\} : \mathrm{istate}_4(q) = \sigma\}.$$

**(2)** Let $q$ be executable. The *history* of $q$, denoted by $\mathrm{hist}_4(q)$, is defined by:

$$\mathrm{hist}_4(q) = \begin{cases} \langle (a_i, \sigma_i') \rangle_{i \in \nu} \\ \qquad \text{if } q = \langle (\sigma_i, a_i, \sigma_i') \rangle_{i \in \nu}, \\ \langle (a_i, \sigma_i') \rangle_{i \in k} \cdot \langle \surd \rangle \\ \qquad \text{if } q = \langle (\sigma_i, a_i, \sigma_i') \rangle_{i \in k} \cdot \langle (\sigma_k, \surd) \rangle, \\ \langle (a_i, \sigma_i') \rangle_{i \in k} \cdot \langle\ \delta\ \rangle \\ \qquad \text{if } q = \langle (\sigma_i, a_i, \sigma_i') \rangle_{i \in k} \cdot \langle (\sigma_k, \delta(\Gamma)) \rangle. \end{cases}$$ ▮

The next lemma is shown in a similar fashion to Lemma 7.7.

**Lemma 7.20** (Another Formulation of Abstraction Function $\mathcal{A}_4$)

**(1)** *For* $p \in \mathbf{P}_4^*, \sigma \in \Sigma$, *one has*

$$\mathcal{A}_4(p)(\sigma) = \{\mathrm{hist}_4(q) : q \in p \cap \mathbf{E}_4\langle\sigma\rangle\}.$$

**(2)** $\forall k \geq 1, \forall p_1, \ldots, p_k \in \mathbf{P}_4^*, \forall \sigma [\ \mathcal{A}_4(\bigcup_{i \in \bar{k}} [p_i])(\sigma) = \bigcup_{i \in \bar{k}} [\mathcal{A}_4(p_i)(\sigma)]\ ]$. ▮

By means of this lemma, one has the correctness of $\mathcal{M}_4$.

**Lemma 7.21** (Correctness of $\mathcal{M}_4$ w.r.t. $\mathcal{O}_4$)

**(1)** $\mathcal{A}_4 \circ \mathcal{C}_4 = \mathcal{O}_4$.

**(2)** $\mathcal{A}_4 \circ \mathcal{M}_4 = \mathcal{O}_4$.

**(3)** $\forall s_1, s_2 \in \mathcal{L}_4[\emptyset][\ \mathcal{M}_4[\![s_1]\!] = \mathcal{M}_4[\![s_2]\!] \Rightarrow$
$\forall X \in \mathcal{X}_{\mathcal{P}}, \forall S \in \mathcal{L}_4[X][\ \mathcal{O}_4[\![S[s_1/X]]\!] = \mathcal{O}_4[\![S[s_2/X]]\!]\ ]]$. ∎

**Proof.** **(1)** By showing that $\mathcal{A}_4 \circ \mathcal{C}_4$ is the fixed-point of $\Psi_4^{\mathcal{O}}$ defined in Definition 7.20.
**(2)** Immediate from part (1) and Lemma 7.19 (2).
**(3)** Immediate from part (2) and the compositional definition of $\mathcal{M}_4$. ∎

### 7.3.5 Full Abstractness of $\mathcal{M}_4$ with respect to $\mathcal{O}_4$

As for $\mathcal{L}_3$, we present the following lemma to establish the full abstractness of $\mathcal{M}_4$:

**Lemma 7.22** (Uniform Distinction Lemma for $\mathcal{L}_4$) *If either* **V** *or* **Chan** *is inifite, then the following hold for every* $\mathcal{X} \in \wp_{+\text{fin}}(\mathcal{IV})$.

**(1)** *For every* $r \in (\Sigma_{\mathcal{X}} \times \mathbf{A}_4 \times \Sigma_{\mathcal{X}})^{<\omega}$,

$$\begin{aligned}&\forall p_1, p_2 \in \mathbf{P}_4^*[\ p_1[r] \neq \emptyset \wedge p_2[r] = \emptyset \Rightarrow \\ &\quad \forall \sigma \in \Sigma_{\mathcal{X}}, \exists T \in \mathcal{L}_4[\emptyset][ \\ &\qquad \mathcal{A}_4(p_1 \tilde{\|}_4 \mathcal{M}_4[\![T]\!])(\sigma) \setminus \mathcal{A}_4(p_2 \tilde{\|}_4 \mathcal{M}_4[\![T]\!])(\sigma) \neq \emptyset\ ]].\end{aligned} \tag{7.77}$$

**(2)** *For every* $q \in (\Sigma_{\mathcal{X}} \times \mathbf{A}_4 \times \Sigma_{\mathcal{X}})^{<\omega} \cdot \mathbf{B}_4$,

$$\begin{aligned}&\forall p_1, p_2 \in \mathbf{P}_4^*[\ q \in p_1 \setminus p_2 \Rightarrow \\ &\quad \forall \sigma \in \Sigma_{\mathcal{X}}, \exists T \in \mathcal{L}_4[\emptyset][ \\ &\qquad \mathcal{A}_4(p_1 \tilde{\|}_4 \mathcal{M}_4[\![T]\!])(\sigma) \setminus \mathcal{A}_4(p_2 \tilde{\|}_4 \mathcal{M}_4[\![T]\!])(\sigma) \neq \emptyset\ ]].\ \blacksquare\end{aligned} \tag{7.78}$$

The proof of this lemma is given later. First, note that the full abstractness of $\mathcal{M}_4$ follows immediately from Lemma 7.21 and Lemma 7.22, in the same way as Theorem 7.1 follows from Lemma 7.8 and Lemma 7.9.

**Theorem 7.2** (Full Abstractness of $\mathcal{M}_4$ w.r.t. $\mathcal{O}_4$) *If either* **V** *or* **Chan** *is infinite, then for every* $s_1, s_2 \in \mathcal{L}_4[\emptyset]$, *one has*

$$\begin{aligned}&\mathcal{M}_4[\![s_1]\!] = \mathcal{M}_4[\![s_2]\!] \Leftrightarrow \\ &\forall X \in \mathcal{X}_{\mathcal{P}}, \forall S \in \mathcal{L}_4[X][\ \mathcal{O}_4[\![S[s_1/X]]\!] = \mathcal{O}_4[\![S[s_2/X]]\!]\ ].\ \blacksquare\end{aligned}$$

We present the following lemma as a preliminary to the proof of Lemma 7.22. For its proof we assume that either **V** or **Chan** is infinite.

**Lemma 7.23** (Testing Lemma for $\mathcal{L}_4$) *If either* **V** *or* **Chan** *is infinite, then for* $\mathcal{X} \in \wp_{+\text{fin}}(\mathcal{IV})$, $p \in \mathbf{P}_4^*$, $(\sigma', a, \sigma'') \in (\Sigma_{\mathcal{X}} \times \mathbf{A}_4 \times \Sigma_{\mathcal{X}})$, $\sigma_0 \in \Sigma_{\mathcal{X}}$, *there are two finite sequences* $r_1, r_2 \in (\Sigma_{\mathcal{X}} \times \mathbf{A}_4 \times \Sigma_{\mathcal{X}})^{<\omega}$ *such that the following hold:*

**(1)** $r_1 \cdot \langle(\sigma', a, \sigma'')\rangle \cdot r_2 \in \mathbf{E}_4\langle\sigma_0\rangle$.

**(2)** *For every tester* $T' \in \mathcal{L}_4[\emptyset]$, *there exists another tester* $T \in \mathcal{L}_4[\emptyset]$ *such that the following hold:*

$$\textbf{(i)} \qquad \mathcal{M}_4[\![T]\!][r_1 \cdot r_2] = \mathcal{M}_4[\![T']\!], \tag{7.79}$$

$$\textbf{(ii)} \qquad \forall q' \in \mathbf{Q}_4[\ r_1 \cdot \langle(\sigma', a, \sigma'')\rangle \cdot r_2 \cdot q' \in p\tilde{\|}_4\mathcal{M}_4[\![T]\!] \Rightarrow \\ p[\langle(\sigma', a, \sigma'')\rangle] \neq \emptyset \wedge q' \in p[\langle(\sigma', a, \sigma'')\rangle]\tilde{\|}_4\mathcal{M}_4[\![T']\!]\ ]. \blacksquare \tag{7.80}$$

**Proof.** The proof is formulated by supposing that $\mathcal{X}$ is reduced to one variable: $\mathcal{X} = \{x\}$, as Lemma 7.10. However, the lemma still holds when $\mathcal{X}$ is composed of more than one variable, as Lemma 7.10.

We distinguish two cases according to which of **V** and **Chan** is infinite.

**Case 1.** Suppose that **V** is infinite. For $v \in \mathbf{V}$, let $\bar{\sigma}(v)$ be defined as in Lemma 7.10. The proof is given by distinguishing two cases according to whether $\sigma_0(x) = \sigma'(x)$.

**Subcase 1.** When $\sigma_0(x) = \sigma'(x)$, we can easily construct two sequences $r_1$, $r_2$ satisfying (1), (2) of Lemma 7.23 as follows: Let

$$r_1 = \epsilon, \quad r_2 = \langle(\sigma'', \tau, \bar{\sigma}(v_1))\rangle,$$

where $v_1$ is chosen such that

$$\begin{cases} \textbf{(i)} & v_1 \neq \sigma''(x), \\ \textbf{(ii)} & v_1 \notin \{v \in \mathbf{V} : \langle(\sigma', a, \sigma''), (\sigma'', \tau, \bar{\sigma}(v))\rangle \in p_{[2]}\}. \end{cases} \tag{7.81}$$

Note that the right-hand side of (7.81) (ii) is finite by Definition 7.25, and therefore, there is $v_1$ satisfying (7.81). It is shown that (1) and (2) of Lemma 7.23 hold in a similar fashion to the corresponding part in the proof of Lemma 7.10.

**Subcase 2.** When $\sigma_0(x) \neq \sigma'(x)$, we can construct two sequences $r_1, r_2$, satisfying (1) and (2) of Lemma 7.23 as follows: Let

$$r_1 = \langle(\sigma_0, \tau, \sigma')\rangle, \quad r_2 = \langle(\sigma'', \tau, \bar{\sigma}(v_1))\rangle,$$

where $v_1$ is chosen such that

$$\begin{cases} \textbf{(i)}\ v_1 \notin \{v \in \mathbf{V} : \langle(\sigma_0, \tau, \sigma''), (\sigma', a, \sigma''), (\sigma'', \tau, \bar{\sigma}(v))\rangle \in p_{[3]}\}, \\ \textbf{(ii)}\ v_1 \neq \sigma'(x), \quad \textbf{(iii)}\ v_1 \neq \sigma''(x), \\ \textbf{(iv)}\ v_1 \notin \{v \in \mathbf{V} : \langle(\sigma', a, \sigma''), (\sigma'', \tau, \bar{\sigma}(v))\rangle \in p_{[2]}\}. \end{cases} \tag{7.82}$$

Note that the right-hand sides of (7.82) (i) and (iv) are finite by Definition 7.25, and therefore, there is $v_1$ satisfying (7.82). In this case also, it is shown that (1) and (2) of Lemma 7.23 hold in a similar fashion to the corresponding part in the proof of Lemma 7.10.

**Case 2.** Suppose that **Chan** is infinite. Let $c \in Chan$ such that

$$c! \notin \text{sort}[\text{act}_4(p[\langle(\sigma_0, \tau, \sigma')\rangle], \sigma')], \tag{7.83}$$

$$c! \notin \text{sort}[\text{act}_4(p, \sigma')], \tag{7.84}$$

$$c! \notin \text{sort}[\text{act}_4(p[\langle(\sigma', a, \sigma'')\rangle], \sigma'')], \tag{7.85}$$

$$c! \neq \text{sort}(a). \tag{7.86}$$

(Note that such a channel $c$ exists, because **Chan** is infinite and the three sets

$$\{c' \in \mathbf{Chan} : c'! \in \mathrm{sort}[\mathrm{act}_4(p[\langle(\sigma_0, \tau, \sigma')\rangle], \sigma')]\},$$

$$\{c' \in \mathbf{Chan} : c'! \in \mathrm{sort}[\mathrm{act}_4(p, \sigma')]\},$$

$$\{c' \in \mathbf{Chan} : c'! \in \mathrm{sort}[\mathrm{act}_4(p[\langle(\sigma', a, \sigma'')\rangle], \sigma'')]\}$$

are finite by the fact ASFin($p$).) Let

$$r_1 = \langle(\sigma_0, \tau, \bar{\sigma}[\sigma'(x)]), (\sigma', c!\bar{v}, \sigma')\rangle, \quad r_2 = \langle(\sigma'', c!\bar{v}, \sigma'')\rangle.$$

Then, obviously (1) of Lemma 7.23 holds. We will show (2) of Lemma 7.23. For every $T'$, let

$$T \equiv (x := \sigma'(x)); c!\bar{v}; c!\bar{v}; T'.$$

Then obviously, one has (7.79). We will show (7.80). Let $q' \in \mathbf{Q}_4$ such that

$$\begin{aligned} & r_1 \cdot \langle(\sigma', a, \sigma'')\rangle \cdot r_2 \cdot q' \\ & = \langle(\sigma_0, \tau, \bar{\sigma}[\sigma'(x)]), (\sigma', c!\bar{v}, \sigma'), (\sigma', a, \sigma''), (\sigma'', c!\bar{v}, \sigma'')\rangle \cdot q' \\ & \in p\tilde{\|}_4 \mathcal{M}_4[\![T]\!]. \end{aligned}$$

Then, the first step $(\sigma_0, \tau, \bar{\sigma}[\sigma'(x)])$ must stem from $T$, because if this is not the case, it must stem from $p$, and therefore, then the second step $(\sigma', c!\bar{v}, \sigma')$ must also stem from $p$, but this is impossible by (7.83). Also, the second step $(\sigma', c!\bar{v}, \sigma')$ must stem from $T$ by (7.84). The third action $a$ cannot stem from $T$ by (7.86), and therefore, it must stem from $p$. Finally, the forth step $(\sigma'', c!\bar{v}, \sigma'')$ cannot stem from $p$ by (7.85), and thus, this must stem from $T$. Summarizing the above, one obtains (7.80). ■

The following proposition follows immediately from Lemma 7.23 as Corollary 7.1 followed from Lemma 7.10; this corollary is to play a central role in the proof of Lemma 7.22.

**Corollary 7.2** *Let either* **V** *or* **Chan** *be infinite, and let* $\mathcal{X} \in \wp_{+\mathrm{fin}}(\mathcal{IV})$, $p \in \mathbf{P}_4^*$, $(\sigma', a, \sigma'') \in (\Sigma_{\mathcal{X}} \times \mathbf{A}_4 \times \Sigma_{\mathcal{X}})$, *and* $\sigma_0 \in \Sigma_{\mathcal{X}}$. *Then, there are* $\rho_1, \rho_2 \in (\mathbf{A}_4 \times \Sigma_{\mathcal{X}})^{<\omega}$ *such that for every* $T' \in \mathcal{L}_4[\emptyset]$ *there exists* $T \in \mathcal{L}_4[\emptyset]$ *such that, putting* $\sigma_1 = \mathrm{last}(\rho_1 \cdot \sigma'' \cdot \rho_2)$, *one has the following:*

**(1)** *For every* $p' \in \mathbf{P}_4^*$, *one has*

$$\begin{aligned} & \forall \rho' \in \mathbf{R}_4[\ p'[\langle(\sigma', a, \sigma'')\rangle] \neq \emptyset \\ & \quad \wedge \rho' \in \mathcal{A}_4(p'[\langle(\sigma', a, \sigma'')\rangle]\tilde{\|}_4 \mathcal{M}_4[\![T']\!])(\sigma_1) \\ & \quad \Rightarrow \rho_1 \cdot \sigma'' \cdot \rho_2 \cdot \rho' \in \mathcal{A}_4(p'\tilde{\|}_4 \mathcal{M}_4[\![T]\!])(\sigma_0)\ ]. \end{aligned} \tag{7.87}$$

**(2)** *For* $p' = p$, *one has the converse of (7.88). That is,*

$$\begin{aligned} & \forall \rho' \in \mathbf{R}_4[\ \rho_1 \cdot \sigma'' \cdot \rho_2 \cdot \rho' \in \mathcal{A}_4(p\tilde{\|}_4 \mathcal{M}_4[\![T]\!])(\sigma_0) \Rightarrow \\ & \quad p[\langle(\sigma', a, \sigma'')\rangle] \neq \emptyset \\ & \quad \wedge \rho' \in \mathcal{A}_4(p[\langle(\sigma', a, \sigma'')\rangle]\tilde{\|}_4 \mathcal{M}_4[\![T']\!])(\sigma_1)\ ]. \end{aligned} \tag{7.88}$$
■

**Proof of Lemma 7.22.** Let $\mathcal{X} \in \wp_{+\text{fin}}(\mathcal{IV})$.
**Part (1).** The first part is proved by means of Corollary 7.2, as Lemma 7.9 (1) was proved by means of Corollary 7.1, by induction on the length of $r \in (\Sigma_{\mathcal{X}} \times \mathbf{A}_4 \times \Sigma_{\mathcal{X}})^{<\omega}$.
**Part (2).** We will prove that (7.78) holds for every $q \in (\Sigma_{\mathcal{X}} \times \mathbf{A}_4 \times \Sigma_{\mathcal{X}})^{<\omega} \cdot \mathbf{B}_4$, by induction on the length of $q$. The proof is similar to the proof of the corresponding part of Lemma 7.9 except for the induction base, which is established by means of the method of [BKO 88] with some adaptation to the present setting; the induction step can be established using the testing method (Corollary 7.2).
**Induction Base:** Let $\text{lgt}(q) = 1$, say $q = \langle(\sigma, \Upsilon)\rangle$. Let $p_1, p_2 \in \mathbf{P}_4^*$ such that $q \in p_1 \setminus p_2$, and let $\Sigma_{\mathcal{X}}$. Since $\mathcal{X}$ is finite and nonempty, we can put $\mathcal{X} = \{x_1, \ldots, x_r\}$. We distinguish two cases according to the type of $\Upsilon$.
**Case 1.** Suppose $\Upsilon = \surd$. Then as in Case 1 in the induction base of the proof of Lemma 7.9 (2), we can construct an appropriate tester $T$ for every $\sigma \in \Sigma_{\mathcal{X}}$ such that

$$\mathcal{A}_4(p_1 \tilde{\|}_4 \mathcal{M}_4[\![T]\!])(\sigma) \setminus \mathcal{A}_4(p_2 \tilde{\|}_4 \mathcal{M}_4[\![T]\!])(\sigma) \neq \emptyset.$$

**Case 2.** Suppose $\Upsilon = \delta(\Gamma')$ for some $\Gamma' \in \wp(\mathbf{CS}_4^{\surd})$ with $\surd \in \Gamma'$. We will construct a tester $T$ such that

$$\mathcal{A}_4(p_1 \tilde{\|}_4 \mathcal{M}_4[\![T]\!])(\sigma) \setminus \mathcal{A}_4(p_2 \tilde{\|}_4 \mathcal{M}_4[\![T]\!])(\sigma) \neq \emptyset. \tag{7.89}$$

Since $p_2$ satisfies the disjointness inaction condition, there exists $\mathcal{R}_2$ such that

$$\mathcal{R}_2 \subseteq \wp(\text{sact}_4(p_2, \sigma) \cap \mathbf{CS}_4^{\surd}), \tag{7.90}$$

and

$$\forall \Gamma \in \wp(\mathbf{CS}_4^{\surd})[\ \langle(\sigma', \Gamma)\rangle \in p_2 \Leftrightarrow \exists R \in \mathcal{R}_2[\ \Gamma \cap R = \emptyset\ ]]. \tag{7.91}$$

Fix such $\mathcal{R}_2$, and let

$$\Gamma'' = \text{sact}_4(p_2, \sigma) \cap \Gamma' \cap \mathbf{CS}_4. \tag{7.92}$$

By (7.91) and the fact that $q \notin p_2$, one has

$$\forall R' \in \mathcal{R}_2[\ \Gamma' \cap R' \neq \emptyset\ ].$$

The set $\text{sact}_4(p_2, \sigma)$ is finite since $\text{ASFin}(p_2)$, which implies that $\Gamma''$ is finite. Let

$$\Gamma'' = \{\gamma_1, \ldots, \gamma_n\}.$$

Let us set

$$\begin{aligned} &T \equiv (x_1 := \sigma'(x_1)); \cdots; (x_r := \sigma'(x_r)); T', \\ &T' \equiv \mathbf{e} + \phi(\overline{\gamma_1}) + \cdots + \phi(\overline{\gamma_n}), \end{aligned}$$

where

$$\phi(\gamma) = \begin{cases} (c!\ v) & \text{if } \gamma = c! \text{ with } v \in \mathbf{V} \text{ arbitrary}, \\ (c?\ x) & \text{if } \gamma = c? \text{ with } x \in \mathcal{IV} \text{ arbitrary}. \end{cases}$$

With this tester $T$, we will show that

$$\langle(\tau,\sigma'_1),\ldots,(\tau,\sigma'_r),\ \delta\,\rangle \in \mathcal{A}_4(p_1\tilde{\|}_4\mathcal{M}_4[\![T]\!])(\sigma)\setminus\mathcal{A}_4(p_2\tilde{\|}_4\mathcal{M}_4[\![T]\!])(\sigma),$$

where

$$\sigma'_i=\sigma[\langle\sigma'(x_1),\ldots,\sigma'(x_i)\rangle/\langle x_1,\ldots,x_i\rangle]\quad (i\in r+1).$$

First, let us show

$$\langle(\tau,\sigma'_1),\ldots,(\tau,\sigma'_r),\ \delta\,\rangle \in \mathcal{A}_4(p_1\tilde{\|}_4\mathcal{M}_4[\![T]\!])(\sigma).$$

Under the assumption that $q\in p_1$, one has

$$\langle(\sigma',\delta(\Gamma'))\rangle\in p_1. \tag{7.93}$$

Moreover, by the definition of $T'$, one has

$$\langle(\sigma',\mathbf{CS}_4\setminus(\overline{\Gamma''})\rangle =\in \mathcal{M}_4[\![T']\!]. \tag{7.94}$$

Moreover,

$$\begin{aligned}&(\mathbf{CS}_4\setminus\Gamma')\cap\overline{(\mathbf{CS}_4\setminus(\mathbf{CS}_4\setminus\overline{\Gamma''}))}=(\mathbf{CS}_4\setminus\Gamma')\cap\Gamma''\\&=(\mathbf{CS}_4\setminus\Gamma')\cap\mathrm{sact}_4(p_2,\sigma)\cap\Gamma\cap\mathbf{CS}'_4\quad(\text{by }(7.92))\\&=\emptyset.\end{aligned} \tag{7.95}$$

Further, $\langle\surd\rangle\in\Gamma'$. By this, (7.93), (7.94), (7.95), and the definitions of $\tilde{\|}_4$ and $\tilde{\|}_4^{\delta}$, one has

$$\langle(\sigma'_0,\tau,\sigma'_1),\ldots,(\sigma'_{r-1},\tau,\sigma'_r),(\sigma',\emptyset),\delta(\{\surd\})\rangle\in p_1\tilde{\|}_4\mathcal{M}_4[\![T]\!],$$

i.e.,

$$\langle(\tau,\sigma'_1),\ldots,(\tau,\sigma'_r),\ \delta\,\rangle \in \mathcal{A}_4(p_1\tilde{\|}_4\mathcal{M}_4[\![T]\!])(\sigma). \tag{7.96}$$

Next let us show, by contradiction, that

$$\langle(\tau,\sigma'_1),\ldots,(\tau,\sigma'_r),\ \delta\,\rangle \notin \mathcal{A}_4(p_2\tilde{\|}_4\mathcal{M}_4[\![T]\!])(\sigma).$$

Assume, to the contrary,

$$\langle(\tau,\sigma'_1),\ldots,(\tau,\sigma'_r),\ \delta\,\rangle \in \mathcal{A}_4(p_2\tilde{\|}_4\mathcal{M}_4[\![T]\!])(\sigma). \tag{7.97}$$

Then, by the definition of $\mathcal{A}_4$, one has

$$\langle(\sigma'_0,\tau,\sigma'_1),\ldots,(\sigma'_{r-1},\tau,\sigma'_r),(\sigma',\delta(\{\surd\}))\rangle\in p_2\tilde{\|}_4\mathcal{M}_4[\![T]\!]. \tag{7.98}$$

Hence,

$$\begin{aligned}\langle(\sigma',\delta(\{\surd\}))\rangle&\in p_2\tilde{\|}_4\mathcal{M}_4[\![T]\!][\langle(\sigma'_0,\tau,\sigma'_1),\ldots,(\sigma'_{r-1},\tau,\sigma'_r)\rangle]\\&=p_2\tilde{\|}_4\mathcal{M}_4[\![T']\!].\end{aligned}$$

By this and the definitions of $\tilde{\|}_4$ and $\tilde{\|}_4^{\delta}$, there exist $\Gamma_1,\Gamma_2\in\wp(\mathbf{CS}_4^{\checkmark})$ such that

$$\begin{array}{l}\textbf{(i)}\ \langle(\sigma',\Gamma_1)\rangle \in p_2,\quad \textbf{(ii)}\ \langle(\sigma',\Gamma_2)\rangle \in \mathcal{M}_4[\![T']\!],\\ \textbf{(iii)}\ (\mathbf{CS}_4\setminus\Gamma_1)\cap\overline{(\mathbf{CS}_4\setminus\Gamma_2)}=\emptyset,\\ \textbf{(iv)}\ \{\surd\}\subseteq\Gamma_1 \vee \{\surd\}\in\Gamma_2.\end{array} \tag{7.99}$$

By (7.99) (ii) and the form of $T'$, $\surd \notin \Gamma_2$. From this and (7.99), it follows that

$$\{\surd\} \subseteq \Gamma_1. \tag{7.100}$$

By (7.99) (i), there exists $R' \in \mathcal{R}_2$ such that $\Gamma_1 \cap R' = \emptyset$. Fix such $R'$. Then,

$$\mathbf{CS}_4^{\checkmark}\setminus\Gamma_1 = \mathbf{CS}_4^{\checkmark}\setminus\Gamma_1 \quad (\text{by } (7.100)) \quad \supseteq R'. \tag{7.101}$$

By the fact that $\langle(\sigma',\Gamma')\rangle \notin p_2$, one has

$$\Gamma' \cap R' \neq \emptyset. \tag{7.102}$$

By (7.99) (ii), one has $\Gamma_2 \cap \overline{\Gamma''} = \emptyset$, i.e., $\mathbf{CS}_4\setminus\Gamma_2 \supseteq \overline{\Gamma''}$, and therefore,

$$\overline{\mathbf{CS}_4\setminus\Gamma_2} \supseteq \Gamma''. \tag{7.103}$$

Thus

$$\begin{array}{l}(\mathbf{CS}_4\setminus\Gamma_1)\cap\overline{\mathbf{CS}_4\setminus\Gamma_2} \supseteq R'\cap\Gamma'' \quad (\text{by } (7.101) \text{ and } (7.103))\\ = R'\cap(\mathrm{sact}_4(p_2,\sigma)\cap\Gamma'\cap\mathbf{CS}_4) \quad (\text{by } (7.92))\\ \subseteq= R'\cap(\mathrm{sact}_4(p_2,\sigma)\cap\Gamma'\cap\mathbf{CS}_4^{\checkmark})\\ = R'\cap\Gamma' \quad (\text{since } R'\subseteq \mathrm{sact}_4(p_2,\sigma)\cap\mathbf{CS}_4^{\checkmark} \text{ by } (7.90))\\ \neq\emptyset \quad (\text{by } (7.102)).\end{array}$$

This contradicts (7.99) (iii). Hence (7.97) is false, and therefore, one has

$$\langle(\tau,\sigma_1'),\ldots,(\tau,\sigma_r'),\ \delta\,\rangle \notin \mathcal{A}_4(p_2\tilde{\|}_4\mathcal{M}_4[\![T]\!])(\sigma).$$

By this and (7.94), one has

$$\langle(\tau,\sigma_1'),\ldots,(\tau,\sigma_r'),\ \delta\,\rangle \in \mathcal{A}_4(p_1\tilde{\|}_4\mathcal{M}_4[\![T]\!])(\sigma)\setminus\mathcal{A}_4(p_2\tilde{\|}_4\mathcal{M}_4[\![T]\!])(\sigma).$$

**Case 3.** Suppose $\Upsilon \in \wp(\mathbf{CS}_4)$. Then, by Lemma 7.16 (2), one has

$$\langle(\sigma',\delta(\Gamma\cup\{\surd\}))\rangle \in (p_1\tilde{\|}_4\tilde{\mathbf{0}}_4)\setminus(p_2\tilde{\|}_4\tilde{\mathbf{0}}_4).$$

Thus this case is reduced to Case 2.
**Induction Step:** By means of Corollary 7.2, the induction step is established, in a similar fashion to the induction step of the proof of Lemma 7.9 (1). ■

### 7.3.6 Full Abstractness of $\mathcal{M}_4$ w.r.t. $\mathcal{O}_4^*$

**Connection between $\mathcal{O}_4$ and $\mathcal{O}_4^*$**

An abstraction function $\mathcal{A}_4^*$ from the range of $\mathcal{O}_4$ to that of $\mathcal{O}_4^*$ is defined as follows:

**Definition 7.29** (Abstraction Function $\mathcal{A}_4^*$)

**(1)** First, a function $\hat{\mathcal{A}}_4^* : \mathbf{R}_4 \to \mathbf{R}_4^*$ is defined as follows: For every $\rho \in \mathbf{R}_4$,

$$\hat{\mathcal{A}}_4^*(\rho) = \begin{cases} \langle a_n \rangle_{n\in\omega} & \text{if } \rho = \langle (\sigma_n, a_n) \rangle_{n\in\omega}, \\ \langle a_i \rangle_{i\in n} \cdot \langle \surd \rangle & \text{if } \rho = \langle (a_i, \sigma_i) \rangle_{i\in n} \cdot \langle \surd \rangle, \\ \langle a_i \rangle_{i\in n} \cdot \langle\, \delta \,\rangle & \text{if } \rho = \langle (a_i, \sigma_i) \rangle_{i\in n} \cdot \langle\, \delta \,\rangle. \end{cases}$$

**(2)** By means of $\hat{\mathcal{A}}_4^*$, the function $\mathcal{A}_4^* : \wp(\mathbf{R}_4) \to \wp(\mathbf{R}_4^*)$ is defined as follows: For every $R \in \wp(\mathbf{R}_4^*)$,

$$\mathcal{A}_4^*(R) = \{\hat{\mathcal{A}}_4^*(\rho) : \rho \in R\}. \blacksquare$$

Having defined $\mathcal{A}_4^*$, the following fact is obtained immediately from Lemma 7.20 and the definition of $\mathcal{O}_4^*$.

**Lemma 7.24** (Relative Abstraction of $\mathcal{O}_4$ w.r.t. $\mathcal{O}_4^*$) $\mathcal{O}_4^* = \mathcal{A}_4^* \circ \mathcal{O}_4$. $\blacksquare$

Let us establish the completeness of $\mathcal{O}_4^*$ w.r.t. $\mathcal{O}_4$ (proposition (7.3) stated in § 7.1). First, we present the proposition in another way taking the contrapositive of (7.3) and restricting contexts to the ones of the form "$P \,\|\, T$".

**Lemma 7.25** (Completeness of $\mathcal{O}_4^*$ w.r.t. $\mathcal{O}_4$) *If either* **V** *or* **Chan** *is infinite, then, for every* $s_0, s_1 \in \mathcal{L}_4[\emptyset]$, *the following holds:*

$$\mathcal{O}_4[\![s_0]\!] \neq \mathcal{O}_4[\![s_1]\!] \Rightarrow \exists T \in \mathcal{L}_4[\emptyset][\ \mathcal{O}_4^*[\![s_0 \,\|\, T]\!] \neq \mathcal{O}_4^*[\![s_1 \,\|\, T]\!]\ ]. \blacksquare \qquad (7.104)$$

Several preliminaries are necessary for the proof of this lemma.

**Definition 7.30** Let $o \in (\Sigma \to \wp_{\mathrm{fin}}(\mathbf{R}_4))$.

**(1)** For $\mathcal{X} \in \wp_{\mathrm{fin}}(\mathcal{IV})$, let

$$\bar{\sigma}_{\mathcal{X}} = (\lambda x \in (\mathcal{IV} \setminus \mathcal{X}).\ \bar{v}).$$

**(2)** We use the variable $\mathbf{u}$ ranging over $(\mathcal{IV} \not\to \mathbf{V})$. For $\mathbf{u} \in (\mathcal{IV} \not\to \mathbf{V})$, let

$$[\mathbf{u}] = \mathbf{u} \cup \bar{\sigma}_{\mathcal{X}},$$

where $\mathcal{X} = \mathrm{dom}(\mathbf{u})$.

**(3)** For $n \in \omega$, $(\mathrm{FIRN}_4')^{(n)}(o)$ iff there exists $\mathcal{X}_0 \in \wp_{\mathrm{fin}}(\mathcal{IV})$ such that for every $\mathcal{X} \in \wp_{\mathrm{fin}}(\mathcal{IV})$ with $\mathcal{X}_0 \subseteq \mathcal{X}$, the following holds:

$$\begin{aligned} &\forall \sigma \in \Sigma, \forall \langle (a_i, \sigma_i) \rangle_{i\in n} \in (\mathbf{A}_4 \times \Sigma)^n [\ \langle (a_i, \sigma_i) \rangle_{i\in n} \in (o(\sigma))_{[n]} \\ &\quad \Leftrightarrow \forall i \in n[\ \sigma_i \quad (\mathcal{IV} \setminus \mathcal{X}) = \sigma \quad (\mathcal{IV} \setminus \mathcal{X})\ ] \\ &\qquad \wedge \langle (a_i, [\sigma_i \quad \mathcal{X}]) \rangle_{i\in n} \in (o([\sigma \quad \mathcal{X}]))_{[n]}\ ]. \end{aligned}$$

**(4)** $\mathrm{FIRN}_4'(o)$ iff $\forall n \in \omega[\ (\mathrm{FIRN}_4')^{(n)}(o)\ ]$.

**(5)** For $n \in \omega$, $(\mathrm{FIRT}_4')^{(n)}(o)$ iff there exists $\mathcal{X}_0 \in \wp_{\mathrm{fin}}(\mathcal{IV})$ such that for every $\mathcal{X} \in \wp_{\mathrm{fin}}(\mathcal{IV})$ with $\mathcal{X}_0 \subseteq \mathcal{X}$, the following holds:

$$
\begin{aligned}
&\forall \sigma \in \Sigma, \forall \langle (a_i, \sigma_i) \rangle_{i \in n} \in (\mathbf{A}_4 \times \Sigma)^n, \forall \chi \in \{\surd, \delta\}[\\
&\quad \langle (a_i, \sigma_i) \rangle_{i \in n} \cdot \langle \chi \rangle \in o(\sigma)\\
&\quad \Leftrightarrow \forall i \in n[\ \sigma_i \quad (\mathcal{IV} \setminus \mathcal{X}) = \sigma \quad (\mathcal{IV} \setminus \mathcal{X})\ ]\\
&\qquad \wedge \langle (a_i, [\sigma_i \quad \mathcal{X}]) \rangle_{i \in n} \cdot \langle \chi \rangle \in o([\sigma \quad \mathcal{X}])\ ].
\end{aligned}
$$

**(6)** $\mathrm{FIRT}'_4(o)$ iff $\forall n \in \omega[\ (\mathrm{FIRT}'_4)^{(n)}(o)\ ]$.

**(7)** $\mathrm{FIR}'_4(o)$ iff $\mathrm{FIRN}'_4(o)$ and $\mathrm{FIRT}'_4(o)$. ∎

By Lemma 7.17 (4), one has $\mathrm{FIR}_4(\mathcal{M}_4[\![s]\!])$ for every $s \in \mathcal{L}_4[\emptyset]$. From this and Lemma 7.21 (2), the following proposition immediately follows.

**Proposition 7.1** *For every* $s \in \mathcal{L}_4[\emptyset]$, *one has* $\mathrm{FIR}'_4(\mathcal{O}_4[\![s]\!])$. ∎

We introduce a format for testers to be used in the proof of Lemma 7.25.

**Definition 7.31** Let $c \in \mathbf{Chan}$ and $\mathcal{X} \in \wp_{+\mathrm{fin}}(\mathcal{IV})$. Put $k = \sharp(\mathcal{X})$, and let $x_0, \ldots, x_{k-1} \in \mathcal{IV}$ such that $\{x_0, \ldots, x_{k-1}\} = \mathcal{X}$ and $x_0 <_{\mathrm{iv}} \ldots <_{\mathrm{iv}} x_{k-1}$.

**(1)** Let $\mathbf{u} \in (\mathcal{X} \to \mathbf{V})$ and $s \in \mathcal{L}_4[\emptyset]$.

**(i)** For $v \in \mathbf{V}$,

$$H^{c}_{\mathcal{X}}(\mathbf{u}, v, s) \equiv \mathbf{if}(\mathcal{X} = \mathbf{u}, (c!\ v); s, \mathbf{0}),$$

where "$\mathcal{X} = \mathbf{u}$" is an abbreviation of the Boolean expression

$$x_0 = \mathbf{u}(x_0) \wedge \cdots \wedge x_{k-1} = \mathbf{u}(x_{k-1}).$$

**(ii)** For $v', v'' \in \mathbf{V}$, let

$$G^{c}_{\mathcal{X}}(\mathbf{u}, v', v'', s) \equiv H^{c}_{\mathcal{X}}(\mathbf{u}, v', H^{c}_{\mathcal{X}}(\mathbf{u}, v'', s)).$$

**(iii)** For $c' \in \mathbf{Chan}$, let

$$\tilde{G}_{\mathcal{X}}(\mathbf{u}, c', s) \equiv \mathbf{if}(\mathcal{X} = \mathbf{u}, (c'!\bar{v}; s), \mathbf{0}).$$

**(2)** For $n \in \omega$, $\vec{\mathbf{u}} \in (\mathcal{X} \to \mathbf{V})^n$, $s \in \mathcal{L}_4[\emptyset]$, and $\mathbf{v}', \mathbf{v}'' \in \mathbf{V}^n$, let $F^{c}_{\mathcal{X}}(\vec{\mathbf{u}}, \mathbf{v}', \mathbf{v}'', s)$ be defined by induction on $n$ as follows:

**(i)** $F^{c}_{\mathcal{X}}(\epsilon, \epsilon, \epsilon, s) \equiv s$,

**(ii)** For $n \in \omega$, $\mathbf{u} \in (\mathcal{X} \to \mathbf{V})$, $\vec{\mathbf{u}} \in (\mathcal{X} \to \mathbf{V})^n$, $v', v'' \in \mathbf{V}$, $\mathbf{v}', \mathbf{v}'' \in \mathbf{V}^n$,

$$
\begin{aligned}
&F^{c}_{\mathcal{X}}(\langle \mathbf{u} \rangle \cdot \vec{\mathbf{u}},\ \langle v' \rangle \cdot \mathbf{v}',\ \langle v'' \rangle \cdot \mathbf{v}'', s)\\
&\equiv G^{c}_{\mathcal{X}}(\mathbf{u}, v', v'', F^{c}_{\mathcal{X}}(\vec{\mathbf{u}}, \mathbf{v}', \mathbf{v}'', s)).
\end{aligned}
$$

**(3)** For $n \in \omega$, $\vec{\mathbf{u}} \in (\mathcal{X} \to \mathbf{V})^n$, $s \in \mathcal{L}_4[\emptyset]$, and $\vec{c} \in \mathbf{Chan}^n$, let $\tilde{F}_{\mathcal{X}}(\vec{\mathbf{u}}, \vec{c}, s)$ be defined by induction on $n$ as follows:

**(i)** $\tilde{F}_{\mathcal{X}}(\epsilon, \epsilon, \epsilon, s) \equiv s$,

**(ii)** For $n \in \omega$, $\mathbf{u} \in (\mathcal{X} \to \mathbf{V})$, $\vec{\mathbf{u}} \in (\mathcal{X} \to \mathbf{V})^n$, $c' \in \mathbf{Chan}$, $\vec{c} \in \mathbf{Chan}^n$,

$$\tilde{F}_{\mathcal{X}}(\langle \mathbf{u} \rangle \cdot \vec{\mathbf{u}},\ \langle c' \rangle \cdot \vec{c}, s) \equiv \tilde{G}_{\mathcal{X}}(\mathbf{u}, c', \tilde{F}_{\mathcal{X}}(\vec{\mathbf{u}}, \vec{c}, s)). ∎$$

The following lemma is given as a preliminary to the proof of Lemma 7.25.

**Lemma 7.26** *Let* $n \geq 1$, $\mathcal{L}' \in \wp_{\mathrm{fin}}(\mathcal{L}_4[\emptyset])$, $\langle a_0, \ldots, a_{n-1} \rangle \in (\mathbf{A}_4)^n$, $\langle \sigma_0, \ldots, \sigma_{n-1} \rangle \in \Sigma^n$, *and* $\sigma \in \Sigma$.

**(1)** *Let* $\mathbf{V}$ *be infinite, and* $c \in \mathbf{Chan}$. *Then there exist,* $\mathcal{X}_0 \in \wp_{+\mathrm{fin}}(\mathcal{IV})$ *and* $\mathbf{v}' = \langle v'_0, \ldots, v'_{n-1} \rangle$, $\mathbf{v}'' = \langle v''_0, \ldots, v''_{n-1} \rangle \in \mathbf{V}^n$ *such that for every* $\mathcal{X} \in \wp_{+\mathrm{fin}}(\mathcal{IV})$ *with* $\mathcal{X}_0 \subseteq \mathcal{X}$, *the following holds:*

$$\begin{aligned}
&\forall s \in \mathcal{L}', \forall \hat{s} \in \mathcal{L}_4[\emptyset], \forall (s', \sigma') \in \mathcal{L}_4[\emptyset] \times \Sigma[\\
&\quad (s \,\|\, F^c_{\mathcal{X}}(\langle \sigma_i \ \ \mathcal{X} \rangle_{i \in n}, \mathbf{v}', \mathbf{v}'', \hat{s}), [\sigma \ \ \mathcal{X}])\\
&\quad \xrightarrow{\langle a_0, c!v'_0, c!v''_0 \rangle \cdot \cdots \cdot \langle a_{n-1}, c!v'_{n-1}, c!v''_{n-1} \rangle}_{4*} (s', \sigma') \Rightarrow\\
&\quad \exists \langle s_0, \ldots, s_{n-1} \rangle \in (\mathcal{L}_4[\emptyset])^n[\\
&\qquad (s, [\sigma \ \ \mathcal{X}]) \xrightarrow{a_0}_4 (s_0, [\sigma_0 \ \ \mathcal{X}]) \xrightarrow{a_1}_4 \cdots \xrightarrow{a_{n-1}}_4 (s_{n-1}, [\sigma_{n-1} \ \ \mathcal{X}])\\
&\qquad \wedge s' \equiv s_{n-1} \,\|\, \hat{s} \,]\\
&\quad \wedge \sigma' = [\sigma_{n-1} \ \ \mathcal{X}] \,].
\end{aligned}$$

**(2)** *Let* $\mathbf{Chan}$ *be infinite. Then there exist* $\mathcal{X}_0 \in \wp_{+\mathrm{fin}}(\mathcal{IV})$ *and* $\vec{c} = \langle c'_0, \ldots, c'_{n-1} \rangle$, *such that for every* $\mathcal{X} \in \wp_{+\mathrm{fin}}(\mathcal{IV})$ *with* $\mathcal{X}_0 \subseteq \mathcal{X}$, *the following holds:*

$$\begin{aligned}
&\forall s \in \mathcal{L}', \forall \hat{s} \in \mathcal{L}_4[\emptyset], \forall (s', \sigma') \in \mathcal{L}_4[\emptyset] \times \Sigma[\\
&\quad (s \,\|\, \tilde{F}_{\mathcal{X}}(\langle \sigma_i \ \ \mathcal{X} \rangle_{i \in n}, \vec{c}, \hat{s}), [\sigma \ \ \mathcal{X}])\\
&\quad \xrightarrow{\langle a_0, c'_0!\bar{v} \rangle \cdot \cdots \cdot \langle a_{n-1}, c'_{n-1}!\bar{v} \rangle}_{4*} (s', \sigma') \Rightarrow\\
&\quad \exists \langle s_0, \ldots, s_{n-1} \rangle \in (\mathcal{L}_4[\emptyset])^n[\\
&\qquad (s, [\sigma \ \ \mathcal{X}]) \xrightarrow{a_0}_4 (s_0, [\sigma_0 \ \ \mathcal{X}]) \xrightarrow{a_1}_4 \cdots \xrightarrow{a_{n-1}}_4 (s_{n-1}, [\sigma_{n-1} \ \ \mathcal{X}])\\
&\qquad \wedge s' \equiv s_{n-1} \,\|\, \hat{s} \,]\\
&\quad \wedge \sigma' = [\sigma_{n-1} \ \ \mathcal{X}] \,]. \blacksquare
\end{aligned}$$

The proof of this lemma is given later. First Lemma 7.25 is derived from it.

**Proof of Lemma 7.25.** By using Lemma 7.26 (1), we will prove that (7.104) holds for every $s_0, s_1 \in \mathcal{L}_4[\emptyset]$, under the assumption that $\mathbf{V}$ *is infinite.* (By using Lemma 7.26 (2), the same conclusion can be obtained in a similar fashion under the assumption that **Chan** *is infinite.*)

Let $s_0, s_1 \in \mathcal{L}_4[\emptyset]$, and suppose that

$$\mathcal{O}_4[\![s_0]\!] \neq \mathcal{O}_4[\![s_1]\!].$$

Then there exists $\sigma \in \Sigma$ such that either

$$\exists \rho \in \mathbf{R}_4[\ \rho \in \mathcal{O}_4[\![s_0]\!](\sigma) \wedge \rho \notin \mathcal{O}_4[\![s_1]\!](\sigma)\ ], \tag{7.105}$$

or

$$\exists \rho \in \mathbf{R}_4[\ \rho \notin \mathcal{O}_4[\![s_0]\!](\sigma) \wedge \rho \in \mathcal{O}_4[\![s_1]\!](\sigma)\ ].$$

We consider the first case; the result can be obtained in the second case in the same fashion. The proof is given by distinguishing two cases according to whether $\rho$ is infinite or finite.

**Case 1.** Suppose $\rho$ is infinite. First, let us show by contradiction that there is $n \in \omega$ such that

$$\neg\exists\tilde{\rho} \in \mathbf{R}_4[\ (\rho \quad n) \cdot \tilde{\rho} \in \mathcal{O}_4[\![s_1]\!](\sigma)\ ]. \tag{7.106}$$

Assume that

$$\forall n \in \omega, \exists\tilde{\rho} \in \mathbf{R}_4[\ (\rho \quad n) \cdot \tilde{\rho} \in \mathcal{O}_4[\![s_1]\!](\sigma)\ ].$$

Then by the closedness of $\mathcal{O}_4[\![s_1]\!](\sigma)$, one has $\rho \in \mathcal{O}_4[\![s_1]\!](\sigma)$, which contradicts the fact that $\rho \notin \mathcal{O}_4[\![s_1]\!](\sigma)$. Thus there is $n \in \omega$ such that (7.106) holds. Fix such a number $n$, and let $\hat{\rho} = \rho \quad n$. Further, let $\langle(a_i, \sigma_i)\rangle_{i\in n} = \hat{\rho}$.

By (7.105),

$$\hat{\rho} \in \mathcal{O}_4[\![s_0]\!](\sigma) \quad n. \tag{7.107}$$

Also, by (7.106),

$$\hat{\rho} \notin \mathcal{O}_4[\![s_1]\!](\sigma) \quad n. \tag{7.108}$$

By Proposition 7.1, one has $(\mathrm{FIRN}_4')^{(n)}(\mathcal{O}_4[\![s_j]\!])$ $(j \in 2)$. Thus, there exists $\mathcal{X}_0 \in \wp_{+\mathrm{fin}}(\mathcal{IV})$ such that for every $\mathcal{X} \in \wp_{+\mathrm{fin}}(\mathcal{IV})$ with $\mathcal{X}_0 \subseteq \mathcal{X}$, the following holds:

$$\begin{aligned}
&\forall j \in 2, \forall\sigma \in \Sigma, \forall\langle(a_i, \sigma_i)\rangle_{i\in n} \in (\mathbf{A}_4 \times \Sigma)^n[\\
&\quad \langle(a_i, \sigma_i)\rangle_{i\in n} \in \mathcal{O}_4[\![s_j]\!](\sigma) \quad n\\
&\quad \Leftrightarrow \langle(a_i, [\sigma_i \quad \mathcal{X}])\rangle_{i\in n} \in \mathcal{O}_4[\![s_j]\!]([\sigma \quad \mathcal{X}] \quad n\ ]].
\end{aligned} \tag{7.109}$$

Fix such a set $\mathcal{X}_0$. Then, from (7.107) and (7.108), it follows that

$$\forall\mathcal{X} \in \wp_{+\mathrm{fin}}[\ \mathcal{X}_0 \subseteq \mathcal{X} \Rightarrow \langle(a_i, [\sigma_i \quad \mathcal{X}])\rangle_{i\in n} \in \mathcal{O}_4[\![s_0]\!]([\sigma \quad \mathcal{X}]) \quad n\ ]. \tag{7.110}$$

Also, from (7.108) and (7.109), it follows that

$$\forall\mathcal{X} \in \wp_{+\mathrm{fin}}[\ \mathcal{X}_0 \subseteq \mathcal{X} \Rightarrow \langle(a_i, [\sigma_i \quad \mathcal{X}])\rangle_{i\in n} \notin \mathcal{O}_4[\![s_1]\!]([\sigma \quad \mathcal{X}]) \quad n\ ]. \tag{7.111}$$

Fix $c \in \mathbf{Chan}$. Then, by Lemma 7.26, there exist

$$\mathbf{v}' = \langle v_0', \ldots, v_{n-1}'\rangle \in \mathbf{V}^n,\ \ \mathbf{v}'' = \langle v_0'', \ldots, v_{n-1}''\rangle \in \mathbf{V}^n,\ \ \mathcal{X}_1 \in \wp_{+\mathrm{fin}}(\mathcal{IV})$$

such that the following holds for every $\mathcal{X} \in \wp_{+\mathrm{fin}}(\mathcal{IV})$ with $\mathcal{X}_1 \subseteq \mathcal{X}$:

$$\begin{aligned}
&\forall(s', \sigma') \in \mathcal{L}_4[\emptyset] \times \Sigma[\\
&\quad (s_1 \,\|\, F_{\mathcal{X}}^{c}(\langle\sigma_i \quad \mathcal{X}\rangle_{i\in n}, \mathbf{v}', \mathbf{v}'', \mathbf{0}), [\sigma \quad \mathcal{X}])\\
&\quad \xrightarrow{\langle a_0, c!v_0', c!v_0'', \ldots, a_{n-1}, c!v_{n-1}', c!v_{n-1}''\rangle}_{4*} (s', \sigma')\\
&\quad \Rightarrow \exists s_0', \ldots, s_{n-1}' \in \mathcal{L}_4[\emptyset][\\
&\qquad\quad (s_1, [\sigma \quad \mathcal{X}]) \xrightarrow{a_0}_4 (s_0', [\mathbf{u}_0])\\
&\qquad\quad \xrightarrow{a_1}_4 \cdots \xrightarrow{a_{n-1}}_4 (s_{n-1}', [\sigma_{n-1} \quad \mathcal{X}])\\
&\qquad\quad \wedge s' \equiv s_{n-1}' \,\|\, \mathbf{0}\ ]\\
&\qquad \wedge \sigma' = [\sigma_{n-1} \quad \mathcal{X}]\ ].
\end{aligned} \tag{7.112}$$

Let $\mathcal{X} = \mathcal{X}_0 \cup \mathcal{X}_1$, $\mathbf{u} = \sigma \quad \mathcal{X}$, and for $i \in n$, let $\vec{\mathbf{u}}_i = \langle\sigma_i \quad \mathcal{X}, \ldots, \sigma_{n-1} \quad \mathcal{X}\rangle$.

First, let us show that

$$\exists\tilde{\eta} \in \mathbf{R}_4^*[\ \langle a_0, c!v_0', c!v_0'', \ldots, a_{n-1}, c!v_{n-1}', c!v_{n-1}''\rangle \cdot \tilde{\eta} \\ \in \mathcal{O}_4^*[\![s_0 \parallel F_{\mathcal{X}}^{c}(\vec{\mathbf{u}}_0, \mathbf{v}', \mathbf{v}'', \mathbf{0})]\!]([\mathbf{u}])\ ]. \tag{7.113}$$

By (7.110) and Lemma 7.14, one has

$$\exists s_0', \ldots, s_{n-1}' \in \mathcal{L}_4[\emptyset][ \\ (s_0, [\mathbf{u}]) \xrightarrow{a_0}_4 (s_0', [\sigma_0 \quad \mathcal{X}]) \xrightarrow{a_1}_4 \cdots \xrightarrow{a_{n-1}}_4 (s_{n-1}', [\sigma_{n-1} \quad \mathcal{X}])\ ]. \tag{7.114}$$

For $i \in n$, let

$$\mathbf{v}_i' = \langle v_i', \ldots, v_{n-1}'\rangle, \ \ \mathbf{v}_i'' = \langle v_i'', \ldots, v_{n-1}''\rangle. \tag{7.115}$$

Then, by (7.114) and the definition of $F_{\mathcal{X}}^{c}(\vec{\mathbf{u}}_0, \mathbf{v}', \mathbf{v}'', \mathbf{0})$, one has

$$\begin{array}{l}(s_0 \parallel F_{\mathcal{X}}^{c}(\vec{\mathbf{u}}_0, \mathbf{v}_0', \mathbf{v}_0'', \mathbf{0}), [\mathbf{u}]) \\ \xrightarrow{a_0}_4 (s_0' \parallel F_{\mathcal{X}}^{c}(\vec{\mathbf{u}}_0, \mathbf{v}_0', \mathbf{v}_0'', \mathbf{0}), [\sigma_0 \quad \mathcal{X}]) \\ \xrightarrow{(c!v_0', c!v_0'')}_{4*} (s_1' \parallel F_{\mathcal{X}}^{c}(\vec{\mathbf{u}}_1, \mathbf{v}_1', \mathbf{v}_1'', \mathbf{0}), [\sigma_0 \quad \mathcal{X}]) \\ \vdots \\ \xrightarrow{a_{n-1}}_4 (s_{n-1}' \parallel F_{\mathcal{X}}^{c}(\vec{\mathbf{u}}_{n-1}, \mathbf{v}_{n-1}', \mathbf{v}_{n-1}'', \mathbf{0}), [\sigma_{n-1} \quad \mathcal{X}]) \\ \xrightarrow{(c!v_{n-1}', c!v_{n-1}'')}_{4*} (s_{n-1}' \parallel \mathbf{0}, [\sigma_{n-1} \quad \mathcal{X}]).\end{array}$$

By this and the definition of $\mathcal{O}_4^*$, one has (7.113) with $\tilde{\eta} \in \mathcal{O}_4^*[\![s_n' \parallel \mathbf{0}]\!]([\mathbf{u}_n])$.
Next let us show, by contradiction, that

$$\neg\exists\tilde{\eta} \in \mathbf{R}_4^*[\ \langle a_0, c!v_0', c!v_0'', \ldots, a_{n-1}, c!v_{n-1}', c!v_{n-1}''\rangle \cdot \tilde{\eta} \\ \in \mathcal{O}_4^*[\![s_1 \parallel F_{\mathcal{X}}^{c}(\vec{\mathbf{u}}_0, \mathbf{v}', \mathbf{v}'', \mathbf{0})]\!]([\mathbf{u}])\ ]. \tag{7.116}$$

Assume, to the contrary, that

$$\exists\tilde{\eta} \in \mathbf{R}_4^*[\ \langle a_0, c!v_0', c!v_0'', \ldots, a_{n-1}, c!v_{n-1}', c!v_{n-1}''\rangle \cdot \tilde{\eta} \\ \in \mathcal{O}_4^*[\![s_1 \parallel F_{\mathcal{X}}^{c}(\vec{\mathbf{u}}_0, \mathbf{v}', \mathbf{v}'', \mathbf{0})]\!]([\mathbf{u}])\ ].$$

Then by the definition of $\mathcal{O}_4^*$, there exists $(s', \sigma') \in \mathcal{L}_4[\emptyset] \times \Sigma$ such that

$$(s_1 \parallel F_{\mathcal{X}}^{c}(\vec{\mathbf{u}}_0, \mathbf{v}', \mathbf{v}'', \mathbf{0}), [\mathbf{u}]) \xrightarrow{\langle a_0, c!v_0', c!v_0'', \ldots, a_{n-1}, c!v_{n-1}', c!v_{n-1}''\rangle}_{4*} (s', \sigma').$$

By this and (7.112), there exist $s_0', \ldots, s_{n-1}' \in \mathcal{L}_4[\emptyset]$ such that

$$(s_1, [\mathbf{u}]) \xrightarrow{a_0}_4 (s_0', [\mathbf{u}_0]) \xrightarrow{a_1}_4 \cdots \xrightarrow{a_{n-1}}_4 (s_{n-1}', [\mathbf{u}_{n-1}]).$$

From this and Lemma 7.14, it follows that

$$\exists\tilde{\rho} \in \mathbf{R}_4[\ \langle (a_0, [\mathbf{u}_0]), \ldots, (a_{n-1}, [\mathbf{u}_{n-1}])\rangle \cdot \tilde{\rho} \in \mathcal{O}_4[\![s_1]\!]([\mathbf{u}])\ ],$$

which contradicts (7.111). Thus one has (7.116).
By (7.113) and (7.116), one has

$$\mathcal{O}_4^*[\![s_0 \parallel F_{\mathcal{X}}^{c}(\vec{\mathbf{u}}_0, \mathbf{v}', \mathbf{v}'', \mathbf{0})]\!]([\mathbf{u}]) \neq \mathcal{O}_4^*[\![s_1 \parallel F_{\mathcal{X}}^{c}(\vec{\mathbf{u}}_0, \mathbf{v}', \mathbf{v}'', \mathbf{0})]\!]([\mathbf{u}])$$

and therefore, one has the right-hand side of (7.104).

**Case 2.** Suppose $\rho$ is finite, and let

$$\rho = \langle (a_i, \sigma_i \rangle_{i \in n} \cdot \langle \chi \rangle.$$

By Proposition 7.1, one has $(\text{FIRT}'_4)^{(n)}(\mathcal{O}_4[\![s_j]\!])$ $(j \in 2)$. Thus, there exists exists $\mathcal{X}_2 \in \wp_{+\text{fin}}(\mathcal{IV})$ such that for every $\mathcal{X} \in \wp_{+\text{fin}}(\mathcal{IV})$ with $\mathcal{X}_2 \subseteq \mathcal{X}$, the following holds:

$$\begin{array}{l} \forall j \in 2, \forall \sigma \in \Sigma, \forall \langle (a_i, \sigma_i) \rangle_{i \in n} \in (\mathbf{A}_4 \times \Sigma)^n, \forall \chi \in \{\surd, \delta\}[ \\ \quad \langle (a_i, \sigma_i) \rangle_{i \in n} \cdot \langle \chi \rangle \in \mathcal{O}_4[\![s_j]\!](\sigma) \\ \quad \Leftrightarrow \langle (a_i, [\sigma_i \ \ \mathcal{X}]) \rangle_{i \in n} \cdot \langle \chi \rangle \in \mathcal{O}_4[\![s_j]\!]([\sigma \ \ \mathcal{X}]) \ ]. \end{array} \tag{7.117}$$

Fix such a set $\mathcal{X}_2$. Then, propositions (7.118) and (7.119) bellow follow from (7.105).

$$\begin{array}{l} \forall \mathcal{X} \in \wp_{+\text{fin}}(\mathcal{IV})[\ \mathcal{X}_2 \subseteq \mathcal{X} \Rightarrow \\ \quad \langle (a_i, [\sigma_i \ \ \mathcal{X}]) \rangle_{i \in n} \cdot \langle \chi \rangle \in \mathcal{O}_4[\![s_0]\!]([\sigma \ \ \mathcal{X}]) \ ]. \end{array} \tag{7.118}$$

$$\begin{array}{l} \forall \mathcal{X} \in \wp_{+\text{fin}}(\mathcal{IV})[\mathcal{X}_2 \subseteq \mathcal{X} \Rightarrow \\ \quad \langle (a_i, [\sigma_i \ \ \mathcal{X}]) \rangle_{i \in n} \cdot \langle \chi \rangle \notin \mathcal{O}_4[\![s_1]\!]([\sigma \ \ \mathcal{X}]) \ ]. \end{array} \tag{7.119}$$

We distinguish two subcases according to whether $\chi = \surd$ or $\chi = \delta$.

**Subcase 2.1.** Suppose that $\chi = \surd$. Then, proposition (7.120) below follows from the definition of $\mathcal{O}_4$ and (7.118).

$$\begin{array}{l} \forall \mathcal{X} \in \wp_{+\text{fin}}(\mathcal{IV})[\ \mathcal{X}_2 \subseteq \mathcal{X} \Rightarrow \\ \quad \exists s'_0, \ldots, s'_{n-1} \in \mathcal{L}_4[\emptyset][ \\ \quad (s_0, [\sigma \ \ \mathcal{X}]) \xrightarrow{a_0}_4 (s'_0, [\sigma_0 \ \ \mathcal{X}]) \\ \quad \xrightarrow{a_1}_4 \cdots \xrightarrow{a_{n-1}}_4 (s'_{n-1}, [\sigma_{n-1} \ \ \mathcal{X}]) \\ \quad \wedge \surd \in \text{act}_4(s'_{n-1}, [\sigma_{n-1} \ \ \mathcal{X}]) \ ]]. \end{array} \tag{7.120}$$

Also, proposition (7.121) below follows from the definition of $\mathcal{O}_4$ and (7.119).

$$\begin{array}{l} \forall \mathcal{X} \in \wp_{+\text{fin}}(\mathcal{IV})[\ \mathcal{X}_2 \subseteq \mathcal{X} \Rightarrow \\ \quad \forall s'_0, \ldots, s'_{n-1} \in \mathcal{L}_4[\emptyset][ \\ \quad (s_1, [\sigma \ \ \mathcal{X}]) \xrightarrow{a_0}_4 (s'_0, [\sigma_0 \ \ \mathcal{X}]) \\ \quad \xrightarrow{a_1}_4 \cdots \xrightarrow{a_{n-1}}_4 (s'_{n-1}, [\sigma_{n-1} \ \ \mathcal{X}]) \\ \quad \Rightarrow \surd \notin \text{act}_4(s'_{n-1}, [\sigma_{n-1} \ \ \mathcal{X}]) \ ]]. \end{array} \tag{7.121}$$

Fix $c \in$ **Chan**. Then, by Lemma 7.26, there exist

$$\mathbf{v}' = \langle v'_0, \ldots, v'_{n-1} \rangle \in \mathbf{V}^n, \ \ \mathbf{v}'' = \langle v''_0, \ldots, v''_{n-1} \rangle \in \mathbf{V}^n, \ \ \mathcal{X}_3 \in \wp_{+\text{fin}}(\mathcal{IV})$$

such that for every $\mathcal{X} \in \wp_{+\text{fin}}(\mathcal{IV})$ with $\mathcal{X}_3 \subseteq \mathcal{X}$, the following holds:

$$\begin{aligned}
&\forall (s', \sigma') \in \mathcal{L}_4[\emptyset] \times \Sigma[\\
&\quad (s_1 \parallel F^{\mathrm{c}}_{\mathcal{X}}(\langle \sigma_i\ \ \mathcal{X}\rangle_{i \in n}, \mathbf{v}', \mathbf{v}'', \mathbf{e}), [\sigma\ \ \mathcal{X}])\\
&\quad \xrightarrow{\langle a_0, c!v'_0, c!v''_0, \ldots, a_{n-1}, c!v'_{n-1}, c!v''_{n-1}\rangle}{}_{4*} (s', \sigma')\\
&\quad \Rightarrow \exists s'_0, \ldots, s'_{n-1} \in \mathcal{L}_4[\emptyset][\\
&\qquad (s_1, [\sigma\ \ \mathcal{X}]) \xrightarrow{a_0}{}_4 (s'_1, [\mathbf{u}_0])\\
&\qquad \xrightarrow{a_1}{}_4 \cdots \xrightarrow{a_{n-1}}{}_4 (s'_{n-1}, [\sigma_{n-1}\ \ \mathcal{X}])\\
&\qquad \wedge s' \equiv s'_{n-1} \parallel \mathbf{0}\ ]\\
&\quad \wedge \sigma' = [\sigma_{n-1}\ \ \mathcal{X}]\ ].
\end{aligned} \tag{7.122}$$

Let $\mathcal{X} = \mathcal{X}_2 \cup \mathcal{X}_3$, $\mathbf{u} = \sigma\ \ \mathcal{X}$, and for $i \in n$, let $\mathbf{u}_i = \sigma_i\ \ \mathcal{X}$. Further for $i \in n$, let $\mathbf{v}'_i$, $\mathbf{v}''_i$, $\vec{\mathbf{u}}_i$ be defined as in (7.115). Then, as in Case 1, one has

$$\begin{aligned}
&(s_0 \parallel F^{\mathrm{c}}_{\mathcal{X}}(\vec{\mathbf{u}}_0, \mathbf{v}'_0, \mathbf{v}''_0, \mathbf{e}), \mathbf{u})\\
&\xrightarrow{a_0}{}_4 (s'_0 \parallel F^{\mathrm{c}}_{\mathcal{X}}(\vec{\mathbf{u}}_0, \mathbf{v}'_0, \mathbf{v}''_0, \mathbf{e}), [\mathbf{u}_0])\\
&\xrightarrow{(c!v'_0, c!v''_0)}{}_{4*} (s'_0 \parallel F^{\mathrm{c}}_{\mathcal{X}}(\vec{\mathbf{u}}_1, \mathbf{v}'_1, \mathbf{v}''_1, \mathbf{e}), [\mathbf{u}_0])\\
&\ \vdots\\
&\xrightarrow{a_{n-1}}{}_4 (s'_{n-1} \parallel F^{\mathrm{c}}_{\mathcal{X}}(\vec{\mathbf{u}}_{n-1}, \mathbf{v}'_{n-1}, \mathbf{v}''_{n-1}, \mathbf{e}), [\mathbf{u}_{n-1}])\\
&\xrightarrow{(c!v'_{n-1}, c!v''_{n-1})}{}_{4*} (s'_{n-1} \parallel \mathbf{e}, [\mathbf{u}_{n-1}]).
\end{aligned}$$

Moreover $\surd \in \mathrm{act}_4(s'_{n-1} \parallel \mathbf{e}, [\mathbf{u}_{n-1}])$ by (7.120). Thus, by the definition of $\mathcal{O}^*_4$, one has

$$\begin{aligned}
&\langle a_0, c!v'_0, c!v''_0, \ldots, a_{n-1}, c!v'_{n-1}, c!v''_{n-1}, \surd\rangle\\
&\in \mathcal{O}^*_4[\![s_0 \parallel F^{\mathrm{c}}_{\mathcal{X}}(\vec{\mathbf{u}}, \mathbf{v}', \mathbf{v}'', \mathbf{e})]\!]([\mathbf{u}]).
\end{aligned} \tag{7.123}$$

Next let us show by contradiction that

$$\begin{aligned}
&\langle a_0, c!v'_0, c!v''_0, \ldots, a_{n-1}, c!v'_{n-1}, c!v''_{n-1}, \surd\rangle\\
&\notin \mathcal{O}^*_4[\![s_1 \parallel F^{\mathrm{c}}_{\mathcal{X}}(\vec{\mathbf{u}}, \mathbf{v}', \mathbf{v}'', \mathbf{e})]\!]([\mathbf{u}]).
\end{aligned} \tag{7.124}$$

Assume, to the contrary, that

$$\begin{aligned}
&\langle a_0, c!v'_0, c!v''_0, \ldots, a_{n-1}, c!v'_{n-1}, c!v''_{n-1}, \surd\rangle\\
&\in \mathcal{O}^*_4[\![s_1 \parallel F^{\mathrm{c}}_{\mathcal{X}}(\vec{\mathbf{u}}, \mathbf{v}', \mathbf{v}'', \mathbf{e})]\!]([\mathbf{u}]).
\end{aligned}$$

Then by the definition of $\mathcal{O}^*_4$, there exists $(s', \sigma') \in \mathcal{L}_4[\emptyset] \times \Sigma$ such that

$$\begin{cases}
\text{(i)} & (s_1 \parallel F^{\mathrm{c}}_{\mathcal{X}}(\vec{\mathbf{u}}, \mathbf{v}', \mathbf{v}'', \mathbf{e}), [\mathbf{u}])\\
& \quad \xrightarrow{\langle a_0, c!v'_0, c!v''_0, \ldots, a_{n-1}, c!v'_{n-1}, c!v''_{n-1}\rangle}{}_{4*} (s', \sigma'),\\
\text{(ii)} & \surd \in \mathrm{act}_4(s', \sigma').
\end{cases} \tag{7.125}$$

From (7.125) (i) and (7.122), it follows that

$$\begin{aligned}
&\exists s'_0, \ldots, s'_{n-1} \in \mathcal{L}_4[\emptyset][\\
&\quad (s_1, [\mathbf{u}]) \xrightarrow{a_0}{}_4 (s'_1, [\mathbf{u}_0]) \xrightarrow{a_1}{}_4 \cdots \xrightarrow{a_{n-1}}{}_4 (s'_{n-1}, [\mathbf{u}_{n-1}])\\
&\quad \wedge s' \equiv s'_{n-1} \parallel \mathbf{0}\ ].
\end{aligned}$$

From this and (7.121), it follows that

$$\surd \notin \mathrm{act}_4(s'_{n-1} \parallel \mathrm{e}, [\mathbf{u}_{n-1}]),$$

which contradicts (7.125) (ii). Thus one has (7.124).

By (7.123) and (7.124), one has the right-hand side of (7.104).

**Subcase 2.2.** Suppose that $\chi = \delta$. Then, it can be shown, as in Subcase 2.1, that there exist

$$\mathbf{v}' = \langle v'_0, \ldots, v'_{n-1} \rangle, \ \mathbf{v}'' = \langle v''_0, \ldots, v''_{n-1} \rangle \in \mathbf{V}^n, \ \mathcal{X} \in \wp_{+\mathrm{fin}}(\mathcal{IV})$$

such that propositions (7.126) and (7.127) below hold.

$$\begin{array}{l} \langle a_0, c!v'_0, c!v''_0, \ldots, a_{n-1}, c!v'_{n-1}, c!v''_{n-1}, \delta \rangle \\ \in \mathcal{O}^*_4[\![ s_0 \parallel F^{\mathrm{c}}_{\mathcal{X}}(\langle \sigma_i \quad \mathcal{X} \rangle_{i \in n}, \mathbf{v}', \mathbf{v}'', \mathrm{e}) ]\!]([\sigma \quad \mathcal{X}]). \end{array} \tag{7.126}$$

$$\begin{array}{l} \langle a_0, c'!v_0, c!v''_0, \ldots, a_{n-1}, c!v'_{n-1}, c!v''_{n-1}, \delta \rangle \\ \notin \mathcal{O}^*_4[\![ s_1 \parallel F^{\mathrm{c}}_{\mathcal{X}}(\langle \sigma_i \quad \mathcal{X} \rangle_{i \in n}, \mathbf{v}', \mathbf{v}'', \mathrm{e}) ]\!]([\sigma \quad \mathcal{X}]). \end{array} \tag{7.127}$$

Thus, one has the right-hand side of (7.104). ■

For the proof of Lemma 7.26, we need Proposition 7.2, Proposition 7.3, Lemma 7.27 below.

**Proposition 7.2** *For $n \geq 1$ and $s \in \mathcal{L}_4[\emptyset]$, there exists $\mathcal{X}_0 \in \wp_{\mathrm{fin}}(\mathcal{IV})$ such that for every $\mathcal{X} \in \wp_{\mathrm{fin}}(\mathcal{IV})$ with $\mathcal{X}_0 \subseteq \mathcal{X}$ the following holds:*

$$\begin{array}{l} \forall s_0, \ldots, s_{n-1} \in \mathcal{L}_4[\emptyset], \forall a_0, \ldots, a_{n-1} \in \mathbf{A}_4, \\ \forall \sigma, \sigma_0, \ldots, \sigma_{n-1} \in \Sigma, \forall \bar{\sigma} \in ((\mathcal{IV} \setminus \mathcal{X}) \to \mathbf{V})[ \\ \quad (s, \sigma) \xrightarrow{a_0}_4 (s_0, \sigma_0) \xrightarrow{a_1}_4 \cdots \xrightarrow{a_{n-1}}_4 (s_{n-1}, \sigma_{n-1}) \Leftrightarrow \\ \quad \forall i \in n[\ \sigma \quad (\mathcal{IV} \setminus \mathcal{X}) = \sigma_i \quad (\mathcal{IV} \setminus \mathcal{X})\ ] \\ \quad \wedge (s, (\sigma \quad \mathcal{X}) \cup \bar{\sigma}) \xrightarrow{a_0}_4 (s_0, (\sigma_0 \quad \mathcal{X}) \cup \bar{\sigma}) \\ \qquad \xrightarrow{a_1}_4 \cdots \xrightarrow{a_{n-1}}_4 (s_{n-1}, (\sigma_{n-1} \quad \mathcal{X}) \cup \bar{\sigma})\ ]. \blacksquare \end{array}$$

**Proof. Induction Base.** The claim for $n = 1$ can be shown by induction on the structure of $s \in \mathcal{L}_4[\emptyset]$, using the guardedness restriction.

**Induction Step.** Let $k \geq 1$. Then, we can show that if the claim holds for $n = k$, then it also holds for $n = k + 1$, using the result of induction base and the image finiteness of the transition system $\langle \xrightarrow{\alpha}_4 : \alpha \in \mathbf{A}^{\surd}_4 \rangle$. ■

**Proposition 7.3** *Let $s, s' \in \mathcal{L}_4[\emptyset]$, $\sigma, \sigma' \in \Sigma$, and $c \in$ **Chan**, and $v \in \mathbf{V}$. Then*

$$(s, \sigma) \xrightarrow{c?v}_4 (s', \sigma') \Rightarrow \exists x \in \mathcal{IV}[\ v = \sigma'(x)\ ]. \blacksquare$$

**Proof.** Immediate from the definition of $\xrightarrow{a}_4$ in Definition 7.18. ■

**Lemma 7.27** *Let $\mathcal{L}' \in \wp_{\mathrm{fin}}(\mathcal{L}_4[\emptyset])$, $a \in \mathbf{A}_4$, and $\sigma, \sigma_0 \in \Sigma$.*

**(1)** *Let* $\mathbf{V}$ *be infinite and* $c \in$ **Chan**. *Then there exist,* $\mathcal{X}_0 \in \wp_{+\text{fin}}(\mathcal{IV})$ *and* $v', v'' \in \mathbf{V}$ *such that for every* $\mathcal{X} \in \wp_{+\text{fin}}(\mathcal{IV})$ *with* $\mathcal{X}_0 \subseteq \mathcal{X}$, *the following holds:*

$$\begin{array}{l} \forall s \in \mathcal{L}', \forall \hat{s} \in \mathcal{L}_4[\emptyset], \forall (s', \sigma') \in \mathcal{L}_4[\emptyset] \times \Sigma[ \\ \quad (s \,\|\, G^{c}_{\mathcal{X}}(\sigma_0 \;\; \mathcal{X}, v', v'', \hat{s}), [\sigma \;\; \mathcal{X}]) \xrightarrow{\langle a, c!v', c!v'' \rangle}_{4*} (s', \sigma') \Rightarrow \\ \quad \exists s_0 \in \mathcal{L}_4[\emptyset][\ (s, [\sigma \;\; \mathcal{X}]) \xrightarrow{a}_4 (s_0, [\sigma_0 \;\; \mathcal{X}]) \wedge \ s' \equiv s_0 \,\|\, \hat{s}\ ] \\ \quad \wedge\, \sigma' = [\sigma_0 \;\; \mathcal{X}]\ ]. \end{array}$$

**(2)** *Let* **Chan** *be infinite. Then there exist,* $\mathcal{X}_0 \in \wp_{+\text{fin}}(\mathcal{IV})$ *and* $c' \in$ **Chan** *such that for every* $\mathcal{X} \in \wp_{+\text{fin}}(\mathcal{IV})$ *with* $\mathcal{X}_0 \subseteq \mathcal{X}$, *the following holds:*

$$\begin{array}{l} \forall s \in \mathcal{L}', \forall \hat{s} \in \mathcal{L}_4[\emptyset], \forall (s', \sigma') \in \mathcal{L}_4[\emptyset] \times \Sigma[ \\ \quad (s \,\|\, \tilde{G}_{\mathcal{X}}(\sigma_0 \;\; \mathcal{X}, c', \hat{s}), [\sigma \;\; \mathcal{X}]) \xrightarrow{\langle a, c'!\bar{v} \rangle}_{4*} (s', \sigma') \Rightarrow \\ \quad \exists s_0 \in \mathcal{L}_4[\emptyset][\ (s, [\sigma \;\; \mathcal{X}]) \xrightarrow{a}_4 (s_0, [\sigma_0 \;\; \mathcal{X}]) \wedge \ s' \equiv s_0 \,\|\, \hat{s}\ ] \\ \quad \wedge\, \sigma' = [\sigma_0 \;\; \mathcal{X}]\ ]. \blacksquare \end{array}$$

**Proof.** We will prove only part (1); part 2 can be proved more easily in a similar fashion.

Let $\mathcal{L}' \in \wp_{\text{fin}}(\mathcal{L}_4[\emptyset])$, $c \in$ **Chan**, $a \in \mathbf{A}_4$, $\sigma, \sigma_0 \in \Sigma$. Then, by Proposition 7.2, there exists $\mathcal{X}_0 \in \wp_{+\text{fin}}(\mathcal{IV})$ such that the following folds for every $n \in [1..3]$ and $\mathcal{X} \in \wp_{+\text{fin}}(\mathcal{IV})$ with $\mathcal{X}_0 \subseteq \mathcal{X}$:

$$\begin{array}{l} \forall s \in \mathcal{L}', \forall s_0, \ldots, s_{n-1} \in \mathcal{L}_4[\emptyset], \forall a_0, \ldots, a_{n-1} \in \mathbf{A}_4, \\ \forall \sigma, \sigma_0, \ldots, \sigma_{n-1} \in \Sigma, \forall \bar{\sigma} \in ((\mathcal{IV} \setminus \mathcal{X}) \to \mathbf{V})[ \\ \quad (s, \sigma) \xrightarrow{a_0}_4 (s_0, \sigma_0) \xrightarrow{a_1}_4 \cdots \xrightarrow{a_{n-1}}_4 (s_{n-1}, \sigma_{n-1}) \Leftrightarrow \\ \quad \forall i \in n[\ \sigma \;\; (\mathcal{IV} \setminus \mathcal{X}) = \sigma_i \;\; (\mathcal{IV} \setminus \mathcal{X})\ ] \\ \quad \wedge\, (s, (\sigma \;\; \mathcal{X}) \cup \bar{\sigma}) \xrightarrow{a_0}_4 (s_0, (\sigma_0 \;\; \mathcal{X}) \cup \bar{\sigma}) \\ \qquad \xrightarrow{a_1}_4 \cdots \xrightarrow{a_{n-1}}_4 (s_{n-1}, (\sigma_{n-1} \;\; \mathcal{X}) \cup \bar{\sigma})\ ]. \end{array} \tag{7.128}$$

From the above property for $n = 1$, it immediately follows that

$$\begin{array}{l} \forall s \in \mathcal{L}', \forall a \in \mathbf{A}_4, \forall \sigma, \sigma' \in \Sigma, \forall s' \in \mathcal{L}_4[\emptyset], \forall x \in \mathcal{IV}[ \\ \quad (s, \sigma) \xrightarrow{a}_4 (s', \sigma') \wedge \sigma(x) \neq \sigma'(x) \\ \quad \Rightarrow x \in \mathcal{X}\ ]. \end{array} \tag{7.129}$$

Fix such a set $\mathcal{X}_0$.

First, let us choose $v' \in \mathbf{V}$ satisfying the following:

$$\begin{array}{ll} \textbf{(i)} & a \neq c!v' \wedge v' \neq \bar{v}, \\ \textbf{(ii)} & v' \notin \{\sigma(x) : x \in \mathcal{X}_0\} \cup \{\sigma_0(x) : x \in \mathcal{X}_0\}, \\ \textbf{(iii)} & v' \notin \{v \in \mathbf{V} : \exists s \in \mathcal{L}', \exists s', s'' \in \mathcal{L}_4[\emptyset], \exists \sigma' \in \Sigma[ \\ & \qquad (s, \sigma) \xrightarrow{a}_4 (s', \sigma') \xrightarrow{c!v}_4 (s'', \sigma')\ ]\}. \end{array} \tag{7.130}$$

We remark that the right-hand side of (7.130) (iii) is finite, since the transition relation is image finite (cf. Lemma 7.13 (1)). Therefore there exists such an element $v' \in \mathbf{V}$.

Furthermore, let us choose $v'' \in \mathbf{V}$ satisfying the following:

$$\begin{array}{ll} \text{(i)} & v'' \notin \{v \in \mathbf{V} : \exists s \in \mathcal{L}', \exists s', s'', s''' \in \mathcal{L}_4[\emptyset], \exists \sigma' \in \Sigma[ \\ & \quad (s, \sigma) \xrightarrow{c?v'}_4 (s', \sigma') \\ & \quad \xrightarrow{c!v'}_4 (s'', \sigma') \xrightarrow{c!v}_4 (s''', \sigma') \,]\}, \\ \text{(ii)} & v'' \neq v', \\ \text{(iii)} & v'' \notin \{v \in \mathbf{V} : \exists s \in \mathcal{L}', \exists s', s'' \in \mathcal{L}_4[\emptyset], \exists \sigma' \in \Sigma[ \\ & \quad (s, \sigma) \xrightarrow{a}_4 (s', \sigma') \xrightarrow{c!v}_4 (s'', \sigma') \,]\}. \end{array} \tag{7.131}$$

We remark that the right-hand sides of (7.131) (i) and (iii) are finite, and therefore, there exists such an element $v'' \in \mathbf{V}$.

We will show that $\mathcal{X}_0$, and $v', v''$ have the desired property.

Let $\mathcal{X} \in \wp_{+\text{fin}}(\mathcal{IV})$ and suppose $\mathcal{X}_0 \subseteq \mathcal{X}$. Further let $s \in \mathcal{L}'$, $\hat{s} \in \mathcal{L}_4[\emptyset]$, $(s', \sigma') \in \mathcal{L}_4[\emptyset] \times \Sigma$, and suppose

$$(s \,\|\, G^{c}_{\mathcal{X}}(\sigma_0 \ \mathcal{X}, v', v'', \hat{s}), [\sigma \ \mathcal{X}]) \xrightarrow{\langle a, c!v', c!v'' \rangle}_{4*} (s', \sigma'). \tag{7.132}$$

Let us show that

$$\begin{array}{l} \exists s_1 \in \mathcal{L}_4[\emptyset][\, (s, [\sigma \ \mathcal{X}]) \xrightarrow{a}_4 (s_1, [\sigma_0 \ \mathcal{X}]) \wedge s' \equiv s_1 \,\|\, \hat{s} \,] \\ \wedge\, \sigma' = [\sigma_0 \ \mathcal{X}]. \end{array} \tag{7.133}$$

It follows from (7.128) and (7.130) (iii) that

$$\begin{array}{l} v' \notin \{v \in \mathbf{V} : \exists s \in \mathcal{L}', \exists s', s'' \in \mathcal{L}_4[\emptyset], \exists \sigma' \in \Sigma[ \\ \quad (s, [\sigma \ \mathcal{X}]) \xrightarrow{a}_4 (s', \sigma') \xrightarrow{c!v}_4 (s'', \sigma') \,]\}. \end{array} \tag{7.134}$$

Also, by (7.128) and (7.131) (i), one has

$$\begin{array}{l} v'' \notin \{v \in \mathbf{V} : \exists s \in \mathcal{L}', \exists s', s'', s''' \in \mathcal{L}_4[\emptyset], \exists \sigma' \in \Sigma[\ . \\ \quad (s, [\sigma \ \mathcal{X}]) \xrightarrow{c?v'}_4 (s', \sigma') \\ \quad \xrightarrow{c!v'}_4 (s'', \sigma') \xrightarrow{c!v}_4 (s''', \sigma') \,]\}. \end{array} \tag{7.135}$$

Likewise, by (7.128) and (7.131) (iii), one has

$$\begin{array}{l} v'' \notin \{v \in \mathbf{V} : \exists s \in \mathcal{L}', \exists s', s'' \in \mathcal{L}_4[\emptyset], \exists \sigma' \in \Sigma[ \\ \quad (s, [\sigma \ \mathcal{X}]) \xrightarrow{a}_4 (s', \sigma') \xrightarrow{c!v}_4 (s'', \sigma') \,]\}. \end{array} \tag{7.136}$$

First, we observe that the first action $a$ in (7.132) cannot stem from the right-hand side of $s \,\|\, G^{c}_{\mathcal{X}}((\sigma_0 \ \mathcal{X}), v', v'', \hat{s})$, by (7.130) (i).

Assume, for the sake of contradiction, that

*the first action $a$ in (7.132) is the synchronization of two actions stemming from the left- and right-hand sides of $s \,\|\, G^{c}_{\mathcal{X}}((\sigma_0 \ \mathcal{X}), v', v'', \hat{s})$.* (7.137)

Then, there exist $s_1' \in \mathcal{L}_4[\emptyset]$ and $x \in \mathcal{IV}$ such that

$$\begin{array}{ll} \text{(i)} & (s, [\sigma \ \mathcal{X}]) \xrightarrow{c?v'}_4 (s_1', [\sigma \ \mathcal{X}][v'/x]), \\ \text{(ii)} & (s_1' \,\|\, \hat{s}', [\sigma \ \mathcal{X}][v'/x]) \xrightarrow{\langle c!v', c!v'' \rangle}_{4*} (s', \sigma'), \end{array} \tag{7.138}$$

where

$$\hat{s}' \equiv \mathbf{if}(\mathcal{X} = (\sigma_0 \ \ \mathcal{X}),\ (c!\ v'');\hat{s},\ \mathbf{0}).$$

By (7.138) (i) and (7.128), one has

$$(s, [\sigma \ \ \mathcal{X}_0]) \xrightarrow{c?v'} (s_1', [[\sigma \ \ \mathcal{X}'] \ \ \mathcal{X}_0]).$$

We will show, by contradiction, that

$$x \in \mathcal{X}_0. \tag{7.139}$$

Assume, for the sake of contradiction, that (7.139) does not hold. Then

$$[[\sigma \ \ \mathcal{X}][v'/x] \ \ \mathcal{X}_0] = [\sigma \ \ \mathcal{X}_0].$$

Thus, by Proposition 7.3, one has either

$$\textbf{(i)}\ v' = \bar{v} \quad \text{or} \quad \textbf{(ii)}\ v' \in \{\sigma(x) : x \in \mathcal{X}_0\}. \tag{7.140}$$

However, both (7.140) (i) and (ii) are impossible by (7.130) (i) and (ii), respectively. Thus, (7.139) must hold.

The first action $c!v'$ in (7.138) (ii) cannot stem from the right-hand side of $s_1' \parallel \hat{s}'$ by (7.131) (ii). Hence it must stem from the left-hand side of $s_1' \parallel \hat{s}'$, and therefore, there exists $s_1''$ such that

$$\begin{array}{ll} \textbf{(i)} & (s_1', [\sigma \ \ \mathcal{X}][v'/x]) \xrightarrow{c!v'}_4 (s_1'', [\sigma \ \ \mathcal{X}][v'/x]), \\ \textbf{(ii)} & (s_1'' \parallel \hat{s}', [\sigma \ \ \mathcal{X}][v'/x]) \xrightarrow{c!v''}_4 (s', \sigma'). \end{array} \tag{7.141}$$

The action $c!v''$ in (7.141) (ii) cannot stem from the left-hand side of $s_1'' \parallel \hat{s}'$ by (7.135). Hence it must stem from the right-hand side $\hat{s}'$, and therefore, there exists $\hat{s}''$ such that

$$(\hat{s}', [(\sigma \ \ \mathcal{X})][v'/x]) \xrightarrow{c!v''}_4 (\hat{s}'', \sigma').$$

But this is impossible by (7.130) (ii), (7.139), and the form of $\hat{s}'$. Thus the assumption (7.137) must be false.

Summarizing, the first action $a$ in (7.132) must stem from the left-hand side of $s \parallel G^c_{\mathcal{X}}(\sigma_0, v', v'', \hat{s})$. Consequently there exist $s_1 \in \mathcal{L}_4[\emptyset]$ and $\sigma_1' \in \Sigma$ such that

$$\begin{array}{ll} \textbf{(i)} & (s, [(\sigma \ \ \mathcal{X})]) \xrightarrow{a}_4 (s_1, \sigma_1'), \\ \textbf{(ii)} & (s_1 \parallel G^c_{\mathcal{X}}(\sigma_1', v', v'', \hat{s}), \sigma_1') \xrightarrow{(c!v', c!v'')}_{4*} (s', \sigma'). \end{array} \tag{7.142}$$

We observe the first action of (7.142) (ii) cannot stem from the left-hand side of $s_1 \parallel G^c_{\mathcal{X}}(\sigma_1', v', v'', \hat{s})$ by (7.134). Thus it must stem from the right-hand side $G^c_{\mathcal{X}}(\sigma_1', v', v'', \hat{s})$, and therefore, one has

$$(s_1 \parallel \hat{s}', \sigma_1') \xrightarrow{c!v''}_4 (s', \sigma'). \tag{7.143}$$

Finally, we observe that the action $c!v''$ in (7.143) cannot stem from the left-hand side of $s_1 \parallel \hat{s}'$ by (7.136). Thus it must stem from the right-hand side $\hat{s}'$, and therefore,

$$\text{(i) } \sigma'_1 \;\; \mathcal{X} = \sigma_0 \;\; \mathcal{X}, \;\; \text{(ii) } \sigma' = \sigma'_1, \;\; \text{(iii) } s' \equiv s_1 \parallel \hat{s}. \tag{7.144}$$

Also by (7.129),

$$\forall x \in (\mathcal{IV} \setminus \mathcal{X})[\; \sigma'_1(x) = \bar{v} \;].$$

By this and (7.144) (i), $\sigma'_1 = [\sigma_0 \;\; \mathcal{X}]$, and therefore,

$$\sigma' = \sigma'_1 = [\sigma_0 \;\; \mathcal{X}]$$

By this and (7.144) (iii), one has (7.133). ■

Using Lemma 7.27, Lemma 7.26 is proved as follows:

**Proof of Lemma 7.26.** We will prove part (1) by using Lemma 7.27 (1). (Part (2) can be proved by using Lemma 7.27 (2) more easily in a similar fashion.)

Let $c \in \mathbf{Chan}$. Setting $\Phi(n)$ as in (7.145) below, let us prove that $\Phi(n)$ holds for every $n \geq 1$ by induction.

$$\begin{aligned}
&\Phi(n) \Leftrightarrow \\
&\forall \mathcal{L}' \in \wp_{\text{fin}}(\mathcal{L}), \forall \sigma \in \Sigma, \\
&\forall \langle \sigma_0, \ldots, \sigma_{n-1} \rangle \in \Sigma^n, \forall \langle a_0, \ldots, a_{n-1} \rangle \in (\mathbf{A}_4)^n, \\
&\exists \mathcal{X}_0 \in \wp_{+\text{fin}}(\mathcal{IV}), \\
&\exists \mathbf{v}' = \langle v'_0, \ldots, v'_{n-1} \rangle, \mathbf{v}'' = \langle v''_0, \ldots, v''_{n-1} \rangle \in \mathbf{V}^n[ \\
&\quad \forall \mathcal{X} \in \wp_{+\text{fin}}(\mathcal{IV})[\; \mathcal{X}_0 \subseteq \mathcal{X} \Rightarrow \\
&\qquad \forall s \in \mathcal{L}', \forall \hat{s} \in \mathcal{L}_4[\emptyset], \forall (s', \sigma') \in \mathcal{L}_4[\emptyset] \times \Sigma[ \\
&\qquad\quad (s \parallel F^c_{\mathcal{X}}(\langle \sigma_i \;\; \mathcal{X} \rangle_{i \in n}, \mathbf{v}', \mathbf{v}'', \hat{s}), [\sigma \;\; \mathcal{X}]) \\
&\qquad\quad \xrightarrow{\langle a_0, c!v'_0, c!v''_0, \ldots, a_{n-1}, c!v'_{n-1}, c!v''_{n-1} \rangle}_{4*} (s', \sigma') \\
&\qquad\quad \Rightarrow \exists \langle s_0, \ldots, s_{n-1} \rangle \in (\mathcal{L}_4[\emptyset])^n[ \\
&\qquad\qquad\quad (s, [\sigma \;\; \mathcal{X}]) \xrightarrow{a_0}_4 (s_0, [\sigma_0 \;\; \mathcal{X}]) \\
&\qquad\qquad\quad \xrightarrow{a_1}_4 \cdots \xrightarrow{a_{n-1}}_4 (s_{n-1}, [\sigma_{n-1} \;\; \mathcal{X}]) \\
&\qquad\qquad\quad \wedge\; s' \equiv s_{n-1} \parallel \hat{s} \;] \\
&\qquad\quad \wedge \sigma' = [\sigma_{n-1} \;\; \mathcal{X}] \;]]].
\end{aligned} \tag{7.145}$$

The proposition $\Phi(1)$ follows immediately from Lemma 7.27.

Fix $n \geq 1$, and suppose $\Phi(n)$ holds. Let us prove $\Phi(n+1)$.

Let $\mathcal{L}' \in \wp_{\text{fin}}(\mathcal{L}_4)$, $\sigma \in \Sigma$, $\vec{\sigma} = \langle \sigma_0, \ldots, \sigma_n \rangle \in \Sigma^{n+1}$, and $\langle a_0, \ldots, a_n \rangle \in \mathbf{A}_4^{n+1}$. By Lemma 7.27, there exist $\mathcal{X}_0 \in \wp_{+\text{fin}}(\mathcal{IV})$ and $v'_0, v''_0 \in \mathbf{V}$ such that

$$\begin{aligned}
&\forall \mathcal{X} \in \wp_{+\text{fin}}(\mathcal{IV})[\; \mathcal{X}_0 \subseteq \mathcal{X} \Rightarrow \\
&\quad \forall s \in \mathcal{L}', \forall T \in \mathcal{L}_4[\emptyset], \forall (s', \sigma') \in \mathcal{L}_4[\emptyset] \times \Sigma[ \\
&\qquad (s \parallel G^c_{\mathcal{X}}(\sigma_0 \;\; \mathcal{X}, v'_0, v''_0, T), [\sigma \;\; \mathcal{X}]) \xrightarrow{(a_0, c!v'_0, c!v''_0)}_{4*} (s', \sigma') \Rightarrow \\
&\qquad \exists s_0 \in \mathcal{L}_4[\emptyset][\; (s, [\sigma \;\; \mathcal{X}]) \xrightarrow{a_0}_4 (s_0, [\sigma_0 \;\; \mathcal{X}]) \wedge s' \equiv s_0 \parallel T \;] \\
&\qquad \wedge \sigma' = [\sigma_0 \;\; \mathcal{X}] \;]].
\end{aligned} \tag{7.146}$$

Also by Proposition 7.2, there exists $\mathcal{X}_1 \in \wp_{+\text{fin}}(\mathcal{IV})$ such that

$$\begin{array}{l} \forall \mathcal{X} \in \wp_{+\text{fin}}(\mathcal{IV})[\ \mathcal{X}_0 \subseteq \mathcal{X} \Rightarrow \\ \quad \forall s \in \mathcal{L}', \forall s_0 \in \mathcal{L}_4[\emptyset], \forall a_0 \in \mathbf{A}_4, \\ \quad \forall \sigma, \sigma_0 \in \Sigma, \forall \bar{\sigma} \in ((\mathcal{IV} \setminus \mathcal{X}) \to \mathbf{V})[ \\ \quad\quad (s, \sigma) \xrightarrow{a_0}_4 (s_0, \sigma_0) \Leftrightarrow \\ \quad\quad \sigma \;\; (\mathcal{IV} \setminus \mathcal{X}) = \sigma_0 \;\; (\mathcal{IV} \setminus \mathcal{X}) \\ \quad\quad \wedge (s, (\sigma \;\; \mathcal{X}) \cup \bar{\sigma}) \xrightarrow{a_0}_4 (s_0, (\sigma_0 \;\; \mathcal{X}) \cup \bar{\sigma})\ ]]. \end{array} \tag{7.147}$$

Let $\mathcal{X}_2 = \mathcal{X}_0 \cup \mathcal{X}_1$.

Further let

$$\mathcal{L}^{(0)} = \{s' \in \mathcal{L}_4[\emptyset] : \exists s \in \mathcal{L}'[\ (s, \sigma) \xrightarrow{a_0}_4 (s', \sigma_0)\ ]\},$$

Then, $\mathcal{L}^{(0)}$ is finite by Lemma 7.13. Thus by the induction hypothesis, there exist $\mathbf{v}_1' = \langle v_1', \ldots, v_n' \rangle$, $\mathbf{v}_1'' = \langle v_1'', \ldots, v_n'' \rangle \in \mathbf{V}^n$, $\mathcal{X}_3 \in \wp_{+\text{fin}}(\mathcal{IV})$ such that

$$\begin{array}{l} \forall \mathcal{X} \in \wp_{+\text{fin}}(\mathcal{IV})[\ \mathcal{X}_3 \subseteq \mathcal{X} \Rightarrow \\ \quad \forall s_0 \in \mathcal{L}^{(0)}, \forall (s', \sigma') \in \mathcal{L}_4[\emptyset] \times \Sigma[ \\ \quad\quad (s_0 \parallel F^{\mathrm{c}}_{\mathcal{X}}(\langle \sigma_{i+1} \rangle_{i \in n}, \mathbf{v}_1', \mathbf{v}_1'', \hat{s}), [\sigma_0 \;\; \mathcal{X}]) \\ \quad\quad \xrightarrow{\langle a_1, c!v_1', c!v_1'', \ldots, a_n, c!v_n', c!v_n'' \rangle}_{4*} (s', \sigma') \\ \quad\quad \Rightarrow \exists \langle s_1, \ldots, s_n \rangle \in (\mathcal{L}_4[\emptyset])^n[ \\ \quad\quad\quad (s_0, [\sigma_0 \;\; \mathcal{X}]) \xrightarrow{a_1}_4 (s_1, [\sigma_1 \;\; \mathcal{X}]) \\ \quad\quad\quad \xrightarrow{a_2}_4 \cdots \xrightarrow{a_n}_4 (s_n, [\sigma_n \;\; \mathcal{X}]) \\ \quad\quad\quad \wedge\ s' \equiv s_n \parallel \hat{s}\ ]\ \wedge\ \sigma' = [\sigma_n \;\; \mathcal{X}]\ ]]. \end{array} \tag{7.148}$$

Put $\mathbf{v}' = \langle v_0' \rangle \cdot \mathbf{v}_1'$, $\mathbf{v}'' = \langle v_0'' \rangle \cdot \mathbf{v}_1''$, and $\mathcal{X}_4 = \mathcal{X}_0 \cup \mathcal{X}_3$. It suffices to show

$$\begin{array}{l} \forall \mathcal{X} \in \wp_{+\text{fin}}(\mathcal{IV})[\ \mathcal{X}_4 \subseteq \mathcal{X} \Rightarrow \\ \quad \forall s \in \mathcal{L}', \forall \hat{s} \in \mathcal{L}_4[\emptyset], \forall (s', \sigma') \in \mathcal{L}_4[\emptyset] \times \Sigma[ \\ \quad\quad (s \parallel F^{\mathrm{c}}_{\mathcal{X}}(\langle \sigma_i \rangle_{i \in (n+1)}, \mathbf{v}', \mathbf{v}'', \hat{s}), [\sigma \;\; \mathcal{X}]) \\ \quad\quad \xrightarrow{\langle a_0, c!v_0', c!v_0'', \ldots, a_n, c!v_n', c!v_n'' \rangle}_4 (s', \sigma') \Rightarrow \\ \quad \exists \langle s_0, s_1, \ldots, s_n \rangle \in (\mathcal{L}_4[\emptyset])^{n+1}[ \\ \quad\quad (s, [\sigma \;\; \mathcal{X}]) \xrightarrow{a_0}_4 (s_0, [\sigma_0 \;\; \mathcal{X}]) \xrightarrow{a_1}_4 \cdots \xrightarrow{a_n}_4 (s_n, [\sigma_n \;\; \mathcal{X}]) \\ \quad\quad \wedge\ s' \equiv s_n \parallel \hat{s}\ ] \\ \quad \wedge\, \sigma' = [\sigma_n \;\; \mathcal{X}]\ ]]. \end{array} \tag{7.149}$$

Let $\mathcal{X} \in \wp_{+\text{fin}}(\mathcal{IV})$ with $\mathcal{X}_4 \subseteq \mathcal{X}$. Further let $s \in \mathcal{L}'$, $\hat{s} \in \mathcal{L}_4[\emptyset]$, $(s', \sigma') \in \mathcal{L}_4[\emptyset] \times \Sigma$, and suppose

$$(s \parallel F^{\mathrm{c}}_{\mathcal{X}}(\langle \sigma_i \rangle_{i \in (n+1)}, \mathbf{v}', \mathbf{v}'', \hat{s}), [\sigma \;\; \mathcal{X}]) \xrightarrow{\langle a_0, c!v_0', c!v_0'', \ldots, a_n, c!v_n', c!v_n'' \rangle}_{4*} (s', \sigma').$$

Since

$$\begin{array}{l} F^{\mathrm{c}}_{\mathcal{X}}(\langle \sigma_i \rangle_{i \in (n+1)}, \mathbf{v}', \mathbf{v}'', \hat{s}) \\ = G^{\mathrm{c}}_{\mathcal{X}}(\sigma_0 \;\; \mathcal{X}, v_0', v_0'', F^{\mathrm{c}}_{\mathcal{X}}(\langle \sigma_{i+1} \rangle_{i \in n}, \mathbf{v}_1', \mathbf{v}_1'', \hat{s})), \end{array}$$

by the definition of $F^{\mathrm{c}}_{\mathcal{X}}$ (cf. Definition 7.31) there exists $(s_0', \sigma_0') \in \mathcal{L}_4[\emptyset] \times \Sigma$ such that

$$\begin{aligned} &\textbf{(i)} \quad (s \,\|\, G^{c}_{\mathcal{X}}(\sigma_0 \quad \mathcal{X}, v_0', v_0'', F^{c}_{\mathcal{X}}(\langle \sigma_{i+1} \quad \mathcal{X}\rangle_{i\in n}, \mathbf{v}_1', \mathbf{v}_1'', \hat{s})), [\sigma \quad \mathcal{X}]) \\ &\qquad \xrightarrow{\langle a_0, c!v_0', c!v_0'' \rangle}_{4*} (s_0', \sigma_0'), \\ &\textbf{(ii)} \quad (s_0', \sigma_0') \xrightarrow{\langle a_1, c!v_1', c!v_1'', \ldots, a_n, c!v_n', c!v_n'' \rangle}_{4*} (s', \sigma'). \end{aligned} \tag{7.150}$$

By (7.150) (i) and (7.146), there exist $s_0 \in \mathcal{L}_4[\emptyset]$ such that

$$\textbf{(i)}\ (s, [\sigma \quad \mathcal{X}]) \xrightarrow{a_0}_4 (s_0, [\sigma_0 \quad \mathcal{X}]), \quad \textbf{(ii)}\ s_0' \equiv s_0 \,\|\, F^{c}_{\mathcal{X}}(\vec{\sigma}, \mathbf{v}', \mathbf{v}'', \hat{s}), \tag{7.151}$$

and moreover, $\sigma_0' = [\sigma_0 \quad \mathcal{X}]$. By (7.147), one has

$$\begin{aligned} \mathcal{L}^{(0)} &= \{ s' \in \mathcal{L}_4[\emptyset] : \exists s \in \mathcal{L}'[\ (s, \sigma) \xrightarrow{a_0}_4 (s', \sigma_0)\ ] \} \\ &= \{ s' \in \mathcal{L}_4[\emptyset] : \exists s \in \mathcal{L}'[\ (s, [\sigma \quad \mathcal{X}]) \xrightarrow{a_0}_4 (s', [\sigma_0 \quad \mathcal{X}])\ ] \}. \end{aligned}$$

Thus, by (7.151) (i) and the definition of $\mathcal{L}^{(0)}$, one has $s_0 \in \mathcal{L}^{(0)}$. Moreover, by (7.151) (ii) and (7.150) (ii), one has

$$(s_0 \,\|\, F^{c}_{\mathcal{X}}(\langle \sigma_{i+1} \rangle_{i\in n}, \mathbf{v}_1', \mathbf{v}_1'', \hat{s}), [\sigma_0 \quad \mathcal{X}]) \xrightarrow{\langle a_1, c!v_1', c!v_1'', \ldots, a_n, c!v_n', c!v_n'' \rangle}_{4*} (s', \sigma').$$

Thus by (7.148), there exist $\langle s_1, \ldots, s_n \rangle \in (\mathcal{L}_4[\emptyset])^n$ such that

$$(s_0, [\sigma_0 \quad \mathcal{X}]) \xrightarrow{a_1}_4 (s_1, [\sigma_1 \quad \mathcal{X}]) \xrightarrow{a_2}_4 \cdots \xrightarrow{a_n}_4 (s_n, [\sigma_n \quad \mathcal{X}]) \wedge s' \equiv s_n \,\|\, \hat{s},$$

and moreover, $\sigma' = [\sigma_n \quad \mathcal{X}]$. Thus one has (7.149). ■

### Full Abstractness of $\mathcal{M}_4$ w.r.t. $\mathcal{O}_4^*$

In § 7.3.4, the correctness of $\mathcal{M}_4$ w.r.t. $\mathcal{O}$ was established. By this and the results in § 7.3.6, the full abstractness of $\mathcal{M}$ w.r.t. $\mathcal{O}_4^*$ is established, as sketched in § 7.1.

From Lemma 7.21 and Lemma 7.24, the correctness of $\mathcal{M}_4$ w.r.t. $\mathcal{O}_4^*$ immediately follows:

**Lemma 7.28** (Correctness of $\mathcal{M}_4$ w.r.t. $\mathcal{O}_4^*$)

**(1)** $(\mathcal{A}_4^* \circ \mathcal{A}_4) \circ \mathcal{M}_4 = \mathcal{O}_4^*$.

**(2)** *The model $\mathcal{M}_4$ correct w.r.t. $\mathcal{O}_4^*$, i.e., the following holds for every $s_1, s_2 \in \mathcal{L}$:*

$$\begin{aligned} &\mathcal{M}_4[\![s_1]\!] = \mathcal{M}_4[\![s_2]\!] \Rightarrow \\ &\forall X \in \mathcal{X}_{\mathcal{P}}, \forall S \in \mathcal{L}_4[X][\ \mathcal{O}_4^*[\![S[s_1/X]]\!] = \mathcal{O}_4^*[\![S[s_2/X]]\!]\ ]. \end{aligned}$$ ■

From Theorem 7.2 and Lemma 7.25, the full abstractness of $\mathcal{M}_4$ w.r.t. $\mathcal{O}_4^*$ follows:

**Theorem 7.3** (Full Abstractness of $\mathcal{M}_4$ w.r.t. $\mathcal{O}_4^*$) *If either* **V** *or* **Chan** *is infinite, then for every $s_1, s_2 \in \mathcal{L}$, the following holds:*

$$\begin{aligned} &\mathcal{M}_4[\![s_1]\!] = \mathcal{M}_4[\![s_2]\!] \Leftrightarrow \\ &\forall X \in \mathcal{X}_{\mathcal{P}}, \forall S \in \mathcal{L}_4[X][\ \mathcal{O}_4^*[\![S[s_1/X]]\!] = \mathcal{O}_4^*[\![S[s_2/X]]\!]\ ]. \end{aligned} \tag{7.152}$$ ■

**Proof.** Let $s_1, s_2 \in \mathcal{L}$.

The $\Rightarrow$-part of (7.152) is the statement of Lemma 7.28 (2).

The $\Leftarrow$-part of (7.152) can be shown by applying Proposition 3.5 (2). Here we will show this directly using Theorem 7.2 and Lemma 7.25 for the sake of better understanding. It suffices to show the contrapositive:

$$\begin{array}{l} \mathcal{M}_4[\![s_1]\!] \neq \mathcal{M}_4[\![s_2]\!] \Rightarrow \\ \exists X \in \mathcal{X}_{\mathcal{P}}, \exists S \in \mathcal{L}_4[X][\ \mathcal{O}_4^*[\![S[s_1/X]]\!] \neq \mathcal{O}_4^*[\![S[s_2/X]]\!]\ ]. \end{array} \tag{7.153}$$

Suppose $\mathcal{M}_4[\![s_1]\!] \neq \mathcal{M}_4[\![s_2]\!]$. Then by Theorem 7.2, there is $T \in \mathcal{L}$ such that

$$\mathcal{O}_4[\![s_1 \,\|\, T]\!] \neq \mathcal{O}_4[\![s_2 \,\|\, T]\!].$$

By this and Lemma 7.25, there is $T^* \in \mathcal{L}$ such that

$$\mathcal{O}_4^*[\![(s_1 \,\|\, T) \,\|\, T^*]\!] \neq \mathcal{O}_4^*[\![(s_2 \,\|\, T) \,\|\, T^*]\!].$$

Thus one has the desired result (7.153). ■

### 7.3.7 Comparison of $\mathcal{M}_4$ and Roscoe's Model for Occam

Roscoe, in [Ros 84], constructed a denotational model for a large subset of *occam*. The language in [Ros 84] is similar to $\mathcal{L}_4$ in many respects. However there are several differences between the two: One major difference is that, unlike individual variables in $\mathcal{L}_4$, variables in occam (except read-only ones) are not shared by two or more parallel processes, and therefore, intermediate states of one process cannot directly affect another process. Thus, in [Ros 84], a denotational model $\mathcal{C}$ can be constructed (for the language) without taking account of intermediate states: The model $\mathcal{C}$ is constructed as a hybrid of the failures model for CSP (proposed in [BHR 85] and improved in [BR 84]), and the conventional model for sequential languages which defines the meaning of a program as a relation between initial and final states. We expect that a model for $\mathcal{L}_4$ can be constructed along the lines of $\mathcal{C}$, and it will be more abstract than $\mathcal{M}_4$ in nature. However it will not be compositional w.r.t. $\|$, since processes of $\mathcal{L}_4$ have shared variables.

## 7.4 Concluding Remarks

We conclude this chapter with some remarks about possible extensions of the reported results and related works. There are two directions for such extensions. One is to investigate fully abstract models for other languages, e.g., a nonuniform concurrent language with *process creation* and (a form of) *local variables* as the language $\mathcal{L}_3$ in [BR 91]. The other is to investigate fully abstract denotational models for the same language $\mathcal{L}_3$ (or $\mathcal{L}_4$) w.r.t. other operational models, which might be more abstract than the one treated in this chapter.

For instance, it might be possible to construct a fully abstract denotational model for an operational model $\mathcal{B}'$ for $\mathcal{L}_3$ which is defined slightly modifying $\mathcal{B}$ in § 7.2.6 as follows: For every statement $s$ and state $\sigma$,

$$\begin{aligned}\mathcal{B}'[\![s]\!](\sigma) = \{\sigma' : \exists s'[\ (s,\sigma)(\xrightarrow{\tau}_3)^*(s',\sigma') \xrightarrow{\checkmark}_3\ ]\} \\ \cup\{\bot : \exists\langle(s_n,\sigma_n)\rangle_{n\in\omega}[\ (s_0,\sigma_0) = (s,\sigma) \\ \wedge \forall n \in \omega[\ (s_n,\sigma_n) \to (s_{n+1},\sigma_{n+1})]]\}.\end{aligned}$$

It was shown in [AP 86] that there is no fully abstract denotational model w.r.t. $\mathcal{B}'$ if the language has *countable nondeterminism.* However it is still to be investigated whether there is a fully abstract denotational model w.r.t. $\mathcal{B}'$, since the language $\mathcal{L}_3$ does not have countable nondeterminism. It seems that $\mathcal{M}_3$ is not fully abstract w.r.t. $\mathcal{B}'$; at least, we cannot establish the full abstractness w.r.t. $\mathcal{B}'$ as we have done w.r.t. $\mathcal{O}_3$, since there are $s_1, s_2 \in \mathcal{L}_3[\emptyset]$ such that $\mathcal{M}_3[\![s_1]\!] \neq \mathcal{M}_3[\![s_2]\!]$ but $\forall T \in \mathcal{L}_3[\emptyset][\ \mathcal{B}'[\![s_1 \parallel T]\!] = \mathcal{B}'[\![s_2 \parallel T]\!]\ ]$. This is easily verified by putting $s_1 \equiv \mathbf{0}$ and $s_2 \equiv (x := x); \mathbf{0}$.

For $\mathcal{L}_4$, a language for communicating concurrent systems, there are several possible operational models besides $\mathcal{O}_4$ defined in § 7.3. There are several dimensions for classifying operational models for such a language; such a classification and comparative study of those models were presented in [Gla 90]. One of those dimensions is the dichotomy of *linear time* versus *branching time*: A model is called a *linear time model*, if it identifies processes differing only in the branching structure of their execution paths; otherwise it is called *branching time.* Another dimension is the dichotomy of *weak* versus *strong*: A model is called *weak*, if it identifies processes differing only in their internal or silent actions (denoted by $\tau$ in this thesis); otherwise it is called *strong.* Also, there are two kinds of languages, i.e., *uniform* languages and *nonuniform* languages. By combination of these criteria, one has eight types of operational models, and for each of them, one has the problem to construct a fully abstract denotational model, or to characterize somehow the fully abstract compositional model. The obtained results on these problems so far are summarized in Table 9.1 on page 354.

## 7.A Proof of Lemma 7.6

By Lemma 7.3 (2), it is sufficient to show

$$\begin{aligned}&\forall k \geq 1, \forall p_0, \ldots, p_{k-1}, p' \in \mathbf{P}_3[ \\ &\quad \tilde{\|}_3(\textstyle\bigcup_{i\in k}[p_i], p') = \bigcup_{i\in k}[\tilde{\|}_3(p_i, p')]\ ].\end{aligned} \tag{7.154}$$

Let

$$\mathbf{M}_3^{\mathrm{dis}} = ((\wp_+(k) \to \mathbf{P}_3) \times \mathbf{P}_3 \to \mathbf{P}_3).$$

Let $F, G \in \mathbf{M}_3^{\mathrm{dis}}$ be defined as follows: For $I \in \wp_+(k)$, $\langle p_i\rangle_{i\in I} \in (I \to \mathbf{P}_3)$, and $p' \in \mathbf{P}_3$,

$$\begin{aligned}F(\langle p_i\rangle_{i\in I}, p') &= \tilde{\|}_3(\textstyle\bigcup_{i\in I}[p_i], p'), \\ G(\langle p_i\rangle_{i\in I}, p') &= \textstyle\bigcup_{i\in I}[\tilde{\|}_3(p_i, p')].\end{aligned}$$

By the definition of $\tilde{\|}_3$, one has

$$F(\langle p_i\rangle_{i\in I}, p') = \tilde{\|\!\lfloor}_3(\bigcup_{i\in I}[p_i], p') \cup \tilde{\|\!\lfloor}_3(p', \bigcup_{i\in I}[p_i]) \cup \tilde{\|}_3^{\surd}(\bigcup_{i\in I}[p_i], p') \cup \tilde{\|}_3^{\delta}(\bigcup_{i\in I}[p_i], p'). \tag{7.155}$$

For $J \subseteq I$, let

$$A(J) = \{(\sigma, \sigma') : \forall i \in J[p_i[\langle(\sigma,\sigma')\rangle] \neq \emptyset] \wedge \forall i \in (I \setminus J)[p_i[\langle(\sigma,\sigma')\rangle] = \emptyset]\}.$$

Then

$$\begin{aligned}
&\tilde{\|\!\lfloor}_3(\bigcup_{i\in I}[p_i], p') \\
&= \bigcup\{\langle(\sigma,\sigma')\rangle \cdot \tilde{\|}_3(\bigcup_{i\in I}[p_i][\langle(\sigma,\sigma')\rangle], p') : \bigcup_{i\in I}[p_i][\langle(\sigma,\sigma')\rangle] \neq \emptyset\} \\
&= \bigcup_{J\in\wp_+(I)}[\bigcup_{(\sigma,\sigma')\in A(J)}[\langle(\sigma,\sigma')\rangle \cdot \tilde{\|}_3(\bigcup_{i\in I}[p_i][\langle(\sigma,\sigma')\rangle], p')]] \\
&= \bigcup_{J\in\wp_+(I)}[\bigcup_{(\sigma,\sigma')\in A(J)}[\langle(\sigma,\sigma')\rangle \cdot F(\langle p_i[\langle(\sigma,\sigma')\rangle]\rangle_{i\in J}, p')]].
\end{aligned}$$

Let $\mathcal{F} : \mathbf{M}_3^{\mathrm{dis}} \to \mathbf{M}_3^{\mathrm{dis}}$ be defined as follows: For $f \in \mathbf{M}_3^{\mathrm{dis}}$,

$$\begin{aligned}
&\mathcal{F}(f)(\langle p_i\rangle_{i\in I}, p') \\
&= \bigcup_{J\in\wp_+(I)}[\bigcup_{(\sigma,\sigma')\in A(J)}[\langle(\sigma,\sigma')\rangle \cdot f(\langle p_i[\langle(\sigma,\sigma')\rangle]\rangle_{i\in J}, p')]] \\
&\quad \cup \bigcup\{\langle(\sigma,\sigma')\rangle \cdot f(\langle p_i\rangle_{i\in I}, p'[\langle(\sigma,\sigma')\rangle]) : p'[\langle(\sigma,\sigma')\rangle] \neq \emptyset\} \\
&\quad \cup \tilde{\|}_3^{\surd}(\bigcup_{i\in I}[p_i], p') \cup \tilde{\|}_3^{\delta}(\bigcup_{i\in I}[p_i], p').
\end{aligned} \tag{7.156}$$

Then $\mathcal{F}$ is a contraction and $F = \mathcal{F}(F)$, i.e., $F = \mathrm{fix}(\mathcal{F})$.

Next, let us show that $G = \mathcal{F}(G)$.

$$\begin{aligned}
&G(\langle p_i\rangle_{i\in I}, p') \\
&= \bigcup_{i\in I}[\tilde{\|\!\lfloor}_3(p_i, p') \cup \tilde{\|\!\lfloor}_3(p', p_i) \cup \tilde{\|}_3^{\surd}(p_i, p') \cup \tilde{\|}_3^{\delta}(p_i, p')] \\
&= \bigcup_{i\in I}[\tilde{\|\!\lfloor}_3(p_i, p')] \cup \bigcup_{i\in I}[\tilde{\|\!\lfloor}_3(p', p_i)] \\
&\quad \cup \bigcup_{i\in I}[\tilde{\|}_3^{\surd}(p_i, p')] \cup \bigcup_{i\in I}[\tilde{\|}_3^{\delta}(p_i, p')] \\
&= \bigcup_{i\in I}[\tilde{\|\!\lfloor}_3(p_i, p')] \\
&\quad \cup \bigcup\{\langle(\sigma,\sigma')\rangle \cdot G(\langle p_i\rangle_{i\in I}, p'[\langle(\sigma,\sigma')\rangle]) : p'[\langle(\sigma,\sigma')\rangle] \neq \emptyset\} \\
&\quad \cup \tilde{\|}_3^{\surd}(\bigcup_{i\in I}[p_i], p') \cup \tilde{\|}_3^{\delta}(\bigcup_{i\in I}[p_i], p').
\end{aligned} \tag{7.157}$$

Thus it suffices to show that

$$\begin{aligned}
&\bigcup_{i\in I}[\tilde{\|\!\lfloor}_3(p_i, p')] \\
&= \bigcup_{J\in\wp_+(I)}[\bigcup_{(\sigma,\sigma')\in A(J)}[\langle(\sigma,\sigma')\rangle \cdot G(\langle p_i[\langle(\sigma,\sigma')\rangle]\rangle_{i\in J}, p')]].
\end{aligned} \tag{7.158}$$

For $H : I \times \wp_+(I) \to \mathbf{P}_3$, it immediately follows that

$$\begin{aligned}
&\bigcup_{i\in I}[\bigcup\{H(i, J) : J \in \wp_+(I) \wedge i \in J\}] \\
&= \bigcup_{J\in\wp_+(I)}[\bigcup_{i\in J}[H(i, J)]].
\end{aligned} \tag{7.159}$$

Hence

$$\begin{aligned}
&\bigcup_{i\in I}[\tilde{\|\!\lfloor}_3(p_i, p')] \\
&= \bigcup_{i\in I}[\bigcup\{\langle(\sigma,\sigma')\rangle \cdot \tilde{\|}_3(p_i[\langle(\sigma,\sigma')\rangle], p') : p_i[\langle(\sigma,\sigma')\rangle] \neq \emptyset\}]
\end{aligned}$$

$$= \bigcup_{i\in I}[\bigcup\{\bigcup_{(\sigma,\sigma')\in A(J)}[\langle(\sigma,\sigma')\rangle \cdot \tilde{\|}_3(p_i[\langle(\sigma,\sigma')\rangle],p')] : J \in \wp_+(I) \wedge i \in J\}]$$
$$= \bigcup_{J\in\wp_+(I)}[\bigcup_{i\in J}[\bigcup_{(\sigma,\sigma')\in A(J)}[\langle(\sigma,\sigma')\rangle \cdot \tilde{\|}_3(p_i[\langle(\sigma,\sigma')\rangle],p')]]]$$

(by (7.159) with

$$H(i,J) = \bigcup_{(\sigma,\sigma')\in A(J)}[\langle(\sigma,\sigma')\rangle \cdot \tilde{\|}_3(p_i[\langle(\sigma,\sigma')\rangle],p')]$$

)

$$= \bigcup_{J\in\wp_+(I)}[\bigcup_{(\sigma,\sigma')\in A(J)}[\bigcup_{i\in J}[\langle(\sigma,\sigma')\rangle \cdot \tilde{\|}_3(p_i[\langle(\sigma,\sigma')\rangle],p')]]]$$
$$= \bigcup_{J\in\wp_+(I)}[\bigcup_{(\sigma,\sigma')\in A(J)}[\langle(\sigma,\sigma')\rangle \cdot \bigcup_{i\in J}[\tilde{\|}_3(p_i[\langle(\sigma,\sigma')\rangle],p')]]]$$
$$= \bigcup_{J\in\wp_+(I)}[\bigcup_{(\sigma,\sigma')\in A(J)}[\langle(\sigma,\sigma')\rangle \cdot G(\langle p_i[\langle(\sigma,\sigma')\rangle]\rangle_{i\in J},p')]].$$

Thus, one has (7.158). Thus, by (7.157), one has

$$G(\langle p_i\rangle_{i\in I},p') = \mathcal{F}(G)(\langle p_i\rangle_{i\in I},p').$$

Hence $G = \mathrm{fix}(\mathcal{F}) = F$. ■

## 7.B Proof of Lemma 7.7

**(1)** Let $\mathcal{A}_3' : \mathbf{P}_3 \to (\Sigma \to \wp_{+\mathrm{cl}}(\mathbf{R}_3))$ be defined as follows: For $p \in \mathbf{P}_3$ and $\sigma \in \Sigma$,

$$\mathcal{A}_3'(p)(\sigma) = \{\mathrm{hist}_3(q) : q \in p \wedge \mathrm{istate}_3(q) = \sigma \wedge \mathrm{Exec}_3(q)\}. \tag{7.160}$$

The set $\mathcal{A}_3'(p)(\sigma)$ is closed, because $p$ is closed. Moreover $\mathcal{A}_3'(p)(\sigma)$ is nonempty, because $p$ satisfies the disabled-$\tau$ condition and is closed w.r.t. disabed actions. Thus $\mathcal{A}_3'(p)(\sigma) \in \wp_{+\mathrm{cl}}(\Sigma^{\leq\omega})$.

Let $\mathbf{M}_3^{\mathcal{A}}$ and $\Delta_3$ as in Definition 7.11. By Definition 7.11, it is sufficient to show

$$\mathcal{A}_3' = \Delta_3 \circ \mathcal{A}_3'. \tag{7.161}$$

Let $p \in \mathbf{P}_3$ and $\sigma \in \Sigma$. By the definition of $\Delta_3$, one has

$$\begin{aligned}&(\Delta_3 \circ \mathcal{A}_3')(p)(\sigma)\\ &= \bigcup\{\sigma' \cdot \{\mathrm{hist}_3(q') : q' \in p[\langle(\sigma,\sigma')\rangle] \cap \mathbf{E}_3\langle\sigma'\rangle\} : \\ &\qquad p[\langle(\sigma,\sigma')\rangle] \neq \emptyset\ \}\\ &\quad \cup \mathrm{if}(\langle\sigma\rangle \in p, \{\epsilon\}, \emptyset).\end{aligned} \tag{7.162}$$

First, by (7.160), (7.162), it is straightforward that

$$\epsilon \in \mathcal{A}_3'(p)(\sigma) \Leftrightarrow \epsilon \in (\Delta_3 \circ \mathcal{A}_3')(p)(\sigma).$$

Next, let us show, for $\sigma' \in \Sigma$ and $\rho' \in \mathbf{R}_3$, that

$$\sigma' \cdot \rho' \in \mathcal{A}'_3(p)(\sigma) \Leftrightarrow \sigma' \cdot \rho' \in (\Delta_3 \circ \mathcal{A}'_3)(p)(\sigma). \tag{7.163}$$

($\Rightarrow$) Suppose $\sigma' \cdot \rho' \in \mathcal{A}'_3(p)(\sigma)$. Then there exists $q \in p$ such that

$$\sigma' \cdot \rho' = \mathrm{hist}_3(q) \land \mathrm{istate}_3(q) = \sigma \land \mathrm{Exec}_3(q).$$

For such $q$, there exists $q'$ such that

$$q = (\sigma, \sigma') \cdot q' \land \rho' = \mathrm{hist}_3(q') \land \mathrm{istate}_3(q') = \sigma' \land \mathrm{Exec}_3(q').$$

For such $q'$, $q' \in p[\langle(\sigma, \sigma')\rangle]$, and therefore, $p[\langle(\sigma, \sigma')\rangle] \neq \emptyset$. Hence

$$\begin{aligned}&\sigma' \cdot \rho' \\ &\in \bigcup\{\sigma'' \cdot \{\mathrm{hist}_3(q'') : q'' \in p[\langle(\sigma, \sigma'')\rangle] \cap \mathbf{E}_3\langle\sigma''\rangle\} : \\ &\qquad p[\langle(\sigma, \sigma'')\rangle] \neq \emptyset\ \} \\ &\subseteq (\Delta_3 \circ \mathcal{A}'_3)(p)(\sigma).\end{aligned}$$

($\Leftarrow$) Suppose

$$\sigma' \cdot \rho' \in (\Delta_3 \circ \mathcal{A}'_3)(p)(\sigma).$$

Then

$$\begin{aligned}&\sigma' \cdot \rho' \\ &\in \bigcup\{\sigma'' \cdot \{\mathrm{hist}_3(q'') : q'' \in p[\langle(\sigma, \sigma'')\rangle] \cap \mathbf{E}_3\langle\sigma''\rangle\} : \\ &\qquad p[\langle(\sigma, \sigma'')\rangle] \neq \emptyset\ \}.\end{aligned}$$

Hence $p[\langle(\sigma, \sigma')\rangle] \neq \emptyset$ and there exists $q' \in p[\langle(\sigma, \sigma')\rangle]$ such that

$$\rho' = \mathrm{hist}_3(q') \land q' \in \mathbf{E}_3\langle\sigma'\rangle.$$

For such $q'$, let $q = \langle(\sigma, \sigma')\rangle \cdot q'$. Then

$$\begin{aligned}&q \in p \land q \in \mathbf{E}_3\langle\sigma\rangle \\ &\land \mathrm{hist}_3(q) = \sigma' \cdot \mathrm{hist}_3(q') = \sigma' \cdot \rho'.\end{aligned}$$

Hence

$$\begin{aligned}\sigma' \cdot \rho' &\in \{\mathrm{hist}_3(q) : q \in p \land \mathrm{istate}_3(q) = \sigma \land \mathrm{Exec}_3(q)\} \\ &= \mathcal{A}'_3(p).\end{aligned}$$

Thus one has (7.163).
**(2)** This part follows immediately from **(1)**. ■

## 7.C Proof of Lemma 7.10 with $\sharp(\mathcal{X}) \geq 2$

Roughly, the proof of Lemma 7.10 with $\sharp(\mathcal{X}) \geq 2$ is given by $n$-times iteration of the method introduced in Subsection 7.2.5 for the proof with $\sharp(\mathcal{X}) = 1$. Let $\mathcal{X} \in \wp_{+\mathrm{fin}}(\mathcal{IV})$, $p \in \mathbf{P}^*_3$, and $\sigma', \sigma'', \sigma_0 \in \Sigma_{\mathcal{X}}$. Further let $n = \sharp(\mathcal{X})$, and $\mathcal{X} = \{x_1, \ldots, x_n\}$ with $x_1 <_{\mathrm{iv}} \cdots <_{\mathrm{iv}} x_n$.

First, we need some preliminary definitions.

**Definition 7.32**

**(1)** Let $k \geq 1$ and $\mathbf{v} \in \mathbf{V}^{[1..k]}$. For $i \in [1..k]$, $\mathbf{v}(i)$ is the $i$-th component of $\mathbf{v}$. For $i \in [0..k]$, let

$$\mathbf{v}[i] = \mathbf{v} \quad {}_{[1..i]} = (\lambda j \in [1..i].\ \mathbf{v}(j)).$$

Note that $\mathbf{v}[0] = \epsilon$.

**(2)** For $kin[1..n]$ and $\mathbf{v} \in \mathbf{V}^{[1..k]}$,

$$[\mathbf{v}] = (\lambda x \in \mathcal{IV}.\ \text{if}(x \in \{x_i : i \in [1..k]\}, \mathbf{v}(i), \bar{v})). \blacksquare$$

**Definition 7.33** (Format for Testers with $\sharp(\mathcal{X}) \geq 2$) For $k \in [1..n]$, $\mathbf{v}, \mathbf{v}', \mathbf{v}_2 \in \mathbf{V}^{[1..k]}$, $v_1 \in \mathbf{V}$, $\mathbf{v}_3 \in \mathbf{V}^{[1..(k-1)]}$, and $T' \in \mathcal{L}_3[\emptyset]$, the *tester* $F_k^{(n)}(\mathbf{v}, \mathbf{v}', v_1, \mathbf{v}_2, \mathbf{v}_3, T')$ is defined by induction $k$ as follows:

**(1)** If $\mathbf{v}'(1) = \mathbf{v}(1)$, let

$$F_1^{(n)}(\mathbf{v}, \mathbf{v}', v_1, \mathbf{v}_2, \epsilon, T') \equiv (x_1 := v_1); T'.$$

Otherwise, let

$$\begin{aligned} F_1^{(n)}(\mathbf{v}, \mathbf{v}', v_1, \mathbf{v}_2, \epsilon, T') \equiv \mathbf{if}(&x_1 = \mathbf{v}(1), \\ &(x_1 := \mathbf{v}'(1)); (x_1 := v_1); T', \\ &(x_1 := \mathbf{v}_2(1)); \mathbf{0}\ ). \end{aligned}$$

**(2)** Let $k \in [1..(n-1)]$. If $\mathbf{v}'(k+1) = \mathbf{v}(k+1)$, let

$$\begin{aligned} &F_{k+1}^{(n)}(\mathbf{v}, \mathbf{v}', v_1, \mathbf{v}_2, \mathbf{v}_3, T') \\ &\equiv\ (x_k := \mathbf{v}_3(k)); \\ &\quad F_k^{(n)}(\mathbf{v}[\mathbf{v}_3(k)/k][k], \mathbf{v}'[k], v_1, \mathbf{v}_2[k], \mathbf{v}_3[k-1], T'). \end{aligned}$$

Otherwise, let

$$\begin{aligned} &F_{k+1}^{(n)}(\mathbf{v}, \mathbf{v}', v_1, \mathbf{v}_2, \mathbf{v}_3, T') \\ &\equiv \mathbf{if}(x_{k+1} = \mathbf{v}(k+1), \\ &\qquad (x_{k+1} := \mathbf{v}'(k+1)); (x_k := \mathbf{v}_3(k)); \\ &\qquad F_k^{(n)}(\mathbf{v}[\mathbf{v}_3(k)/k][k], \mathbf{v}'[k], v_1, \mathbf{v}_2[k], \mathbf{v}_3[k-1], T'), \\ &\qquad (x_{k+1} := \mathbf{v}_2(k+1)); \mathbf{0}\ ). \blacksquare \end{aligned}$$

**Definition 7.34** (Testing Sequence with $\sharp(\mathcal{X}) \geq 2$) Let $k \in [1..n]$, $\mathbf{v} \in \mathbf{V}^{[1..n]}$, $\mathbf{v}' \in \mathbf{V}^{[1..k]}$, and $\mathbf{v}_3 \in \mathbf{V}^{[1..(k-1)]}$. The *testing sequence* $r_k^{(n)}(\mathbf{v}, \mathbf{v}', \mathbf{v}_3)$ is defined by induction on $k$ as follows:

**(1)** If $\mathbf{v}(1) = \mathbf{v}'(1)$, let

$$r_1^{(n)}(\mathbf{v}, \mathbf{v}', \epsilon) = \epsilon.$$

Otherwise, let

$$r_1^{(n)}(\mathbf{v}, \mathbf{v}', \epsilon) = \langle([\mathbf{v}], [\mathbf{v}[\mathbf{v}'(1)/1]])\rangle.$$

**(2)** Let $k \in [1..(n-1)]$. If $\mathbf{v}'(k+1) = \mathbf{v}(k+1)$, let

$$\begin{aligned} & r_{k+1}^{(n)}(\mathbf{v}, \mathbf{v}', \mathbf{v}_3) \\ & = \langle([\mathbf{v}], [\mathbf{v}[\mathbf{v}_3(k)/k]])\rangle \cdot r_k^{(n)}(\mathbf{v}[\mathbf{v}_3(k)/k], \mathbf{v}'[k], \mathbf{v}_3[k-1]). \end{aligned}$$

Otherwise, let

$$\begin{aligned} & r_{k+1}^{(n)}(\mathbf{v}, \mathbf{v}', \mathbf{v}_3) \\ & = \langle([\mathbf{v}], [\mathbf{v}[\mathbf{v}'(k+1)/(k+1)]])\rangle \\ & \quad \cdot \langle([\mathbf{v}[\mathbf{v}'(k+1)/(k+1)]], [\mathbf{v}[\mathbf{v}'(k+1)/(k+1)][\mathbf{v}_3(k)/k]])\rangle \\ & \quad \cdot r_k^{(n)}(\mathbf{v}[\mathbf{v}'(k+1)/(k+1)][\mathbf{v}_3(k)/k], \mathbf{v}'[k], \mathbf{v}_3[k-1]). \blacksquare \end{aligned}$$

By means of the format for testers and the testing sequences, we can establish the following lemma, from which the Testing Lemma with $\sharp(\mathcal{X}) \geq 2$ follows immediately.

**Lemma 7.29** *Let* $\mathbf{V}$ *be infinite, and let* $p \in \mathbf{P}_3^*$, $\mathbf{v}', \mathbf{v}'', \mathbf{v}_0 \in \mathbf{V}^{[1..n]}$. *For every* $v_1 \in \mathbf{V}$, $\mathbf{v}_2 \in \mathbf{V}^{[1..n]}$, *and* $\mathbf{v}_3 \in \mathbf{V}^{[1..(n-1)]}$, *the following hold:*

**(1)** *The sequence*

$$r_n^{(n)}(\mathbf{v}_0, \mathbf{v}', \mathbf{v}_3) \cdot \langle([\mathbf{v}'], [\mathbf{v}'']), ([\mathbf{v}''], [\mathbf{v}''[v_1/1]])\rangle$$

*is executable and its initial state is* $[\mathbf{v}_0]$, *i.e.,*

$$r_n^{(n)}(\mathbf{v}_0, \mathbf{v}', \mathbf{v}_3) \cdot \langle([\mathbf{v}'], [\mathbf{v}'']), ([\mathbf{v}''], [\mathbf{v}''[v_1/1]])\rangle \in \mathbf{E}_3\langle[\mathbf{v}_0]\rangle$$

**(2)** *For every* $T' \in \mathcal{L}_3[\emptyset]$, *two conditions (7.164) and (7.164) below are satisfied:*

$$\mathcal{M}_3[\![F_n^{(n)}(\mathbf{v}_0, \mathbf{v}', v_1, \mathbf{v}_2, \mathbf{v}_3, T')]\!][r] = \mathcal{M}_3[\![T']\!], \tag{7.164}$$

*where* $r = r_n^{(n)}(\mathbf{v}_0, \mathbf{v}', \mathbf{v}_3) \cdot \langle([\mathbf{v}''], [\mathbf{v}''[v_1/1]])\rangle$;

$$\begin{aligned} & \forall q' \in \mathbf{Q}_3[\ p[\langle([\mathbf{v}'], [\mathbf{v}''])\rangle] \neq \emptyset \\ & \quad \wedge q' \in p[\langle([\mathbf{v}'], [\mathbf{v}''])\rangle] \,\tilde{\|}_3\, \mathcal{M}_3[\![T']\!] \Rightarrow \\ & \quad r_n^{(n)}(\mathbf{v}_0, \mathbf{v}', \mathbf{v}_3) \cdot \langle([\mathbf{v}'], [\mathbf{v}'']), ([\mathbf{v}''], [\mathbf{v}''[v_1/1]])\rangle \cdot q' \\ & \quad \in p \,\tilde{\|}_3\, \mathcal{M}_3[\![F_n^{(n)}(\mathbf{v}_0, \mathbf{v}', v_1, \mathbf{v}_2, \mathbf{v}_3, T')]\!]\ ]. \blacksquare \end{aligned} \tag{7.165}$$

**Proof.** Let $v_1 \in \mathbf{V}$, $\mathbf{v}_2 \in \mathbf{V}^{[1..n]}$, and $\mathbf{v}_3 \in \mathbf{V}^{[1..(n-1)]}$.
**(1)** It is shown immediately by induction on $k \in [1..n]$ that for every $k \in [1..n]$, the following holds:

$$\begin{aligned} & \forall \mathbf{v} \in \mathbf{V}^{[1..n]}[\ \forall i \in ([1..n] \setminus [1..k])[\ \mathbf{v}(i) = \mathbf{v}'(i)\ ] \Rightarrow \\ & \quad r_k^{(n)}(\mathbf{v}, \mathbf{v}'[k], \mathbf{v}_3[k-1]) \cdot \langle(\mathbf{v}', \mathbf{v}''), (\mathbf{v}'', \mathbf{v}''[v_1/1])\rangle \in \mathbf{E}_3\langle[\mathbf{v}]\rangle\ ]. \end{aligned}$$

Putting $k = n$ and $\mathbf{v} = \mathbf{v}_0$, one has the desired result.
**(2)** Fix $T' \in \mathcal{L}_3[\emptyset]$. Let us show (7.164) and (7.165).

First, we will show (7.164). It is shown immediately by induction on $k \in [1..n]$ that for $k \in [1..n]$ the following holds:

$$\forall \mathbf{v} \in \mathbf{V}^{[1..n]}[\ \mathcal{M}_3[\![F_k^{(n)}(\mathbf{v}[k], \mathbf{v}'[k], v_1, \mathbf{v}_2[k], \mathbf{v}_3[k-1], T')]\!][r] = \mathcal{M}_3[\![T']\!]\ ],$$

where

$$r = r_k^{(n)}(\mathbf{v}, \mathbf{v}'[k], \mathbf{v}_3[k-1]) \cdot \langle([\mathbf{v}''], [\mathbf{v}''[v_1/1]])\rangle.$$

Putting $k = n$ and $\mathbf{v} = \mathbf{v}_0$, one has the desired reslut.

Next, we will show (7.165). It is shown immediately by induction on $k \in [1..n]$ that for $k \in [1..n]$ the following holds:

$$\begin{aligned}
&\forall \mathbf{v} \in \mathbf{V}^{[1..n]}, \forall q' \in \mathbf{Q}_3[\\
&\quad p[\langle([\mathbf{v}'], [\mathbf{v}''])\rangle] \neq \emptyset \wedge q' \in p[\langle([\mathbf{v}'], [\mathbf{v}''])\rangle]\ \tilde{\|}_3\ \mathcal{M}_3[\![T']\!] \Rightarrow\\
&\quad r_k^{(n)}(\mathbf{v}, \mathbf{v}'[k], \mathbf{v}_3[k-1]) \cdot \langle([\mathbf{v}'], [\mathbf{v}'']), ([\mathbf{v}''], [\mathbf{v}''[v_1/1]])\rangle \cdot q'\\
&\quad \in p\ \tilde{\|}_3\ \mathcal{M}_3[\![F_k^{(n)}(\mathbf{v}[k], \mathbf{v}'[k], v_1, \mathbf{v}_2[k], \mathbf{v}_3[k-1], T')]\!]\ ].
\end{aligned}$$

Putting $k = n$ and $\mathbf{v} = \mathbf{v}_0$, one has the desired reslut. ∎

**Lemma 7.30** *Let* $\mathbf{V}$ *be infinite, and let* $p \in \mathbf{P}_3^*$, $\mathbf{v}', \mathbf{v}'', \mathbf{v} \in \mathbf{V}^{[1..n]}$. *There exist* $v_1 \in \mathbf{V}$, $\mathbf{v}_2 \in \mathbf{V}^{[1..n]}$, *and* $\mathbf{v}_3 \in \mathbf{V}^{[1..(n-1)]}$ *such that for every* $T' \in \mathcal{L}_3[\emptyset]$, *the following holds:*

$$\begin{aligned}
&\forall q' \in \mathbf{Q}_3[\\
&\quad r_n^{(n)}(\mathbf{v}, \mathbf{v}', \mathbf{v}_3) \cdot \langle([\mathbf{v}'], [\mathbf{v}'']), ([\mathbf{v}''], [\mathbf{v}''[v_1/1]])\rangle \cdot q'\\
&\quad \in p\ \tilde{\|}_3\ \mathcal{M}_3[\![F_n^{(n)}(\mathbf{v}, \mathbf{v}', v_1, \mathbf{v}_2, \mathbf{v}_3, T')]\!]\\
&\quad \Rightarrow p[\langle([\mathbf{v}'], [\mathbf{v}''])\rangle] \neq \emptyset \wedge q' \in p[\langle([\mathbf{v}'], [\mathbf{v}''])\rangle]\ \tilde{\|}_3\ \mathcal{M}_3[\![T']\!]\ ].\ ∎
\end{aligned}$$

**Proof.** Let us show by induction on $i \in [1..n]$ that for every $i \in [1..n]$ the following holds:

$$\begin{aligned}
&\forall \mathbf{v} \in \mathbf{V}^{[1..n]}, \exists v_1 \in \mathbf{V}, \exists \mathbf{v}_2 \in \mathbf{V}^i, \exists \mathbf{v}_3 \in \mathbf{V}^{[1..(i-1)]}, \forall q' \in \mathbf{Q}_3[\\
&\quad r_i^{(n)}(\mathbf{v}, \mathbf{v}'[i], \mathbf{v}_3) \cdot \langle([\mathbf{v}'], [\mathbf{v}'']), ([\mathbf{v}''], [\mathbf{v}''[v_1/1]])\rangle \cdot q'\\
&\quad \in p\tilde{\|}_3\mathcal{M}_3[\![F_i^{(n)}(\mathbf{v}[i], \mathbf{v}'[i], v_1, \mathbf{v}_2, \mathbf{v}_3, T')]\!]\\
&\quad \Rightarrow p[\langle([\mathbf{v}'], [\mathbf{v}''])\rangle] \neq \emptyset\\
&\qquad \wedge q' \in p[\langle([\mathbf{v}'], [\mathbf{v}''])\rangle]\ \tilde{\|}_3\ \mathcal{M}_3[\![T']\!]\ ].
\end{aligned} \tag{7.166}$$

**Induction Base:** We can show (7.166) with $i = 1$ in a similar fashion to the proof of Lemma 7.10 in Subsection 7.2.5. Let $\mathbf{v} \in \mathbf{V}^{[1..n]}$. We distinguish two cases according to whether $\mathbf{v}(1) = \mathbf{v}'(1)$.

**Case 1.** Suppose $\mathbf{v}(1) = \mathbf{v}'(1)$. Then, by the definitions of $r_1^{(n)}(\cdots)$ and $F_1^{(n)}(\cdots)$, it is sufficient to show that

$$\begin{aligned}
&\exists v_1 \in \mathbf{V}, \forall q' \in \mathbf{Q}_3[\\
&\quad \langle([\mathbf{v}'], [\mathbf{v}'']), ([\mathbf{v}''], [\mathbf{v}''[v_1/1]])\rangle \cdot q'\\
&\quad \in p\tilde{\|}_3\mathcal{M}_3[\![(x_1 := v_1); T']\!]\\
&\quad \Rightarrow p[\langle([\mathbf{v}'], [\mathbf{v}''])\rangle] \neq \emptyset \wedge q' \in p\ \tilde{\|}_3\ \mathcal{M}_3[\![T']\!]\ ].
\end{aligned} \tag{7.167}$$

We can establish this by choosing $v_1$ such that

$$\begin{cases} v_1 \neq \mathbf{v}''(1), \\ v_1 \notin \{v \in \mathbf{V} : \langle([\mathbf{v}'],[\mathbf{v}'']),([\mathbf{v}''],[\mathbf{v}''[v/1]])\rangle \in p_{[2]}\}. \end{cases}$$

**Case 2.** Suppose $\mathbf{v}(1) \neq \mathbf{v}'(1)$. Then, by the definitions of $r_1^{(n)}(\cdots)$ and $F_1^{(n)}(\cdots)$, it is sufficient to show that

$$\begin{array}{l} \exists v_1, v_2 \in \mathbf{V}[ \\ \quad \langle([\mathbf{v}],[\mathbf{v}[\mathbf{v}'(1)/1]]),([\mathbf{v}'],[\mathbf{v}'']),([\mathbf{v}''],[\mathbf{v}''[v_1/1]])\rangle \cdot q' \\ \quad \in p \,\tilde{\|}_3\, \mathcal{M}_3[\![T]\!] \\ \quad \Rightarrow p[\langle([\mathbf{v}'],[\mathbf{v}''])\rangle] \neq \emptyset \wedge q' \in p \,\tilde{\|}_3\, \mathcal{M}_3[\![T']\!] \,], \end{array} \tag{7.168}$$

where

$$\begin{array}{l} T \equiv \mathbf{if}(x_1 = \mathbf{v}(1), \\ \qquad (x_1 := \mathbf{v}'(1)); (x_1 := v_1); T', \\ \qquad (x_1 := v_2; \mathbf{0}). \end{array}$$

We can establish this by choosing $v_1$ and $v_2$ such that

$$\begin{cases} \textbf{(i)}\ v_1 \notin \{v \in \mathbf{V} : \\ \qquad \langle([\mathbf{v}],[\mathbf{v}[\mathbf{v}'(1)/1]]),([\mathbf{v}'],[\mathbf{v}'']),([\mathbf{v}''],[\mathbf{v}''[v/1]])\rangle \in p_{[3]}\}, \\ \textbf{(ii)}\ v_1 \neq \mathbf{v}'(1), \\ \textbf{(iii)}\ v_1 \neq \mathbf{v}''(1), \\ \textbf{(iv)}\ v_1 \notin \{v \in \mathbf{V} : \langle([\mathbf{v}'],[\mathbf{v}'']),([\mathbf{v}''],[\mathbf{v}''[v/1]])\rangle \in p_{[2]}\}, \end{cases}$$

and

$$\begin{cases} \textbf{(i)}\ v_2 \neq \mathbf{v}''(1), \\ \textbf{(ii)}\ v_2 \neq v_1. \end{cases}$$

**Induction Step:** Assume that (7.166) with $i = k$ holds for some $k \in [1..(n-1)]$. We will show that (7.166) with i $= k+1$ holds. Let $\mathbf{v} \in \mathbf{V}^{[1..n]}$. We distinguish two cases according to whether $\mathbf{v}'(k+1) = \mathbf{v}(k+1)$.

**Case 1.** Suppose $\mathbf{v}'(k+1) = \mathbf{v}(k+1)$. Choose $v_3 \in \mathbf{V}$ such that

$$v_3 \notin \{v \in \mathbf{V} : \langle([\mathbf{v}],[\mathbf{v}[v/k]])\rangle \in p_{[1]}\}, \tag{7.169}$$

and let $v_2$ be an arbitrary element of $\mathbf{V}$.

By the induction hypothesis, there exist $v_1 \in \mathbf{V}$, $\mathbf{v}_2 \in \mathbf{V}^{[1..k]}$, and $\mathbf{v}_3 \in \mathbf{V}^{[1..(k-1)]}$ such that

$$\begin{array}{l} \forall q' \in \mathbf{Q}_3[ \\ \quad r_k^{(n)}(\mathbf{v}[v_3/k],\mathbf{v}'[k],\mathbf{v}_3) \\ \quad \cdot\langle([\mathbf{v}'],[\mathbf{v}'']),([\mathbf{v}''],[\mathbf{v}''[v_1/1]])\rangle \cdot q' \\ \quad \in p \tilde{\|}_3 \mathcal{M}_3[\![F_k^{(n)}(\mathbf{v}[v_3/k][k],\mathbf{v}'[k],v_1,\mathbf{v}_2,\mathbf{v}_3,T')]\!] \\ \quad \Rightarrow p[\langle([\mathbf{v}'],[\mathbf{v}''])\rangle] \neq \emptyset \\ \qquad \wedge\, q' \in p[\langle([\mathbf{v}'],[\mathbf{v}''])\rangle] \,\tilde{\|}_3\, \mathcal{M}_3[\![T']\!] \,]. \end{array} \tag{7.170}$$

Put

$$\tilde{\mathbf{v}}_2 = \mathbf{v}_2 \cup \{(k+1, v_2)\}, \quad \tilde{\mathbf{v}}_3 = \mathbf{v}_3 \cup \{(k, v_3)\}.$$

We will show

$$\begin{aligned}
&\forall q' \in \mathbf{Q}_3[\\
&\quad r_{k+1}^{(n)}(\mathbf{v}, \mathbf{v}'[k+1], \tilde{\mathbf{v}}_3) \cdot \langle([\mathbf{v}'], [\mathbf{v}'']), ([\mathbf{v}''], [\mathbf{v}''[v_1/1]])\rangle \cdot q'\\
&\quad \in p \tilde{\|}_3 \mathcal{M}_3[\![F_{k+1}^{(n)}(\mathbf{v}[k+1], \mathbf{v}'[k+1], v_1, \tilde{\mathbf{v}}_2, \tilde{\mathbf{v}}_3, T')]\!]\\
&\quad \Rightarrow p[\langle([\mathbf{v}'], [\mathbf{v}''])\rangle] \neq \emptyset \wedge q' \in p[\langle([\mathbf{v}'], [\mathbf{v}''])\rangle] \tilde{\|}_3 \mathcal{M}_3[\![T']\!] \;].
\end{aligned} \tag{7.171}$$

Let $q' \in \mathbf{Q}_3$, and suppose

$$\begin{aligned}
&r_{k+1}^{(n)}(\mathbf{v}, \mathbf{v}'[k+1], \tilde{\mathbf{v}}_3) \cdot \langle([\mathbf{v}'], [\mathbf{v}'']), ([\mathbf{v}''], [\mathbf{v}''[v_1/1]])\rangle \cdot q'\\
&\in p \tilde{\|}_3 \mathcal{M}_3[\![F_{k+1}^{(n)}(\mathbf{v}[k+1], \mathbf{v}'[k+1], v_1, \tilde{\mathbf{v}}_2, \tilde{\mathbf{v}}_3, T')]\!].
\end{aligned}$$

Then, by (7.169), the first step of $r_{k+1}^{(n)}(\mathbf{v}, \mathbf{v}'[k+1], \tilde{\mathbf{v}}_3)$ cannot stem from $p$, and therefore, it must stem from $\mathcal{M}_3[\![F_{k+1}^{(n)}(\mathbf{v}[k+1], \mathbf{v}'[k+1], v_1, \tilde{\mathbf{v}}_2, \tilde{\mathbf{v}}_3, T')]\!]$. Thus one has

$$\begin{aligned}
&r_k^{(n)}(\mathbf{v}[v_3/k], \mathbf{v}'[k], \mathbf{v}_3) \cdot \langle([\mathbf{v}'], [\mathbf{v}'']), ([\mathbf{v}''], [\mathbf{v}''[v_1/1]])\rangle \cdot q'\\
&\quad \in p \tilde{\|}_3 \mathcal{M}_3[\![F_k^{(n)}(\mathbf{v}[v_3/k][k], \mathbf{v}'[k], v_1, \mathbf{v}_2, \mathbf{v}_3, T')]\!].
\end{aligned}$$

By this and (7.170), one has

$$p[\langle([\mathbf{v}'], [\mathbf{v}''])\rangle] \neq \emptyset \wedge q' \in p[\langle([\mathbf{v}'], [\mathbf{v}''])\rangle] \tilde{\|}_3 \mathcal{M}_3[\![T']\!].$$

Thus one has (7.171).

**Case 2.** Suppose $\mathbf{v}'(k+1) \neq \mathbf{v}(k+1)$. Choose $v_2 \in \mathbf{V}$ such that

$$v_2 \neq \mathbf{v}'(k+1). \tag{7.172}$$

Also choose $v_3 \in \mathbf{V}$ such that

$$\begin{cases}
(\mathbf{i})\ v_3 \notin \{v \in \mathbf{V} : \langle([\mathbf{v}], [\mathbf{v}[\mathbf{v}'(k+1)/(k+1)]]),\\
\qquad\qquad ([\mathbf{v}[\mathbf{v}'(k+1)/(k+1)]],\\
\qquad\qquad [\mathbf{v}[\mathbf{v}'(k+1)/(k+1)][v/k]])\rangle\\
\qquad\qquad \in p_{[2]}\},\\
(\mathbf{ii})\ v_3 \notin \{v \in \mathbf{V} : \langle([\mathbf{v}[\mathbf{v}'(k+1)/(k+1)]],\\
\qquad\qquad [\mathbf{v}[\mathbf{v}'(k+1)/(k+1)][v/k]])\rangle\\
\qquad\qquad \in p_{[1]}\}.
\end{cases} \tag{7.173}$$

By the induction hypothesis, there exist $v_1 \in \mathbf{V}$, $\mathbf{v}_2 \in \mathbf{V}^{[1..k]}$, and $\mathbf{v}_3 \in \mathbf{V}^{[1..(k-1)]}$ such that

$$\begin{aligned}
&\forall q' \in \mathbf{Q}_3[\\
&\quad r_k^{(n)}(\mathbf{v}[\mathbf{v}'(k+1)/(k+1)][v_3/k], \mathbf{v}'[k], \mathbf{v}_3)\\
&\quad \cdot\langle([\mathbf{v}'], [\mathbf{v}'']), ([\mathbf{v}''], [\mathbf{v}''[v_1/1]])\rangle \cdot q'\\
&\quad \in p \tilde{\|}_3 \mathcal{M}_3[\![F_k^{(n)}(\mathbf{v}[v_3/k][k], \mathbf{v}'[k], v_1, \mathbf{v}_2, \mathbf{v}_3, T')]\!]\\
&\quad \Rightarrow p[\langle([\mathbf{v}'], [\mathbf{v}''])\rangle] \neq \emptyset\\
&\qquad \wedge q' \in p[\langle([\mathbf{v}'], [\mathbf{v}''])\rangle] \tilde{\|}_3 \mathcal{M}_3[\![T']\!] \;].
\end{aligned} \tag{7.174}$$

Let $\tilde{\mathbf{v}}_3 = \mathbf{v}_3 \cup \{(k, v_3)\}$ and $\tilde{\mathbf{v}}_2 = \mathbf{v}_2 \cup \{(k+1, v_2)\}$. We will show

$$\begin{array}{l}\forall q' \in \mathbf{Q}_3[\\ \quad r^{(n)}_{k+1}(\mathbf{v}, \mathbf{v}'[k+1], \tilde{\mathbf{v}}_3) \cdot \langle([\mathbf{v}'], [\mathbf{v}'']), ([\mathbf{v}''], [\mathbf{v}''[v_1/1]])\rangle \cdot q' \\ \quad \in p\tilde{\|}_3 \mathcal{M}_3[\![F^{(n)}_{k+1}(\mathbf{v}[k+1], \mathbf{v}'[k+1], v_1, \tilde{\mathbf{v}}_2, \tilde{\mathbf{v}}_3, T')]\!] \\ \quad \Rightarrow p[\langle([\mathbf{v}'], [\mathbf{v}''])\rangle] \neq \emptyset \wedge q' \in p[\langle([\mathbf{v}'], [\mathbf{v}''])\rangle] \tilde{\|}_3 \mathcal{M}_3[\![T']\!] \,].\end{array} \tag{7.175}$$

Let $q' \in \mathbf{Q}_3$, and suppose

$$\begin{array}{l}r^{(n)}_{k+1}(\mathbf{v}, \mathbf{v}'[k+1], \tilde{\mathbf{v}}_3) \cdot \langle([\mathbf{v}'], [\mathbf{v}'']), ([\mathbf{v}''], [\mathbf{v}''[v_1/1]])\rangle \cdot q' \\ \in p\tilde{\|}_3 \mathcal{M}_3[\![F^{(n)}_{k+1}(\mathbf{v}[k+1], \mathbf{v}'[k+1], v_1, \tilde{\mathbf{v}}_2, \tilde{\mathbf{v}}_3, T')]\!].\end{array}$$

Let $t = \mathcal{M}_3[\![F^{(n)}_{k+1}(\mathbf{v}[k+1], \mathbf{v}'[k+1], v_1, \tilde{\mathbf{v}}_2, \tilde{\mathbf{v}}_3, T')]\!]$.

By (7.173) (i), it is impossible that the first and second steps of

$$r^{(n)}_{k+1}(\mathbf{v}, \mathbf{v}'[k+1], \tilde{\mathbf{v}}_3)$$

stem from $p$.

Also, by (7.172), it is impossible that the first two steps stem from $p$ and $t$, respectively.

Thus the first step must stem from $t$. Moreover, by (7.173) (ii), the second step cannot stem from $p$. Hence the first and second steps must stem from $t$. Thus one has

$$\begin{array}{l}r^{(n)}_{k}(\mathbf{v}[\mathbf{v}'(k+1)/(k+1)][v_3/k], \mathbf{v}'[k], \mathbf{v}_3) \\ \cdot\langle([\mathbf{v}'], [\mathbf{v}'']), ([\mathbf{v}''], [\mathbf{v}''[v_1/1]])\rangle \cdot q' \\ \in (p\tilde{\|}_3 \\ \mathcal{M}_3[\![F^{(n)}_{k}(\mathbf{v}[v_3/k][k], \mathbf{v}'[k], v_1, \mathbf{v}_2, \mathbf{v}_3, T')]\!]).\end{array}$$

By this and (7.174), one has

$$p[\langle([\mathbf{v}'], [\mathbf{v}''])\rangle] \neq \emptyset \wedge q' \in p[\langle([\mathbf{v}'], [\mathbf{v}''])\rangle] \tilde{\|}_3 \mathcal{M}_3[\![T']\!].$$

Thus one has (7.175). ■

**Proof of Lemma 7.10 with $\sharp(\mathcal{X}) \geq 2$.** Take

$$v_1 \in \mathbf{V}, \quad \mathbf{v}_2 \in \mathbf{V}^{[1..n]}, \quad \mathbf{v}_3 \in \mathbf{V}^{[1..(n-1)]}$$

as in Lemma 7.30, and let

$$r_1 = r^{(n)}_n(\mathbf{v}_0, \mathbf{v}', \mathbf{v}_3), \quad r_2 = \langle(\mathbf{v}'', \mathbf{v}''[v_1/1])\rangle.$$

Then by Lemmas 7.29 and 7.30, one has the desired result. ■

## 7.D Proof of Lemma 7.18

The proof is similar to that of Lemma 7.6. By Lemma 7.16, it is sufficient to show

$$\forall k \geq 1, \forall p_0, \ldots, p_{k-1}, p' \in \mathbf{P}_4^*[ \\ \tilde{\|}_4(\bigcup_{i\in k}[p_i], p') = \bigcup_{i\in k}[\tilde{\|}_4(p_i, p')]]. \tag{7.176}$$

Let

$$\mathbf{M}_4^{\text{dis}} = ((\wp_+(\bar{k}) \to \mathbf{P}_4^*) \times \mathbf{P}_4^* \to \mathbf{P}_4^*),$$

and let $F, G \in \mathbf{M}_4^{\text{dis}}$ be defined as follows: For $I \in \wp_+(k)$, $\langle p_i \rangle_{i\in I} \in (I \to \mathbf{P}_4^*)$, and $p' \in \mathbf{P}_4^*$,

$$F(\langle p_i \rangle_{i\in I}, p') = \tilde{\|}_4(\bigcup_{i\in I}[p_i], p'),$$

$$G(\langle p_i \rangle_{i\in I}, p') = \bigcup_{i\in I}[\tilde{\|}_4(p_i, p')].$$

By the definition of $\tilde{\|}_4$, one has

$$\begin{aligned} &F(\langle p_i \rangle_{i\in I}, p') \\ &= \tilde{\lfloor\!\lfloor}_4(\bigcup_{i\in I}[p_i], p') \cup \tilde{\lfloor\!\lfloor}_4(p', \bigcup_{i\in I}[p_i]) \\ &\quad \cup \tilde{\lfloor}_4(\bigcup_{i\in I}[p_i], p') \cup \tilde{\lfloor}_4(p', \bigcup_{i\in I}[p_i]) \\ &\quad \cup \tilde{\|}_4^{\checkmark}(\bigcup_{i\in I}[p_i], p'). \cup \tilde{\|}_4^{\delta}(\bigcup_{i\in I}[p_i], p'). \end{aligned} \tag{7.177}$$

For $J \subseteq I$, let

$$\begin{aligned} A(J) = \{(\sigma, a, \sigma') : &\forall i \in J[p_i[\langle(\sigma, a, \sigma')\rangle] \neq \emptyset] \\ &\wedge \forall i \in (I \setminus J)[p_i[\langle(\sigma, a, \sigma')\rangle] = \emptyset]\}. \end{aligned}$$

Then, as in Lemma 7.6, one has

$$\begin{aligned} &\tilde{\lfloor\!\lfloor}_4(\bigcup_{i\in I}[p_i], p') \\ &= \bigcup_{J\in\wp_+(I)}[\bigcup_{(\sigma,a,\sigma')\in A(J)}[\langle(\sigma, a, \sigma')\rangle \cdot F(\langle p_i[\langle(\sigma, a, \sigma')\rangle]\rangle_{i\in J}, p')]]. \end{aligned} \tag{7.178}$$

Let

$$\begin{aligned} C(J) = \{(\sigma, c, v, \sigma') : \quad &\forall i \in J[p_i[\langle(\sigma, c!v, \sigma)\rangle] \neq \emptyset] \\ &\wedge \forall i \in (I \setminus J)[p_i[\langle(\sigma, c!v, \sigma)\rangle] = \emptyset] \\ &\wedge p'[\langle(\sigma, c?v, \sigma')\rangle] \neq \emptyset \}. \end{aligned}$$

Then

$$\begin{aligned} &\tilde{\lfloor}_4(\bigcup_{i\in I}[p_i], p') \\ &= \bigcup\{\langle(\sigma, \tau, \sigma')\rangle \cdot \tilde{\|}_4(\bigcup_{i\in I}[p_i][\langle(\sigma, c!v, \sigma)\rangle], p'[\langle(\sigma, c?v, \sigma')\rangle]) : \\ &\qquad \bigcup_{i\in I}[p_i][\langle(\sigma, c!v, \sigma)\rangle] \neq \emptyset \wedge p'[\langle(\sigma, c?v, \sigma')\rangle] \neq \emptyset \} \\ &\qquad\quad \text{(Taking closure is omitted, since} \\ &\qquad\quad \text{ASFin}(\bigcup_{i\in I}[p_i]), \\ &\qquad\quad \text{OVFin}(\bigcup_{i\in I}[p_i]), \quad \text{and} \quad \text{ASFin}(p') \\ &\qquad\quad \text{by Lemma 7.17)} \\ &= \bigcup_{J\in\wp_+(I)}[\bigcup_{(\sigma,c,v,\sigma')\in C(J)}[ \\ &\qquad \langle(\sigma, \tau, \sigma')\rangle \cdot \tilde{\|}_4(\bigcup_{i\in I}[p_i][\langle(\sigma, c!v, \sigma)\rangle], p'[\langle(\sigma, c?v, \sigma')\rangle])]] \\ &= \bigcup_{J\in\wp_+(I)}[\bigcup_{(\sigma,c,v,\sigma')\in C(J)}[ \\ &\qquad \langle(\sigma, \tau, \sigma')\rangle \cdot F(\langle p_i[\langle(\sigma, c!v, \sigma)\rangle]\rangle_{i\in J}, p'[\langle(\sigma, c?v, \sigma')\rangle])]]. \end{aligned} \tag{7.179}$$

Let

$$C'(J) = \{(\sigma, c, v, \sigma') : \begin{array}{l} \forall i \in J[p_i[\langle(\sigma, c?v, \sigma')\rangle] \neq \emptyset] \\ \wedge \forall i \in (I \setminus J)[p_i[\langle(\sigma, c?v, \sigma')\rangle] = \emptyset] \\ \wedge p'[\langle(\sigma, c!v, \sigma)\rangle] \neq \emptyset \ \}. \end{array}$$

Then as (7.179), one has

$$\begin{array}{l} \tilde{\lfloor}_4(p', \bigcup_{i\in I}[p_i]) \\ = \bigcup_{J\in\wp_+(I)}[\bigcup_{(\sigma,c,v,\sigma')\in C'(J)}[ \\ \quad \langle(\sigma, \tau, \sigma')\rangle \cdot F(\langle p_i[\langle(\sigma, c?v, \sigma')\rangle]\rangle_{i\in J}, p'[\langle(\sigma, c!v, \sigma)\rangle])]]. \end{array} \tag{7.180}$$

Let $\mathcal{F} : \mathbf{M}_4^{\text{dis}} \to \mathbf{M}_4^{\text{dis}}$ be defined as follows: For $f \in \mathbf{M}_4^{\text{dis}}$,

$$\begin{array}{l} \mathcal{F}(f)(\langle p_i\rangle_{i\in I}, p') \\ = \ \bigcup_{J\in\wp_+(I)}[\bigcup_{(\sigma,a,\sigma')\in A(J)}[\langle(\sigma, a, \sigma')\rangle \cdot f(\langle p_i[\langle(\sigma, a, \sigma')\rangle]\rangle_{i\in J}, p')]] \\ \quad \cup \bigcup\{\langle(\sigma, a, \sigma')\rangle \cdot f(\langle p_i\rangle_{i\in I}, p'[\langle(\sigma, a, \sigma')\rangle]) : p'[\langle(\sigma, a, \sigma')\rangle] \neq \emptyset\} \\ \quad \cup \bigcup_{J\in\wp_+(I)}[\bigcup_{(\sigma,c,v,\sigma')\in C(J)}[ \\ \qquad \langle(\sigma, \tau, \sigma')\rangle \cdot f(\langle p_i[\langle(\sigma, c!v, \sigma)\rangle]\rangle_{i\in J}, p'[\langle(\sigma, c?v, \sigma')\rangle])]] \\ \quad \cup \bigcup_{J\in\wp_+(I)}[\bigcup_{(\sigma,c,v,\sigma')\in C'(J)}[ \\ \qquad \langle(\sigma, \tau, \sigma')\rangle \cdot f(\langle p_i[\langle(\sigma, c?v, \sigma')\rangle]\rangle_{i\in J}, p'[\langle(\sigma, c!v, \sigma)\rangle])]] \\ \quad \cup \tilde{\|}_4^{\checkmark}(\bigcup_{i\in I}[p_i], p') \cup \tilde{\|}_4^{\delta}(\bigcup_{i\in I}[p_i], p'). \end{array} \tag{7.181}$$

Then $\mathcal{F}$ is a contraction; by (7.178), (7.179), and (7.180), one has $F = \mathcal{F}(F)$, i.e., $F = \text{fix}(\mathcal{F})$.

Next, let us show that $G = \mathcal{F}(G)$.

$$\begin{array}{l} G(\langle p_i\rangle_{i\in I}, p') \\ = \bigcup_{i\in I}[\tilde{\lfloor\!\lfloor}_4(p_i, p') \cup \tilde{\lfloor\!\lfloor}_4(p', p_i) \cup \tilde{\lfloor}_4(p_i, p') \cup \tilde{\lfloor}_4(p', p_i) \\ \qquad \cup \tilde{\|}_4^{\checkmark}(p_i, p') \cup \tilde{\|}_4^{\delta}(p_i, p')] \\ = \bigcup_{i\in I}[\tilde{\lfloor\!\lfloor}_4(p_i, p')] \cup \bigcup_{i\in I}[\tilde{\lfloor\!\lfloor}_4(p', p_i)] \\ \quad \cup \bigcup_{i\in I}[\tilde{\lfloor}_4(p_i, p')] \cup \bigcup_{i\in I}[\tilde{\lfloor}_4(p', p_i)] \\ \quad \cup \bigcup_{i\in I}[\tilde{\|}_4^{\checkmark}(p_i, p')]. \cup \bigcup_{i\in I}[\tilde{\|}_4^{\delta}(p_i, p')]. \end{array} \tag{7.182}$$

As in the proof of Lemma 7.6, one has

$$\begin{array}{l} \bigcup_{i\in I}[\tilde{\lfloor\!\lfloor}_4(p_i, p')] \\ = \bigcup_{J\in\wp_+(I)}[\bigcup_{(\sigma,a,\sigma')\in A(J)}[\langle(\sigma, a, \sigma')\rangle \cdot G(\langle p_i[\langle(\sigma, a, \sigma')\rangle]\rangle_{i\in J}, p')]]. \end{array} \tag{7.183}$$

Moreover

$$\begin{array}{l} \bigcup_{i\in I}[\tilde{\lfloor}_4(p_i, p')] \\ = \bigcup_{i\in I}[\bigcup\{\langle(\sigma, \tau, \sigma')\rangle \cdot \tilde{\|}_4(p_i[\langle(\sigma, c!v, \sigma')\rangle], p'[\langle(\sigma, c?v, \sigma')\rangle]) : \\ \quad p_i[\langle(\sigma, c!v, \sigma')\rangle] \neq \emptyset \ \wedge \ p'[\langle(\sigma, c?v, \sigma')\rangle] \ \}] \\ = \bigcup_{i\in I}[\bigcup\{\bigcup_{(\sigma,c,v,\sigma')\in C(J)}[ \\ \qquad \langle(\sigma, \tau, \sigma')\rangle \cdot \tilde{\|}_4(p_i[\langle(\sigma, c!v, \sigma)\rangle], p'[\langle(\sigma, c?v, \sigma')\rangle])] : \\ \qquad J \in \wp_+(I) \ \wedge \ i \in J \ \}] \end{array}$$

$$= \bigcup_{J \in \wp_+(I)} [\bigcup_{i \in J} [\bigcup_{(\sigma,c,v,\sigma') \in C(J)} [\langle(\sigma,\tau,\sigma')\rangle \cdot \tilde{\|}_4 (p_i[\langle(\sigma, c!v, \sigma)\rangle], p'[\langle(\sigma, c?v, \sigma')\rangle])]]]$$
$$= \bigcup_{J \in \wp_+(I)} [\bigcup_{(\sigma,c,v,\sigma') \in C(J)} [\bigcup_{i \in J} [\langle(\sigma,\tau,\sigma')\rangle \cdot \tilde{\|}_4 (p_i[\langle(\sigma, c!v, \sigma)\rangle], p'[\langle(\sigma, c?v, \sigma')\rangle])]]]$$
$$= \bigcup_{J \in \wp_+(I)} [\bigcup_{(\sigma,c,v,\sigma') \in C(J)} [\langle(\sigma,\tau,\sigma')\rangle \cdot \bigcup_{i \in J} [\tilde{\|}_4 (p_i[\langle(\sigma, c!v, \sigma)\rangle], p'[\langle(\sigma, c?v, \sigma')\rangle])]]].$$

Thus

$$\begin{aligned} &\bigcup_{i \in I} [\tilde{\lfloor}_4 (p_i, p')] \\ &= \bigcup_{J \in \wp_+(I)} [\bigcup_{(\sigma,c,v,\sigma') \in C(J)} [ \\ &\quad \langle(\sigma,\tau,\sigma')\rangle \cdot G(\langle p_i[\langle(\sigma, c!v, \sigma)\rangle]\rangle_{i \in J}, p'[\langle(\sigma, c?v, \sigma')\rangle])]]. \end{aligned} \tag{7.184}$$

Likewise, one has

$$\begin{aligned} &\bigcup_{i \in I} [\tilde{\lfloor}_4 (p', p_i)] \\ &= \bigcup_{J \in \wp_+(I)} [\bigcup_{(\sigma,c,v,\sigma') \in C'(J)} [ \\ &\quad \langle(\sigma,\tau,\sigma')\rangle \cdot G(\langle p_i[\langle(\sigma, c?v, \sigma')\rangle]\rangle_{i \in J}, p'[\langle(\sigma, c!v, \sigma)\rangle])]]. \end{aligned} \tag{7.185}$$

By the definition of $\tilde{\|\!\lfloor}_4$, one has

$$\begin{aligned} &\bigcup_{i \in I} [\tilde{\|\!\lfloor}_4 (p', p_i)] \\ &= \bigcup_{i \in I} [\bigcup \{\langle(\sigma,a,\sigma')\rangle \cdot (p'[\langle(\sigma,a,\sigma')\rangle] \tilde{\|}_4 p_i) : p'[\langle(\sigma,a,\sigma')\rangle] \neq \emptyset\}] \\ &= \bigcup \{\langle(\sigma,a,\sigma')\rangle \cdot \bigcup_{i \in I} [p'[\langle(\sigma,a,\sigma')\rangle] \tilde{\|}_4 p_i] : p'[\langle(\sigma,a,\sigma')\rangle] \neq \emptyset\} \\ &= \bigcup \{\langle(\sigma,a,\sigma')\rangle \cdot G(\langle p_i\rangle_{i \in I}, p'[\langle(\sigma,a,\sigma')\rangle]) : p'[\langle(\sigma,a,\sigma')\rangle] \neq \emptyset\}. \end{aligned} \tag{7.186}$$

Also by the definitions of $\tilde{\|}_4^{\checkmark}$ and $\tilde{\|}_4^{\delta}$, one has

$$\begin{aligned} &\bigcup_{i \in I} [\tilde{\|}_4^{\checkmark} (p', p_i)] = \tilde{\|}_4^{\checkmark} (p', \bigcup_{i \in I} [p_i]), \\ &\bigcup_{i \in I} [\tilde{\|}_4^{\delta} (p', p_i)] = \tilde{\|}_4^{\delta} (p', \bigcup_{i \in I} [p_i]). \end{aligned} \tag{7.187}$$

Thus, by (7.182), (7.183), (7.184), (7.185), (7.186), and (7.187), one has

$$G(\langle p_i\rangle_{i \in I}, p') = \mathcal{F}(G)(\langle p_i\rangle_{i \in I}, p').$$

Hence $G = \text{fix}(\mathcal{F}) = F$. ■

# Chapter 8

# A Fully Abstract Model for a Nonuniform Concurrent Language with Parameterization and Locality

The full abstractness of a denotational model w.r.t. operational ones for a concurrent language $\mathcal{L}_5$ is investigated. The language is *nonuniform* in that the meaning of atomic statements generally depends on the current state; it has *parameterization* with *channel-* and *value-parameters* and *locality* in the form of *local variables* and *local channels*, in addition to more conventional constructs: *value assignments* to variables, *parallel composition* with CSP/CCS-like communication, *nondeterministic choice*, and *recursion*. First two operational models $\mathcal{O}_5$ and $\mathcal{O}_5^*$ for $\mathcal{L}_5$ are introduced in terms of a Plotkin-style transition system. Both models are *linear* in that they map each statement to the set of its possible execution paths of a certain kind; the second model $\mathcal{O}_5^*$ is more abstract than the first one in that $\mathcal{O}_5^*$ ignores states whereas $\mathcal{O}_5$ involves them. Then a denotational model $\mathcal{M}_5$ is defined compositionally using interpreted operations of the language, with meanings of recursive programs as fixed-points in an appropriate complete metric space. The *full abstractness* of $\mathcal{M}_5$ w.r.t. $\mathcal{O}_5$ and $\mathcal{O}_5^*$ is established. That is, it is shown for $\mathcal{O} = \mathcal{O}_5, \mathcal{O}_5^*$ that $\mathcal{M}_5$ is most abstract of those models $\mathcal{C}$ which are compositional and more distinctive than $\mathcal{O}$.

## 8.1 Introduction

We investigate full abstractness of a denotational model w.r.t. operational ones for a concurrent language $\mathcal{L}_5$. The language is *nonuniform* in that the meaning of atomic statements generally depends on the current state; it has *parameterization* with *channel-* and *value-parameters* and *locality* in the form of *local variables* and *local channels*, in addition to more conventional constructs *value assignments to variables*, *parallel composition with CSP/CCS-like communication*, *nondeterministic choice*, and *recursion*.

First, two operational models $\mathcal{O}_5$ and $\mathcal{O}_5^*$ for $\mathcal{L}_5$ are introduced in terms of a Plotkin-style transition system ([Plo 81]). Both models are *linear* in that they map each statement to the set of its possible execution paths of a certain kind; the second model $\mathcal{O}_5^*$ is more abstract than the first one in that $\mathcal{O}_5^*$ ignores states whereas $\mathcal{O}_5$ involves them. Then, a denotational model $\mathcal{M}_5$ is defined as an extension of the model $\mathcal{M}_4$ defined in Chapter 7 for a sublanguage $\mathcal{L}_4$ (of $\mathcal{L}_5$) without parameterization and locality.

In Chapter 7, we defined the model $\mathcal{M}_4$ compositionally using interpreted operations of the language, with meanings of recursive programs as fixed-points in an appropriate complete metric space, and established the full abstractness of $\mathcal{M}_4$ w.r.t. $\mathcal{O}_4$. (The work Chapter 7, in turn, was inspired by earlier ones [BKO 88] and [Rut 89] treating a similar full abstractness problem for a uniform concurrent language.) The present paper extends this result to the language $\mathcal{L}_5$ with parameterization and locality. Namely, the model $\mathcal{M}_5$ for $\mathcal{L}_5$ is defined as an extension of $\mathcal{M}_4$ so that the extended model $\mathcal{M}_5$ is still fully abstract w.r.t. $\mathcal{O}_5$. From this, the full abstractness of $\mathcal{M}_5$ w.r.t. $\mathcal{O}_5^*$ is also obtained.

The main body of the present paper consists of §§ 8.2 to 8.6.

In § 8.2 we introduce the language $\mathcal{L}_5$, which is the result of extending the language $\mathcal{L}_4$ treated in Chapter 7 so as to include parameterization and locality. A major motive for the extension is our desire to enhance the *expressive power* of $\mathcal{L}_4$ from the viewpoint of *software development*, rather than from that of computability, preserving the desirable properties of its semantics such as *full abstractness*. The new features are important from the viewpoint of *software development*: Parameterization is useful in describing *abstract* programs which can be instantiated with particular parameters suitable for a particular situation; locality is useful in *modular* development of software, where the interaction between two modules should be minimized.

There are two directions for extending a *pure language* like pure CCS so as to include value-passing: One is to define an *applicative language* based on the pure language introducing *parameterization*; the other is to define a *nonuniform language* introducing *individual variables* and *value assignment* to the variables. In much of the literature, only one of these directions is adopted; e.g., the former is adopted in [HI 90], and the latter in Chapter 7. From a practical point of view, however, both directions are useful; actually many practical languages, such as Ada, C, and Common Lisp, have both *individual variables* and *parameterization*. We design our language $\mathcal{L}_5$ so as to include both features. The language $\mathcal{L}_5$ is obtained by extending a base language, a subset of pure CCS, in the above two directions, in two stages: First, the base language is extended to a *nonuniform* language, which is the one treated in Chapter 7; then the resulting nonuniform language is extended to an *applicative nonuniform* language, which we name $\mathcal{L}_5$.

In § 8.3, three operational models $\mathcal{O}_5$, $\mathcal{O}_5^*$, and $\mathcal{C}_5$ for $\mathcal{L}_5$ are introduced in terms of a Plotkin-style transition system ([Plo 81]). The first two models $\mathcal{O}_5$ and $\mathcal{O}_5^*$ are *linear* in the sense explained above; the third model $\mathcal{C}_5$ is a variant of the *failures model* which was first introduced in [BHR 85]. The last model is shown, in the following sections, to be equivalent to the denotational model $\mathcal{M}_5$.

In § 8.4, the denotational model $\mathcal{M}_5$ is defined compositionally using semantic operations which are interpretations of the syntactic constructs of $\mathcal{L}_5$, with meanings of recursive programs as fixed-points in an appropriate complete metric space.

Then, in § 8.5, the semantic equivalence between $\mathcal{C}_5$ and $\mathcal{M}_5$ is established, by showing that $\mathcal{C}_5$ is compositional w.r.t. all semantic operations of $\mathcal{L}_5$. From a technical point of view, the *compositionality result* is the *key* of the present paper; from this the semantic equivalence immediately follows, and the full abstractness results in § 8.6 follows from this equivalence and the corresponding results in Chapter 7.

Finally in § 8.7, we give some remarks on related work and directions for future study.

The Appendices include two mathematical definitions and description examples in $\mathcal{L}_5$.

## 8.2 A Nonuniform Concurrent Language $\mathcal{L}_5$

In this section, a nonuniform concurrent language $\mathcal{L}_5$ is introduced. It is an extension of the language $\mathcal{L}_4$ treated in Chapter 7. The language $\mathcal{L}_4$ has value assignments to individual variables, parallel composition with CSP/CCS-like communication, nondeterministic choice, and recursion. In addition to these constructs, $\mathcal{L}_5$ has *parameterization* with *channel-* and *value-parameters*, and *locality* in the form of *local variables* and *local channels.*

As a preliminary to the definition of $\mathcal{L}_5$, *value-expressions* and *channel-expressions* are defined with several related notions by:

**Definition 8.1** (Value Expressions and Channel Expressions)

**(1)** First, the following sets are assumed to be given:

**(i)** An infinite set ($\mathbf{c} \in$) **Chan** of *channels*,

**(ii)** A set ($v \in$) $\mathbf{V}$ of values,

**(iii)** A set ($x \in$) $\mathcal{IV}$ of *individual variables.*

It is also assumed that $\mathbf{V}$ contains a distinguished element **nil** standing for the logical false value.[1] Further, we assume that **Chan** is partitioned

[1]It is possible to introduce *Boolean* expressions, instead of introducing the value **nil** to define conditional statements. The present approach of introducing **nil**, as in Lisp and the language C, is adopted for simplifying the semantic definitions in § 8.4.

into two disjoint sets $\mathbf{Chan}_0$ and $\mathbf{Chan}_1$, with $\mathbf{Chan}_1$ being infinite. (This assumption is made for convenience in semantic interpretation of local channels in §§ 8.3 and 8.4.) We fix an enumeration $\langle \mathbf{c}_n^* \rangle_{n \in \omega}$ of $\mathbf{Chan}_1$, with $\mathbf{Chan}_1 = \{\mathbf{c}_n^* : n \in \omega\}$ and $\mathbf{c}_i^* \neq \mathbf{c}_j^*$ for distinct indexes $i, j$.

**(2)** Let $(\xi \in)\ \mathcal{X}_\mathcal{V}$ and $(\eta \in)\ \mathcal{X}_\mathcal{C}$ be the set of *value variables* and the set of *channel variables*, respectively. For $i \geq 1$, let us use variables $\vec{\eta}^{\langle i \rangle}$ and $\vec{\xi}^{\langle i \rangle}$ ranging over $(i \rightarrowtail \mathcal{X}_\mathcal{C})$ and $(i \rightarrowtail \mathcal{X}_\mathcal{V})$, respectively.
Note that elements of $\mathcal{X}_\mathcal{V}$ have very different nature from that of *individual variables* $x \in \mathcal{IV}$: Individual variables are used to store values, whereas elments of $\mathcal{X}_\mathcal{V}$ are used as parameters, e.g., as $\xi$ in "$(\lambda\xi.\ E)$".

**(3)** We assume a signature

$$\mathbf{Sig}_\mathrm{b} = (\{\mathcal{V}, \mathcal{F}, \mathcal{C}, \mathcal{H}\}, \mathbf{Fun}_\mathrm{b}, \mathrm{type}_\mathrm{b})$$

in the sense of many-sorted algebra to be given, where $\mathcal{V}$, $\mathcal{F}$, $\mathcal{C}$, and $\mathcal{H}$ are sorts of *values*, *value-expressions*, *channels*, and *channels expressions*, respectively, $\mathbf{Fun}_\mathrm{b}$ is a set of function symbols, and $\mathrm{type}_\mathrm{b}(\cdot)$ is a function assigning a type $(\in \{\mathcal{V}, \mathcal{F}, \mathcal{C}, \mathcal{H}\}^{<\omega} \times \{\mathcal{F}, \mathcal{H}\})$ to each function symbol in $\mathbf{Fun}_\mathrm{b}$. The types $\mathcal{V}$, $\mathcal{F}$, $\mathcal{C}$, and $\mathcal{H}$ correspond to semantic domains $\mathbf{V}$, $\mathbf{F}$, $\mathbf{Chan}$, and $\mathbf{H}$, respectively, where where $\mathbf{H} = (\Sigma \to \mathbf{Chan})$, $\mathbf{F} = (\Sigma \to \mathbf{V})$. It is convenient, for later purposes, to postulate that $v \in \mathbf{Fun}_\mathrm{b}$, $x \in \mathbf{Fun}_\mathrm{b}$, $\mathbf{c} \in \mathbf{Fun}_\mathrm{b}$ with $\mathrm{type}_\mathrm{b}(v) = (\epsilon, \mathcal{V})$, $\mathrm{type}_\mathrm{b}(x) = (\epsilon, \mathcal{F})$, $\mathrm{type}_\mathrm{b}(v) = (\epsilon, \mathcal{C})$, for every $v \in \mathbf{V}$, $x \in \mathcal{IV}$, $\mathbf{c} \in \mathbf{Chan}$. Further, we postulate that there are two function symbols $\iota_\mathrm{v}, \iota_\mathrm{c} \in \mathbf{Fun}_\mathrm{b}$ with $\mathrm{type}_\mathrm{b}(\iota_\mathrm{v}) = (\langle \mathcal{V} \rangle, \mathcal{F})$ and $\mathrm{type}_\mathrm{b}(\iota_\mathrm{v}) = (\langle \mathcal{V} \rangle, \mathcal{F})$; the symbol $\iota_\mathrm{v}$ (resp. $\iota_\mathrm{c}$) converts a value (resp. a channel) to a corresponding value-expression (resp. a corresponding channel-expression). Moreover, we assume, for convenience in the definition of semantic models in §§ 8.3 and 8.4, that

$$\mathrm{type}_\mathrm{b}(F) \in \{\mathcal{F}\}^{<\omega} \times \{\mathcal{F}, \mathcal{H}\},$$

for every non-constant function symbol $F \in (\mathbf{Fun}_\mathrm{b} \setminus \{\iota_\mathrm{v}, \iota_\mathrm{c}\})$.
Let $(E \in)\ \tilde{\mathcal{E}}_\mathcal{V}$, $(\tilde{E} \in)\ \tilde{\mathcal{E}}_\mathcal{F}$, $(G \in)\ \tilde{\mathcal{E}}_\mathcal{C}$, $(\tilde{G} \in)\ \tilde{\mathcal{E}}_\mathcal{H}$ be the sets of terms generated by $\mathbf{Sig}_\mathrm{b}$, $\mathcal{X}_\mathcal{V}$, $\mathcal{X}_\mathcal{C}$ of sort $\mathcal{V}$, $\mathcal{F}$, $\mathcal{C}$, $\mathcal{H}$, respectively. Let

$$(e \in)\ \mathcal{E}_\mathcal{V} = \{E \in \tilde{\mathcal{E}}_\mathcal{V} : \mathrm{FV}(E) = \emptyset\},$$

where $\mathrm{FV}(E)$ is the set of elements of $\mathcal{X}_\mathcal{V} \cup \mathcal{X}_\mathcal{C}$ contained in $E$. Likewise, let

$$(\tilde{e} \in)\ \mathcal{E}_\mathcal{F} = \{\tilde{E} \in \tilde{\mathcal{E}}_\mathcal{F} : \mathrm{FV}(\tilde{E}) = \emptyset\},$$

$$(g \in)\ \mathcal{E}_\mathcal{C} = \{G \in \tilde{\mathcal{E}}_\mathcal{C} : \mathrm{FV}(G) = \emptyset\}.$$

$$(\tilde{g} \in)\ \mathcal{E}_\mathcal{H} = \{\tilde{G} \in \tilde{\mathcal{E}}_\mathcal{H} : \mathrm{FV}(\tilde{G}) = \emptyset\}.$$

We use variables $\vec{e}^{\,(i)}$ (resp. $\vec{g}^{\,(i)}$) ranging over $(\mathcal{E}_\mathcal{F})^i$ (resp. $(\mathcal{E}_\mathcal{H})^i$) for each $i \in \omega$. (One has $\mathbf{V} \subseteq \mathcal{E}_\mathcal{V}, \mathcal{IV} \subseteq \mathcal{E}_\mathcal{F}$, and $\mathbf{Chan} \subseteq \mathcal{E}_\mathcal{C}$, because $\mathbf{V}, \mathcal{IV}, \mathbf{Chan} \subseteq \mathbf{Fun}_\mathrm{b}$.)

**(4)** We define the set $\Sigma$ of *states* by:

$$(\sigma \in)\ \Sigma = (\mathcal{IV} \to \mathbf{V}).$$

For each $e \in \mathcal{E_V}$, $\tilde{e} \in \mathcal{E_F}$, $h \in \mathcal{E_V}$, $\tilde{g} \in \mathcal{E_H}$, their *evaluations* $[\![e]\!] \in \mathbf{V}$, $[\![\tilde{e}]\!] \in (\Sigma \to \mathbf{V})$, $[\![g]\!] \in (\Sigma \to \mathbf{Chan})$, $[\![\tilde{g}]\!] \in (\Sigma \to \mathbf{Chan})$, are assumed to be given. We assume for every $e \in \mathcal{E_V}$ and $h \in \mathcal{E_C}$ that

$$[\![\iota_v(e)]\!] = (\lambda\sigma \in \Sigma.\ [\![e]\!]), \quad [\![\iota_v(h)]\!] = (\lambda\sigma \in \Sigma.\ [\![h]\!]).$$

We assume for each function symbol $F \in \mathbf{Fun}_b$ with $\mathrm{type}_b(F) \in \{\mathcal{V}, \mathcal{F}, \mathcal{C}, \mathcal{H}\}^+ \times \{\mathcal{H}\}$ that $\mathrm{ran}([\![F(\cdots)]\!]) \subseteq \mathbf{Chan}_0$, for convenience in semantic interpretation of local channels. ∎

By means of the sets $\tilde{\mathcal{E}}_\mathcal{F}$ and $\tilde{\mathcal{E}}_\mathcal{H}$, the language $\mathcal{L}_5$ is defined by:

**Definition 8.2** (Language $\mathcal{L}_5$)

**(1)** First, let $(X^{(0,0)} \in)\ \mathcal{X}_\mathcal{P}^{(0,0)}$ be the set of statement variables. Also let $(X^{(i,j)} \in)\ \mathcal{X}_\mathcal{P}^{(i,j)}$ be the set of *parameterized statement variables* with $i$ channel-parameters and $j$ value-parameters, for $(i,j) \neq (0,0)$. In the sequel, we sometimes omit the superscript $^{(0,0)}$, and write $X$ (resp. $\mathcal{X}_\mathcal{P}$) for $X^{(0,0)}$ (resp. $\mathcal{X}_\mathcal{P}^{(0,0)}$) for brevity. Further, let

$$(Z \in)\ \mathcal{X}_\mathcal{P}^* = \bigcup\{\mathcal{X}_\mathcal{P}^{(i,j)} : (i,j) \in \omega^2\}.$$

**(2)** First, the language $(S \in)\ \widetilde{\mathcal{L}}_5$ *without guardedness condition* is defined simultaneously with the sets of *parameterized statements* $(S^{(i,j)} \in)\ \widetilde{\mathcal{L}}_5^{(i,j)}$ $((i,j) \neq (0,0))$ as follows:

**(i)**
$$\begin{aligned} S ::= {} & X \mid \mathbf{0} \mid \mathbf{e} \mid \mathbf{asg}(x,E) \mid \mathbf{out}(H,E) \mid \\ & \mathbf{in}(H,x) \mid \mathbf{in}'(H,S^{(0,1)}) \mid (S_1;S_2) \mid (S_1+S_2) \mid (S_1 \parallel S_2) \mid \\ & \partial_C(S) \mid \mathbf{if}(E,S_1,S_2) \mid \mathbf{let}(x,E,S) \mid \mathbf{LC}(S^{(1,0)}) \mid \\ & (\mu X.\ S) \mid S^{(i,j)}(\vec{H}^{(i)} \cdot \vec{E}^{(j)}), \end{aligned}$$

where $X$ and $x$ range over $\mathcal{X_P}$ and $\mathcal{IV}$, respectively; $H$ and $E$ range over $\tilde{\mathcal{E}}_\mathcal{H}$ and $\tilde{\mathcal{E}}_\mathcal{F}$, respectively; $\vec{H}^{(i)}$ and $\vec{E}^{(j)}$ range over $(\tilde{\mathcal{E}}_\mathcal{H})^i$ and $(\tilde{\mathcal{E}}_\mathcal{F})^j$, respectively; $C$ ranges over $\wp(\mathbf{Chan})$.

Each construct above has the following intuitive meaning: **(a)** The constants $\mathbf{0}$ and $\mathbf{e}$ represent *inaction* and (successful) *termination*, respectively; **(b)** $(\mathbf{asg}(x,E);S)$ represents an assignment of a value-expression $E$ to a variable $x$; **(c)** $\mathbf{out}(H,E)$ represents an *output* of a value $E$ to a channel $H$; **(d)** $\mathbf{in}(H,x)$ represents an *input* from a channel $H$; **(e)** $\mathbf{in}'(H,S^{(0,1)})$ also represents an *input* from a channel $H$, with $S^{(0,1)}$ applied to the input value; **(f)** the combinators ';', '+', '$\parallel$', and $\partial_C(\cdot)$ represent *sequential composition*, *alternative choice*, *parallel composition*, and *action restriction*, respectively; **(g)** $\mathbf{if}(\cdot,\cdot,\cdot)$ is the usual conditional construct; **(h)** $\mathbf{let}(x,E,S)$ is a statement with a *local variable* $x$ whose initial value is $E$; **(i)** $\mathbf{LC}(S^{(1,0)})$ is a statement with a *local channel*; **(j)** $(\mu X.\ S)$ represents a recursive statement, which intuitively stands for a solution to the equation $X = S$; **(k)** $S^{(i,j)}(\vec{H}^{(i)} \cdot \vec{E}^{(j)})$ is the *application* of a parameterized statement $S^{(i,j)}$ to actual arguments $\vec{H}^{(i)} \cdot \vec{E}^{(j)}$.

**(ii)** For each $(i,j) \neq (0,0)$,

$$S^{(i,j)} ::= X^{(i,j)} \mid (\lambda \vec{\eta}^{\langle i \rangle} \cdot \vec{\xi}^{\langle j \rangle}.\ S) \mid (\mu X^{(i,j)}.\ S^{(i,j)}),$$

where $X^{(i,j)}$ ranges over $\mathcal{X}_{\mathcal{P}}^{(i,j)}$, and we put $\vec{\eta}^{\langle 0 \rangle} = \vec{\xi}^{\langle 0 \rangle} = \epsilon$.

For notational convenience, we put $\widetilde{\mathcal{L}}_5^{(0,0)} = \widetilde{\mathcal{L}}_5$, and let us use the variable $S^{(0,0)}$ ranging over $\widetilde{\mathcal{L}}_5^{(0,0)}$; let

$$(\mathbf{S} \in)\ \widetilde{\mathcal{L}}_5^* = \bigcup \{\widetilde{\mathcal{L}}_5^{(i,j)} : (i,j) \in \omega^2\}.$$

**(3)** The constructs "$(\lambda \vec{\eta}^{\langle i \rangle} \cdot \vec{\xi}^{\langle j \rangle}.\ \cdots)$", "$(\mu X.\ \cdots)$", "$(\mu X^{(i,j)}.\ \cdots)$" have the usual binding property. Let $(\zeta \in)\ \mathcal{X}^*$ be the set of all variables, i.e., let $\mathcal{X}^* = \mathcal{X}_{\mathcal{V}} \cup \mathcal{X}_{\mathcal{C}} \cup \mathcal{X}_{\mathcal{P}}^*$. For $\mathbf{S} \in \widetilde{\mathcal{L}}_5^*$, let $\mathrm{FV}(\mathbf{S})$ be the set of free variables in $\mathbf{S}$, i.e., the set of elements of $\mathcal{X}^*$ having a free occurrence in $\mathbf{S}$. For $(i,j) \in \omega^2$, the language $\mathcal{L}_5^{(i,j)}$ is defined to be the set of $\mathbf{S} \in \widetilde{\mathcal{L}}_5^{(i,j)}$ satisfying the following *guardedness* condition:

*For each subexpression* $(\mu Z.\ \mathbf{S}')$ *of* $\mathbf{S}$, *each free occurrence of* $Z$ *in* $\mathbf{S}'$ *occurs in a subexpression of* $\mathbf{S}'$ *of the form* $(\mathbf{asg}(x,E); \mathbf{S}'')$, $(\mathbf{out}(H,E); \mathbf{S}'')$, $(\mathbf{in}(H,x); \mathbf{S}'')$, *or* $(\mathbf{in}'(H,\mathbf{S}''); \mathbf{S}'')$. (8.1)

Let

$$\mathcal{L}_5 = \mathcal{L}_5^{(0,0)}, \quad \mathcal{L}_5^* = \bigcup \{\mathcal{L}_5^{(i,j)} : (i,j) \in \omega^2\}.$$

(See § 8.A for an inductive formulation of the guardedness condition and a more formal definition of $\mathcal{L}_5^{(i,j)}$ for $(i,j) \in \omega^2$.)

**(4)** For $\mathcal{Y} \subseteq \mathcal{X}^*$ and $\mathcal{L}' \subseteq \widetilde{\mathcal{L}}_5^*$, let

$$\mathcal{L}'[\mathcal{Y}] = \{S \in \mathcal{L}' : \mathrm{FV}(S) \subseteq \mathcal{Y}\}.$$

Further, for $\zeta \in \mathcal{X}^*$, let

$$\mathcal{L}'[\zeta] = \mathcal{L}'[\{\zeta\}].$$

We use the variable $s^{(i,j)}$ ranging over $\mathcal{L}_5^{(i,j)}[\emptyset]$ $((i,j) \in \omega^2)$, and the variable $\mathbf{s}$ (resp. $s$) ranging over $\mathcal{L}_5^*[\emptyset]$ (resp. $\mathcal{L}_5[\emptyset]$). ∎

For description examples in $\mathcal{L}_5$, see § 8.C. In the examples, we omit $\iota_{\mathrm{v}}$ and $\iota_{\mathrm{c}}$, and simply write $e$ and $h$ instead of $\iota_{\mathrm{v}}(e)$ and $\iota_{\mathrm{c}}(h)$ ($e \in \mathcal{E}_{\mathcal{V}}$ and $h \in \mathcal{E}_{\mathcal{C}}$), when they appear as arguments of a function symbol in $\mathbf{Fun}_5$.

We also characterize the language $\mathcal{L}_5$ in another way: Along the lines of the standard typed $\lambda$-calculus (cf. [Mit 90]), we characterize $\widetilde{\mathcal{L}}_5^{(i,j)}$ as the set of terms (of type $(i,j)$) generated by a *signature* $\mathbf{Sig}_5$ $((i,j) \in \omega^2)$:

**Definition 8.3** (Signature $\mathbf{Sig}_5$) The signature $\mathbf{Sig}_5$ is defined by:

$$\mathbf{Sig}_5 = (\{\mathcal{V}, \mathcal{F}, \mathcal{C}, \mathcal{H}, \mathcal{P}\}, \mathbf{Fun}_5, \mathrm{type}_5),$$

where $\mathcal{V}$, $\mathcal{F}$, $\mathcal{C}$, $\mathcal{H}$, $\mathcal{P}$ are the *base types* of values, value-expressions, channels, channel-expressions, *statements*, respectively, and $\mathbf{Fun}_5$ is a set of function symbols with $\mathrm{type}_5$ being a mapping assigning a type to each function symbol. The types $\mathcal{V}$, $\mathcal{F}$, $\mathcal{C}$, $\mathcal{H}$, and $\mathcal{P}$ correspond to the semantic domains $(v \in)$ $\mathbf{V}$, $(f \in)$ $\mathbf{F}$, $(\mathbf{c} \in)$ $\mathbf{Chan}$, $(h \in)$ $\mathbf{H}$, and $(p \in)$ $\mathbf{P}_4$, respectively. For $(i,j) \in \omega^2$, let

$$(p^{(i,j)} \in)\ \mathbf{P}_4^{(i,j)} = (\mathbf{Chan}^i \cdot \mathbf{V}^j \to \mathbf{P}_4).$$

We identify $\mathbf{P}_4^{(0,0)}$ with $\mathbf{P}_4$, and let

$$(\mathbf{p} \in)\ \mathbf{P}_4^* = \bigcup\{\mathbf{P}_4^{(i,j)} : (i,j) \in \omega^2\}.$$

The set of types of $\mathbf{Sig}_5$ is

$$\{\mathcal{V}, \mathcal{F}, \mathcal{C}, \mathcal{H}\} \cup \{\mathcal{P}^{(i,j)} : (i,j) \in \omega^2\},$$

where $\mathcal{P}^{(i,j)}$ corresponds to the semantic domain $\mathbf{P}_4^{(i,j)}$ $((i,j) \in \omega^2)$. The type $\mathcal{P}^{(0,0)}$ is identified with $\mathcal{P}$. It is assumed that for each $F \in \mathbf{Fun}_5$,

$$\mathrm{type}_5(F) = (\mathcal{H}^i \cdot \mathcal{F}^j \cdot \mathcal{P}^k \cdot (\mathcal{P}^{(0,1)})^\ell \cdot (\mathcal{P}^{(1,0)})^m, \mathcal{P}),$$

with $i, j, k, \ell, m \in \omega$. For $i, j, k, \ell, m \in \omega$, let

$$\mathbf{Fun}_5^{(i,j,k,\ell,m)} = \mathrm{type}_5^{-1}[\{(\mathcal{H}^i \cdot \mathcal{F}^j \cdot \mathcal{P}^k \cdot (\mathcal{P}^{(0,1)})^\ell \cdot (\mathcal{P}^{(1,0)})^m, \mathcal{P})\}].$$

We define $\mathbf{Fun}_5$ as follows:

**(i)** $\mathbf{Fun}_5^{(0,0,0,0,0)} = \{\mathbf{0}, \mathbf{e}\}$;

**(ii)** $\mathbf{Fun}_5^{(0,1,0,0,0)} = \{\mathbf{asg}(x, \cdot) : x \in \mathcal{IV}\}$;

**(iii)** $\mathbf{Fun}_5^{(0,1,1,0,0)} = \{\mathbf{let}(x, \cdot, \cdot) : x \in \mathcal{IV}\}$;

**(iv)** $\mathbf{Fun}_5^{(1,1,0,0,0)} = \{\mathbf{out}(\cdot, \cdot)\}$;

**(v)** $\mathbf{Fun}_5^{(1,0,0,0,0)} = \{\mathbf{in}(\cdot, x) : x \in \mathcal{X_V}\}$;

**(vi)** $\mathbf{Fun}_5^{(0,0,1,0,0)} = \{\partial_C(\cdot) : C \subseteq \wp(\mathbf{Chan})\}$;

**(vii)** $\mathbf{Fun}_5^{(1,0,0,1,0)} = \{\mathbf{in}'(\cdot, \cdot)\}$;

**(viii)** $\mathbf{Fun}_5^{(0,0,0,0,1)} = \{\mathbf{LC}(\cdot)\}$;

**(ix)** $\mathbf{Fun}_5^{(0,0,2,0,0)} = \{+, \|\}$;

**(x)** $\mathbf{Fun}_5^{(0,1,2,0,0)} = \{\mathbf{if}(\cdot, \cdot, \cdot)\}$;

**(xi)** $\mathbf{Fun}_5^{(i,j,k,\ell,m)} = \emptyset$ for the other indexes $(i, j, k, \ell, m)$. ∎

Having defined $\mathbf{Sig}_5$, it is easy to check that the sets $\widetilde{\mathcal{L}}_5^{(i,j)}$ can be characterized as the set of terms of type $\mathcal{P}^{(i,j)}$ generated by $\mathbf{Sig}_5$, $\mathcal{X}^*$, $\tilde{\mathcal{E}}_{\mathcal{H}}$, and $\tilde{\mathcal{E}}_{\mathcal{F}}$ ($(i,j) \in \omega^2$). Thus the syntax of $\widetilde{\mathcal{L}}_5^*$ fits into the framework of the typed $\lambda$-calculus. On the other hand, the semantics for it does not quite fit into the calculus because of its *nonuniform* nature, as we will see in the following two sections (see, e.g., Lemma 8.6 (5)).

**Notation 8.1** The function $\mathrm{type}_5(\cdot)$ is extended so that it is defined on $\tilde{\mathcal{E}}_{\mathcal{H}} \cup \tilde{\mathcal{E}}_{\mathcal{F}} \cup \widetilde{\mathcal{L}}_5^*$ as usual. For $\phi \in (\mathcal{X}^* \nrightarrow (\tilde{\mathcal{E}}_{\mathcal{H}} \cup \tilde{\mathcal{E}}_{\mathcal{F}} \cup \widetilde{\mathcal{L}}_5^*))$, we say $\phi$ respects the type iff

$$\forall \zeta \in \mathrm{dom}(\phi)[\ \mathrm{type}_5(\phi(\zeta)) = \mathrm{type}_5(\zeta)\ ].$$

For $\mathbf{S} \in \widetilde{\mathcal{L}}_5^*$ and a type respecting mapping $\phi \in (\mathcal{X}^* \nrightarrow \widetilde{\mathcal{L}}_5^*)$, we denote by $\mathbf{S}[\phi]$ the result of simultaneously replacing all free occurrences of $\zeta$ by $\phi(\zeta)$ $(\zeta \in \mathrm{dom}(\phi))$. ∎

## 8.3 Operational Models $\mathcal{O}_5$, $\mathcal{O}_5^*$, $\mathcal{C}_5$ for $\mathcal{L}_5$

In this section, three operational models $\mathcal{O}_5$, $\mathcal{O}_5^*$, and $\mathcal{C}_5$ for $\mathcal{L}_5$ are introduced in terms of a Plotkin-style transition system ([Plo 81]). The first two models $\mathcal{O}_5$ and $\mathcal{O}_5^*$ are *linear* in that both of them map each statement to the set of its possible execution paths of a certain kind; the second model $\mathcal{O}_5^*$ is more abstract than the first one in that $\mathcal{O}_5^*$ ignores states whereas $\mathcal{O}_5$ involves them. The third model $\mathcal{C}_5$ is a variant of the *failures model* which was first introduced in [BHR 85]. The last model is shown, in § 8.5, to be equivalent to a denotational model $\mathcal{M}_5$ defined in § 8.4.

In this chapter, we use the the following notations with meanings defined in Chapter 7:

**Notation 8.2** **(1)** The sets $C!$, $C?$, $C!?$ for $C \in \wp(\mathbf{Chan})$, and $\mathbf{CS}_4$;

**(2)** the symbols $\surd$ and $\delta$, the sets $(a \subset)\ \mathbf{A}_4$, $(\alpha \subset)\ \mathbf{A}_4^{\surd}$, the variable $c$ ranging over $\mathbf{CS}_4 \times \mathbf{V}$, $\bar{c}$ for $c \in \mathbf{CS}_4 \times \mathbf{V}$, and the notation $\delta(\Gamma)$ denoting $\langle \delta, \Gamma \rangle$;

**(3)** the set $(A \in)$ **ASort** of actions sorts;

**(4)** the function sort : $\mathbf{A}_4 \to \mathbf{ASort}$;

**(5)** the function $\mathrm{chan}(\cdot) : \mathbf{CS}_4 \to \mathbf{Chan}$.

**(6)** The set $\mathbf{R}_4$.

**(7)** The set $(\Upsilon \in)\ \mathbf{B}_4$. ∎

As a preliminary to the definition of *transition relations*, the notion of *syntactic sort* is defined by:

**Definition 8.4** (Syntactic Sort) For $\mathbf{S} \in \widetilde{\mathcal{L}}_5^*$, the *syntactic sort* of $\mathbf{S}$, written $\mathcal{S}(\mathbf{S})$, is the set of function symbols of sort $\mathcal{H}$ occurring in $\mathbf{S}$. Formally, it is defined by induction on the structure of $\mathbf{S}$ as follows:

**(i)** If $\mathbf{S} \in \mathcal{X}_{\mathcal{P}}^*$ or $\mathbf{S} \equiv \mathbf{0}$, then $\mathcal{S}(\mathbf{S}) = \emptyset$.

**(ii)** If $\mathbf{S} \equiv (\mathbf{asg}(x, E); \mathbf{S}')$, $\mathbf{S} \equiv \mathbf{let}(x, E, \mathbf{S}')$, $\mathbf{S} \equiv \mathbf{LC}(\mathbf{S}')$, $\mathbf{S} \equiv (\mu Z.\ \mathbf{S}')$, $\mathbf{S} \equiv \mathbf{S}'(\vec{H} \cdot \vec{E})$, or $\mathbf{S} \equiv (\lambda\vec{\eta} \cdot \vec{\xi}.\ \mathbf{S}')$, then $\mathcal{S}(\mathbf{S}) = \mathcal{S}(\mathbf{S}')$.

**(iii)** If $\mathbf{S} \equiv \partial_C(\mathbf{S}')$, then $\mathcal{S}(\mathbf{S}) = \mathcal{S}(\mathbf{S}') \setminus C$.

**(iv)** If $\mathbf{S} \equiv (\mathbf{S}_1 + \mathbf{S}_2)$, $\mathbf{S} \equiv (\mathbf{S}_1 \parallel \mathbf{S}_2)$, $\mathbf{S} \equiv \mathbf{if}(E, \mathbf{S}_1, \mathbf{S}_2)$, then $\mathcal{S}(\mathbf{S}) = \mathcal{S}(\mathbf{S}_1) \cup \mathcal{S}(\mathbf{S}_2)$.

**(v)** If $\mathbf{S} \equiv (\mathbf{out}(G, E); \mathbf{S}')$, $\mathbf{S} \equiv (\mathbf{in}(G, x); \mathbf{S}')$, or $\mathbf{S} \equiv \mathbf{in}'(G, \mathbf{S}')$, then $\mathcal{S}(\mathbf{S}) = \mathcal{S}(\mathbf{S}') \cup \mathcal{S}(G)$. ∎

By definition, $\mathcal{S}(\mathbf{S})$ is finite for every $\mathbf{S} \in \widetilde{\mathcal{L}}_5^*$.

Next, let us define the transition relations $\xrightarrow{a}_5 \subseteq (\mathcal{L}_5[\emptyset] \times \Sigma) \times (\mathcal{L}_5[\emptyset] \times \Sigma)$ $(a \in \mathbf{A}_4)$, and $\xrightarrow{\surd}_5 \subseteq \mathcal{L}_5[\emptyset] \times \Sigma$. For $s_1, s_2 \in \mathcal{L}_5[\emptyset]$, $\sigma_1, \sigma_2 \in \Sigma$, and $a \in \mathbf{A}_4$, we write $(s_1, \sigma_1) \xrightarrow{a}_5 (s_2, \sigma_2)$ (resp. $(s_1, \sigma_1) \xrightarrow{\tau}_5$) for $((s_1, \sigma_1), (s_2, \sigma_2)) \in \xrightarrow{a}_5$ (resp. for $(s_1, \sigma_1) \in \xrightarrow{\tau}_5$), as usual. Whenever $(s, \sigma) \xrightarrow{a}_5 (s', \sigma')$, we call $(a, (s', \sigma'))$ (resp. $(s', \sigma')$) an *immediate derivative* (resp. an immediate $a$-derivative) of $(s, \sigma)$ as in [Mil 89]. For $\mathbf{c}!, \mathbf{c}? \in \mathbf{CS}_4$ and $v \in \mathbf{V}$, we sometimes write $\mathbf{c}!v$ and $\mathbf{c}?v$ for $(\mathbf{c}!, v)$ and $(\mathbf{c}?, v)$, respectively.

**Definition 8.5** (Transition Relations $\xrightarrow{\alpha}_5$)

**(1)** $\quad \mathbf{e} \xrightarrow{\surd}_5$ .

**(2)** $\quad (\mathbf{asg}(x, e), \sigma) \xrightarrow{\tau}_5 (\mathbf{e}, \sigma[[\![e]\!](\sigma)/x])$.

**(3)** $\quad (\mathbf{out}(g, e), \sigma) \xrightarrow{([\![g]\!](\sigma)!, [\![e]\!](\sigma))}_5 (\mathbf{e}, \sigma)$.

**(4)** For every $v \in \mathbf{V}$,

$$(\mathbf{in}(g, x), \sigma) \xrightarrow{([\![g]\!](\sigma)?, v)}_5 (\mathbf{e}, \sigma[v/x]),$$

where for the notation $\sigma[v/x]$ see Notation 2.3.

**(5)** For every $v \in \mathbf{V}$,

$$(\mathbf{in}'(g, s^{(0,1)}), \sigma) \xrightarrow{([\![g]\!](\sigma)?, v)}_5 (s^{(0,1)}(v), \sigma).$$

**(6.1)**

$$\frac{(s_1, \sigma) \xrightarrow{a}_5 (s, \sigma')}{\begin{array}{l}(s_1 + s_2, \sigma) \xrightarrow{a}_5 (s, \sigma') \\ (s_2 + s_1, \sigma) \xrightarrow{a}_5 (s, \sigma')\end{array}} \quad (a \in \mathbf{A}_4).$$

**(6.2)**

$$\frac{(s_1, \sigma) \xrightarrow{\surd}_5}{\begin{array}{l}(s_1 + s_2, \sigma) \xrightarrow{\surd}_5 \\ (s_2 + s_1, \sigma) \xrightarrow{\surd}_5\end{array}}.$$

**(7.1)**

$$\frac{(s_1, \sigma) \xrightarrow{a}_5 (s, \sigma')}{\begin{array}{c}(s_1 \parallel s_2, \sigma) \xrightarrow{a}_5 (s \parallel s_2, \sigma') \\ (s_2 \parallel s_1, \sigma) \xrightarrow{a}_5 (s_2 \parallel s, \sigma')\end{array}} \quad (a \in \mathbf{A}_4.)$$

**(7.2)** For every $\mathbf{c} \in \mathbf{Chan}$, $v \in \mathbf{V}$,

$$\frac{(s_1, \sigma) \xrightarrow{\mathbf{c}!v}_5 (s_1', \sigma), \ \ (s_2, \sigma) \xrightarrow{\mathbf{c}?v}_5 (s_2', \sigma')}{\begin{array}{c}(s_1 \parallel s_2, \sigma) \xrightarrow{\tau}_5 (s_1' \parallel s_2', \sigma') \\ (s_2 \parallel s_1, \sigma) \xrightarrow{\tau}_5 (s_2' \parallel s_1', \sigma')\end{array}}.$$

**(7.3)**

$$\frac{(s_1, \sigma) \xrightarrow{\surd}_5, \ \ (s_2, \sigma) \xrightarrow{\surd}_5}{(s_1 \parallel s_2, \sigma) \xrightarrow{\surd}_5}.$$

**(8.1)** For every $\mathbf{c} \in \mathbf{Chan}$, $v \in \mathbf{V}$,

$$\frac{s \xrightarrow{a}_5 s'}{\partial_C(s) \xrightarrow{a}_5 \partial_C(s')}.$$

**(8.2)**

$$\frac{s \xrightarrow{\surd}_5}{\partial_C(s) \xrightarrow{\surd}_5}.$$

**(9.1)**

$$\frac{(s_1, \sigma) \xrightarrow{a}_5 (s, \sigma')}{(\mathbf{if}(e, s_1, s_2), \sigma) \xrightarrow{a}_5 (s, \sigma')} \quad ([\![e]\!](\sigma) \neq \mathbf{nil}).$$

**(9.2)**

$$\frac{(s_1, \sigma) \xrightarrow{\surd}_5}{(\mathbf{if}(e, s_1, s_2), \sigma) \xrightarrow{\surd}_5} \quad ([\![e]\!](\sigma) \neq \mathbf{nil}).$$

**(9.3)**

$$\frac{(s_2, \sigma) \xrightarrow{a}_5 (s, \sigma')}{(\mathbf{if}(e, s_1, s_2), \sigma) \xrightarrow{a}_5 (s, \sigma')} \quad ([\![e]\!](\sigma) = \mathbf{nil}).$$

**(9.4)**

$$\frac{(s_2, \sigma) \xrightarrow{\surd}_5}{(\mathbf{if}(e, s_1, s_2), \sigma) \xrightarrow{\surd}_5} \quad ([\![e]\!](\sigma) = \mathbf{nil}).$$

**(10.1)**

$$\frac{(s, \sigma[\llbracket e \rrbracket(\sigma)/x]) \xrightarrow{a}_5 (s', \sigma')}{(\mathbf{let}(x, e, s), \sigma) \xrightarrow{a}_5 (\mathbf{let}(x, \sigma'(x), s'), \sigma'[\sigma(x)/x])}.$$

Intuitively, this rule is explained as follows: To infer the immediate derivatives of the configuration $(\mathbf{let}(x, e, s), \sigma)$, we first need to find the immediate derivatives of the configuration $(s, \sigma[\llbracket e \rrbracket(\sigma)/x])$, because in $(\mathbf{let}(x, e, s), \sigma)$ the statement $s$ acts as if it is in the state $\sigma[\llbracket e \rrbracket(\sigma)/x]$ which is the same as $\sigma$ except that it binds $x$ to the value $\llbracket e \rrbracket(\sigma)$ of $e$ in $\sigma$, forgetting the current value $\sigma(x)$ for a moment for later restoration. From $(s, \sigma[\llbracket e \rrbracket(\sigma)/x]) \xrightarrow{a}_5 (s', \sigma')$, it is inferred that the configuration $(\mathbf{let}(x, e, s), \sigma)$ has an immediate $a$-derivative $(\mathbf{let}(x, \sigma'(x), s'), \sigma'[\sigma(x)/x])$, where the statement $\mathbf{let}(x, \sigma'(x), s')$ is obtained from $s'$ by putting it in the $\mathbf{let}(\cdots)$ construct *locally* binding $x$ to $\sigma'(x)$, and the state $\sigma'[\sigma(x)/x]$ is obtained from $\sigma'$ by restoring the original value $\sigma(x)$ of $x$; the restoration is applied, because in $\mathbf{let}(x, e, s)$, the variable $x$ is *localized*, and so, the (global) value of $x$ never changes whatever action the statement $\mathbf{let}(x, e, s)$ may perform. (See § 8.7 for related work on the semantics of local variables.)

**(10.2)**

$$\frac{(s, \sigma[\llbracket e \rrbracket(\sigma)/x]) \xrightarrow{\surd}_5}{(\mathbf{let}(x, e, s), \sigma) \xrightarrow{\surd}_5}.$$

**(11)**

(i) $$\frac{(\partial_{\{\mathbf{c}_m^*\}}(s^{(1,0)}(\mathbf{c}_m^*)), \sigma) \xrightarrow{a}_5 (s', \sigma')}{(\mathbf{LC}(s^{(1,0)}), \sigma) \xrightarrow{a}_5 (s', \sigma')},$$

(ii) $$\frac{(\partial_{\{\mathbf{c}_m^*\}}(s^{(1,0)}(\mathbf{c}_m^*)), \sigma) \xrightarrow{\surd}_5}{(\mathbf{LC}(s^{(1,0)}), \sigma) \xrightarrow{\surd}_5},$$

where $m = \min\{n \in \omega : \mathbf{c}_n^* \notin \mathcal{S}(s^{(1,0)})\}$. Note that the set

$$\{n \in \omega : \mathbf{c}_n^* \notin \mathcal{S}(s^{(1,0)})\}$$

is nonempty, since $\mathcal{S}(s^{(1,0)})$ is finite by definition. (See Remark 8.1 below for the motivation for this rule.)

**(12.1)** The following rules are called the *recursion rules*:

(i) $$\frac{(S[(\mu X.\ S)/X], \sigma) \xrightarrow{a}_5 (s', \sigma')}{((\mu X.\ S), \sigma) \xrightarrow{a}_5 (s', \sigma')},$$

(ii) $$\frac{(S[(\mu X.\ S)/X], \sigma) \xrightarrow{\surd}_5}{((\mu X.\ S), \sigma) \xrightarrow{\surd}_5}.$$

**(12.2)** The following rules are called the *parameterized recursion rules*:

$$\text{(i)}\quad \frac{(S^{(i,j)}[(\mu X^{(i,j)}.\ S^{(i,j)})/X^{(i,j)}](\vec{g}^{(i)}\cdot\vec{e}^{(j)}),\sigma) \xrightarrow{a}_5 (s',\sigma')}{\langle(\mu X^{(i,j)}.\ S^{(i,j)})(\vec{g}^{(i)}\cdot\vec{e}^{(j)}),\sigma\rangle \xrightarrow{a}_5 (s',\sigma')},$$

$$\text{(ii)}\quad \frac{(S^{(i,j)}[(\mu X^{(i,j)}.\ S^{(i,j)})/X^{(i,j)}](\vec{g}^{(i)}\cdot\vec{e}^{(j)}),\sigma) \xrightarrow{\surd}_5}{\langle(\mu X^{(i,j)}.\ S^{(i,j)})(\vec{g}^{(i)}\cdot\vec{e}^{(j)}),\sigma\rangle \xrightarrow{\surd}_5},$$

where $(i,j)\neq(0,0)$.

**(13)** The following two rules are called the *λ-rules*:

$$\text{(i)}\quad \frac{(S[(\vec{g}^{(i)}[\![\sigma]\!]\cdot\vec{e}^{(j)}[\![\sigma]\!])/(\vec{\eta}^{\langle i\rangle}\cdot\vec{\xi}^{\langle j\rangle})],\sigma) \xrightarrow{a}_5 (s',\sigma')}{((\lambda\vec{\eta}^{\langle i\rangle}\cdot\vec{\xi}^{\langle j\rangle}.\ S)(\vec{g}^{(i)}\cdot\vec{e}^{(j)}),\sigma) \xrightarrow{a}_5 (s',\sigma')},$$

$$\text{(ii)}\quad \frac{(S[(\vec{g}^{(i)}[\![\sigma]\!]\cdot\vec{e}^{(j)}[\![\sigma]\!])/(\vec{\eta}^{\langle i\rangle}\cdot\vec{\xi}^{\langle j\rangle})],\sigma) \xrightarrow{\surd}_5}{((\lambda\vec{\eta}^{\langle i\rangle}\cdot\vec{\xi}^{\langle j\rangle}.\ S)(\vec{g}^{(i)}\cdot\vec{e}^{(j)}),\sigma) \xrightarrow{\surd}_5},$$

where $(i,j)\neq(0,0)$ and

$$[\![\vec{g}^{(i)}]\!] = \langle[\![\vec{g}^{(i)}(k)]\!]\rangle_{k\in i},\quad [\![\vec{e}^{(j)}]\!](\sigma) = \langle[\![\vec{e}^{(j)}(k)]\!](\sigma)\rangle_{k\in j}. \blacksquare$$

**Remark 8.1** The present treatment of local channels is very similar to Milner's treatment of auxiliary channels introduced for defining *the linking combinator* '$\frown$' (cf. [Mil 89] § 3.3). This combinator is defined by:

$$s_1 \frown s_2 = \partial_{\{\mathbf{c}\}}(s_1[\mathbf{c}/\mathbf{c}'] \parallel s_1[\mathbf{c}/\mathbf{c}'']),$$

where $\mathbf{c}'$ and $\mathbf{c}''$ are predefined channels for output and input, respectively, $\mathbf{c}$ is an auxiliary channel which is fresh, i.e., chosen so that $\mathbf{c}\in\mathcal{S}(s_1)\cup\mathcal{S}(s_2)$, and $[\mathbf{c}/\mathbf{c}']$ is the *renaming combinator* which replaces $\mathbf{c}'$ by $\mathbf{c}$. Strictly speaking, however, the combinator '$\frown$' is not defined in the calculus, i.e., there are not a term $S$ and variables $X_1$, $X_2$ such that $s_1\frown s_2 \equiv S[(s_1,s_2)/(X_1,X_2)]$ for all $s_1$, $s_2$. In our language, $\mathbf{LC}(\cdot)$ is a predefined function symbol, for which transition rules in Definition 8.5 (11) are prescribed in terms of an auxiliary channel $\mathbf{c}^*_m$ which is fresh. See Example 8.3 in § 8.C for a description example using $\mathbf{LC}(\cdot)$. ∎

By the above definition and the guardedness condition in Definition 8.2, we have the following lemma stating several useful properties of the transition system $(\mathcal{L}_5[\emptyset], \mathbf{A}_4^{\surd}, \{\xrightarrow{\alpha}_5 : \alpha\in\mathbf{A}_4^{\surd}\})$.

**Lemma 8.1** *For every $s\in\mathcal{L}_5[\emptyset]$ and $\sigma\in\Sigma$, the following hold:*

**(1)** *The transition system is* image-finite *in that for every $a\in\mathbf{A}_4$, the* image set $\{(s',\sigma') : (s,\sigma)\xrightarrow{a}_5 (s',\sigma')\}$ *is finite.*

**(2)** sort[act$_5(s,\sigma)$] *is finite.*

**(3)** *For every* $\mathbf{c} \in \mathbf{Chan}$, *the set* $\{v \in \mathbf{V} : (\mathbf{c}!, v) \in \mathrm{act}_5(s, \sigma)\}$ *is finite.* ∎

From Definition 8.5, we obtain the following lemma which states that immediate derivatives of a configuration of the form $(S(\vec{g} \cdot \vec{e}), \sigma)$ are the same as those of $(S([\![\vec{g}]\!](\sigma) \cdot [\![\vec{e}]\!](\sigma)), \sigma)$.

**Lemma 8.2** *Let* $(i, j) \neq (0, 0)$, $\mathbf{S} \in \mathcal{L}_5^{(i,j)}[\emptyset]$, $\vec{g} \in (\mathcal{E}_{\mathrm{C}})^i$, *and* $\vec{e} \in (\mathcal{E}_{\mathrm{V}})^j$. *Then, for every* $\sigma \in \Sigma$, *the following hold:*

**(1)** $$\forall a \in \mathbf{A}_4, \forall (s', \sigma') \in \mathcal{L}_5[\emptyset] \times \Sigma[\ (\mathbf{S}(\vec{g} \cdot \vec{e}), \sigma) \xrightarrow{a}_5 (s', \sigma') \Leftrightarrow (\mathbf{S}([\![\vec{g}]\!](\sigma) \cdot [\![\vec{e}]\!])(\sigma), \sigma) \xrightarrow{a}_5 (s', \sigma')\ ].$$

**(2)** $$(\mathbf{S}(\vec{g} \cdot \vec{e}), \sigma) \xrightarrow{\surd}_5 \Leftrightarrow (\mathbf{S}([\![\vec{g}]\!](\sigma) \cdot [\![\vec{e}]\!])(\sigma), \sigma) \xrightarrow{\surd}_5 .$$ ∎

In terms of the transition relations $\xrightarrow{\alpha}_5$ $(\alpha \in \mathbf{A}_4^{\surd})$ three operational models $\mathcal{O}_5$, $\mathcal{O}_5^*$, and $\mathcal{C}_5$ are defined. The first two models $\mathcal{O}_5$ and $\mathcal{O}_5^*$ are *linear*, and the third one $\mathcal{C}_5$ is a variant of the *failures model*. The models $\mathcal{O}_5$, $\mathcal{O}_5^*$, $\mathcal{C}_5$ are defined in the same fashion as the definition of $\mathcal{O}_4$, $\mathcal{O}_4^*$, and $\mathcal{C}_4$ except that the domain of $\mathcal{O}_5$, $\mathcal{O}_5^*$, $\mathcal{C}_5$ is $\mathcal{L}_5[\emptyset]$ instead of $\mathcal{L}_4[\emptyset]$ (cf. Definitions 7.20, 7.21, 7.26).

From the definition of $\mathcal{C}_5$, the following lemma immediately follows.

**Lemma 8.3** *Let* $(i, j) \neq (0, 0)$, $\mathrm{s} \in \mathcal{L}_5^{(i,j)}[\emptyset]$, $\vec{g} \in (\mathcal{E}_{\mathrm{C}})^i$, *and* $\vec{e} \in (\mathcal{E}_{\mathrm{V}})^j$. *Then,*

$$\mathcal{C}_5[\![\mathrm{s}(\vec{g} \cdot \vec{e})]\!] = \bigcup_{\sigma \in \Sigma} [\mathcal{C}_5[\![\mathrm{s}([\![\vec{g}]\!](\sigma) \cdot [\![\vec{e}]\!](\sigma))]\!]\langle\sigma\rangle].$$ ∎

We define two *abstraction functions* $\mathcal{A}_4$ and $\mathcal{A}_4^*$ which relate $\mathcal{O}_5$ and $\mathcal{O}_5^*$ with $\mathcal{C}_5$, respectively.

We use in this section the abstraction functions $\mathcal{A}_4$ and $\mathcal{A}_4^*$ defined in Chapter 7 (cf. Definitions 7.27 and 7.29).

Having defined $\mathcal{A}_4$ and $\mathcal{A}_4^*$, the following lemma immediately follows from the definitions of $\mathcal{O}_5$, $\mathcal{O}_5^*$, and $\mathcal{C}_5$.

**Lemma 8.4** (Correctness of $\mathcal{C}_5$ w.r.t. $\mathcal{O}_5$ and $\mathcal{O}_5^*$)
**(1)** $\mathcal{O}_5 = \mathcal{A}_4 \circ \mathcal{C}_5$. **(2)** $\mathcal{O}_5^* = \mathcal{A}_4^* \circ \mathcal{C}_5$. ∎

## 8.4 A Denotational Model $\mathcal{M}_5$ for $\mathcal{L}_5$

In this section, we define a denotational model $\mathcal{M}_5$ compositionally using interpreted operations of the language $\mathcal{L}_5$, with meanings of recursive programs as fixed-points in the complete metric space $\mathbf{P}_4$; it turns out that $\mathcal{M}_5$ is equal to the failures model $\mathcal{C}_5$, as stated earlier. The model $\mathcal{M}_5$ is a *conservative extension* of the model $\mathcal{M}_4$ presented in Chapter 7, for a sublanguage $\mathcal{L}_4$ of $\mathcal{L}_5$: That is, in $\mathcal{M}_5$, the semantic operations corresponding to the syntactic constructs in $\mathcal{L}_4$ are the same as those in $\mathcal{M}_4$, and $\mathcal{M}_5[\![s]\!] = \mathcal{M}_4[\![s]\!]$ for $s \in \mathcal{L}_4$.

As a basis for the definition of $\mathcal{M}_5$, we will define an *interpretation* $\mathcal{I}_5$ which maps each syntactic combinator introduced in Definition 8.3 to a corresponding semantic operation on $\mathbf{P}_4$. We use the semantic operations $\tilde{\mathbf{0}}_4, \tilde{\mathbf{e}}_4 \in \mathbf{P}_4$, and $\tilde{;}_4, \tilde{+}_4, \tilde{\|}_4 : \mathbf{P}_4 \times \mathbf{P}_4 \to \mathbf{P}_4$ defined in Chapter 7 (cf. Definition 7.23). Semantic operations corresponding to the combinators **asg**, $\partial_C(\cdot)$, $\mathbf{let}(x, \cdot, \cdot)$, and $\mathbf{LC}(\cdot)$ will be defined later. First, semantic operations corresponding to the other combinators are defined as in Definition 7.23 by:

**Definition 8.6** (Semantic Operations)

**(1)** For each $x \in \mathcal{IV}$, the semantic operation $\text{asg}_5^{(x)} : \mathbf{F} \to \mathbf{P}_4$ corresponding to the construct $\mathbf{asg}(x, \cdot)$ is defined as follows: For $f \in \mathbf{F}$,

$$\text{asg}_5^{(x)}(f) = \bigcup \{\langle(\sigma, \tau, \sigma[f(\sigma)/x])\rangle : \sigma \in \Sigma\} \cdot \tilde{\mathbf{e}}_4.$$

**(2)** The semantic operation $\text{out}_5 : \mathbf{H} \times \mathbf{F} \to \mathbf{P}_4$ corresponding to the construct $\mathbf{out}(\cdot, \cdot)$ is defined as follows: For $h \in \mathbf{H}$, $f \in \mathbf{F}$,

$$\begin{aligned}\text{out}_5(h, f) = &\{\langle(\sigma, (h(\sigma)!, f(\sigma)), \sigma)\rangle : \sigma \in \Sigma\} \cdot \tilde{\mathbf{e}}_4 \\ &\cup \{\langle(\sigma, \delta(\Gamma))\rangle : \sigma \in \Sigma \wedge \Gamma \in \wp(\mathbf{CS}_4^{\checkmark} \setminus \{h(\sigma)!\})\}.\end{aligned}$$

**(3)** For each $x \in \mathcal{IV}$, the semantic operation $\text{in}_5^{(x)} : \mathbf{H} \to \mathbf{P}_4$ corresponding to the construct $\mathbf{in}(\cdot, x)$ is defined as follows: For $h \in \mathbf{H}$,

$$\begin{aligned}\text{in}_5^{(x)}(h, p) = &\bigcup \{\langle(\sigma, (h(\sigma)?, v), \sigma[v/x])\rangle : \sigma \in \Sigma \wedge v \in \mathbf{V}\} \cdot \tilde{\mathbf{e}}_4 \\ &\cup \{\langle(\sigma, \delta(\Gamma))\rangle : \sigma \in \Sigma \wedge \Gamma \subseteq \mathbf{CS}_4 \setminus \{h(\sigma)?\}\}.\end{aligned}$$

**(4)** The semantic operation $\text{in}'_5 : \mathbf{H} \times \mathbf{P}_4^{(0,1)} \to \mathbf{P}_4$ corresponding to the construct $\mathbf{in}'(\cdot, \cdot)$ is defined as follows: For $h \in \mathbf{H}$ and $\mathbf{p} \in \mathbf{P}_4^{(0,1)}$,

$$\begin{aligned}\text{in}'_5(h, \mathbf{p}) = &\bigcup \{\langle(\sigma, (h(\sigma)?, v), \sigma)\rangle \cdot \mathbf{p}(v) : \sigma \in \Sigma \wedge v \in \mathbf{V}\} \\ &\cup \{\langle(\sigma, \delta(\Gamma))\rangle : \sigma \in \Sigma \wedge \Gamma \subseteq \mathbf{CS}_4 \setminus \{h(\sigma)?\}\}.\end{aligned}$$

**(5)** For $q \in \mathbf{Q}_4 \cup (\Sigma \times \mathbf{A}_4 \times \Sigma)^+$, let $\text{istate}_4(q) = \sigma$ if $\exists a, \sigma'[\ q = \langle(\sigma, a, \sigma')\rangle \cdot q'\ ]$; otherwise $\exists \Gamma[\ q = \langle(\sigma'', \delta(\Gamma))\rangle\ ]$ and let $\text{istate}_4(q) = \sigma''$. Also for $p \in \mathbf{P}_4$, $\sigma \in \Sigma$, let $p\langle\sigma\rangle = \{q \in p : \text{istate}_4(q) = \sigma\}$. The semantic operation $\text{if}_5$ : $\mathbf{F} \times \mathbf{P}_4 \times \mathbf{P}_4 \to \mathbf{P}_4$ is defined as follows: For $f \in \mathbf{F}$, $p_1, p_2 \in \mathbf{P}_4$,

$$\begin{aligned}\text{if}_5(f, p_1, p_2) = &\bigcup \{p_1\langle\sigma\rangle : \sigma \in \Sigma \wedge f(\sigma) \neq \mathbf{nil}\} \\ &\cup \bigcup \{p_2\langle\sigma\rangle : \sigma \in \Sigma \wedge f(\sigma) = \mathbf{nil}\}. \blacksquare\end{aligned}$$

A semantic operation $\tilde{\partial}_5^{(C)}(\cdot)$ corresponding to the construct $\partial_C(\cdot)$ is defined by:

**Definition 8.7** (Semantic Operation $\tilde{\partial}_5^{(C)}$ Corresponding to $\partial_C$) The operation $\tilde{\partial}_5^{(C)}(\cdot) : \mathbf{P}_4 \to \mathbf{P}_4$ is defined as follows: For $p \in \mathbf{P}_4$,

$$
\begin{aligned}
&\tilde{\partial}_5^{(C)}(p) = \\
&\{\langle(\sigma_i, a_i, \sigma_i')\rangle_{i\in\omega} \in (\Sigma \times \mathbf{A}_4 \times \Sigma)^\omega \cap p : \forall i \in \omega[\ \mathrm{sort}(a_i) \notin C!?\ ]\} \\
&\cup\{q : \exists n \in \omega, \exists\langle\sigma_i\rangle_{i\in(n+1)}, \langle\sigma_i'\rangle_{i\in(n+1)} \in \Sigma^{n+1}, \\
&\qquad \exists\langle a_i\rangle_{i\in n} \in (\mathbf{A}_4)^n, \exists\Gamma \in \wp(\mathbf{CS}_4)[ \\
&\qquad\quad q = \langle(\sigma_i, a_i, \sigma_i')\rangle_{i\in n} \cdot \langle(\sigma_n, \delta(\Gamma))\rangle \\
&\qquad\quad \wedge \forall i \in n[\ \mathrm{sort}(a_i) \notin C!?\ ] \\
&\qquad\quad \wedge \langle(\sigma_i, a_i, \sigma_i')\rangle_{i\in n} \cdot \langle(\sigma_n, \delta(\Gamma \setminus C!?))\rangle \in p\ ]\} \\
&\cup\{q : \exists n \in \omega, \exists\langle\sigma_i\rangle_{i\in(n+1)}, \langle\sigma_i'\rangle_{i\in(n+1)} \in \Sigma^{n+1}, \\
&\qquad \exists\langle a_i\rangle_{i\in n} \in (\mathbf{A}_4)^n[ \\
&\qquad\quad q = \langle(\sigma_i, a_i, \sigma_i')\rangle_{i\in n} \cdot \langle(\sigma_n, \surd)\rangle \\
&\qquad\quad \wedge \forall i \in n[\ \mathrm{sort}(a_i) \notin C!?\ ] \\
&\qquad\quad \wedge \langle(\sigma_i, a_i, \sigma_i')\rangle_{i\in n} \cdot \langle(\sigma_n, \surd)\rangle \in p\ ]\}. \blacksquare
\end{aligned}
$$

A semantic operation corresponding to the construct $\mathbf{let}(\cdot,\cdot,\cdot)$ is defined by:

**Definition 8.8** (Semantic Operation $\mathrm{let}_5^{(x)}$ corresponding to $\mathbf{let}_x$) Let $x \in \mathcal{IV}$. First, we define an auxiliary function $\text{let-base}_5^{(x)} : \mathbf{F} \times \mathbf{P}_4 \to \mathbf{P}_4$ as follows: For every $(f,p) \in \mathbf{F} \times \mathbf{P}_4$,

$$
\begin{aligned}
&\text{let-base}_5^{(x)}(f,p) \\
&= \{\langle(\sigma,\surd)\rangle : \sigma \in \Sigma \wedge \langle(\sigma[f(\sigma)/x], \surd\rangle \in p\} \\
&\quad \cup \{\langle(\sigma,\delta(\Gamma))\rangle \in (\Sigma \times \wp(\mathbf{CS}_4))^1 : \langle(\sigma[f(\sigma)/x], \delta(\Gamma))\rangle \in p\}.
\end{aligned}
$$

By means of $\text{let-base}_5^{(x)}$, the operation $\mathrm{let}_5^{(x)} : \mathbf{F} \times \mathbf{P}_4 \to \mathbf{P}_4$ is defined recursively as follows: For every $(f,p) \in \mathbf{F} \times \mathbf{P}_4$,

$$
\begin{aligned}
&\mathrm{let}_5^{(x)}(f,p) \\
&= \text{let-base}_5^{(x)}(f,p) \cup \\
&\quad (\bigcup\{\langle(\sigma, a, \sigma'[\sigma(x)/x])\rangle \cdot \mathrm{let}_5^{(x)}(\sigma'(x), p[\langle(\sigma[f(\sigma)/x], a, \sigma')\rangle]) : \\
&\qquad p[\langle(\sigma[f(\sigma)/x], a, \sigma')\rangle] \neq \emptyset\})^{\mathrm{cls}}.
\end{aligned}
\tag{8.2}
$$

Formally, $\mathrm{let}_5^{(x)}$ is defined as the fixed-point of a higher-order contraction as follows: First let $\mathbf{M}_\ell = (\mathbf{F} \times \mathbf{P}_4) \to \mathbf{P}_4$, and let us define $\Phi_\ell : \mathbf{M}_\ell \to \mathbf{M}_\ell$ as follows: For $F \in \mathbf{M}_\ell$, and $(f,p) \in \mathbf{F} \times \mathbf{P}_4$,

$$
\begin{aligned}
&\Phi_\ell(F)(f,p) \\
&= \ \text{let-base}_5^{(x)}(f,p) \\
&\quad \cup (\bigcup\{\langle(\sigma, a, \sigma'[\sigma(x)/x])\rangle \cdot F(\sigma'(x), p[\langle(\sigma[f(\sigma)/x], a, \sigma')\rangle]) : \\
&\qquad p[\langle(\sigma[f(\sigma)/x], a, \sigma')\rangle] \neq \emptyset\})^{\mathrm{cls}}.
\end{aligned}
$$

It is easy to verify that $\Phi_\ell$ is a contraction from $\mathbf{M}_\ell$ to itself. Let

$$\mathrm{let}_5^{(x)} = \mathrm{fix}(\Phi_\ell).$$

Then one has (8.2) by definition. For later use, we put

$$\begin{aligned}&\text{let-act}_5^{(x)}(f,p)\\&= (\bigcup\{\langle(\sigma,a,\sigma'[\sigma(x)/x])\rangle\cdot \text{let}_5^{(x)}(\sigma'(x),p[\langle(\sigma[f(\sigma)/x],a,\sigma')\rangle]) :\\&\qquad p[\langle(\sigma[f(\sigma)/x],a,\sigma')\rangle]\neq\emptyset\})^{\text{cls}},\end{aligned} \tag{8.3}$$

obtaining

$$\text{let}_5^{(x)}(f,p) = \text{let-base}_5^{(x)}(f,p)\cup\text{let-act}_5^{(x)}(f,p)$$

by definition. ∎

The following example illustrates how the elements of $\text{let}_5^{(x)}(f,p)$ are obtained from those of $p$:

**Example 8.1** Let $x\in\mathcal{IV}$ and $q$ be a finite element of $\mathbf{Q}_4$ of the form

$$\langle(\sigma_i,a_i,\sigma_i')\rangle_{i\in n}\cdot\langle(\sigma_n,\Upsilon)\rangle,\quad\text{or}\quad\langle(\sigma_i,a_i,\sigma_i')\rangle_{i\in\omega}.$$

We say $q$ is *$x$-stabilized* iff

$$\forall i\in\text{lgt}(q)[\ i+1<\text{lgt}(q)\ \Rightarrow\ \sigma_i(x)=\sigma_i'\ ];$$

also we say $q$ is *$x$-connected* iff

$$\forall i\in\text{lgt}(q)[\ i+1<\text{lgt}(q)\ \Rightarrow\ \sigma_i'(x)=\sigma_{i+1}\ ].$$

Let $v\in\mathbf{V}$ and $p\in\mathbf{P}_4$. It immediately follows from the definition of $\text{let}_5^{(x)}$ that all elements of $\text{let}_\text{x}(p)$ are $x$-stabilized. Elements of $\text{let}_\text{x}(p)$ are obtained from those elements of $p$ that are $x$-connected. That is, for finite elements of $\text{let}_5^{(x)}(p)$ we have (8.4) below for every $n>1$, and for its infinite elements we have (8.5):

$$\begin{aligned}&\langle(\sigma_i,a_i,\sigma_i'[v_i/x])\rangle_{i\in n}\cdot\langle(\sigma_n,\Upsilon)\rangle\in\text{let}_5^{(x)}(v,p)\\&\Leftrightarrow\ \langle(\sigma_i[v_i'/x],a_i,\sigma_i')\rangle_{i\in n}\cdot\langle(\sigma_n[v_n'/x],\Upsilon)\rangle\in p,\end{aligned}\tag{8.4}$$

$$\langle(\sigma_i,a_i,\sigma_i'[v_i/x])\rangle_{i\in\omega}\in\text{let}_5^{(x)}(v,p)\ \Leftrightarrow\ \langle(\sigma_i[v_i'/x],a_i,\sigma_i')\rangle_{i\in\omega}\in p,\tag{8.5}$$

where $v_i=\sigma_i(x)$, $v_i'=\sigma_{i-1}'(x)$ for $i>0$, and $v_0'=v$. Note that the sequences $\langle(\sigma_i,a_i,\sigma_i'[v_i/x])\rangle_{i\in n}\cdot\langle(\sigma_n,\Upsilon)\rangle$ and $\langle(\sigma_i,a_i,\sigma_i'[v_i/x])\rangle_{i\in\omega}$ (resp. $\langle(\sigma_i[v_i'/x],a_i,\sigma_i')\rangle_{i\in n}\cdot\langle(\sigma_n[v_n'/x],\Upsilon)\rangle$ and $\langle(\sigma_i[v_i'/x],a_i,\sigma_i')\rangle_{i\in\omega}$) are $x$-stabilized (resp. $x$-connected). Here we show only (8.4) for $n=1$ below (a proof that (8.4) holds for every $n>0$ and a proof of (8.5) can be obtained by induction in a similar fashion).

$$\begin{aligned}&\langle(\sigma_0,a_0,\sigma_0'[v_0/x]),(\sigma_1,\Upsilon)\rangle\in\text{let}_5^{(x)}(v,p)\\&\Leftrightarrow\ p[\langle(\sigma_0[v_0'/x],a_0,\sigma_0')\rangle]\neq\emptyset\\&\qquad\wedge\langle(\sigma_1,\Upsilon)\rangle\in\text{let}_5^{(x)}(v_1',p[\langle(\sigma_0[v_0'/x],a_0,\sigma_0')\rangle])\\&\Leftrightarrow\ p[\langle(\sigma_0[v_0'/x],a_0,\sigma_0')\rangle]\neq\emptyset\\&\qquad\wedge\langle(\sigma_1[v_1'/x],\Upsilon)\rangle\in p[\langle(\sigma_0[v_0'/x],a_0,\sigma_0')\rangle]\\&\Leftrightarrow\ \langle(\sigma_0[v_0'/x],a_0,\sigma_0'),(\sigma_1[v_1'/x],\Upsilon)\rangle\in p.\ \blacksquare\end{aligned}$$

As a preliminary to the definition of a semantic operation LC corresponding the construct $\mathbf{LC}(\cdot)$, we introduce the following semantic operation which *renames* channels in processes:

**Definition 8.9** (Renaming Operation) Let $\phi : \mathbf{Chan} \to \mathbf{Chan}$. First, we define $\widetilde{\phi} : \mathbf{A}_4 \to \mathbf{A}_4$ as follows: For $a \in \mathbf{A}_4$,

$$\widetilde{\phi}(a) = \begin{cases} (\phi(\mathbf{c})!, v) & \text{if } a = (\mathbf{c}!, v), \\ (\phi(\mathbf{c})?, v) & \text{if } a = (\mathbf{c}?, v), \\ a & \text{otherwise.} \end{cases}$$

By means of $\widetilde{\phi}$, a *renaming operation* $\widetilde{\Theta}_\phi : \mathbf{P}_4 \to \mathbf{P}_4$ is defined as follows: For $p \in \mathbf{P}_4$,

$$\begin{aligned}
&\widetilde{\Theta}_\phi(p) = \\
&\{q \in \mathbf{Q}_4 : \exists \langle (\sigma_i, a_i, \sigma'_i) \rangle_{i \in \omega} \in (\Sigma \times \mathbf{A}_4 \times \Sigma)^\omega \cap p[ \\
&\quad q = \langle (\sigma_i, \widetilde{\phi}(a_i), \sigma'_i) \rangle_{i \in \omega} \;]\} \\
&\cup \{q \in \mathbf{Q}_4 : \exists n \in \omega, \exists \langle (\sigma_i, \widetilde{\phi}(a_i), \sigma'_i) \rangle_{i \in n} \in (\Sigma \times \mathbf{A}_4 \times \Sigma)^n, \\
&\quad \exists (\sigma, \Gamma) \in \Sigma \times \wp(\mathbf{CS}_4)[\; \langle (\sigma_i, a_i, \sigma'_i) \rangle_{i \in n} \cdot \langle (\sigma, \delta(\widetilde{\phi}^{-1}[\Gamma])) \rangle \in p \\
&\quad \wedge q = \langle (\sigma_i, \widetilde{\phi}(a_i), \sigma'_i) \rangle_{i \in n} \cdot \langle (\sigma, \delta(\Gamma)) \rangle \;]\} \\
&\cup \{q \in \mathbf{Q}_4 : \exists n \in \omega, \exists \langle (\sigma_i, \widetilde{\phi}(a_i), \sigma'_i) \rangle_{i \in n} \in (\Sigma \times \mathbf{A}_4 \times \Sigma)^n, \\
&\quad \exists \sigma \in \Sigma[\; \langle (\sigma_i, a_i, \sigma'_i) \rangle_{i \in n} \cdot \langle (\sigma, \surd) \rangle \in p \\
&\quad \wedge q = \langle (\sigma_i, \widetilde{\phi}(a_i), \sigma'_i) \rangle_{i \in n} \cdot \langle (\sigma, \surd) \rangle \;]\}. \blacksquare
\end{aligned}$$

By means of the renaming operation $\widetilde{\Theta}_\phi$, a semantic operation corresponding to the construct $\mathbf{LC}(\cdot)$ is defined by:

**Definition 8.10** (Semantic Operation $\mathrm{LC}(\cdot)$ Corresponding to $\mathbf{LC}(\cdot)$)

**(1)** For $p \in \mathbf{P}_4$, the *semantic sort* of $p$, written $\tilde{\mathcal{S}}(p)$, is defined by:

$$\begin{aligned}
&\tilde{\mathcal{S}}(p) \\
&= \{\; \mathbf{c} \in \mathbf{Chan} : \\
&\qquad \exists w_1 \in (\Sigma \times \mathbf{A}_4 \times \Sigma)^{<\omega}, \exists (\sigma, a, \sigma') \in \Sigma \times \mathbf{A}_4 \times \Sigma[ \\
&\qquad\quad a \neq \tau \wedge \mathbf{c} = \mathrm{chan}(\mathrm{sort}(a)) \wedge p[w_1 \cdot \langle (\sigma, a, \sigma') \rangle] \neq \emptyset \;]\}.
\end{aligned}$$

**(2)** For $\mathbf{c}, \mathbf{c}'$, let $[\mathbf{c}'/\mathbf{c}]$ be the mapping from **Chan** to **Chan** such that $[\mathbf{c}'/\mathbf{c}](\mathbf{c}) = \mathbf{c}'$ and $[\mathbf{c}'/\mathbf{c}](\mathbf{c}'') = \mathbf{c}''$ for every $\mathbf{c}'' \in \mathbf{Chan} \setminus \{\mathbf{c}\}$. The operation $\mathrm{LC} : \mathbf{P}_4^{(1,0)} \to \mathbf{P}_4$ is defined as follows: Let $\mathbf{p} \in \mathbf{P}_4^{(1,0)}$. If there exists $C \in \wp_{\mathrm{f}}(\mathbf{Chan}_1)$ such that

$$\begin{aligned}
&\forall \mathbf{c}, \mathbf{c}' \in (\mathbf{Chan}_1 \setminus C)[ \\
&\quad \widetilde{\Theta}_{[\mathbf{c}'/\mathbf{c}]}(\mathbf{p}(\mathbf{c})) = \mathbf{p}(\mathbf{c}') \wedge (\mathbf{c} \neq \mathbf{c}' \Rightarrow \mathbf{c}' \notin \tilde{\mathcal{S}}(\mathbf{p}(\mathbf{c}))) \;],
\end{aligned}$$

then let $\mathrm{LC}(\mathbf{p}) = \tilde{\partial}_5^{\{\mathbf{c}^*_m\}}(\mathbf{p}(\mathbf{c}^*_m))$, where

$$\begin{aligned}
m = \min\{i \in \omega : \exists C \in \wp_{\mathrm{f}}(\mathbf{Chan}_1)[\; &\mathbf{c}^*_i \notin C \wedge \\
&\forall \mathbf{c}, \mathbf{c}' \in (\mathbf{Chan}_1 \setminus C)[\; \widetilde{\Theta}_{[\mathbf{c}'/\mathbf{c}]}(\mathbf{p}(\mathbf{c})) = \mathbf{p}(\mathbf{c}') \\
&\quad \wedge (\mathbf{c} \neq \mathbf{c}' \Rightarrow \mathbf{c}' \notin \tilde{\mathcal{S}}(\mathbf{p}(\mathbf{c}))) \;]]\}.
\end{aligned}$$

Otherwise, let $\text{LC}(\mathbf{p}) = \widetilde{\mathbf{0}}$. ∎

Let $\mathcal{I}_5$ be an *interpretation* of $\mathbf{Sig}_5$, a mapping which maps each syntactic combinator to its corresponding semantic operation:

**Definition 8.11** (Interpretation $\mathcal{I}_5$ for $\mathbf{Sig}_5$) Representing a mapping as a set of pairs as usual, we define $\mathcal{I}_5$ by:

$$\begin{aligned}\mathcal{I}_5 = \ & \{(\mathbf{0}, \widetilde{\mathbf{0}}), (\mathbf{e}, \widetilde{\mathbf{e}}), (\mathbf{out}(\cdot,\cdot), \text{out}_5), (\mathbf{in}'(\cdot,\cdot), \text{in}'_5)\} \\ & \cup\{(\mathbf{LC}(\cdot), \text{LC}), (+, \tilde{+}_4), (\|, \tilde{\|}_4), (;, \tilde{;}_4), (\mathbf{if}(\cdot,\cdot,\cdot), \text{if}_5)\} \\ & \cup\{(\mathbf{asg}(x,\cdot), \text{asg}_5^{(x)}) : x \in \mathcal{IV}\} \\ & \cup\{(\mathbf{let}(x,\cdot,\cdot), \text{let}_5^{(x)}) : x \in \mathcal{IV}\} \\ & \cup\{(\mathbf{in}(\cdot,x), \text{in}_5^{(x)}) : x \in \mathcal{IV}\} \\ & \cup\{(\partial_C(\cdot), \tilde{\partial}_5^{(C)}(\cdot)) : C \in \wp(\mathbf{Chan})\}. \ \blacksquare\end{aligned}$$

The semantic operations in $\mathcal{I}_5$ have the following useful metric properties, which are to be employed in the definition of the denotational model $\mathcal{M}_5$:

**Lemma 8.5** (Metric Properties of Semantic Operations)

**(1)** Let $(i,j,k,l,m) \in \omega^5$, $F \in \mathbf{Fun}_5^{(i,j,k,l,m)}$, $\vec{h} \in \mathbf{H}^i$, $\vec{f} \in \mathbf{F}^i$. Then, the mapping

$$(\lambda(\vec{p}_1, \vec{\mathbf{p}}, \vec{\mathbf{p}}') \in \mathbf{P}_4^k \times (\mathbf{P}_4^{(0,1)})^\ell \times (\mathbf{P}_4^{(1,0)})^m.\ \mathcal{I}_5(F)(\vec{h}\cdot\vec{f}\cdot\vec{p}_1\cdot\vec{\mathbf{p}}\cdot\vec{\mathbf{p}}'))$$

is a nonexpansive mapping from $\mathbf{P}_4^k \times (\mathbf{P}_4^{(0,1)})^\ell \times (\mathbf{P}_4^{(1,0)})^m$ to $\mathbf{P}_4$.

**(2)** For $x \in \mathcal{IV}$, $f \in \mathbf{F}$, $h \in \mathbf{H}$, the three mappings $(\lambda p \in \mathbf{P}_4.\ (\text{asg}_5^{(x)}(f) \tilde{;}_4 p))$, $(\lambda p \in \mathbf{P}_4.\ (\text{out}_5(h,f) \tilde{;}_4 p))$, and $(\lambda p \in \mathbf{P}_4.\ (\text{in}_5^{(x)}(h) \tilde{;}_4 p))$ are contractions from $\mathbf{P}_4$ to $\mathbf{P}_4$ with coefficient $\kappa$.

**(3)** Fore every $h \in \mathbf{H}$, $f \in \mathbf{F}$, $(\lambda \mathbf{p} \in \mathbf{P}_4^{(0,1)}.\ \text{in}'_5(h,\mathbf{p}))$ is a contraction from $\mathbf{P}_4^{(0,1)}$ to $\mathbf{P}_4$ with coefficient $\kappa$. ∎

In terms of the interpretation $\mathcal{I}_5$, the denotational model $\mathcal{M}_5$ is defined compositionally, with meanings of recursive programs as fixed-points in the complete metric space $\mathbf{P}_4$. For the definition of $\mathcal{M}_5$, a few preliminary definitions are needed.

**Definition 8.12** (Semantic and Syntactic Valuations)

**(1)** For $\mathbf{S} \in \widetilde{\mathcal{L}}_5^*$, let $\deg(\mathbf{S})$ denote the *depth of nesting* of function symbols (including constant symbols), $\mu$-recursions, function applications, and $\lambda$-abstractions in $\mathbf{S}$.

**(2)** For $\tilde{\mathbf{p}} \in (\mathcal{X}_{\mathcal{P}}^* \to \mathbf{P}_4^*)$, we say $\tilde{\mathbf{p}}$ respects the type iff for each $Z \in \mathcal{X}_{\mathcal{P}}^*$ the type of $\tilde{\mathbf{p}}(Z)$ is the same as that of $Z$. Let

$$\text{SeVal} = \{\tilde{\mathbf{p}} \in (\mathcal{X}_{\mathcal{P}}^* \to \mathbf{P}_4^*) : \tilde{\mathbf{p}} \text{ respects the type}\}.$$

For $\theta \in (\mathcal{X}_{\mathcal{P}}^* \nrightarrow \mathcal{L}_5^*[\emptyset])$, we say $\theta$ respects the type iff for each $Z \in \text{dom}(\theta)$ the type of $\theta(Z)$ is the same as that of $Z$. Let

$$\mathrm{SyVal} = \{\theta \in (\mathcal{X}^*_{\mathcal{P}} \nrightarrow \mathcal{L}^*_5[\emptyset]) : \theta \text{ respects the type}\}.$$

**(3)** For $\mathbf{S} \in \mathcal{L}^*_5[\mathcal{X}^*_{\mathcal{P}}]$ and $\theta \in \mathrm{SyVal}$, let $\mathbf{S}[\theta]$ denote the result of simultaneously replacing all the free occurrences of $Z \in \mathrm{dom}(\theta)$ by $\theta(Z)$. ∎

For $\mathbf{S} \in \mathcal{L}^*_5[\mathcal{X}^*_{\mathcal{P}}]$, and $\tilde{\mathbf{p}} \in \mathrm{SeVal}$, we can define $\mathcal{M}_5[\![\mathbf{S}]\!](\tilde{\mathbf{p}})$, *the meaning of* $\mathbf{S}$ *in* $\mathcal{M}_5$ *under the evaluation* $\tilde{\mathbf{p}}$, so that the conditions in the next lemma are satisfied. Moreover, it can be shown that there is a unique mapping $\mathcal{M}_5 : \mathcal{L}^*_5[\mathcal{X}^*_{\mathcal{P}}] \to (\mathrm{SeVal} \to \mathbf{P}^*_4)$ satisfying the conditions. (See § 8.B for the formal definition of $\mathcal{M}_5$.)

**Lemma 8.6** (Characterization of Denotational Model $\mathcal{M}_5$)

**(1)** *For* $\mathbf{S} \in \mathcal{X}^*_{\mathcal{P}}$, *one has* $\mathcal{M}_5[\![\mathbf{S}]\!](\tilde{\mathbf{p}}) = \tilde{\mathbf{p}}(\mathbf{S})$.

**(2)** *For* $\mathbf{S} \equiv F(\vec{g} \cdot \vec{e} \cdot \vec{S} \cdot \vec{T} \cdot \vec{U})$ *with* $F \in \mathbf{Fun}_5^{(i,j,k,\ell,m)}$, $\vec{g} \in (\mathcal{E}_{\mathcal{H}})^i$, $\vec{e} \in (\mathcal{E}_{\mathcal{F}})^j$, $\vec{S} \in \mathcal{L}^k_5$, $\vec{T} \in (\mathcal{L}_5^{(0,1)})^\ell$, $\vec{U} \in (\mathcal{L}_5^{(1,0)})^m$, *one has*

$$\mathcal{M}_5[\![\mathbf{S}]\!](\tilde{\mathbf{p}}) = \mathcal{I}_5(F)([\![\vec{g}]\!] \cdot [\![\vec{e}]\!] \cdot \mathcal{M}_5[\![\vec{S}]\!](\tilde{\mathbf{p}}) \cdot \mathcal{M}_5[\![\vec{T}]\!](\tilde{\mathbf{p}}) \cdot \mathcal{M}_5[\![\vec{U}]\!](\tilde{\mathbf{p}})).$$

**(3)** *For* $\mathbf{S} \equiv (\lambda \vec{\eta} \cdot \vec{\xi}.\ \mathbf{S}')$, *with* $\mathrm{lgt}(\vec{\eta}) = i$ *and* $\mathrm{lgt}(\vec{\xi}) = j$, *one has*

$$\mathcal{M}_5[\![\mathbf{S}]\!](\tilde{\mathbf{p}}) = (\lambda \vec{\mathbf{c}} \cdot \vec{v} \in \mathbf{Chan}^i \cdot \mathbf{V}^j.\ \mathcal{M}_5[\![\mathbf{S}'[(\vec{\mathbf{c}} \cdot \vec{v})/(\vec{\eta} \cdot \vec{\xi})]]\!](\tilde{\mathbf{p}})).$$

**(4)** *For* $\mathbf{S} \equiv (\mu X^{(i,j)}.\ S^{(i,j)})$ *with* $(i,j) \in \omega^2$, *the mapping*

$$(\lambda \mathbf{p} \in \mathbf{P}_4^{(i,j)}.\ \mathcal{M}_5[\![S^{(i,j)}]\!](\tilde{\mathbf{p}}[\mathbf{p}/X^{(i,j)}]))$$

*is a contraction from* $\mathbf{P}_4^{(i,j)}$ *to itself (for the notation* $\tilde{\mathbf{p}}[p^{(i,j)}/X^{(i,j)}]$ *see Notation 2.3), and*

$$\mathcal{M}_5[\![\mathbf{S}]\!](\tilde{\mathbf{p}}) = \mathrm{fix}(\lambda \mathbf{p} \in \mathbf{P}_4^{(i,j)}.\ \mathcal{M}_5[\![(S^{(i,j)})]\!](\tilde{\mathbf{p}}[\mathbf{p}/X^{(i,j)}])).$$

*By definition, the following holds for every* $\mathbf{p} \in \mathbf{P}_4^{(i,j)}$:

$$\mathbf{p} = \mathcal{M}_5[\![(\mu X^{(i,j)}.\ S^{(i,j)})]\!](\tilde{\mathbf{p}}) \Leftrightarrow \mathbf{p} = \mathcal{M}_5[\![(S^{(i,j)})]\!](\tilde{\mathbf{p}}[\mathbf{p}/X^{(i,j)}]). \quad (8.6)$$

**(5)** *The interpretation of function application is* nonuniform *in the following sense: For* $\mathbf{S} \equiv S^{(i,j)}(\vec{g}^{(i)} \cdot \vec{e}^{(j)})$,

$$\mathcal{M}_5[\![\mathbf{S}]\!](\tilde{\mathbf{p}}) = \bigcup_{\sigma \in \Sigma} [\,\mathcal{M}_5[\![S^{(i,j)}]\!](\tilde{\mathbf{p}})([\![\vec{g}^{(i)}]\!](\sigma) \cdot [\![\vec{e}^{(j)}]\!](\sigma))\langle\sigma\rangle\,].$$ ∎

From the definition of $\mathcal{M}_5$, we immediately have the following proposition stating that the value $\mathcal{M}_5[\![\mathbf{S}]\!](\tilde{\mathbf{p}})$ does not depend on $\tilde{\mathbf{p}}(Z)$ for $Z \notin \mathrm{FV}(\mathbf{S})$:

**Proposition 8.1** *Let* $\mathbf{S} \in \mathcal{L}^*_5$. *Then for every* $\tilde{\mathbf{p}}_1, \tilde{\mathbf{p}}_2 \in \mathrm{SeVal}$ *such that* $\tilde{\mathbf{p}}_1 \ \mathrm{FV}(\mathbf{S}) = \tilde{\mathbf{p}}_2 \ \mathrm{FV}(\mathbf{S})$, *one has* $\mathcal{M}_5[\![\mathbf{S}]\!](\tilde{\mathbf{p}}_1) = \mathcal{M}_5[\![\mathbf{S}]\!](\tilde{\mathbf{p}}_2)$. ∎

Thus, for $\mathbf{s} \in \mathcal{L}^*_5[\emptyset]$, the value $\mathcal{M}_5[\![\mathbf{s}]\!](\tilde{\mathbf{p}})$ does not depend on the choice of $\tilde{\mathbf{p}}$; we denote the value by $\dot{\mathcal{M}}_5[\![\mathbf{s}]\!]$, and we define a function $\dot{\mathcal{M}}_5 : \mathcal{L}^*_5[\emptyset] \to \mathbf{P}_4$ by:

$$\dot{\mathcal{M}}_5^* = (\lambda \mathrm{s} \in \mathcal{L}_5^*[\emptyset].\ \dot{\mathcal{M}}_5[\![\mathrm{s}]\!]).$$

The model $\mathcal{M}_5$ is *compositional* in the following sense:

**Lemma 8.7** (Compositionality of $\mathcal{M}_5$) *Let* $\mathbf{S} \in \mathcal{L}_5^*[\mathcal{X}_\mathcal{P}^*]$, $(i,j) \in \omega^2$, $Z \in \mathcal{X}^{(i,j)}$, $\mathrm{s} \in \mathcal{L}_5^{(i,j)}[\emptyset]$, *and* $\tilde{\mathbf{p}} \in \mathrm{SeVal}$. *Then*

$$\mathcal{M}_5[\![(\mathbf{S}[\mathrm{s}/Z])]\!](\tilde{\mathbf{p}}) = \mathcal{M}_5[\![\mathbf{S}]\!](\tilde{\mathbf{p}}[\dot{\mathcal{M}}_5[\![\mathrm{s}]\!](\tilde{\mathbf{p}})/Z]). \blacksquare$$

## 8.5 Semantic Equivalence between $\mathcal{C}_5$ and $\mathcal{M}_5$

In this section, we establish the semantic equivalence between $\mathcal{C}_5$ and $\mathcal{M}_5$, from which the correctness of $\mathcal{M}_5$ w.r.t. $\mathcal{O}_5$ and $\mathcal{O}_5^*$ immediately follows.

First, it is shown that $\mathcal{C}_5$ is compositional w.r.t. all combinators in $\mathbf{Sig}_5$:

**Lemma 8.8** (Compositionality of $\mathcal{C}_5$) *Let* $i, j, k, \ell, m \in \omega$, *and* $F \in \mathbf{Fun}_5^{(i,j,k,\ell,m)}$. *Then, for every* $\vec{h} \in (\mathbf{E}_\mathcal{H})^i$, $\vec{e} \in (\mathcal{E}_\mathcal{F})^j$, $\vec{s} \in \mathcal{L}_5^k$, $\vec{t} \in (\mathcal{L}_5^{(0,1)})^\ell$, $\vec{u} \in (\mathcal{L}_5^{(1,0)})^m$, *the following holds:*

$$\begin{aligned} &\mathcal{C}_5[\![F(\vec{h}\cdot\vec{e}\cdot\vec{s}\cdot\vec{t}\cdot\vec{u})]\!] \\ &= \mathcal{I}_5(F)([\![\vec{h}\,]\!]\cdot[\![\vec{e}\,]\!]\cdot\mathcal{C}_5[\![\vec{s}\,]\!]\cdot\mathcal{C}_5[\![\vec{t}\,]\!]\cdot\mathcal{C}_5[\![\vec{u}\,]\!]). \blacksquare \end{aligned} \tag{8.7}$$

We will prove claim (8.7) for $F \equiv \mathbf{let}(x,\cdot,\cdot)$ and for $F = \mathbf{LC}(\cdot)$; for the other combinators except $\partial_C(\cdot)$, the claim has been established in Lemma 7.19; for $\partial_C(\cdot)$, the claim can be established in a similar fashion to the proof of Lemma 7.19.

As preliminaries to the proof, we present four lemmas (all of them can be proved easily and their proofs are omitted). The first two state useful properties of the operation $\mathrm{let}_5^{(x)}(\cdot)$; we will employ them for establishing (8.7) for $F \equiv \mathbf{let}(x,\cdot,\cdot)$.

**Lemma 8.9** (Distributivity of $\mathrm{let}_5^{(x)}$) *For* $x \in \mathcal{IV}$ *and* $f \in \mathbf{F}$, *the operation* $\mathrm{let}_5^{(x)}(f,\cdot)$ *is distributive in that*

$$\forall p_1, p_2 \in \mathbf{P}_4[\ \mathrm{let}_5^{(x)}(f, p_1 \cup p_2) = \mathrm{let}_5^{(x)}(f, p_1) \cup \mathrm{let}_5^{(x)}(f, p_2)\ ]. \blacksquare$$

Let us say $p \in \mathbf{P}_4$ is image-finite iff for every $r \in (\Sigma \times \mathbf{A}_4 \times \Sigma)^{<\omega}$, the following holds:

$$p[r] \neq \emptyset \Rightarrow \forall \sigma \in \Sigma, \forall a \in \mathbf{A}_4[\ \{\sigma' \in \Sigma : p[r][\langle(\sigma, a, \sigma')\rangle] \neq \emptyset\} \text{ is finite }].$$

Then, one has the following lemma:

**Lemma 8.10 (1)** *For every* $s \in \mathcal{L}_5[\emptyset]$, $\mathcal{C}_5[\![s]\!]$ *is image-finite.*

**(2)** *For image-finite* $p \in \mathbf{P}_4$, *we can omit taking closure in (8.3).* $\blacksquare$

Next, two lemmas are presented as preliminaries to the proof of (8.7) for $F = \mathbf{LC}(\cdot)$.

**Lemma 8.11** *Let* $\mathbf{s} \in \mathcal{L}_5^{(1,0)}$, *and let* $\mathbf{c}, \mathbf{c}' \in \mathbf{Chan}$ *be such that* $\mathbf{c}, \mathbf{c}' \notin \mathcal{S}(\mathbf{s})$ *and* $\mathbf{c} \neq \mathbf{c}'$. *Then,* $\mathbf{c}' \notin \tilde{\mathcal{S}}(\mathcal{C}_5[\![\mathbf{s}(\mathbf{c})]\!])$ *and*

$$\widetilde{\Theta}_{[\mathbf{c}'/\mathbf{c}]}(\mathcal{C}_5[\![\mathbf{s}(\mathbf{c})]\!]) = \mathcal{C}_5[\![\mathbf{s}(\mathbf{c}')]\!]. \blacksquare$$

**Lemma 8.12** *Let* $p \in \mathbf{P}_4$, $\mathbf{c}, \mathbf{c}' \in \mathbf{Chan}$ *be such that* $\mathbf{c}' \notin \tilde{\mathcal{S}}(p)$. *Then,*

$$\tilde{\partial}_5^{\{\mathbf{c}'\}}(\widetilde{\Theta}_{[\mathbf{c}'/\mathbf{c}]}(p)) = \tilde{\partial}_5^{\{\mathbf{c}\}}(p). \blacksquare$$

For notational convenience, we extend the domain of $\mathcal{C}_5$ by:

**Definition 8.13** **(1)** For $(i,j) \neq (0,0)$ and $\mathbf{s} \in \mathcal{L}_5^{(i,j)}$, let

$$\mathcal{C}_5[\![\mathbf{s}]\!] = (\lambda\vec{\mathbf{c}} \cdot \vec{v} \in \mathbf{Chan}^i \cdot \mathbf{V}^j.\ \mathcal{C}_5[\![\mathbf{s}(\vec{\mathbf{c}} \cdot \vec{v})]\!]).$$

**(2)** For $n \in \omega$ and $\vec{\mathbf{s}} \in (\mathcal{L}_5^*)^n$, let

$$\mathcal{C}_5[\![\vec{\mathbf{s}}]\!] = \langle \mathcal{C}_5[\![\vec{\mathbf{s}}(i)]\!]\rangle_{i\in n}.$$

**(3)** For $\theta \in \mathrm{SyVal}$, let

$$\mathcal{C}_5[\![\theta]\!] = \mathcal{C}_5 \circ \theta = \langle \mathcal{C}_5[\![\theta(Z)]\!]\rangle_{Z\in\mathrm{dom}(\theta)}. \blacksquare$$

**Proof of Lemma 8.8.** **(1)** First, we will prove (8.7) for $F \equiv \mathbf{let}(x,\cdot,\cdot)$, i.e., that the following holds for every $x \in \mathcal{IV}$, $e \in \mathcal{E}_{\mathcal{F}}$, and $s \in \mathcal{L}_5[\emptyset]$:

$$\mathcal{C}_5[\![\mathbf{let}(x,e,s)]\!] = \mathrm{let}_5^{(x)}([\![e]\!], \mathcal{C}_5[\![s]\!]). \tag{8.8}$$

Fix $x \in \mathcal{IV}$, and let

$$\mathbf{K} = (\mathcal{L}_5[\emptyset] \times \mathcal{E}_{\mathcal{F}} \to \mathbf{P}_4).$$

Let us define $F, G \in \mathbf{K}$ as follows: For $(e,s) \in \mathcal{L}_5 \times \mathcal{E}_{\mathcal{F}}$,

$$F(e,s) = \mathcal{C}_5[\![\mathbf{let}(x,e,s)]\!], \quad G(e,s) = \mathrm{let}_5^{(x)}([\![e]\!], \mathcal{C}_5[\![s]\!]).$$

Let us show that $F = G$, which is a reformulation of (8.8), by showing that both $F$ and $G$ are the fixed-point of a higher-order contraction $\mathcal{K} : \mathbf{K} \to \mathbf{K}$ defined as follows: For $K \in \mathbf{K}$, and $(e,s) \in \mathcal{E}_{\mathcal{F}} \times \mathcal{L}_5[\emptyset]$,

$$\begin{aligned}
&\mathcal{K}(K)(e,s) = \\
&\{\langle(\sigma,\surd)\rangle \in (\Sigma \times \{\surd\})^1 : \surd \in \mathrm{act}_5(s, \sigma[[\![e]\!](\sigma)/x])\} \\
&\cup\{\langle(\sigma,\delta(\Gamma))\rangle \in (\Sigma \times \wp(\mathbf{CS}_4))^1 : \mathrm{act}_5(s,\sigma[[\![e]\!](\sigma)/x]) \subseteq \mathbf{CS}_4 \times \mathbf{V} \\
&\quad \wedge \mathrm{act}_5(s,\sigma[[\![e]\!](\sigma)/x]) \cap \Gamma = \emptyset\} \\
&\cup\bigcup\{\langle(\sigma,a,\sigma'[\sigma(x)/x])\rangle \cdot K(\sigma'(x),s') : \sigma,\sigma' \in \Sigma \wedge a \in \mathbf{A}_4 \wedge s' \in \mathcal{L}_5 \\
&\quad\quad \wedge (s,\sigma[[\![e]\!](\sigma)/x]) \xrightarrow{a}_5 (s',\sigma')\}.
\end{aligned}$$

It is easy to verify that $\mathcal{K}$ is a contraction from $\mathbf{K}$ to itself. Also it follows from the definition of $\mathcal{C}_5$ that $F = \mathcal{K}(F)$, i.e.

$$F = \mathrm{fix}(\mathcal{K}). \tag{8.9}$$

Let us show $G = \mathcal{K}(G)$. For $(e, s) \in \mathcal{E}_{\mathcal{F}} \times \mathcal{L}_5[\emptyset]$, it follows from the definition of $\mathrm{let}_5^{(x)}$ that

$$\begin{aligned} G(e,s) &= \mathrm{let}_5^{(x)}([\![e]\!], \mathcal{C}_5[\![s]\!]) \\ &= \text{let-base}_5^{(x)}([\![e]\!], \mathcal{C}_5[\![s]\!]) \cup \text{let-act}_5^{(x)}([\![e]\!], \mathcal{C}_5[\![s]\!]). \end{aligned}$$

Further it follows from the definition of $\text{let-base}_5^{(x)}$ that

$$\begin{aligned} &\text{let-base}_5^{(x)}([\![e]\!], \mathcal{C}_5[\![s]\!]) \\ &= \{\langle(\sigma, \surd)\rangle \in (\Sigma \times \{\surd\})^1 : \surd \in \mathrm{act}_5(s, \sigma[[\![e]\!](\sigma)/x])\} \\ &\quad \cup \{\langle(\sigma, \delta(\Gamma))\rangle \in (\Sigma \times \wp(\mathbf{CS}_4))^1 : \\ &\qquad \mathrm{act}_5(s, \sigma[[\![e]\!](\sigma)/x]) \subseteq \mathbf{CS}_4 \times \mathbf{V} \\ &\qquad \wedge \mathrm{act}_5(s, \sigma[[\![e]\!](\sigma)/x]) \cap \Gamma = \emptyset\}. \end{aligned} \tag{8.10}$$

Also one has

$$\begin{aligned} &\text{let-act}_5^{(x)}([\![e]\!], \mathcal{C}_5[\![s]\!]) \\ &= \bigcup\{\langle(\sigma, a, \sigma'[\sigma(x)/x])\rangle \cdot \mathrm{let}_5^{(x)}(\sigma'(x), (\mathcal{C}_5[\![s]\!])[\langle(\sigma[[\![e]\!](\sigma)/x], a, \sigma')\rangle]) : \\ &\qquad (\mathcal{C}_5[\![s]\!])[\langle(\sigma[[\![e]\!](\sigma)/x], a, \sigma')\rangle] \neq \emptyset\} \\ &\qquad\qquad \text{(by the definition of } \mathrm{let}_5^{(x)}\text{, where taking closure of the} \\ &\qquad\qquad \text{term } \textstyle\bigcup\{\cdots\} \text{ can be omitted by Lemma 8.10 (2))} \\ &= \bigcup\{\langle(\sigma, a, \sigma'[\sigma(x)/x])\rangle \\ &\qquad \cdot \mathrm{let}_5^{(x)}(\sigma'(x), \textstyle\bigcup\{\mathcal{C}_5[\![s']\!] : (s, \sigma[[\![e]\!](\sigma)/x]) \xrightarrow{a}_5 (s', \sigma')\}) : \\ &\qquad \exists s'[\ (s, \sigma[[\![e]\!](\sigma)/x]) \xrightarrow{a}_5 (s', \sigma')\ ]\} \\ &\qquad\qquad \text{(by the definition of } \mathcal{C}_5) \\ &= \bigcup\{\bigcup\{\langle(\sigma, a, \sigma'[\sigma(x)/x])\rangle \\ &\qquad\quad \cdot \mathrm{let}_5^{(x)}(\sigma'(x), \mathcal{C}_5[\![s']\!]) : (s, \sigma[[\![e]\!](\sigma)/x]) \xrightarrow{a}_5 (s', \sigma')\} : \\ &\qquad\quad \exists s'[\ (s, \sigma[[\![e]\!](\sigma)/x]) \xrightarrow{a}_5 (s', \sigma')\ ]\} \\ &\qquad\qquad \text{(by Lemma 8.9)} \\ &= \bigcup\{\langle(\sigma, a, \sigma'[\sigma(x)/x])\rangle \cdot \mathrm{let}_5^{(x)}(\sigma'(x), \mathcal{C}_5[\![s']\!]) : \\ &\qquad (s, \sigma[[\![e]\!](\sigma)/x]) \xrightarrow{a}_5 (s', \sigma')\} \\ &= \bigcup\{\langle(\sigma, a, \sigma'[\sigma(x)/x])\rangle \cdot G(\sigma'(x), s') : \\ &\qquad (s, \sigma[[\![e]\!](\sigma)/x]) \xrightarrow{a}_5 (s', \sigma')\}. \end{aligned}$$

By this and (8.10), one has

$$G(e,s) = \mathrm{let}_x([\![e]\!], s) = \mathcal{K}(G)(e,s).$$

Since $e$ and $s$ are arbitrary, one has $G = \mathcal{K}(G)$, i.e., $G = \mathrm{fix}(\mathcal{K})$. From this and (8.9), the desired result follows.

**(2)** Next, we will prove (8.7) for $F \equiv \mathbf{LF}(\cdot)$, i.e., that the following holds for every $s \in \mathcal{L}_5^{(1,0)}$:

$$\mathcal{C}_5[\![\mathbf{LC}(\mathrm{s})]\!] = \mathrm{LC}(\mathcal{C}_5[\![\mathrm{s}]\!]). \tag{8.11}$$

Let $\mathrm{s} \in \mathcal{L}_5^{(1,0)}$, and set $\mathbf{p} = \mathcal{C}_5[\![\mathrm{s}]\!]$. First, one has

$$\mathcal{C}_5[\![\mathbf{LC}(\mathrm{s})]\!] = \mathcal{C}_5[\![\partial_{\{\mathbf{c}_m^*\}}(\mathrm{s}(\mathbf{c}_m^*))]\!], \tag{8.12}$$

with

$$m = \min\{i \in \omega : \mathbf{c}_i^* \notin \mathcal{S}(\mathrm{s})\},$$

since the only rule by which one can derive transitions of $(\mathrm{s}, \sigma)$ is rule (10) in Definition 8.5. Next one has, by Lemma 8.11,

$$\forall \mathbf{c}, \mathbf{c}' \in (\mathbf{Chan}_1 \setminus \tilde{\mathcal{S}}(\mathbf{p}))[\ \widetilde{\Theta}_{[\mathbf{c}'/\mathbf{c}]}(\mathbf{p}(\mathbf{c})) = \mathbf{p}(\mathbf{c}') \\ \wedge (\mathbf{c} \neq \mathbf{c}' \Rightarrow \mathbf{c}' \notin \tilde{\mathcal{S}}(\mathbf{p}(\mathbf{c})))\ ].$$

Thus, by Definition 8.10, one has

$$\mathrm{LC}(\mathbf{p}) = \tilde{\partial}_5^{\{\mathbf{c}_n^*\}}(\mathbf{p}(\mathbf{c}_n^*)), \tag{8.13}$$

with

$$\begin{aligned} n = \min\{i \in \omega : \exists C \in \wp_{\mathrm{f}}(\mathbf{Chan}_1)[\ \mathbf{c}_i^* \notin C \wedge \\ \forall \mathbf{c}, \mathbf{c}' \in (\mathbf{Chan}_1 \setminus C)[\widetilde{\Theta}_{[\mathbf{c}'/\mathbf{c}]}(\mathbf{p}(\mathbf{c})) = \mathbf{p}(\mathbf{c}') \\ \wedge (\mathbf{c} \neq \mathbf{c}' \Rightarrow \mathbf{c}' \notin \tilde{\mathcal{S}}(\mathbf{p}(\mathbf{c}))) \wedge\ ]\}. \end{aligned} \tag{8.14}$$

By (8.12) and (8.13), it is sufficient to show

$$\mathcal{C}_5[\![\partial_{\{\mathbf{c}_m^*\}}(\mathrm{s}(\mathbf{c}_m^*))]\!] = \tilde{\partial}_5^{\{\mathbf{c}_n^*\}}(\mathbf{p}(\mathbf{c}_n^*)), \tag{8.15}$$

in order to show (8.11). Let us prove (8.15).

By (8.14), there is $C \in \wp_{\mathrm{f}}(\mathbf{Chan}_1)$ such that

$$\mathbf{c}_n^* \notin C \wedge \\ \forall \mathbf{c}, \mathbf{c}' \in (\mathbf{Chan}_1 \setminus C)[\ \widetilde{\Theta}_{[\mathbf{c}'/\mathbf{c}]}(\mathbf{p}(\mathbf{c})) = \mathbf{p}(\mathbf{c}') \\ \wedge (\mathbf{c} \neq \mathbf{c}' \Rightarrow \mathbf{c}' \notin \tilde{\mathcal{S}}(\mathbf{p}(\mathbf{c})))\ ].$$

Let us fix such $C$, and let $\mathbf{c} \in (\mathbf{Chan}_1 \setminus (\mathcal{S}(\mathrm{s}) \cup C))$. Then, one has

$$\begin{aligned} &\mathcal{C}_5[\![\partial_{\{\mathbf{c}_m^*\}}(\mathrm{s}(\mathbf{c}_m^*))]\!] = \tilde{\partial}_5^{\{\mathbf{c}_m^*\}}(\mathcal{C}_5[\![\mathrm{s}(\mathbf{c}_m^*)]\!]) \\ &= \tilde{\partial}_5^{\{\mathbf{c}_m^*\}}(\widetilde{\Theta}_{[\mathbf{c}_m^*/\mathbf{c}]}(\mathcal{C}_5[\![\mathrm{s}(\mathbf{c})]\!])) \quad \text{(by Lemma 8.11)} \\ &= \tilde{\partial}_5^{\{\mathbf{c}\}}(\mathcal{C}_5[\![\mathrm{s}(\mathbf{c})]\!]) \quad \text{(by Lemma 8.12)}. \end{aligned} \tag{8.16}$$

Also one has

$$\begin{aligned} &\tilde{\partial}_5^{\{\mathbf{c}_n^*\}}(\mathbf{p}(\mathbf{c}_n^*)) = \tilde{\partial}_5^{\{\mathbf{c}_n^*\}}(\widetilde{\Theta}_{[\mathbf{c}_n^*/\mathbf{c}]}(\mathbf{p}(\mathbf{c}))) \quad \text{(by Lemma 8.11)} \\ &= \tilde{\partial}_5^{\{\mathbf{c}\}}(\mathbf{p}(\mathbf{c})) \quad \text{(by Lemma 8.12)} \quad = \tilde{\partial}_5^{\{\mathbf{c}\}}(\mathcal{C}_5[\![\mathrm{s}(\mathbf{c})]\!]). \end{aligned} \tag{8.17}$$

By (8.16) and (8.17), one has the desired result (8.15). ■

By means of Lemma 8.8, which states the compositionality of $\mathcal{C}_5$ w.r.t. the combinators in $\mathbf{Sig}_5$, one has the equivalence between $\mathcal{C}_5$ and $\mathcal{M}_5$:

**Lemma 8.13** (Semantic Equivalence between $\mathcal{C}_5$ and $\mathcal{M}_5$)

**(1)** *For* $\mathbf{S} \in \mathcal{L}_5^*[\mathcal{X}_{\mathcal{P}}^*]$, $\tilde{\mathbf{p}} \in \mathrm{SeVal}$, *and* $\theta \in \mathrm{SyVal}$ *with* $\mathrm{FV}(\mathbf{S}) \subseteq \mathrm{dom}(\theta)$, *one has*

$$\mathcal{C}_5[\![\mathbf{S}[\theta]]\!] = \mathcal{M}_5[\![\mathbf{S}]\!](\tilde{\mathbf{p}}[\mathcal{C}_5[\![\theta]\!]]). \tag{8.18}$$

**(2)** *For* $\mathrm{s} \in \mathcal{L}_5^*[\emptyset]$, $\mathcal{C}_5[\![\mathrm{s}]\!] = \mathcal{M}_5[\![\mathrm{s}]\!]$. ∎

**Proof.** **(1)** Let us prove claim (8.18) by induction on deg(**S**).
**Induction Base.** Suppose $\deg(\mathbf{S}) = 0$, i.e., $\mathbf{S} \in \mathcal{X}_{\mathcal{P}}^*$. Then,

$$\mathcal{C}_5[\![\mathbf{S}[\theta]]\!] = \mathcal{C}_5[\![\theta(\mathbf{S})]\!] = \mathcal{M}_5[\![\mathbf{S}]\!](\mathcal{C}_5[\![\theta]\!]).$$

**Induction Step.** Let $n > 0$, and assume claim (8.18) has been proved for **S** with $\deg(\mathbf{S}) < n$. Fix $\mathbf{S} \in \mathcal{L}_5^*[\mathcal{X}_{\mathcal{P}}^*]$ such that $\deg(\mathbf{S}) = n$. Let us show (8.18) for $S$. We distinguish four cases according to the form of $S$.
**Case 1.** Suppose $\mathbf{S} \equiv F(\vec{g} \cdot \vec{e} \cdot \vec{S} \cdot \vec{T} \cdot \vec{U})$ with $F \in \mathbf{Fun}_5^{(i,j,k,\ell,m)}$ and $\vec{g} \in (\mathcal{E}_{\mathcal{H}})^i$, $\vec{e} \in (\mathcal{E}_{\mathcal{F}})^j$, $\vec{S} \in (\mathcal{L}_5[\mathcal{X}_{\mathcal{P}}^*])^k$, $\vec{T} \in (\mathcal{L}_5^{(0,1)}[\mathcal{X}_{\mathcal{P}}^*])^\ell$, $\vec{U} \in (\mathcal{L}_5^{(1,0)}[\mathcal{X}_{\mathcal{P}}^*])^m$. Then,

$$\begin{aligned}
&\mathcal{C}_5[\![\mathbf{S}[\theta]]\!](\tilde{\mathbf{p}}) = \mathcal{I}_5(F)([\![\vec{g}]\!] \cdot [\![\vec{e}]\!] \cdot \mathcal{C}_5[\![\vec{S}[\theta]]\!](\tilde{\mathbf{p}}) \cdot \mathcal{C}_5[\![\vec{T}[\theta]]\!](\tilde{\mathbf{p}}) \cdot \mathcal{C}_5[\![\vec{U}[\theta]]\!](\tilde{\mathbf{p}}))\\
&= \mathcal{I}_5(F)([\![\vec{g}]\!] \cdot [\![\vec{e}]\!] \cdot \mathcal{M}_5[\![\vec{S}]\!](\tilde{\mathbf{p}}[\mathcal{C}_5[\![\theta]\!]]) \cdot \mathcal{M}_5[\![\vec{T}]\!](\tilde{\mathbf{p}}[\mathcal{C}_5[\![\theta]\!]]) \cdot \mathcal{M}_5[\![\vec{U}]\!](\tilde{\mathbf{p}}[\mathcal{C}_5[\![\theta]\!]]))\\
&\qquad \text{(by the induction hypothesis)}\\
&= \mathcal{M}_5[\![S]\!](\mathcal{C}_5[\![\theta]\!]).
\end{aligned}$$

**Case 2.** Suppose $\mathbf{S} \equiv (\lambda\vec{\eta} \cdot \vec{\xi}.\ \mathbf{S}')$. Then,

$$\begin{aligned}
&\mathcal{C}_5[\![\mathbf{S}[\theta]]\!] = \mathcal{C}_5[\![(\lambda\vec{\eta} \cdot \vec{\xi}.\ \mathbf{S}'[\theta])]\!] = (\lambda\vec{\mathbf{c}} \cdot \vec{v}.\ \mathcal{C}_5[\![(\mathbf{S}'[\theta])[(\vec{\mathbf{c}} \cdot \vec{v})/(\vec{\eta} \cdot \vec{\xi})]]\!])\\
&= (\lambda\vec{\mathbf{c}} \cdot \vec{v}.\ \mathcal{C}_5[\![(\mathbf{S}'[(\vec{\mathbf{c}} \cdot \vec{v})/(\vec{\eta} \cdot \vec{\xi})])[\theta]]\!])\\
&\qquad \text{(by the definition of } \mathcal{C}_5)\\
&= (\lambda\vec{\mathbf{c}} \cdot \vec{v}.\ \mathcal{M}_5[\![\mathbf{S}'[(\vec{\mathbf{c}} \cdot \vec{v})/(\vec{\eta} \cdot \vec{\xi})]]\!](\tilde{\mathbf{p}}[\mathcal{C}_5[\![\theta]\!]]))\\
&\qquad \text{(by the induction hypothesis)}\\
&= \mathcal{M}_5[\![(\lambda\vec{\eta} \cdot \vec{\xi}.\ \mathbf{S}')]\!](\tilde{\mathbf{p}}[\mathcal{C}_5[\![\theta]\!]]) \quad \text{(by the definition of } \mathcal{M}_5)\\
&= \mathcal{M}_5[\![\mathbf{S}]\!](\tilde{\mathbf{p}}[\mathcal{C}_5[\![\theta]\!]]).
\end{aligned}$$

**Case 3.** Suppose $\mathbf{S} \equiv (\mu Z.\ \mathbf{S}')$. Then,

$$\begin{aligned}
&\mathcal{C}_5[\![\mathbf{S}[\theta]]\!]\\
&= \mathcal{C}_5[\![(\mu Z.\ (\mathbf{S}')[\theta])]\!] \quad (\text{where } \hat{\theta} = \theta \quad (\mathrm{dom}(\theta) \setminus \{Z\}))\\
&= \mathcal{C}_5[\![(\mathbf{S}'[\theta])[\mathbf{S}[\theta]/Z]]\!]\\
&= \mathcal{C}_5[\![\mathbf{S}'[\theta[\mathbf{S}[\theta]/Z]]]\!] \quad (\text{for the notation } \theta[\mathbf{S}[\theta]/Z] \text{ see Notation 2.3})\\
&= \mathcal{M}_5[\![\mathbf{S}']\!](\tilde{\mathbf{p}}[\mathcal{C}_5[\![\hat{\theta} \cup \{(Z, \mathbf{S}[\theta])\}]\!]]) \quad \text{(by the induction hypothesis)}\\
&= \mathcal{M}_5[\![\mathbf{S}']\!](\tilde{\mathbf{p}}[\mathcal{C}_5[\![\theta]\!]][\mathcal{C}_5[\![\mathbf{S}[\theta]]\!]/Z]).
\end{aligned}$$

Thus,

$$\mathcal{C}_5[\![\mathbf{S}[\theta]]\!] = \mathcal{M}_5[\![\mathbf{S}']\!](\tilde{\mathbf{p}}[\mathcal{C}_5[\![\theta]\!]][\mathcal{C}_5[\![\mathbf{S}[\theta]]\!]/Z]).$$

By this and (8.6), one has

$$\mathcal{C}_5[\![\mathbf{S}[\theta]]\!] = \mathcal{M}_5[\![(\mu Z.\ \mathbf{S}')]\!](\tilde{\mathbf{p}}[\mathcal{C}_5[\![\theta]\!]]) = \mathcal{M}_5[\![\mathbf{S}]\!](\tilde{\mathbf{p}}[\mathcal{C}_5[\![\theta]\!]]).$$

**Case 4.** Suppose $\mathbf{S} \equiv \mathbf{S}'(\vec{g}\cdot\vec{e})$. Then,

$$\mathcal{C}_5[\![\mathbf{S}[\theta]]\!] = \mathcal{C}_5[\![\mathbf{S}'[\theta](\vec{g}\cdot\vec{e})]\!] = \bigcup_{\sigma\in\Sigma}[\mathcal{C}_5[\![\mathbf{S}'[\theta]([\![\vec{g}]\!](\sigma)\cdot[\![\vec{e}]\!](\sigma))]\!]\langle\sigma\rangle]$$

(by Lemma 8.3)

$$= \bigcup_{\sigma\in\Sigma}[\mathcal{C}_5[\![\mathbf{S}'[\theta]]\!]([\![\vec{g}]\!](\sigma)\cdot[\![\vec{e}]\!](\sigma))\langle\sigma\rangle] \quad \text{(by Definition 8.13 (1))}$$

$$= \bigcup_{\sigma\in\Sigma}[\mathcal{M}_5[\![\mathbf{S}']\!](\tilde{\mathbf{p}}[\mathcal{C}_5[\![\theta]\!]])([\![\vec{g}]\!](\sigma)\cdot[\![\vec{e}]\!](\sigma))\langle\sigma\rangle]$$

(by the induction hypothesis)

$$= \mathcal{M}_5[\![\mathbf{S}'(\vec{g}\cdot\vec{e})]\!](\tilde{\mathbf{p}}[\mathcal{C}_5[\![\theta]\!]]). \quad \text{(by Lemma 8.6 (5))}$$

Thus, one has the desired result.

**(2)** This part immediately follows from part (1). ■

From the above lemma and Lemma 8.4, the following lemma immediately follows:

**Corollary 8.1** (Correctness of $\mathcal{M}_5$ w.r.t. $\mathcal{O}_5$ and $\mathcal{O}_5^*$)

**(1)** $\forall s \in \mathcal{L}_5[\emptyset][\ \mathcal{O}_5[\![s]\!] = \mathcal{A}_4(\dot{\mathcal{M}}_5[\![s]\!])\ ]$.

**(2)** $\forall s \in \mathcal{L}_5[\emptyset][\ \mathcal{O}_5^*[\![s]\!] = \mathcal{A}_4^*(\dot{\mathcal{M}}_5[\![s]\!])\ ]$. ■

## 8.6 Full Abstractness of $\mathcal{M}_5$ w.r.t. $\mathcal{O}_5$ and $\mathcal{O}_5^*$

Having established the compositionality of $\mathcal{M}_5$ (Lemma 8.7) and its correctness w.r.t. $\mathcal{O}_5$ and $\mathcal{O}_5^*$ (Corollary 8.1), the full abstractness of $\mathcal{M}_5$ w.r.t. $\mathcal{O}_5$ and $\mathcal{O}_5^*$ immediately follows from the full abstractness results in Chapter 7.

**Theorem 8.1** (Full Abstractness of $\mathcal{M}_5$ w.r.t. $\mathcal{O}_5$ and $\mathcal{O}_5^*$) *If* $\mathbf{V}$ *is infinite, then the following hold:*

**(1)** *For every* $s_1, s_2 \in \mathcal{L}_5[\emptyset]$,

$$\dot{\mathcal{M}}_5[\![s_1]\!] = \dot{\mathcal{M}}_5[\![s_2]\!] \Leftrightarrow \forall X \in \mathcal{X}_{\mathcal{P}}, \forall S \in \mathcal{L}_5[X][\ \mathcal{O}_5[\![S[s_1/X]]\!] = \mathcal{O}_5[\![S[s_2/X]]\!]\ ]. \tag{8.19}$$

**(2)** *For every* $s_1, s_2 \in \mathcal{L}_5[\emptyset]$,

$$\dot{\mathcal{M}}_5[\![s_1]\!] = \dot{\mathcal{M}}_5[\![s_2]\!] \Leftrightarrow \forall X \in \mathcal{X}_{\mathcal{P}}, \forall S \in \mathcal{L}_5[X][\ \mathcal{O}_5^*[\![S[s_1/X]]\!] = \mathcal{O}_5^*[\![S[s_2/X]]\!]\ ]. \ ■ \tag{8.20}$$

In order to prove the above theorem, we employ the following two lemmas, which have been established in Chapter 7.

**Lemma 8.14** *If* $\mathbf{V}$ *is infinite, then*

$$\forall p_1, p_2 \in \mathbf{P}_4[\; p_1 \neq p_2 \;\Rightarrow\; \exists \tilde{s} \in \mathcal{L}_5[\; \mathcal{A}_4(p_1 \,\tilde{\|}\, \dot{\mathcal{M}}_5[\![\tilde{s}]\!]) \neq \mathcal{A}_4(p_2 \,\tilde{\|}\, \dot{\mathcal{M}}_5[\![\tilde{s}]\!]) \;]. \blacksquare$$

**Proof.** See the proof of Theorem 7.2 ■

**Lemma 8.15** *If* $\mathbf{V}$ *is infinite, then*

$$\forall s_1, s_2 \in \mathcal{L}_5[\; \mathcal{O}_5[\![s_1]\!] \neq \mathcal{O}_5[\![s_2]\!] \;\Rightarrow\; \exists \tilde{s} \in \mathcal{L}_5[\; \mathcal{O}_5^*[\![s_1 \,\|\, \tilde{s}]\!] \neq \mathcal{O}_5^*[\![s_2 \,\|\, \tilde{s}]\!] \;]]. \blacksquare$$

**Proof.** See the proof of Theorem 7.3. ■

**Proof of Theorem 8.1.** **(1)** Let $s_1, s_2 \in \mathcal{L}_5[\emptyset]$. The $\Rightarrow$-part of (8.19) follows from Lemma 8.7 and Corollary 8.1 (1) as follows: Suppose $\dot{\mathcal{M}}_5[\![s_1]\!] = \dot{\mathcal{M}}_5[\![s_2]\!]$, let $X \in \mathcal{X}_{\mathcal{P}}$, $S \in \mathcal{L}_5[X]$, and let $\tilde{\mathrm{p}}$ be an arbitrary element of SeVal. Then

$$\begin{aligned}
&\mathcal{C}_5[\![S[s_1/X]]\!] = \mathcal{A}_4(\mathcal{M}_5[\![S[s_1/X]]\!]) \quad \text{(by Corollary 8.1 (1))}\\
&= \mathcal{A}_4(\mathcal{M}_5[\![S]\!](\tilde{\mathrm{p}}[\dot{\mathcal{M}}_5[\![s_1]\!]/X])) \quad \text{(by Lemma 8.7)}\\
&= \mathcal{A}_4(\mathcal{M}_5[\![S]\!](\tilde{\mathrm{p}}[\dot{\mathcal{M}}_5[\![s_2]\!]/X])) \quad \text{(since } \dot{\mathcal{M}}_5[\![s_1]\!] = \dot{\mathcal{M}}_5[\![s_2]\!])\\
&= \mathcal{A}_4(\mathcal{M}_5[\![S[s_2/X]]\!]) = \mathcal{C}_5[\![S[s_2/X]]\!].
\end{aligned}$$

The $\Leftarrow$-part of (8.19) follows from Lemma 8.14 as follows: Let us show the contrapositive. Suppose $\dot{\mathcal{M}}_5[\![s_1]\!] \neq \dot{\mathcal{M}}_5[\![s_1]\!]$. Then, by Lemma 8.14, there is $\tilde{s} \in \mathcal{L}_5[\emptyset]$ such that

$$\mathcal{O}_5[\![s_1 \,\|\, \tilde{s}]\!] = \mathcal{A}_4(\dot{\mathcal{M}}_5[\![s_1]\!] \tilde{\|} \dot{\mathcal{M}}_5[\![s]\!]) \neq \mathcal{A}_4(\dot{\mathcal{M}}_5[\![s_2]\!] \tilde{\|} \dot{\mathcal{M}}_5[\![s]\!]) = \mathcal{O}_5[\![s_2 \,\|\, \tilde{s}]\!].$$

Thus, putting $S \equiv (X \,\|\, \tilde{s})$, one has $\mathcal{C}_5[\![S[s_1/X]]\!] \neq \mathcal{C}_5[\![S[s_2/X]]\!]$. Summing up one has the contrapositive of the $\Leftarrow$-part of (8.19).
**(2)** Let $s_1, s_2 \in \mathcal{L}_5[\emptyset]$. The $\Rightarrow$-part of (8.19) follows from Lemma 8.7 and Corollary 8.1 (2) as in part (1). The $\Leftarrow$-part of (8.20) follows from Lemma 8.14 and Lemma 8.15 as follows: Suppose $\dot{\mathcal{M}}_5[\![s_1]\!] \neq \dot{\mathcal{M}}_5[\![s_1]\!]$. Then, by Lemma 8.14, there is $\tilde{s} \in \mathcal{L}_5[\emptyset]$ such that $\mathcal{O}_5[\![s_1 \,\|\, \tilde{s}]\!] \neq \mathcal{O}_5[\![s_2 \,\|\, \tilde{s}]\!]$. Also by Lemma 8.15, there is $\tilde{s}' \in \mathcal{L}_5$ such that $\mathcal{O}_5^*[\![(s_1 \,\|\, \tilde{s}) \,\|\, \tilde{s}']\!] \neq \mathcal{O}_5^*[\![(s_2 \,\|\, \tilde{s}) \,\|\, \tilde{s}']\!]$. Thus, putting $S \equiv ((X \,\|\, \tilde{s}) \,\|\, \tilde{s}')$, one has $\mathcal{C}_5[\![S[s_1/X]]\!] \neq \mathcal{C}_5[\![S[s_2/X]]\!]$. ■

## 8.7 Concluding Remarks

We conclude this paper with some remarks about related work and directions for future study.

We have shown how the simple nonuniform concurrent language presented in Chapter 7 can be extended so as to incorporate parameterization and locality (local variables and local channels), with the full abstractness result obtained in Chapter 7 being preserved.

Our treatment of parameterization in this chapter is standard except that the interpretation of function application is *nonuniform* in the sense of Lemma 8.6 (5). Our treatment of local channels is very similar to Milner's treatment of auxiliary channels introduced for defining *the linking combinator* '$\frown$' (see Remark 8.1 in § 8.3). There are several approaches to the semantics of local variables. Our treatment of them is similar to that in [BKPR 92] which treats a more general setting with a state space including states with partial information. The treatment in [BKPR 92], in turn, is based on the idea of [SRP 90] of using cylindric algebras (see [HMT 71]) to capture the notion of projecting away information. For other approaches to the problem of giving semantics to local variables especially those using categorical methods, we refer the reader to an expository article [OT 92]

There are two directions for extending the reported results. One is to investigate fully abstract models for extended languages such as the ones including real-time constructs. The other is to investigate fully abstract models for the same language $\mathcal{L}_5$, but w.r.t. more abstract operational models, e.g, linear models which are *weak* in that they ignore internal steps denoted by $\tau$. Such models which are fully abstract w.r.t. weak linear models have been proposed for *uniform* concurrent languages (cf., e.g., [DH 84], [Hen 85], [Hen 88], [HI 90], [Hor 92a]); a similar problem for *nonuniform* concurrent languages is an interesting topic for future study.

## 8.A Formal Definition of Language $\mathcal{L}_5$

In this appendix $\mathcal{L}_5$ is defined formally along the lines of the definition of a simple language $\mathcal{L}_0$ in [KR 90] § 1.1. For each $n \in \omega$, we define $\mathcal{L}_5^{(i,j)}(n)$ and $\mathcal{G}^{(i,j)}(n)$ for $(i,j) \in \omega^2$ simultaneously, by induction on $n$ as follows:

**Definition 8.14** (Definition of Language $\mathcal{L}_5$)

**(1)** For each $(i,j) \in \omega^2$, let $\mathcal{L}_5^{(i,j)}(0) = \emptyset$, and $\mathcal{G}^{(i,j)}(0) = \emptyset$.

**(2)** Let $n \in \omega$. In terms of $\mathcal{L}_5^{(i,j)}(n)$ and $\mathcal{G}^{(i,j)}(n)$ for $(i,j) \in \omega^2$, we define $\mathcal{L}_5^{(i,j)}(n+1)$ as follows: First let

$$
\begin{aligned}
&\mathcal{L}_5^{(0,0)}(n+1) = \\
&\mathcal{X}_{\mathcal{P}}^{(0,0)} \\
&\cup \bigcup \{\{F(\vec{H} \cdot \vec{E} \cdot \vec{S} \cdot \vec{T} \cdot \vec{U}) : F \in \mathbf{Fun}_5^{(i,j,k,\ell,m)} \wedge \vec{H} \in (\tilde{\mathcal{E}}_{\mathcal{H}})^i \\
&\qquad \wedge \vec{E} \in (\tilde{\mathcal{E}}_{\mathcal{F}})^j \wedge \vec{S} \in (\mathcal{L}_5^{(0,0)}(n))^k \\
&\qquad \wedge \vec{T} \in (\mathcal{L}_5^{(0,1)}(n))^\ell \wedge \vec{U} \in (\mathcal{L}_5^{(1,0)}(n))^m\} : \\
&\qquad (i,j,k,\ell,m) \in \omega^5\} \\
&\cup \bigcup \{\{\mathbf{S}(\vec{H} \cdot \vec{E}) : \mathbf{S} \in \mathcal{L}_5^{(i,j)}(n) \wedge \vec{H} \in (\tilde{\mathcal{E}}_{\mathcal{H}})^i \wedge \vec{E} \in (\tilde{\mathcal{E}}_{\mathcal{F}})^j\} : \\
&\qquad (i,j) \neq (0,0)\} \\
&\cup \{(\mu X.\ S).\ X \in \mathcal{X}_{\mathcal{P}} \wedge S \in \mathcal{G}_4(n)\}.
\end{aligned}
$$

Then, for $(i,j) \neq (0,0)$, let

$$\begin{aligned}
&\mathcal{L}_5^{(i,j)}(n+1) = \\
&\mathcal{X}_{\mathcal{P}}^{(i,j)} \\
&\cup \bigcup \{ \{ (\lambda \vec{\eta} \cdot \vec{\xi}.\ \mathbf{S}) : \vec{\eta} \in (i \rightarrowtail \mathcal{X}_{\mathcal{C}}) \wedge \vec{\xi} \in (j \rightarrowtail \mathcal{X}_{\mathcal{F}}) \wedge \mathbf{S} \in \mathcal{L}_5^{(0,0)}(n) \} : \\
&\qquad (i,j) \in \omega^2 \} \\
&\cup \{ (\mu Z.\ \mathbf{S}) : Z \in \mathcal{X}^{(i,j)} \wedge \mathbf{S} \in \mathcal{G}^{(i,j)}(n) \}.
\end{aligned}$$

Also, we define $\mathcal{G}^{(i,j)}(n+1)$ as follows: First let

$$\begin{aligned}
&\mathcal{G}^{(0,0)}(n+1) ::= \\
&(\mathcal{X}^{(0,0)} \setminus \{Z\}) \,\dot{\cup}\, \{ (\mathbf{asg}(x,E); S) : x \in \mathcal{IV} \wedge S \in \mathcal{L}_5^{(0,0)}(n) \} \\
&\cup \{ (\mathbf{out}(H,E); S) : H \in \tilde{\mathcal{E}}_{\mathcal{H}} \wedge E \in \tilde{\mathcal{E}}_{\mathcal{F}} \wedge S \in \mathcal{L}_5^{(0,0)}(n) \} \\
&\cup \{ (\mathbf{in}(H,x); S) : x \in \mathcal{IV} \wedge H \in \tilde{\mathcal{E}}_{\mathcal{H}} \wedge S \in \mathcal{L}_5^{(0,0)}(n) \} \\
&\cup \{ \mathbf{in}'(G,\mathbf{S}) : H \in \tilde{\mathcal{E}}_{\mathcal{H}} \wedge \mathbf{S} \in \mathcal{L}_5^{(0,1)}(n) \} \\
&\cup \bigcup \{ \{ F(\vec{H} \cdot \vec{E} \cdot \vec{S} \cdot \vec{T} \cdot \vec{U}) : F \in \mathbf{Fun}_5^{(i,j,k,\ell,m)} \wedge \vec{H} \in (\tilde{\mathcal{E}}_{\mathcal{H}})^i \\
&\qquad \wedge \vec{E} \in (\tilde{\mathcal{E}}_{\mathcal{F}})^j \wedge \vec{S} \in (\mathcal{G}(n))^k \\
&\qquad \wedge \vec{T} \in (\mathcal{G}^{(0,1)}(n))^\ell \wedge \vec{U} \in (\mathcal{G}^{(1,0)}(n))^m \} : \\
&\qquad (i,j,k,\ell,m) \in \omega^5 \} \\
&\cup \bigcup \{ \{ \mathbf{S}(\vec{H} \cdot \vec{E}) : \mathbf{S} \in \mathcal{G}^{(i,j)}(n) \wedge \vec{H} \in (\tilde{\mathcal{E}}_{\mathcal{H}})^i \wedge \vec{E} \in (\tilde{\mathcal{E}}_{\mathcal{F}})^j \} : \\
&\qquad (i,j) \neq (0,0) \} \\
&\cup \{ (\mu Z'.\ \mathbf{S}) : Z' \in \mathcal{X}_{\mathcal{P}}^{(i,j)} \wedge \mathbf{S} \in \mathcal{G}^{(i,j)}(n) \wedge \mathcal{G}_{Z'}^{(i,j)}(n) \}.
\end{aligned}$$

Then, for $(i,j) \neq (0,0)$, let

$$\begin{aligned}
&\mathcal{G}^{(i,j)}(n+1) ::= \\
&(\mathcal{X}_{\mathcal{P}}^{(i,j)} \setminus \{Z\}) \\
&\cup \{ (\lambda \vec{\eta} \cdot \vec{\xi}.\ \mathbf{S}) : \vec{\eta} \in (i \rightarrowtail \mathcal{X}_{\mathcal{C}}) \wedge \vec{\xi} \in (i \rightarrowtail \mathcal{X}_{\mathcal{F}}) \wedge \mathbf{S} \in \mathcal{G}^{(0,0)} \} \\
&\cup \{ (\mu Z'.\ \mathbf{S}) : Z' \in (\mathcal{X}_{\mathcal{P}})^{(i,j)} \wedge \mathbf{S} \in \mathcal{G}_{Z'}^{(0,0)}(n) \cap \mathcal{G}^{(i,j)}(n) \}.
\end{aligned}$$

It can be shown immediately by induction on $n$ that

$$\begin{aligned}
&\forall n \in \omega, \forall (i,j) \in \omega^2, \forall Z \in \mathcal{X}_{\mathcal{P}}^{*} [\ \mathcal{G}^{(i,j)}(n) \subseteq \mathcal{L}_5^{(i,j)}(n) \\
&\quad \wedge \mathcal{G}^{(i,j)}(n) \subseteq \mathcal{G}^{(i,j)}(n+1) \wedge \mathcal{L}_5^{(i,j)}(n) \subseteq \mathcal{L}_5^{(i,j)}(n+1)\ ].
\end{aligned}$$

**(3)** For each $n \in \omega$, let

$$\mathcal{L}_5^{*}(n) = \bigcup \{ \mathcal{L}_5^{(i,j)}(n) : (i,j) \in \omega^2 \},$$

and

$$\mathcal{G}^{*}(n) = \bigcup \{ \mathcal{G}^{(i,j)}(n) : (i,j) \in \omega^2 \}.$$

Also for each $(i,j) \in \omega^2$, let

$$\mathcal{L}_5^{(i,j)} = \bigcup \{ \mathcal{L}_5^{(i,j)}(n) : n \in \omega \}. \blacksquare$$

## 8.B Definition of Denotational Model $\mathcal{M}_5$

For each $n \in \omega$, $\mathbf{S} \in \mathcal{L}_5^*(n)[\mathcal{X}_\mathcal{P}^*]$, and $\tilde{\mathbf{p}} \in \text{SeVal}$, we will define $\mathcal{M}_5[\![\mathbf{S}]\!](\tilde{\mathbf{p}})$, by induction on $n$. In other words, we will define a family of mappings

$$(\lambda \mathbf{S} \in \mathcal{L}_5^*(n)[\mathcal{X}_\mathcal{P}^*].\ (\lambda \tilde{\mathbf{p}} \in \text{SeVal}.\ \mathcal{M}_5[\![\mathbf{S}]\!](\tilde{\mathbf{p}})))$$

($n \in \omega$), by induction on $n$. We will define this family so that the following two conditions are satisfied for every $n \in \omega$:

$$\begin{aligned}&\forall (i,j) \in \omega^2, \forall \mathbf{S} \in \mathcal{L}_5^{(i,j)}(n)[\mathcal{X}_\mathcal{P}^*][\\ &\quad (\lambda \tilde{\mathbf{p}} \in \text{SeVal}.\ \mathcal{M}_5[\![\mathbf{S}]\!](\tilde{\mathbf{p}})) \text{ is a nonexpansive mapping from}\\ &\quad \text{SeVal to } \mathbf{P}_4^{(i,j)}\ ],\end{aligned} \tag{8.21}$$

$$\begin{aligned}&\forall (k,\ell),(i,j) \in \omega^2, \forall Z \in \mathcal{X}_\mathcal{P}^{(k,\ell)}, \forall \mathbf{S} \in \mathcal{G}_Z^{(i,j)}(n)[\mathcal{X}_\mathcal{P}^*], \forall \tilde{\mathbf{p}} \in \text{SeVal}[\\ &\quad (\lambda \mathbf{p} \in \mathbf{P}_4^{(k,\ell)}.\ \mathcal{M}_5[\![\mathbf{S}]\!](\tilde{\mathbf{p}}[\mathbf{p}/Z])) \text{ is a contraction from } \mathbf{P}_4^{(k,\ell)}\\ &\quad \text{to } \mathbf{P}_4^{(i,j)} \text{ with coefficient } \kappa.\ ]\end{aligned} \tag{8.22}$$

**Step 1.** For $n = 0$, we simply define $(\lambda \mathbf{S} \in \mathcal{L}_5^*(0)[\mathcal{X}_\mathcal{P}^*].\ (\lambda \tilde{\mathbf{p}} \in \text{SeVal}.\ \mathcal{M}_5[\![\mathbf{S}]\!](\tilde{\mathbf{p}})))$ to be the empty mapping, since $\mathcal{L}_5^*(0)[\mathcal{X}_\mathcal{P}^*] = \emptyset$.
**Step 2.** Let $n \in \omega$ and assume that

> For $\mathbf{S} \in \mathcal{L}_5^*(n)[\mathcal{X}_\mathcal{P}^*]$ and $\tilde{\mathbf{p}} \in \text{SeVal}$, $\mathcal{M}_5[\![\mathbf{S}]\!](\tilde{\mathbf{p}})$ has been defined so that the conditions (8.21) and (8.22) are satisfied. (8.23)

**Step 2.1.** First, let us define $\mathcal{M}_5[\![\mathbf{S}]\!](\tilde{\mathbf{p}})$ for $(i,j) \in \omega^2$, $\mathbf{S} \in \mathcal{L}_5^{(i,j)}(n+1)[\mathcal{X}_\mathcal{P}^*]$ and $\tilde{\mathbf{p}} \in \text{SeVal}$. Fix $(i,j) \in \omega^2$. We distinguish two cases according to whether or not $(i,j) = (0,0)$.
**Case 1.** Suppose $(i,j) = (0,0)$. We distinguish several cases according to the form of $\mathbf{S}$.
**Subcase 1.1.** Suppose $\mathbf{S} \in \mathcal{X}_\mathcal{P}$. Then let $\mathcal{M}_5[\![\mathbf{S}]\!](\tilde{\mathbf{p}}) = \tilde{\mathbf{p}}(\mathbf{S})$. Obviously the mapping $(\lambda \tilde{\mathbf{p}} \in \text{SeVal}.\ \mathcal{M}_5[\![\mathbf{S}]\!](\tilde{\mathbf{p}})) = (\lambda \tilde{\mathbf{p}} \in \mathbf{P}_4^1.\ \tilde{\mathbf{p}}(\mathbf{S}))$ is nonexpansive.
**Subcase 1.2.** Suppose $\mathbf{S} \equiv F(\vec{h} \cdot \vec{e} \cdot \vec{S} \cdot \vec{T} \cdot \vec{U})$ for some $(i',j',k',\ell',m')$, $F \in \mathbf{Fun}_5^{(i',j',k',\ell',m')}$, $\vec{g} \in (\mathcal{E}_\mathcal{H})^{i'}$, $\vec{e} \in (\mathcal{E}_\mathcal{F})^{j'}$, $\vec{S} \in (\mathcal{L}_5^{(0,0)}(n)[\mathcal{X}_\mathcal{P}^*])^{k'}$, $\vec{T} \in (\mathcal{L}_5^{(0,1)}(n)[\mathcal{X}_\mathcal{P}^*])^{\ell'}$, $\vec{U} \in (\mathcal{L}_5^{(1,0)}(n)[\mathcal{X}_\mathcal{P}^*])^{m'}$. Then let $\mathcal{M}_5[\![\mathbf{S}]\!](\tilde{\mathbf{p}}) = \mathcal{I}_5(F)([\![\vec{g}]\!] \cdot [\![e]\!] \cdot \mathcal{M}_5[\![\vec{S}]\!](\tilde{\mathbf{p}}) \cdot \mathcal{M}_5[\![\vec{T}]\!](\tilde{\mathbf{p}}) \cdot \mathcal{M}_5[\![\vec{U}]\!](\tilde{\mathbf{p}}))$, where $\mathcal{M}_5[\![\vec{S}]\!](\tilde{\mathbf{p}}) = \langle \mathcal{M}_5[\![\vec{S}(i)]\!](\tilde{\mathbf{p}}) \rangle_{i \in k'}$, and $\mathcal{M}_5[\![\vec{T}]\!](\tilde{\mathbf{p}})$, $\mathcal{M}_5[\![\vec{U}]\!](\tilde{\mathbf{p}})$ are defined in a similar fashion. In this subcase also, the mapping $(\lambda \tilde{\mathbf{p}} \in \text{SeVal}.\ \mathcal{M}_5[\![\mathbf{S}]\!](\tilde{\mathbf{p}}))$ is nonexpansive, by the induction hypothesis (8.23) stating that (8.21) holds and Lemma 8.5 (1).
**Subcase 1.3.** Suppose $\mathbf{S} \equiv \mathbf{S}'(\vec{g} \cdot \vec{e})$ for some $(i',j') \in \omega^2$, $\mathbf{S}' \in \mathcal{L}_5^{(i',j')}(n)[\mathcal{X}_\mathcal{P}^*]$ $\vec{g} \in (\mathcal{E}_\mathcal{H})^{i'}$, $\vec{e} \in (\mathcal{E}_\mathcal{F})^{j'}$. Then let

$$\mathcal{M}_5[\![\mathbf{S}]\!](\tilde{\mathbf{p}}) = \bigcup_{\sigma \in \Sigma} [\mathcal{M}_5[\![\mathbf{S}']\!](\tilde{\mathbf{p}})([\![\vec{g}]\!](\sigma) \cdot [\![\vec{e}]\!](\sigma))\langle \sigma \rangle].$$

It immediately follows from the induction hypothesis (8.23) stating that (8.21) holds and the definition of the metric on $\mathbf{P}_4^{(i',j')}$ that the mapping

$$(\lambda\tilde{\mathbf{p}} \in \text{SeVal}.\ \mathcal{M}_5[\![\mathbf{S}]\!](\tilde{\mathbf{p}}))$$

is nonexpansive.

**Subcase 1.4.** Suppose $\mathbf{S} \equiv (\mu X.\ \mathbf{S}')$ with $\mathbf{S}' \in \mathcal{G}_4(n)[\mathcal{X}_{\mathcal{P}}^*]$. For $\tilde{\mathbf{p}} \in$ SeVal, let $\varphi(\tilde{\mathbf{p}}) = (\lambda p \in \mathbf{P}_4.\ \mathcal{M}_5[\![\mathbf{S}']\!](\tilde{\mathbf{p}}[p/X]))$. By the induction hypothesis (8.23) stating that (8.22) holds, the mapping $\varphi(\tilde{\mathbf{p}})$ is a contraction from $\mathbf{P}_4$ to itself with coefficient $\kappa$. We define $\mathcal{M}_5[\![\mathbf{S}]\!](\tilde{\mathbf{p}}) = \text{fix}(\varphi(\tilde{\mathbf{p}}))$.

It can shown that the mapping $(\lambda\tilde{\mathbf{p}} \in \text{SeVal}.\ \mathcal{M}_5[\![\mathbf{S}]\!](\tilde{\mathbf{p}}))$ is nonexpansive as follows: Let $p_0$ be an arbitrary element of $\mathbf{P}_4$. Then, by Banach's Theorem,

$$\mathcal{M}_5[\![\mathbf{S}]\!](\tilde{\mathbf{p}}) = \text{fix}(\varphi(\tilde{\mathbf{p}})) = \lim_{n\to\infty}[(\varphi(\tilde{\mathbf{p}}))^n(p_0)],$$

where $(\varphi(\tilde{\mathbf{p}}))^n$ is the $n$-th iteration of $\varphi(\tilde{\mathbf{p}})$. Let $\tilde{\mathbf{p}}_1, \tilde{\mathbf{p}}_1 \in$ SeVal. Then, by induction, it can be shown immediately that

$$\forall n \in \omega[\ d((\varphi(\tilde{\mathbf{p}}_1))^n(p_0), (\varphi(\tilde{\mathbf{p}}_2))^n(p_0)) \leq d(\tilde{\mathbf{p}}_1, \tilde{\mathbf{p}}_2)\ ].$$

Thus, one has

$$\begin{aligned}
&d(\mathcal{M}_5[\![\mathbf{S}]\!](\tilde{\mathbf{p}}_1), \mathcal{M}_5[\![\mathbf{S}]\!](\tilde{\mathbf{p}}_2))\\
&= \lim_{n\to\infty}[d((\varphi(\tilde{\mathbf{p}}_1))^n(p_0), (\varphi(\tilde{\mathbf{p}}_2))^n(p_0))]\\
&\leq d(\tilde{\mathbf{p}}_1, \tilde{\mathbf{p}}_2).
\end{aligned}$$

Hence, the mapping $(\lambda\tilde{\mathbf{p}} \in \text{SeVal}.\ \mathcal{M}_5[\![\mathbf{S}]\!](\tilde{\mathbf{p}}))$ is nonexpansive.

**Case 2.** Suppose $(i,j) \neq (0,0)$. In this case also, we distinguish several cases according to the form of $\mathbf{S}$.

**Subcase 2.1.** Suppose $\mathbf{S} \in \mathcal{X}_{\mathcal{P}}^{(i,j)}$ (resp. $\mathbf{S} \equiv (\mu Z.\ \mathbf{S}')$ for some $\mathbf{S}'$). Then, we can define $\mathcal{M}_5[\![\mathbf{S}]\!](\tilde{\mathbf{p}})$ for $\tilde{\mathbf{p}} \in$ SeVal, and show that $(\lambda\tilde{\mathbf{p}} \in \text{SeVal}.\ \mathcal{M}_5[\![\mathbf{S}]\!](\tilde{\mathbf{p}}))$ is nonexpansive, in a similar fashion to Subcase 1.1 (resp. to Subcase 1.3) above.

**Subcase 2.2.** Suppose $\mathbf{S} \equiv (\lambda\vec{\eta}\cdot\vec{\xi}.\ \mathbf{S}')$ with $\vec{\eta} \in (i \rightarrowtail \mathcal{X}_C)$, $\vec{\xi} \in (j \rightarrowtail \mathcal{X}_V)$, $\mathbf{S}' \in \mathcal{L}_5^{(0,0)}(n)[\mathcal{X}_{\mathcal{P}}^*]$. Then let

$$\mathcal{M}_5[\![\mathbf{S}]\!](\tilde{\mathbf{p}}) = (\lambda\vec{c}\cdot\vec{v} \in \mathbf{Chan}^i \cdot \mathbf{V}^j.\ \mathcal{M}_5[\![\mathbf{S}'[(\vec{c}\cdot\vec{v})/(\vec{\eta}\cdot\vec{\eta})]]\!](\tilde{\mathbf{p}})).$$

By the induction hypothesis (8.23) stating that (8.21) holds and the definition of the metric on $\mathbf{P}_4^{(i,j)}$, one has

$$\forall\tilde{\mathbf{p}}_1, \tilde{\mathbf{p}}_2 \in \text{SeVal}[\ d(\mathcal{M}_5[\![\mathbf{S}]\!](\tilde{\mathbf{p}}_1), \mathcal{M}_5[\![\mathbf{S}]\!](\tilde{\mathbf{p}}_2)) \leq d(\tilde{\mathbf{p}}_1, \tilde{\mathbf{p}}_2)\ ].$$

Namely, the mapping $(\lambda\tilde{\mathbf{p}} \in \text{SeVal}.\ \mathcal{M}_5[\![\mathbf{S}]\!](\tilde{\mathbf{p}}))$ is nonexpansive.

**Step 2.2.** Having defined $\mathcal{M}_5[\![\mathbf{S}]\!](\tilde{\mathbf{p}})$ for $\mathbf{S} \in \mathcal{L}_5^{(i,j)}(n+1)[\mathcal{X}_{\mathcal{P}}^*]$ and $\tilde{\mathbf{p}} \in$ SeVal, let us show the following holds for every $(k,\ell), (i,j) \in \omega^2$, $Z \in \mathcal{X}_{\mathcal{P}}^{(k,\ell)}$, $\mathbf{S} \in \mathcal{G}_Z^{(i,j)}(n+1)[\mathcal{X}_{\mathcal{P}}^*]$, and $\tilde{\mathbf{p}} \in$ SeVal:

$$(\lambda\mathbf{p} \in \mathbf{P}_4^{(k,\ell)}.\ \mathcal{M}_5[\![\mathbf{S}]\!](\tilde{\mathbf{p}}[\mathbf{p}/Z]) \text{ is a contraction from } \mathbf{P}_4^{(k,\ell)} \text{ to } \mathbf{P}_4^{(i,j)} \text{ with coefficient } \kappa. \tag{8.24}$$

Fix $(k,\ell),(i,j) \in \omega^2$, $Z \in \mathcal{X}_{\mathcal{P}}^{(k,\ell)}$, $\mathbf{S} \in \mathcal{G}_Z^{(i,j)}(n+1)[\mathcal{X}_{\mathcal{P}}^*]$, and $\tilde{\mathbf{p}} \in \mathrm{SeVal}$. We distinguish two cases according to whether or not $(i,j) = (0,0)$.

**Case 1.** Suppose $(i,j) = (0,0)$. Again, we distinguish several cases according to the form of $\mathbf{S}$.

**Subcase 1.1.** Suppose $\mathbf{S} \in (\mathcal{X}_{\mathcal{P}}^{(0,0)} \setminus \{Z\})$. Then,

$$\forall \mathbf{p} \in \mathbf{P}_4^{(k,\ell)}[\ \mathcal{M}_5[\![\mathbf{S}]\!](\tilde{\mathbf{p}}[\mathbf{p}/Z]) = \tilde{\mathbf{p}}(\mathbf{S})\ ].$$

Thus, the mapping $(\lambda \mathbf{p} \in \mathbf{P}_4^{(k,\ell)}.\ \mathcal{M}_5[\![\mathbf{S}]\!](\tilde{\mathbf{p}}[\mathbf{p}/Z]))$ is a constant mapping, and hence, a contraction from $\mathbf{P}_4^{(k,\ell)}$ to $\mathbf{P}_4^{(i,j)}$. Thus, one has (8.24).

**Subcase 1.2.** Suppose one of the following holds:

(i) $\mathbf{S} \equiv (\mathbf{asg}(x,e); S')$ with $x \in \mathcal{IV}$, $S' \in \mathcal{L}_5^{(0,0)}(n)$,

(ii) $\mathbf{S} \equiv (\mathbf{out}(h,e); S')$ with $h \in \mathcal{E}_{\mathcal{H}}$, $e \in \mathcal{E}_{\mathcal{F}}$, $S' \in \mathcal{L}_5^{(0,0)}(n)$,

(iii) $\mathbf{S} \equiv (\mathbf{in}(h,x); S')$ with $h \in \mathcal{E}_{\mathcal{H}}$, $x \in \mathcal{IV}$, $S' \in \mathcal{L}_5^{(0,0)}(n)$,

(ivf) $\mathbf{S} \equiv \mathbf{in}'(h,\mathbf{S}')$ with $h \in \mathcal{E}_{\mathcal{H}}$, $\mathbf{S}' \in \mathcal{L}_5^{(0,1)}(n)$.

We consider the case **(i)**; in the other cases, the same conclusion is obtained in a similar fashion. For every $\mathbf{p} \in \mathbf{P}_4^{(k,\ell)}$,

$$\mathcal{M}_5[\![\mathbf{S}]\!](\tilde{\mathbf{p}}[\mathbf{p}/Z]) = \mathrm{asg}_5^{(x)}(\mathcal{M}_5[\![S']\!](\tilde{\mathbf{p}}[\mathbf{p}/Z])).$$

Thus, $(\lambda \mathbf{p} \in \mathbf{P}_4^{(k,\ell)}.\ \mathcal{M}_5[\![\mathbf{S}]\!](\tilde{\mathbf{p}}[\mathbf{p}/Z]))$ is a contraction with coefficient $\kappa$, since $\mathrm{asg}_5^{(x)}$ is a contraction with coefficient $\kappa$, and $(\lambda \mathbf{p} \in \mathbf{P}_4^{(k,\ell)}.\ \mathcal{M}_5[\![S']\!](\tilde{\mathbf{p}}[\mathbf{p}/Z]))$ is nonexpansive.

**Subcase 1.3.** Suppose $\mathbf{S} \equiv F(\vec{h}\cdot\vec{e}\cdot\vec{S}\cdot\vec{T}\cdot\vec{U})$ with $F \in \mathbf{Fun}_5^{(i',j',k',\ell',m')}$, $\vec{h} \in (\mathcal{E}_{\mathcal{H}})^{i'}$, $\vec{e} \in (\mathcal{E}_{\mathcal{F}})^{j'}$, $\vec{S} \in (\mathcal{G}_Z(n)[\mathcal{X}_{\mathcal{P}}^*])^{k'}$, $\vec{T} \in (\mathcal{G}_Z^{(0,1)[\mathcal{X}_{\mathcal{P}}^*]}(n))^{\ell'}$, $\vec{U} \in (\mathcal{G}_Z^{(1,0)}(n)[\mathcal{X}_{\mathcal{P}}^*])^{m'}$. Then, one has the desired conclusion that $(\lambda \mathbf{p} \in \mathbf{P}_4^{(\ell,m)}.\ \mathcal{M}_5[\![\mathbf{S}]\!](\tilde{\mathbf{p}}[\mathbf{p}/Z]))$ is a contraction with coefficient $\kappa$, by the induction hypothesis (8.23) stating that (8.22) holds and the fact the semantic operation $\mathcal{I}_5(F)$ is nonexpansive (cf. Lemma 8.5 (1)).

**Subcase 1.4.** Suppose $\mathbf{S} \equiv \mathbf{S}'(\vec{h}\cdot\vec{e})$ with $(i',j') \neq (0,0)$, $\mathbf{S}' \in \mathcal{G}_Z^{(i',j')}(n)[\mathcal{X}_{\mathcal{P}}^*]$, $\vec{h} \in (\mathcal{E}_{\mathcal{H}})^i$, $\vec{e} \in (\mathcal{E}_{\mathcal{F}})^i$. Then, for every $\tilde{\mathbf{p}}' \in \mathrm{SeVal}$,

$$\mathcal{M}_5[\![\mathbf{S}]\!](\tilde{\mathbf{p}}') = \bigcup_{\sigma\in\Sigma}[\mathcal{M}_5[\![\mathbf{S}']\!](\tilde{\mathbf{p}}')([\![\vec{h}]\!](\sigma)\cdot[\![\vec{e}]\!](\sigma))\langle\sigma\rangle].$$

Thus, the desired result that $(\lambda \mathbf{p} \in \mathbf{P}_4^{(k,\ell)}.\ \mathcal{M}_5[\![\mathbf{S}]\!](\tilde{\mathbf{p}}[\mathbf{p}/Z]))$ is a contraction with coefficient $\kappa$ follows from the induction hypothesis (8.23) stating that (8.22) holds and the definition of the metric on the function space $\mathbf{P}_4^{(i,j)}$.

**Subcase 1.5.** Suppose $\mathbf{S} \equiv (\mu Z'.\ \mathbf{S}')$ with $\mathbf{S}' \in \mathcal{G}_Z^{(0,0)}(n)[\mathcal{X}_{\mathcal{P}}^*]$. Thus, by Definition 8.14 (2), one has $\mathbf{S}' \in \mathcal{G}_Z^{(0,0)}(n)[\mathcal{X}_{\mathcal{P}}^*] \cap \mathcal{G}_{Z'}^{(0,0)}(n)[\mathcal{X}_{\mathcal{P}}^*]$. For every $\tilde{\mathbf{p}}' \in \mathrm{SeVal}$,

$$\mathcal{M}_5[\![\mathbf{S}]\!](\tilde{\mathbf{p}}') = \mathrm{fix}(\varphi(\tilde{\mathbf{p}}')), = \lim_{n\to\infty}[(\varphi(\tilde{\mathbf{p}}))^{(n)}(p_0)],$$

where

$$\varphi(\tilde{\mathbf{p}}') = (\lambda p' \in \mathbf{P}_4.\ \mathcal{M}_5[\![\mathbf{S}']\!](\tilde{\mathbf{p}}'[p'/Z'])),$$

and $p_0$ is an arbitrary element of $\mathbf{P}_4$, and $(\varphi(\tilde{\mathbf{p}}))^{(n)}$ is the $n$-th interation of $\varphi(\tilde{\mathbf{p}})$. For $\mathbf{p}_1, \mathbf{p}_2 \in \mathbf{P}_4^{(k,\ell)}$, we will show that for every $n \in \omega$ (8.25) below holds, by induction on $n$, using the induction hypothesis (8.23) (stating that (8.22) holds), and the fact that $d_P$ is an *ultra-metric.*

$$d((\varphi(\tilde{\mathbf{p}}[\mathbf{p}_1/Z]))^{(n)}(p_0), (\varphi(\tilde{\mathbf{p}}[\mathbf{p}_2/Z]))^{(n)}(p_0)) \leq \kappa \cdot d(\mathbf{p}_1, \mathbf{p}_2), \tag{8.25}$$

For $n = 0$, (8.25) obviously holds. Fix $k' \in \omega$, and assume (8.25) holds for $n = k'$. Let us show (8.25) for $n = k' + 1$. One has

$$\begin{aligned}
&d((\varphi(\tilde{\mathbf{p}}[\mathbf{p}_1/Z]))^{(k'+1)}(p_0), (\varphi(\tilde{\mathbf{p}}[\mathbf{p}_2/Z]))^{(k'+1)}(p_0)) \\
&= d(\varphi(\tilde{\mathbf{p}}[\mathbf{p}_1/Z])((\varphi(\tilde{\mathbf{p}}[\mathbf{p}_1/Z]))^{k'}(p_0)), \varphi(\tilde{\mathbf{p}}[\mathbf{p}_2/Z])((\varphi(\tilde{\mathbf{p}}[\mathbf{p}_2/Z]))^{k'}(p_0))) \\
&= d(\mathcal{M}_5[\![\mathbf{S}']\!](\tilde{\mathbf{p}}[\mathbf{p}_1/Z][(\varphi(\tilde{\mathbf{p}}[\mathbf{p}_1/Z]))^{k'}(p_0)/Z']), \\
&\qquad \mathcal{M}_5[\![\mathbf{S}']\!](\tilde{\mathbf{p}}[\mathbf{p}_2/Z][(\varphi(\tilde{\mathbf{p}}[\mathbf{p}_2/Z]))^{k'}(p_0)/Z'])), \\
&\leq \max\{d(\mathcal{M}_5[\![\mathbf{S}']\!](\tilde{\mathbf{p}}[\mathbf{p}_1/Z][(\varphi(\tilde{\mathbf{p}}[\mathbf{p}_1/Z]))^{k'}(p_0)/Z']), \\
&\qquad\qquad \mathcal{M}_5[\![\mathbf{S}']\!](\tilde{\mathbf{p}}[\mathbf{p}_1/Z][(\varphi(\tilde{\mathbf{p}}[\mathbf{p}_2/Z]))^{k'}(p_0)/Z'])), \\
&\qquad\quad d(\mathcal{M}_5[\![\mathbf{S}']\!](\tilde{\mathbf{p}}[\mathbf{p}_1/Z][(\varphi(\tilde{\mathbf{p}}[\mathbf{p}_2/Z]))^{k'}(p_0)/Z']), \\
&\qquad\qquad \mathcal{M}_5[\![\mathbf{S}']\!](\tilde{\mathbf{p}}[\mathbf{p}_2/Z][(\varphi(\tilde{\mathbf{p}}[\mathbf{p}_2/Z]))^{k'}(p_0)/Z']))\} \\
&\qquad \text{(since } d \text{ is an ultra-metric)} \\
&\leq \max\{k^2 \cdot d(\mathbf{p}_1, \mathbf{p}_2), \kappa \cdot d(\mathbf{p}_1, \mathbf{p}_2)\} \\
&\qquad \text{(by the induction hypothesis (8.23) stating that (8.22) holds)} \\
&\leq \kappa \cdot d(\mathbf{p}_1, \mathbf{p}_2).
\end{aligned}$$

Thus (8.25) holds for $n = k' + 1$. Thus, one has

$$\begin{aligned}
&d(\mathcal{M}_5[\![\mathbf{S}]\!](\tilde{\mathbf{p}}[\mathbf{p}_1/Z]), \mathcal{M}_5[\![\mathbf{S}]\!](\tilde{\mathbf{p}}[\mathbf{p}_2/Z])) \\
&= d(\lim_{n\to\infty}[(\varphi(\tilde{\mathbf{p}}[\mathbf{p}_1/Z]))^{(n)}(p_0)], \lim_{n\to\infty}[(\varphi(\tilde{\mathbf{p}}[\mathbf{p}_2/Z]))^{(n)}(p_0)]) \\
&\leq \kappa \cdot d(\mathbf{p}_1, \mathbf{p}_2).
\end{aligned}$$

Since $\mathbf{p}_1, \mathbf{p}_2$ have been chosen arbitrarily, one has the desired conclusion that $(\lambda \mathbf{p} \in \mathbf{P}_4^{(k,\ell)}.\ \mathcal{M}_5[\![\mathbf{S}]\!](\tilde{\mathbf{p}}[\mathbf{p}/Z]))$ is a contraction with coefficient $\kappa$.

**Case 2.** Suppose $(i, j) \neq (0, 0)$. Then, one of the following holds: **(i)** $\mathbf{S} \in \mathcal{X}^{(i,j)}$, **(ii)** $\mathbf{S} \equiv (\lambda\vec{\eta}\cdot\vec{\xi}.\ \mathbf{S}')$, **(iii)** $\mathbf{S} \equiv (\mu Z'.\ \mathbf{S}')$. We consider the case (ii) here; for the other cases (i) and (iii) the same conclusion is obtained as in Subcase 1.1 and Subcase 1.5 above.

By the definition of $\mathcal{M}_5[\![\mathbf{S}]\!](\tilde{\mathbf{p}})$, one has the following for $\mathbf{p} \in \mathbf{P}_4^{(i,j)}$:

$$\begin{aligned}
&\mathcal{M}_5[\![\mathbf{S}]\!](\tilde{\mathbf{p}}[\mathbf{p}/Z]) \\
&= (\lambda\vec{\mathbf{c}}\cdot\vec{v} \in \mathbf{Chan}^i \cdot \mathbf{V}^j.\ \mathcal{M}_5[\![(\mathbf{S}'[(\vec{\mathbf{c}}\cdot\vec{v})/(\vec{\eta}\cdot\vec{\xi})])]\!](\tilde{\mathbf{p}}[\mathbf{p}/Z])).
\end{aligned}$$

Thus by the induction hypothesis (8.23) stating that (8.22) holds, one has the following for $\mathbf{p}_1, \mathbf{p}_2 \in \mathbf{P}_4^{(k,\ell)}$, $(\vec{\mathbf{c}}, \vec{v}) \in \mathbf{Chan}^i \times \mathbf{V}^j$:

$$d(\mathcal{M}_5[\![(\mathbf{S}'[(\vec{\mathbf{c}} \cdot \vec{v})/(\vec{\eta} \cdot \vec{\xi})])]\!](\tilde{\mathbf{p}}[p_1/Z]), \mathcal{M}_5[\![(\mathbf{S}'[(\vec{\mathbf{c}} \cdot \vec{v})/(\vec{\eta} \cdot \vec{\xi})])]\!](\tilde{\mathbf{p}}[p_2/Z])) \\ \leq \kappa \cdot d(\mathbf{p}_1, \mathbf{p}_2).$$

Thus, one has,

$$d(\mathcal{M}_5[\![\mathbf{S}]\!](\tilde{\mathbf{p}}[\mathbf{p}_1/Z]), \mathcal{M}_5[\![\mathbf{S}]\!](\tilde{\mathbf{p}}[\mathbf{p}_2/Z])) \leq \kappa \cdot d(\mathbf{p}_1, \mathbf{p}_2).$$

Since $p_1, p_2$ have been chosen arbitrarily, one has the desired conclusion that

$$(\lambda \mathbf{p} \in \text{SeVal}.\ \mathcal{M}_5[\![\mathbf{S}]\!](\tilde{\mathbf{p}}[\mathbf{p}/Z]))$$

is a contraction with coefficient $\kappa$. ∎

## 8.C Description Examples in $\mathcal{L}_5$

In this appendix, two description examples in $\mathcal{L}_5$ are given.

**Example 8.2** In Figure 8.1, we give a description in $\mathcal{L}_5$ of the server of a typical server-client system supporting an indefinite number of clients.

In the description, the following constructs in $\mathbf{Fun}_b$ are assumed to be predefined: **(a)** a constant **c** of type $\mathcal{C}$ used when clients request the service, **(b)** a constant 'client-indexes' of type $\mathcal{V}$ representing a predefined set of possible client indexes, **(c)** a constant 'sorry' of type $\mathcal{V}$ indicating that all the client indexes are being used, **(d)** a constant 'exit' of type $\mathcal{V}$, **(e)** a function symbol 'service-chan' of type $((\mathcal{V}), \mathcal{C})$, **(f)** a function symbol 'update' of type $(\mathcal{V}^3, \mathcal{V})$ such that $\text{update}(f, n, i) = f[n/i]$ for $f \in (\text{client-ids} \to \omega)$, $n \in \omega$, $i \in \text{client-ids}$, where 'client-ids' is a predefined set of identifiers of clients, **(g)** a function symbol 'bill' of type $(\mathcal{V}^2, \mathcal{V})$ such that $[\![\text{bill}(f, i)]\!] = f(i)$ for $f \in (\text{client-ids} \to \omega)$ and $i \in \text{client-ids}$.

We use three individual variables $x_0$, $x_1$, $x_2$ as *local* variables, and another individual variable $x_3$ as a *global* variable: **(h)** The variable $x_0$ stores the set of client indexes being used; **(i)** the variable $x_1$ stores the *client index* of the current session; **(j)** the variable $x_2$ stores the *bill* of the current session; **(k)** the variable $x_3$ stores the list of bills of all the clients.

Firstly, the server initializes the variable $x_0$ to $\emptyset$ (line 1); then the body in lines 2–20 runs as follows: **(1)** The server receives a service request from a client through the channel **c** with the passed value, the identifier of the client, bound to $\xi$ (lines 3–4). **(2)** If the all the possible client-indexes are being used, i.e., if $\text{client-indexes} \setminus x_0 = \emptyset$, then the server outputs a message 'sorry' (line 6); otherwise the block in lines 7–20 is executed.

The block runs as follows: **(3)** The variable $x_1$ is set to the client-index of the current session (line 7). **(4)** The index is sent to the client through **c**, thereby the client knows which service channel is used in the session (line 8). **(5)** The value of $x_0$ is updated so as to include the value of $x_1$ (line 9).

```
1.  let(x0, ∅,
2.    (μX0.
3.      in'(c,
4.        (λξ.
5.          if(clients-indexes \ x0 = ∅,
6.            out(c, 'sorry'),
7.            let(x1, min(clients-indexes \ x0),
8.              out(c, x1);
9.              asg(x0, x0 ∪ {x1});
10.             (X0 ||
11.               let(x2, account(x3, ξ),
12.                 (μX1.
13.                   in'(service-chan(x1),
14.                     (λξ1.
15.                       if(ξ1 = 'exit',
16.                         (out(service-chan(x1), x2);
17.                          asg(x3, update(x3, bill(x3, ξ) + x2, ξ));
18.                          asg(x0, x0 \ {x1}) ),
19.                         (out(service-chan(x1), retrieve(ξ1));
20.                          asg(x2, x2 + 1); X1 ))))))))))))). ∎
```

Figure 8.1: Description of a Server in $\mathcal{L}_5$

The rest of the block is the parallel composition of the original server, which can receive new service requests, and a subprocess (in lines 11–20) managing the current session. The subprocess runs as follows: First, $x_2$ is initialized to account$(x_3, \xi)$ (line 11). The rest of the subprocess is the iteration of the service transaction (in lines 12–20), which runs as follows: **(6)** The subprocess receives a query through the channel 'service-channel$(x_1)$' with the passed value (the query) is bound to $\xi_1$ (lines 13–14). **(7)** When the passed value is 'exit', the subprocess informs the bill of the session of the client, updates the values of $x_3$ and $x_0$, and then terminates (lines 16–18). Otherwise the subprocess retrieves the answer to the query $\xi_1$, and sends it to the client (line 19); then it increases the value of $x_2$ by 1, and iterates the transaction (line 20). ∎

**Example 8.3** Let us name the parameterized statement given in Figure 8.2 **fact**. For $\mathbf{c} \in \mathbf{Chan}$, $n \geq 0$, the statement $\mathbf{fact}(\mathbf{c}, n)$ computes $n!$ and outputs it through **c**, creating $n$ processes which operate concurrently. ∎

$$
\begin{array}{l}
(\mu X^{(1,1)}. \\
\quad (\lambda(\eta,\xi).\ \mathbf{if}(\xi = 0, (\mathbf{out}(\eta, 1); \mathbf{0}), \\
\qquad\qquad \mathbf{LC}(\lambda\eta'.\ X^{(1,1)}(\eta', \xi - 1) \\
\qquad\qquad \| (\mathbf{in}'(\eta', (\lambda\eta'.\ \mathbf{out}(\eta, \eta \cdot \eta'); \mathbf{0}))) )))).
\end{array}
$$

Figure 8.2: Description of Factorial in $\mathcal{L}_5$

# Chapter 9

# Conclusion

We conclude this thesis with a table which summarizes the results obtained so far on the full abstractness problem for concurrent languages (see Table 9.1 on page 354).

In this thesis, we have investigated the full abstractness problem for concurrent programming languages, in the context of *linear-time* semantics. More specifically, we have investigated how to construct fully abstract models w.r.t. strong/weak linear semantics, both for uniform and nonuniform languages.

As described in §§ 4.1, 5.1, 7.1, fully abstract models for *uniform* languages w.r.t. *linear* operational models (of the *linear-time* variety) were first investigated in [BKO 88], and then investigated in [Rut 89] in the framework of metric semantics (cf. $*1, *2$ in Table 9.1). In Chapter 4, we have constructed fully abstract models w.r.t. *strong linear* semantics for two languages $\mathcal{L}_1$, $\mathcal{L}_2$, of the uniform variety; the first language $\mathcal{L}_1$ is a so-called *pure* concurrent language, and $\mathcal{L}_2$ is an *applicative* language based on $\mathcal{L}_1$.

In Chapter 5, we have constructed fully abstract models w.r.t. three *weak* linear semantics for $\mathcal{L}_1$ within the framework of structural operational semantics in the style of [Plo 81]; in Chapter 6, we have constructed a fully abstract model w.r.t. yet another weak linear semantics also for $\mathcal{L}_1$, in the framework of *denotational semantics* based on cms's and *cpo's*.

In this thesis, we have also investigated similar problems for *nonuniform* languages. The operational models $\mathcal{O}_3$ and $\mathcal{O}_4$ for a nonuniform language introduced in Chapter 7 are *strong* models of the *linear-time* variety; both $\mathcal{O}_3$ and $\mathcal{O}_4$ involve information about *states*. Fully abstract denotational models w.r.t. $\mathcal{O}_3$ and $\mathcal{O}_4$ are presented in Chapter 7 (cf. $*3$ in Table 9.1). Also, we defined a more abstract operational model $\mathcal{O}_4^*$ for $\mathcal{L}_4$ by ignoring *states* in § 7.3, and showed that $\mathcal{M}_4$ is fully abstract w.r.t. $\mathcal{O}_4^*$ as well (cf. $*4$ in Table 9.1).

It seems more difficult to construct fully abstract denotational models w.r.t. *weak* operational models. In the uniform setting, fully abstract models for a CCS-like language were constructed w.r.t. weak linear semantics with divergence, in Chapter 5 and Chapter 6 (cf. $*5$ in Table 9.1).

In the nonuniform setting, the problem seems more difficult, and not much is known on this problem. For nonuniform languages as well, weak operational semantics are naturally defined from strong ones by abstracting from internal actions: For example, a weak operational semantics $\mathcal{O}_4^{**}$ for $\mathcal{L}_4$ is defined from $\mathcal{O}_4^*$ as follows: For every statement $s$ and state $\sigma$,

$$\mathcal{O}_4^{**}[\![s]\!](\sigma) = \{(\rho \setminus \tau) : \rho \in \mathcal{O}_4^*[\![s]\!](\sigma)\},$$

where $(\rho \setminus \tau)$ is the result of ignoring $\tau$'s in $\rho \in (\mathbf{C}_4 \cup \{\tau\})^{\leq\omega}$. It remains for future research to construct fully abstract models w.r.t. weak linear semantics for nonuniform languages like $\mathcal{L}_3$, $\mathcal{L}_4$ which have parallel composition but no coroutine construct (cf. $*7$ in Table 9.1). Hennessy and Plotkin constructed a fully abstract model for a language $\mathcal{L}_{\mathrm{co}}$ which is similar to $\mathcal{L}_3$ but has a *coroutine construct* in addition to parallel composition (cf. $*6$ in Table 9.1). Having the coroutine construct, however, has the effect of making internal actions in weak semantics *visible*, and thus, makes it possible to treat weak semantics as if they were strong ones (cf. § 7.2.6). A related discussion is found in the last section of [BKO 88].

The full abstractness problem has also been investigated in the context of *branching-time* semantics. In [Mil 80], [Mil 85] and [Mil 89], Milner showed that a strong operational model for CCS of the *branching-time* variety is compositional (cf. $*8$ in Table 9.1). Moreover, it was shown in [GV 92] that branching-time and strong operational models are in general compositional under certain conditions (cf. $*9$ in Table 9.1). Denotational models equivalent to those operational models were presented in [Rut 90] for the case when the LTS's on which the operational models are based are finitely branching; this result was extended in [Hor 92b], to the case when the underlying LTS's are possibly infinitely branching. The denotational models are fully abstract w.r.t. the operational models by definition (cf. $*10$, $*11$ in Table 9.1).

In [Mil 80], [Mil 85], [Mil 89], Milner characterized a fully abstract compositional model for CCS w.r.t. *observation equivalence* $\approx$ (cf. $*12$ in Table 9.1). This relation $\approx$ is a weak operational equivalence relation of the *branching-time* variety. Milner characterized *observation congruence* $\approx^c$, which is the coarsest congruence relation included in $\approx$, as follows: For every two statements $s_1$, $s_2$,

$$\begin{aligned} &s_1 \approx^c s_2 \Leftrightarrow \\ &\forall a \in \mathrm{Act}, \textstyle\bigwedge_{(i,j)=(1,2),(2,1)}, \forall s'[\ s_i \xrightarrow{a} s' \Rightarrow \\ &\quad \exists s''[\ s_j(\xrightarrow{\tau})^* \xrightarrow{a} (\xrightarrow{\tau})^* s'' \wedge s' \approx s''\ ]], \end{aligned}$$

where Act is the set of all actions including $\tau$ (cf. [Mil 89], Definition 7.2). The characterization of a fully abstract model in Chapter 5 is an analogue to this characterization, in the context of linear-time semantics.

In the major part of the thesis, in Chapters 4, 7, 8, we use cms's as semantic domains for denotational models, although a cpo is also used in Chapter 6. For more treatment of the full abstractness problem for concurrent languages in an order-theoretic framework, see [HP 79], [BM 87], [MV 88], [MO 90].

In [Hen 88], which is based on [DH 84], [Hen 83], [Hen 85], Hennessy showed in detail the full abstractness of a denotational model **AT** consisting of *acceptance* *trees* equipped with a complete partial order, w.r.t. *testing equivalence*. In § 6.6.3 and § 6.6.4, we showed a variant $\mathbf{AT}_s$ of $\mathbf{AT}$ is isomorphic to a mild variant $\mathcal{D}_{\bar{1}}^{wf}$ of the fully abstract model $\mathcal{D}_1^{wf}$ presented in Chapter 6.

For a survey of the full abstractness problem for sequential languages, see [BCL 85]. In [St 86], the general question concerning the existence of fully abstract models was treated in an algebraic context.

Table 9.1: Results on Fully Abstract Models for Communicating Processes

| | | | |
|---|---|---|---|
| Linear Time | Strong | Uniform | [BKO 88]: Characterization of a fully abstract compositional model.*1<br>[Rut 89], Chap. 4: Construction of a fully abstract denotational model.*2 |
| | | Nonuniform | Chap. 7 ([HBR 90]): Construction of a fully abstract denotational model w.r.t. an operational model *with states*.*3<br>Chap. 7: Construction of a fully abstract denotational model w.r.t. an operational model *without states*.*4 |
| | Weak | Uniform | Chap. 5 ([Hor 92a]), Chap. 6.*5 |
| | | Nonuniform | [HP 79]: For a language with a *coroutine* construct.*6<br>?: For languages with parallel composition like $\mathcal{L}_3$, $\mathcal{L}_4$.*7 |
| Branching Time | Strong | Uniform | [Mil 80], [Mil 85], [Mil 89]: Characterization of a fully abstract compositional model for CCS.*8<br>[GV 92]: Characterization of fully abstract compositional models in general.*9<br>[Rut 90]: Construction of fully abstract denotational models for the case of finitely branching LTS's *10<br>[Hor 92b]: Construction of fully abstract denotational models for the case of infinitely branching LTS's*11 |
| | | Nonuniform | ? |
| | Weak | Uniform | [Mil 80], [Mil 85], [Mil 89]: Characterization of a fully abstract compositional model.*12 |
| | | Nonuniform | ? |

# Appendix A

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# Appendix B

# Index

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# Appendix C

# Curriculum Vitae

**Eiichi Horita** was born on June 8, 1958, in Ishikawa Prefecture, Japan. He received the B. Sc. degree in mathematics from Kyoto University, Kyoto, Japan, in 1982, and the M. Eng. degree in applied mathematics from Kyushu University, Fukuoka, Japan, in 1984. He joined NTT Electrical Communications Laboratories in 1984, where he has been working on formal description and verification of communication protocols, and semantics of concurrent programming languages. At present, he is a research engineer at NTT Software Laboratories. During 1989–1990 he was a visiting researcher at the Center for Mathematics and Computer Science, Amsterdam, the Netherlands.