Semigroup Forum Vol. 38 (1989) 85–89
© 1989 Springer-Verlag New York Inc.

RESEARCH ARTICLE

A NOTE ON COMPACTIFICATIONS OF PRODUCTS OF SEMIGROUPS

M. Hušek and J. de Vries

Communicated by Karl. H. Hofmann

1. Introduction

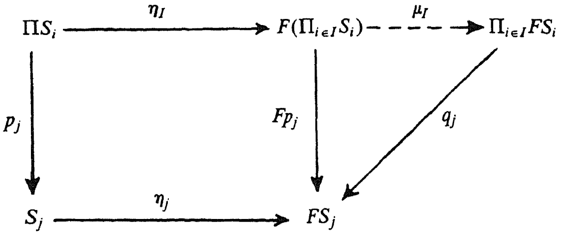
Notation and terminology will be as in [1] except for some minor modifications. All semigroups under consideration are assumed to have an identity. Thus, **STSgp** is the category whose objects are the semitopological semigroups with identity and whose morphisms are the continuous identity preserving homomorphisms. By **TopSgp** (resp. **TopGrp**) we denote the full subcategory of **STSgp** having as objects all topological semigroups with identity (resp. all topological groups), while **CSTSgp**, **CTopSgp** and **CTopGrp** denote the full subcategories of all compact Hausdorff objects in these categories. As is pointed out in [1], it is a straightforward consequence of general results from category theory that all inclusion functors between these categories have left adjoints. Functorial considerations were first introduced in this area in reference [8]. In particular, the following reflectors exist:

STSgp →	**CTopGrp**	:	F^{SAP}
STSgp →	**CTopSgp**	:	F^{AP}
STSgp →	**CSTSgp**	:	F^{WAP}

(Here our notation deviates from [1], where M,A and W are used for F^{SAP}, F^{AP} and F^{WAP}, respectively.). If F is any one of these reflectors, then for each object S of **STSgp** there is an essentially unique "universal arrow", the reflection into the corresponding subcategory, $\eta_S:S\to FS$ which is, in all cases, a morphism with dense range.

We shall consider two additional functors, namely, F^{LUC} and F^{LMC}. These can also be obtained as reflectors, but it is easier to describe them by means of the corresponding universal arrows $\eta_S:S\to FS$(S an object of **STSgp**). Here FS is a compact Hausdorff *right* topological semigroup (i.e., all right translations $\xi\mapsto\xi\xi':FS\mapsto FS$ for $\xi'\in FS$ are continuous), $\eta_S:S\to FS$ is a continuous homomorphism with dense range such that the mapping $(s,\xi)\mapsto\eta_S(s)\xi:S\times FS\to FS$ is continuous (in the case $F=F^{LUC}$) or separately continuous (in the case $F=F^{LMC}$), and η_S is universal for this type of homomorphisms (so we use the characterizations given in Theorems III. 5.5 and III. 4.5 of [1] as definitions).

The following remarks apply to each of the functors mentioned above. If $\{S_i:i\in I\}$ is a set of objects in **STSgp**, then there exists a unique morphism $\mu_I:F(\Pi_{i\in I}S_i)\to\Pi_{i\in I}FS_i$, completing the following commutative diagram for each $j\in I$:



Here $\Pi_{i\in I}S_i$ and $\Pi_{i\in I}FS_i$ denote cartesian products (with coordinate-wise semigroup operations and ordinary product topology; this are just the products in the corresponding categories), the p_j and q denote canonical projections, η_I stands for $\eta_{\Pi S_i}$ and η_j for η_{S_j}. The question is: when is η_I an isomorphism? If μ_I is an isomorphism for all (finite) products, then F is said to *preserve* all (finite) products.

Usually, reflectors do not preserve products (cf. [4] for many examples). In [6] it is shown (generalizing earlier results of De Leeuw and Glicksberg and of Berglund and Milnes) that F^{AP} and F^{SAP} preserve all products, and an example is cited which shows that F^{WAP} doesn't preserve all finite products. In [4] we obtained these properties of F^{AP} and F^{SAP} as consequences of more general results in certain concrete categories, but it seems worthwhile to write down straightforward proofs for F^{AP} and F^{SAP}, the more so as our proofs are very elementary and make no use of function algebras whatsoever. Also, our proof covers all special cases about F^{WAP} and F^{LUC} dealt with in [6].

2. Finite products

Proposition. *The reflectors F^{AP} and F^{SAP} preserve all finite products.*

Proof. Let F be F^{AP} or F^{SAP} and consider two objects S_1 and S_2 in **STSgp**. Let e_1 and e_2 be the identities in S_1 and S_2, respectively, and

$$\alpha_1: x\mapsto(x,e_2): S_1\to S_1\times S_2; \alpha_2: x\mapsto(e_1,x): S_2\to S_1\times S_2$$

the canonical embeddings. Other notation is as in Section 1, but we shall write μ for $\mu_{\{1,2\}}$ and η for $\eta_{\{1,2\}}$.

For $\xi\in F(S_1\times S_2)$ one has, by the definition of $\mu, \mu(\xi)=(Fp_1(\xi), Fp_2(\xi))$. Putting $\xi=\eta(x_1,x_2)$ with $x_i\in S_i$, one sees immediately that $\mu(\eta(x_1,x_2))=(\eta_1(x_1),\eta_2(x_2))$, so $\mu\circ\eta=\eta_1\times\eta_2$. It follows that the range of μ contains the subset $\eta_1[S_1]\times\eta_2[S_2]$, which is dense in $FS_1\times FS_2$. Since the range of μ is compact, it follows that μ is a surjection. Now it is sufficient to show that μ is an injection (for then μ, going from a compact to a Hausdorff space, is a homeomorphism, hence an isomorphism in the category under consideration). To this end, define the mapping $\Phi: FS_1\times FS_2\to F(S_1\times S_2)$ by

$$\Phi(\xi_1,\xi_2):=F\alpha_1(\xi_1)\cdot F\alpha_2(\xi_2),\ (\xi_1,\xi_2)\in FS_1\times FS_2,$$

where the dot denotes the multiplication in the semigroup $F(S_1\times S_2)$ (actually, Φ will turns out to be inverse to μ). In order to show that μ is injective, it is sufficient to prove that $\Phi\circ\mu$ is the identity map on $F(S_1\times S_2)$. Taking into account the observation above that $\mu\circ\eta=\eta_1\times\eta_2$, and the observation that $\Phi(\eta_1(x_1),\eta_2(x_2))=\eta(x_1,e_2)\cdot\eta(e_1,x_2)=\eta(x_1,x_2)$ for $(x_1,x_2)\in S_1\times S_2$, one sees immediately that

$$(\Phi\circ\mu)\circ\eta = \Phi\circ(\eta_1\times\eta_2) = \eta = id_{F(S_1\times S_2)}\circ\eta\ .$$

Since multiplication in $F(S_1\times S_2)$ is continuous, it follows that Φ, hence $\Phi\circ\mu$, is continuous. As η has a dense range, this implies that $\Phi\circ\mu=id_{F(S_1\times S_2)}$. This completes the proof that F preserves all products of two factors. A simple induction procedure shows that F preserves all finite products. □

Remarks. 1. In the proof above (i.e., the case of a product of two factors) one needs only that e_1 is a right identity in S_1 and that e_2 is a left identity in S_2; cf. [2] and [6].

2. The proposition above is valid for any reflector F of **STSgp** into a dense-reflective subcategory of **CTopSgp**: we only needed that the η's have dense range, and that multiplication in $F(S_1\times S_2)$ is simultaneously continuous. Thus, F might be the reflector of **STSgp** into the subcategory of 0-dimensional compact Hausdorff topological semigroups (or groups).

3. In the above proof, continuity of the multiplication in $F(S_1\times S_2)$ is used only to guarantee that the mapping Φ is continuous. Actually, one needs only *continuity of the restriction to* $F\alpha_1[FS_1]\times F\alpha_2[FS_2]$ *of the multiplication map* $(\xi_1,\xi_2)\mapsto\xi_1\xi_2$. Continuity of this restriction, however, can easily be obtained in some additional special cases, so that for those special cases Φ is continuous and products are preserved as well.

Case (a). $F=F^{WAP}$ and FS_1 is algebraically a group. Then for every object S_2 in **STSgp**, $F^{WAP}(S_1\times S_2)=F^{WAP}S_1\times F^{WAP}S_2$. Indeed, $F\alpha_1$, being a section to Fp_1, is an isomorphic embedding, hence $F\alpha_1[FS_1]$ is a closed subgroup of the compact Hausdorff semitopological semigroup $F(S_1\times S_2)$. So Ellis' joint continuity theorem (e.g., as formulated in [7], II. 4.4) implies that the multiplication in $F(S_1\times S_2)$ is jointly continuous on $F\alpha_1[FS_1]\times F(S_1\times S_2)$. Hence Φ is continuous, which implies the desired result. Note, that this situation occurs when S_1 is a dense subsemigroup of a compact Hausdorff topological group G: in that case $F^{WAP}S_1=G$ with $\eta_1: S_1\to G$ the inclusion mapping (this follows from [1], III. 15.7, where it is proved using function algebras; however, we can prove this quite easily by elementary means). This covers the special case mentioned in Corollary 5 of [6].

Case (b). $F=F^{LUC}$ and S_1 is an object of **CTopSgp**. Then for every object S_2 of **STSgp**, $F^{LUC}(S_1 \times S_2)=S_1 \times F^{LUC}S_2=F^{LUC}S_1 \times F^{LUC}S_2$ (the equality $S_1=F^{LUC}S_1$ is trivial for a compact Hausdorff topological semigroup). Indeed, in this case the mapping

$$((s_1,s_2),\xi) \mapsto \eta(s_1,s_2)\xi : (S_1 \times S_2) \times F(S_1 \times S_2) \to F(S_1 \times S_2)$$

is continuous. Put here $s_2=e_2$ and take into account that by assumption $\eta_1 : S_1 \to FS_1$ is an isomorphism. Since $\eta(s_1,e_2)=F\alpha_1(\eta_1(s_1))$ for all $s_1 \in S_1$, it follows that the multiplication mapping of $F(S_1 \times S_2)$ is jointly continuous on $F\alpha_1[FS_1] \times F(S_1 \times S_2)$. This implies the desired result. (Compare this with Corollary 3 of [6].).

Case (c). $F=F^{LMC}$ and S_1 is an object of **CTopGrp**. Then for every object S_2 of **STSgp**, $F^{LMC}(S_1 \times S_2)=S_1 \times F^{LMC}S_2=F^{LMC}S_1 \times F^{LMC}S_2$ (it is obvious that for any semitopological semigroup T one has $F^{LMC}T=T$; this is certainly valid for $T=S_1$). To prove this, first observe that, similar as in (b) above, the multiplication mapping in the right topological semigroup $F(S_1 \times S_2)$ is separately continuous on $F\alpha_1[FS_1] \times F(S_1 \times S_2)$. By the Ellis-Lawson Theorem (cf. [7], II. 4.3), the multiplication is jointly continuous on this set. This implies the desired results (which is, in fact, Theorem 2.6 of [2]).

Case (d). $F=F^{LUC}$ and S_1 is a dense subsemigroup of a compact topological Hausdorff group G. Then for every object S_2 of **STSgp**, $F^{LUC}(S_1 \times S_2)=G \times F^{LUC}S_2=F^{LUC}S_1 \times F^{LUC}S_2$; here $F^{LUC}S_1=G$ with $\eta_1 : S_1 \to G$ the inclusion mapping. To prove this, first observe that $F^{LUC}S_1=G$: this follows from [1], III. 15.4, but an elementary proof, not using function algebras, is possible. Now similar as in case (b) one sees that the multiplication mapping of $F(S_1 \times S_2)$ is jointly continuous on the set $F\alpha_1[\eta_1 S_1] \times F(S_1 \times S_2)$. The following lemma then shows that it is continuous on $F\alpha_1[G] \times F(S_1 \times S_2)$, which is sufficient for the continuity of Φ. Note that this implies the special case, mentioned in Corollary 4 of [6].

LEMMA. *Let T be a compact Hausdorff right topological semigroup, and let T_0 be a subsemigroup such that $H:=\overline{T}_0$ is a topological group. If the multiplication mapping of T is jointly continuous on $T_0 \times T$, then it is also jointly continuous on $H \times T$.*

PROOF. By the Ellis-Lawson Theorem it would be sufficient to show that the multiplication mapping is separately continuous on $H \times T$, but it requires almost no additional effort to prove joint continuity directly. So let $h \in H, t \in T$ and let W be a closed nbd (= neighbourhood) of ht in T. Since $ht=e.ht$ with e (the identity of T) in T_0, there are a nbd U of e in T_0 and an open nbd V of ht in T such that $UV \subseteq W$. So for every $s \in V, Us \subseteq W$, hence $\overline{U}s \subseteq \overline{W}=W$ by continuity of right translations. Thus,

$$\overline{U}.V \subseteq W. \tag{1}$$

Now observe that $U=U' \cap T_0$ for some nbd U' of e in H. Since T_0 is dense in H, it follows that $\overline{U}=\overline{U' \cap T_0}=\overline{U'}$. Replacing U by U', we may and shall assume henceforth that the set U in formula (1) is a nbd of e in H rather than a nbd of e in T_0. Next, recall that V is a nbd of ht in T. There is a nbd U_1 of h in H such that $U_1 t \subseteq V$ and, in addition, there is a nbd U_2 of e in H such that $U_1 \supseteq U_2 h$ and, moreover, $U_2=U_2^{-1}, U_2^2 \subseteq U$. So by (1), $U_2 U_2 V \subseteq W$. Select any $s \in U_2 h \cap T_0 (\neq \varnothing$ because T_0 is dense in H). Then $hs^{-1} \in U_2$ (inverse taken in H), hence

$$U_2 h.s^{-1} V \subseteq U_2 . U_2 . V \subseteq W \tag{2}$$

Here $U_2 h$ is a nbd of h in H. Also, by the choice of U_1 and s we have $t \in s^{-1}V$. As the mapping $\tau \mapsto s\tau : T \to T$ is a bijection (with inverse $\tau \mapsto s^{-1}\tau$) and since it is continuous (for $s \in T_0$), the inverse mapping is continuous as well. In particular, $s^{-1}V$ is an open subset of T, hence a nbd of t. So (2) is just what we want. □

REMARKS (continued). 4. The following shows that F^{WAP} doesn't preserve all finite products (cf. also [2], p. 171, and [5]; we believe our arguments to be much simpler). Let S be a *commutative* topological semigroup with identity. Then the multiplication mapping $\omega : S \times S \to S$ is a morphism in **TopSgp**, so it "extends" uniquely to a morphism

$$\tilde{\omega} := F^{WAP}(\omega) : F^{WAP}(S \times S) \to F^{WAP}S.$$

Now assume that $F^{WAP}(S \times S)=F^{WAP}S \times F^{WAP}S$ (canonically). Then it is easy to see that $\tilde{\omega}$ coincides with the multiplication mapping of $F^{WAP}S$ (which maps $F^{WAP}S \times F^{WAP}S$ into $F^{WAP}S$) on the dense image of $S \times S$. Hence, by a straightforward continuity argument, $\tilde{\omega}$ coincides with this multiplication map on all of $F^{WAP}S \times F^{WAP}S$, and since $\tilde{\omega}$ is jointly continuous it follows that $F^{WAP}S$ is an object in **CTopSgp** rather than **CSTSgp**. Stated otherwise, $F^{WAP}S=F^{AP}S$. Many examples are known where this equality is violated, so those examples must have $F^{WAP}(S \times S) \neq F^{WAP}S \times F^{WAP}S$. In order to keep within the philosophy of this paper, we present an elementary argument (not using (weakly) almost periodic functions) showing that $F^{WAP}S \neq F^{AP}S$ for every non-compact locally compact Hausdorff topological group S. To this end, observe that for such S the one-point compactification $S^* := S \cup \{\infty\}$ is an object in **CSTSgp** (put $\xi.\infty = \infty.\xi = \infty$ for all $\xi \in S^*$). So the embedding $j: S \to S^*$ factorises over the universal arrow $\eta_S: S \to F^{WAP}S$ as $j = \tilde{j} \circ \eta_S$, with $\tilde{j}: F^{WAP}S \to S^*$ a *surjective* morphism. Now suppose that $F^{WAP}S = F^{AP}S$. It is an elementary fact that in the present situation $F^{AP}S$ is a group (even a topological group: by [3], A. 1.5, a compact topological semigroup with a dense group in it is a topological group). So if $\xi \in F^{AP}S$ is such that $\tilde{j}(\xi) = \infty$, then $\tilde{j}(e) = \tilde{j}(\xi.\xi^{-1}) = \infty.\tilde{j}(\xi^{-1}) = \infty$, which is not the case because $\tilde{j}(e) = e \in S$. Hence $F^{WAP}S \neq F^{AP}S$.

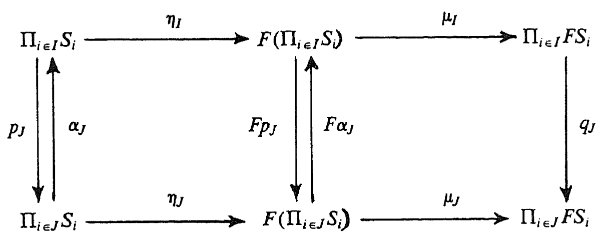
5. The argument in 4 above can be modified so as to show that in 3(a) above the condition that $F^{WAP}S_1$ is a compact topological group cannot be weakened to the condition that S_1 is a compact semitopological semigroup, not even if S_2 is a locally compact topological group. For let S be a commutative semitopological semigroup which is, algebraically, a group. Put $\tilde{S} := F^{WAP}S$, with canonical mapping $\eta: S \to \tilde{S}$. By the Ellis-Lawson theorem (cf. [7], Theorem II. 4.3), the mapping $w: (\xi, s) \mapsto \eta(s)\xi: \tilde{S} \times S \to \tilde{S}$ is continuous. Since $\tilde{S}$ is commutative, w is a morphism in **STSgp**, so it "extends" so a morphism $\tilde{w} := F^{WAP}w: F^{WAP}(\tilde{S} \times S) \to F^{WAP}\tilde{S} = \tilde{S}$. Now again, the assumption that $F^{WAP}(\tilde{S} \times S) = F^{WAP}\tilde{S} \times F^{WAP}S = \tilde{S} \times \tilde{S}$ would lead to the conclusion that $\tilde{w}: \tilde{S} \times \tilde{S} \to \tilde{S}$ is the multiplication mapping of $\tilde{S}$, which would be continuous. This would mean that $F^{WAP}S = F^{AP}S$, which is certainly not true if S is a non-compact locally compact topological group.

6. Whether F^{WAP} preserves a product $S_1 \times S_2$ or not is not a property of $S_1 \times S_2$ alone, but involves the structures of S_1 and S_2. For example, let G be a compact topological group and let H be a non-compact locally compact Hausdorff topological group. By Case (a) of Remark 3, F^{WAP} preserves $S_1 \times S_2$ with $S_1 := G$, $S_2 := H \times H$, but it doesn't preserve $S'_1 \times S'_2$ with $S'_1 := G \times H$, $S'_2 := H$ (see Remark 4), though $S_1 \times S_2$ and $S'_1 \times S'_2$ are topologically isomorphic.

3. INFINITE PRODUCTS

THEOREM. *The reflectors F^{AP} and F^{SAP} preserve all products*

PROOF. Consider a set $\{S_i : i \in I\}$ of objects in **STSgp**. Then for each non-empty subset J of I one has the following diagram



Here p_J and q_J are projections and α_J is the canonical embedding $(x)_{i\in J} \mapsto (\bar{x}_i)_{i\in I}$ with $\bar{x}_i = x_i$ if $i \in J$ and $\bar{x}_i = e_i$ (the identity of S_i) otherwise. As in the proof of the proposition in Section 2 it is sufficient to show that μ_I is injective (having a dense range, μ_I is surjective). For this proof it will be convenient to introduce the following notation: $w_J := \alpha_J \circ p_J$ and $\rho_J := Fw_J = F\alpha_J \circ Fp_J$. In addition, $\mathfrak{F}$ will denote the directed (under $\supseteq$) set of all non-empty finite subsets of I. CLAIM: for every $\xi \in F(\Pi_{i\in I}S_i)$ the net $\{\rho_J(\xi)\}_{J\in\mathfrak{F}}$ converges to ξ.

From this, injectivity of μ_I follows easily: if ξ_1, ξ_2 are in $F(\Pi_{i\in I}S_i)$ and $\xi_1 \neq \xi_2$, then these points have disjoint neighbourhoods, and the claim implies that there is $J \in \mathfrak{F}$ with $\rho_J\xi_1 \neq \rho_J\xi_2$. But then $Fp_J(\xi_1) \neq Fp_J(\xi_2)$, and as μ_J is injective by the main result of Section 2, this implies that $\mu_I(\xi_1) \neq \mu_I(\xi_2)$.

It remains to prove the claim. Assume the contrary: there exists a point ξ_0 in $F(\Pi_{i\in I}S_i)$ which has a neighbourhood U such that the subset

$$\mathfrak{F}_1 := \{J \in \mathfrak{F} : \rho_J(\xi_0) \notin U\}$$

of $\mathfrak{F}$ is cofinal in $\mathfrak{F}$. By compactness, the net $\{\rho_J\xi_0\}_{J\in\mathfrak{F}_1}$ has an accumulation point ζ_0. Then $\zeta_0 \notin U$ and ζ_0 has a neighbourhood V such that $\xi_0 \notin V$. Since multiplication in $F(\Pi_{i\in I}S_i)$ is simultaneously continuous, the equality $\zeta_0 = \zeta_0 \cdot e$ (where e is the identity in $F(\Pi_{i\in I}S_i)$) implies that there are neighbourhoods V' and V_e of ζ_0 and e, respectively, such that $V'.V_e \subseteq V$; replacing V_e by a smaller neighbourhood of e whose closure is contained in V_e (regularity of the topology) shows that one may assume that $V'.\bar{V}_e \subseteq V$. Note, that by the choice of ζ_0, $\mathfrak{F}_2 := \{J \in \mathfrak{F}_1 : \rho_J(\zeta_0) \in V'\}$ is cofinal in $\mathfrak{F}_1$, hence in $\mathfrak{F}$.

By continuity of η_I, there is a neighbourhood W of $(e_i)_{i\in I}$ in $\Pi_{i\in I}S_i$ such that $\eta_I[W]\subseteq V_e$. Let J be a finite subset of I determining a basic neighbourhood of $(e_i)_{i\in I}$ included in W. Then $w_{I\setminus J}(x)\in W$ for all $x\in\Pi_{i\in I}S_i$. Since this J can be replaced by any larger member of $\mathcal{F}$ and $\mathcal{F}_2$ is cofinal in $\mathcal{F}_1$ we may assume that $J\in\mathcal{F}_2$, so that

$$\rho_{I\setminus J}(\eta_I(x)) = \eta_I(w_{I\setminus J}(x))\in\eta_I[W]\subseteq V_e$$

for all $x\in\Pi_{i\in I}S_i$. Stated otherwise, $\rho_{I\setminus J}$ maps the dense (!) range of η_I into V_e. Hence $\rho_{I\setminus J}(\xi)\in\overline{V}_e$ for all $\xi\in F(\Pi_{i\in I}S_i)$. Next, notice that $x=w_J(x).w_{I\setminus J}(x)$ for all $x\in\Pi_{i\in I}S_i$, whence $\xi = \rho_J(\xi)\cdot\rho_{I\setminus J}(\xi)$ for all ξ in the range of η_I. By a continuity argument, this equality holds for all $\xi\in F(\Pi_{i\in I}S_i)$. Taking into account that $J\in\mathcal{F}_2$, this implies in particular that

$$\xi_0 = \rho_J(\xi_0)\cdot\rho_{I\setminus J}(\xi_0)\in V'\cdot\overline{V}_e\subseteq V.$$

This contradicts the choice of V. □

REFERENCES

[1] BERGLUND, J.F. H.D. JUNGHENN and P. MILNES, *Compact right topological semigroups and generalizations of almost periodicity,* Lecture Notes in Math. **663**, Springer-Verlag, Berlin, etc., 1978.

[2] BERGLUND, J.F. and P. MILNES, Algebras of functions on semitopological left-groups, *Trans. Amer. Math. Soc.* **222** (1976), 157-178.

[3] HOFMANN, K.H. and P.S. MOSTERT, *Elements of compact semigroups,* Charles E. Merrill Books, Inc., Columbus (Ohio), 1966.

[4] HUŠEK, M. and J. DE VRIES, Preservation of products by functors close to reflectors, *Topology Appl.* **27** (1987), 171-189.

[5] JUNGHENN, H.D., Tensor products of spaces of almost periodic functions, *Duke Math. J.* **41** (1974), 661-666.

[6] JUNGHENN, H.D., C^*-algebras of functions on direct products of semigroups, *Rocky Mountain J. Math* **10** (1980), 589-597.

[7] RUPPERT, W., *Compact semitopological semigroups: an intrinsic theory,* Lecture Notes in Math. **1079**, Springer-Verlag, Berlin, etc. 1984.

[8] BERGLUND, J.F. and K.H. HOFMANN, *Compact semitopological semigroups and weakly almost periodic functions,* Lecture Notes in Math. **42**, Springer-Verlag, Berlin, etc., 1967.

M. Hušek
Math. Inst. of the Charles University
Sokolovská 83
Prague 8
Czechoslovakia

J. de Vries
CWI
Kruislaan 413
1098 SJ Amsterdam
the Netherlands

Received February 15, 1987