Line arrangements and geometric representations of graphs

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Mark your diaries

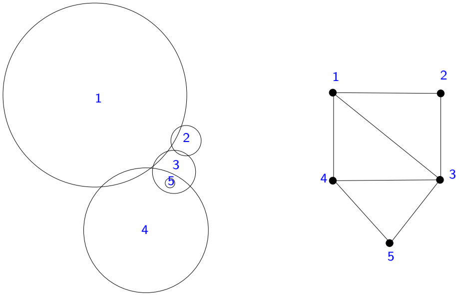
6 December 2010

Workshop “Probabilistic and algebraic methods in combinatorics, optimization and computer science”.

At CWI Amsterdam

Disk graphs

A *disk graph* is a the intersection graph of disks in the plane. That is, we can represent each vertex by a disk in the plane such that the vertices are adjacent iff the corresponding disks intersect.



If we can take all the disks of the same radius then we speak of a *unit disk graph*

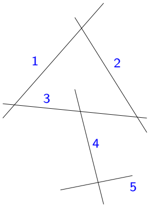
Other dimensions

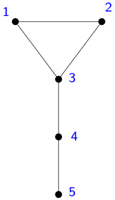
A d*-ball graph* is the intersection graph of balls in d-dimensional space. A d*-unit ball graph* is the intersection graph of unit-radius balls in d-dimensional space.

When $d = 1$ we speak of *interval graphs* resp. *unit interval graphs*.

Segment graphs

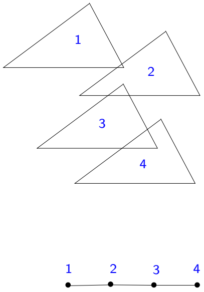
A *segment graph* is the intersection graph of line segments in the plane.

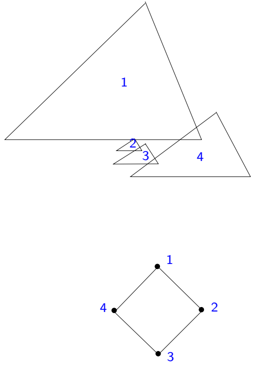




Polygon graphs

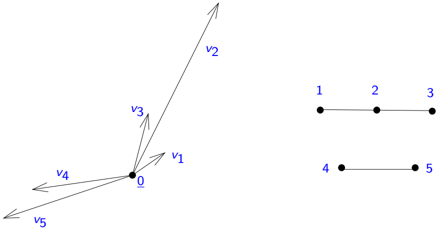
Let P be a convex polygon. A *P-translates graph* is an intersection graph of translates of P, and a *P-homothets graph* is an intersection graph of scaled translates of P.





Dot product graphs

A graph G is a *k-dot product graph* if there are vectors $v_1, \ldots, v_n \in \mathbb{R}^k$ such that $v_i^T v_j \geq 1$ iff $ij \in E(G)$.



1-dot product graphs are also called *threshold graphs*.

Part II: Integer representations

Every disk graph has an integer representation

G is a disk graph iff we can find $(x_1, y_1, r_1, \ldots, x_n, y_n, r_n) \in \mathbb{R}^{3n}$ such that

$$\begin{aligned} (x_i - x_j)^2 + (y_i - y_j)^2 &\leq (r_i + r_j)^2, \quad \text{for all } ij \in E(G), \\ (x_i - x_j)^2 + (y_i - y_j)^2 &> (r_i + r_j)^2, \quad \text{for all } ij \notin E(G). \end{aligned}$$

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By keeping the rs fixed and scaling the xs and ys by a scalar smaller than but very very close to 1, we can make sure equality is never attained in $\leq$.

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Next, we "perturb" all variables very very slightly to get a rational vector $(x_1, y_1, r_1, \ldots, x_n, y_n, r_n) \in \mathbb{Q}^{3n}$.

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Next, we "perturb" all variables very very slightly to get a rational vector $(x_1, y_1, r_1, \ldots, x_n, y_n, r_n) \in \mathbb{Q}^{3n}$.

Finally, we can multiply everything by the product of the denominators to get an integer representation $(x_1, y_1, r_1, \ldots, x_n, y_n, r_n) \in \mathbb{Z}^{3n}$.

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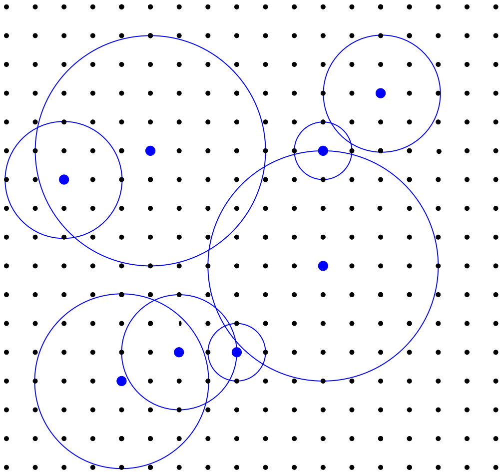

Integer representations of the other graphs classes

Very similar "perturbation and inflation" arguments apply to unit disk graphs, segment graphs, dot-product graphs, ...

Why do we care about integer representations?

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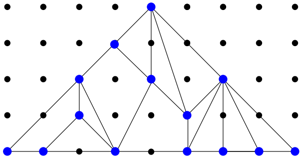
- NP-membership of the recognition problem: smallish integers give a "polynomial certificate".
- If the integers are really small then we can store G using less bits than an adjacency matrix.
- Visualization: How precisely do we need to draw?

Integer representations of planar graphs

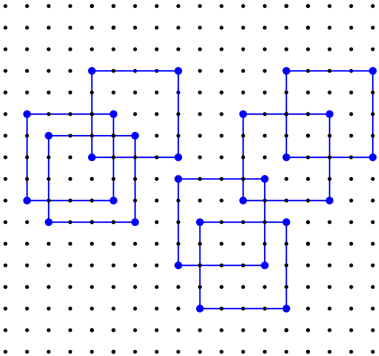
To draw a planar graph in the plane with straight lines we do not need too much space:

Theorem. [Fraysseix-Pach-Pollack,1988] Every planar graph on n vertices has a straight-line embedding with its vertices a subset of the grid $\{1, \ldots, 2n-4\} \times \{1, \ldots, n-2\}$.



Integer representations of unit square graphs

Theorem. [Czyzowicz et al, 1997] Intersection graphs of same-size squares can be represented with all corner points on a $O(n^2) \times O(n^2)$-grid.



Integer representations of P-translate graphs

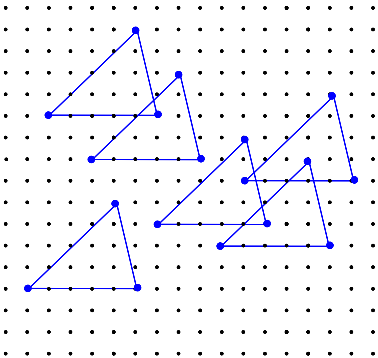
Theorem. [M+Van Leeuwen+Van Leeuwen, 2010+]
Let P be a convex polygon (whose corners have rational coordinates).

(i) If P is a paralellogram then any P-translates graph can be represented on a $O(n^2) \times O(n^2)$-grid, and this is sharp (up to the constant inside the $O(.)$).

(ii) If P is not a parallelogram then any P-translates graph can be represented with all corner points on a $2^{O(n)} \times 2^{O(n)}$-grid, and this is sharp (up to the constant inside the $O(.)$).

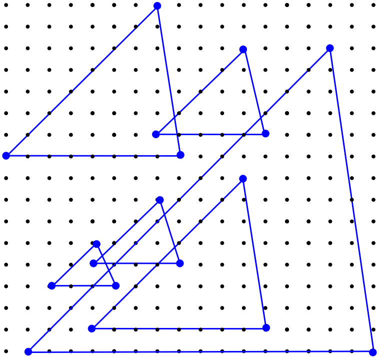
NOTE: while $2^{O(n)}$ is pretty big, we need only $O(n)$ bits to store each coordinate.

Integer representations of P-translate graphs



Integer representations of scaled polygon graphs

Theorem. [M+Van Leeuwen+Van Leeuwen, 2010+]
Fix a convex polygon P (whose corners have rational coordinates). Any P-homothets graph can be represented with all corner points on a $2^{O(n)} \times 2^{O(n)}$ grid, and this is sharp (up to the constant inside $O(.)$).



Some definitions

Let $f_{\mathcal{DG}}(n)$ denote the least k such that every disk graph on n vertices can be represented by disks with centers $\in \{1, \ldots, k\}^2$ and radii $r_1, \ldots, r_n \in \{1, \ldots, k\}$.

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Similarly, let $f_{\mathcal{UDG}}(n)$ denote the least k such that every unit disk graph on n vertices can be represented by disks with centers on $\{1, \ldots, k\}^2$, all of equal radius $r \in \{1, \ldots, k\}$.

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Let $f_{\mathcal{SEG}}(n)$ denote the least k such that all segment graphs on n vertices can be represented by segments with endpoints in $\{1, \dots, k\}^2$.

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Let $f_{\mathcal{SEG}}(n)$ denote the least k such that all segment graphs on n vertices can be represented by segments with endpoints in $\{1, \ldots, k\}^2$.

Let $f_{d\text{-}\mathcal{DPG}}(n)$ denote the least k such that all d-dot product graphs can be represented by vectors in $v_1, \ldots, v_n \in \{-k, \ldots, k\}^d$ together with a threshold $t \in \{1, \ldots, k\}$ such that $ij \in E$ iff $v_i^T v_j \geq t$.

A question

In his book "Efficient representation of graphs" Spinrad asks the following question(s):

Question. [Spinrad, 2003] How large are $f_{\mathcal{DG}}(n)$, $f_{\mathcal{UDG}}(n)$, $f_{d\text{-}\mathcal{DPG}}(n)$? Are they polynomially bounded? Or at least by $2^{O(n^K)}$ for some constant K?

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Van Leeuwen+Van Leeuwen'06 put it more strongly:

Polynomial Representation Hypothesis.[Van Leeuwen+Van Leeuwen, 2006]
$f_{\mathcal{DG}}(n) = 2^{O(n^K)}$ for some constant K.

The Polynomial Representation Hypothesis is false

We will disprove the Polynomial Representation Hypothesis:

Theorem. [McDiarmid+M, 2010+] $f_{\mathcal{DG}}(n) = 2^{2^{\Theta(n)}}$.

Theorem. [McDiarmid+M, 2010+] $f_{\mathcal{UDG}}(n) = 2^{2^{\Theta(n)}}$.

Theorem. [Kang+M, 2010+] $f_{d\text{-}\mathcal{DPG}}(n) = 2^{2^{\Theta(n)}}$ for all $d \geq 2$.

NOTE: here $f(n) = 2^{2^{\Theta(n)}}$ means there exist $c_1, c_2 > 0$ such that $2^{2^{c_1 n}} \leq f(n) \leq 2^{2^{c_2 n}}$.

Integer representations of segment graphs

Theorem. [Kratochvíl+Matoušek,1994] $f_{\mathcal{SEG}}(n) = 2^{2^{\Omega(\sqrt{n})}}$.

We improve this as follows:

Theorem. [McDiarmid+M, 2010+] $f_{\mathcal{SEG}}(n) = 2^{2^{\Theta(n)}}$.

Standard encoding of rationals

A usual convention is that a rational number is stored in the memory of a computer as a pair of integers that are relatively prime.

If n is an integer then its bit-size is:

$$\text{size}(n) = 1 + \lceil \log_2(|n|) \rceil.$$

If $q = \frac{n}{m}$ is a rational with n, m relatively prime then the bit size is

$$\text{size}(q) = \text{size}(n) + \text{size}(m).$$

Implications for bit-size

The results on the previous slides imply (via a small amount of work):

Corollary There exist disk/unit disk/segment/dot product graphs for which any representation using rational coordinates needs exponentially many bits, and exponentially many bits are always enough.

Part III: recognition problems

Recognition problems

If $\mathcal{C}$ is some class of graphs, then the *recognition problem* for $\mathcal{C}$ is the following decision problem:

INPUT:	A graph G (in adjacency matrix form).
OUTPUT:	"YES" if G is a member of $\mathcal{C}$, and "NO" otherwise.

Some results on recognition problems

For one dimensional objects it is usually easy:

Theorem. [Chvátal+Hammer, 1973] Threshold graph recognition is linear.

Theorem. [Booth+Luecker, 1976] Interval graph recognition is linear.

Theorem. [Corneil+Kamura, 1987] Unit interval graph recognition is linear.

Some results on recognition problems in dimension two

Theorem. [Kratochvíl+Matoušek, 1989] Segment graph recognition is NP-hard.

Theorem. [Breu+Kirkpatrick, 1998] Unit disk graph recognition is NP-hard.

Theorem. [Hliněný+Kratochvíl, 2001] Disk graph recognition is NP-hard.

Questions and answers

Conjecture. [Breu+Kirkpatrick,1998] k-unit ball recognition is NP-hard for all $k \geq 2$.

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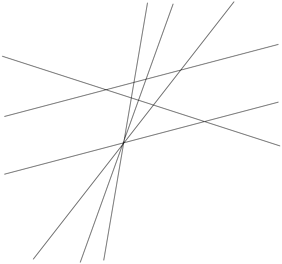
Theorem. [Kang+M,2010+] k-dot product recognition is NP-hard for all $k \geq 2$.

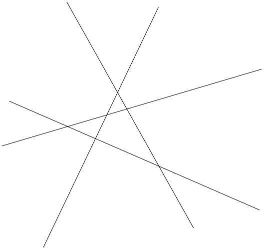
The proofs use a reduction to "simple stretchability" (to be defined later on) and are very similar.

Part IV: Line arrangements

Line arrangements

A *line arrangement* $\mathcal{L} = (\ell_1, \ldots, \ell_n)$ is a family of lines in the plane $\mathbb{R}^2$.

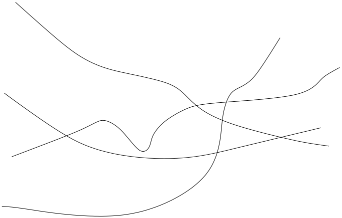




A line arrangement is *simple* if every two lines intersect, and no point lies on more than two lines.

Pseudoline arrangements

A *pseudoline arrangement* $\mathcal{L} = (\ell_1, \ldots, \ell_n)$ is a family of continuous curves in the plane that satisfy some regularity conditions.



A pseudoline arrangement is *simple* if every two pseudolines intersect and no point lies on more than two pseudolines.

Oriented (pseudo-) line arrangements

Let $\mathcal{L} = (\ell_1, \ldots, \ell_n)$ be a (pseudo-) line arrangement.

In an orientation of $\mathcal{L}$, we define one of the two components of $\mathbb{R}^2 \setminus \ell_i$ be the "plus side" (denoted ℓ_i^+) and the other the "minus side" (denoted ℓ_i^-).

For convenience, we shall only deal with oriented (pseudo-) line arrangements from now on.

Combinatorial description

The *sign vector* wrt. $\mathcal{L} = (\ell_1, \ldots, \ell_n)$ associated with a point $p \in \mathbb{R}^2$ is a vector $\sigma(p) \in \{-, 0, +\}^n$, with:

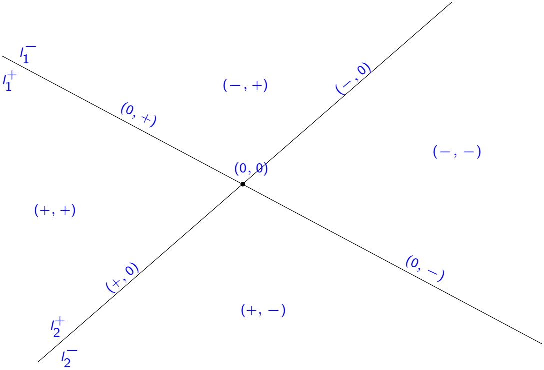
$$(\sigma(p))_i = \begin{cases} - & \text{if } p \in \ell_i^-, \\ 0 & \text{if } p \in \ell_i, \\ + & \text{if } p \in \ell_i^+. \end{cases}$$

The *combinatorial description* of $\mathcal{L}$ is the set of sign vectors

$$\mathcal{D}(\mathcal{L}) := \{\sigma(p) : p \in \mathbb{R}^2\}.$$

If $\mathcal{D}(\mathcal{L}) = \mathcal{D}(\mathcal{L}')$ the we say that $\mathcal{L}$ and $\mathcal{L}'$ are *isomorphic*.

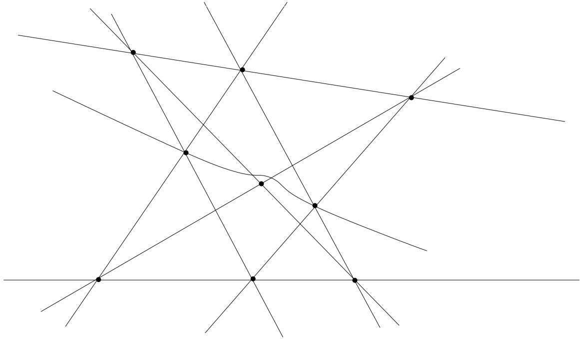
Example sign vectors



Stretchability

We say a pseudoline arrangement is *stretchable* if it is isomorphic to a line arrangement.

An example of a non-stretchable pseudoline arrangement:



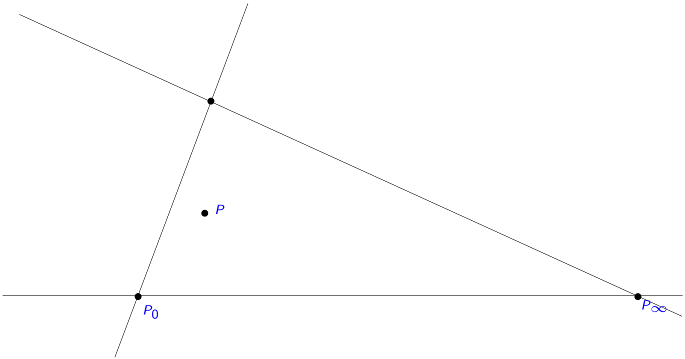
(non-stretchability follows from Pappus' theorem)

The Von Staudt sequences

Von Staudt ("Geometrie der Lage", 1847) invented a way to encode arithmetic operations in line arrangements.

Starting from three lines in general position and one point not on these lines, we "construct" a line arrangement by repeatedly adding a line through two existing (intersection) points.

The Von Staudt sequences – one



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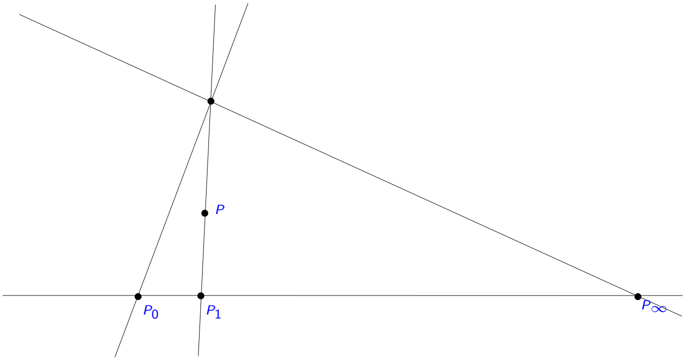
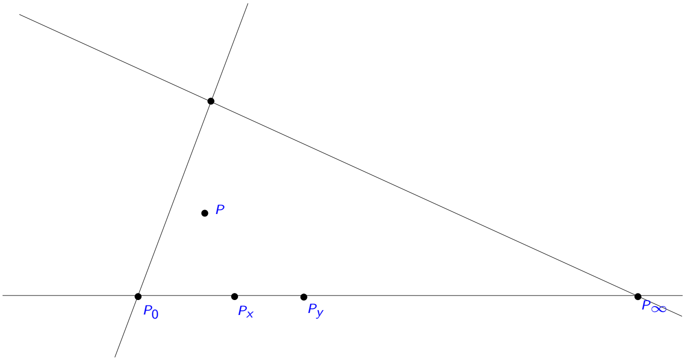


The Von Staudt sequences – addition



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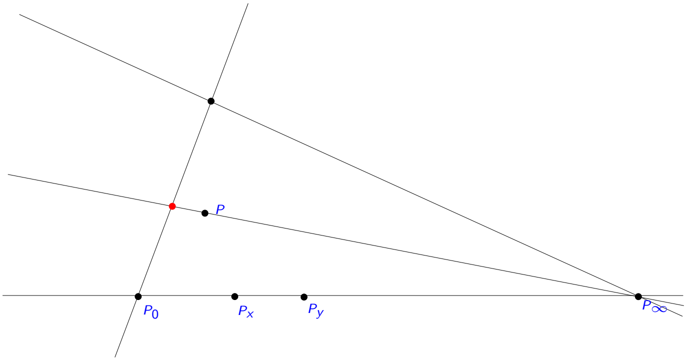

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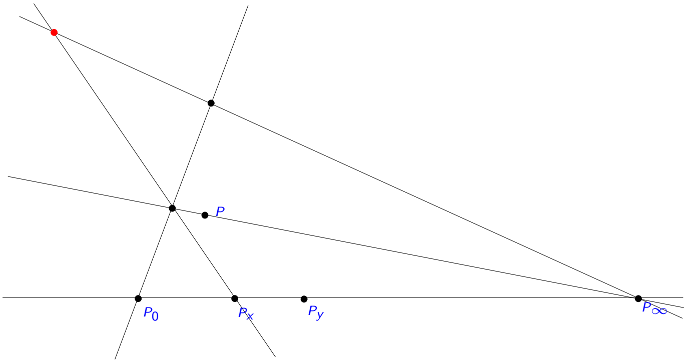

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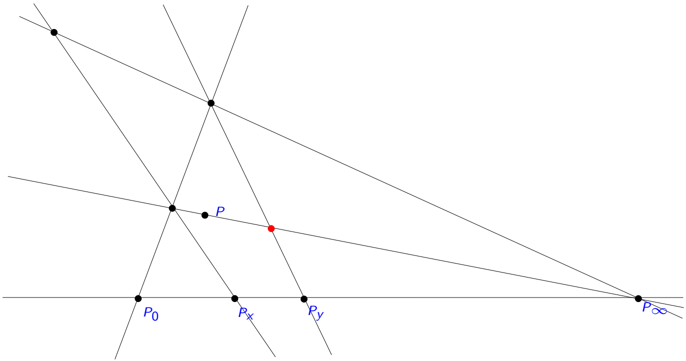

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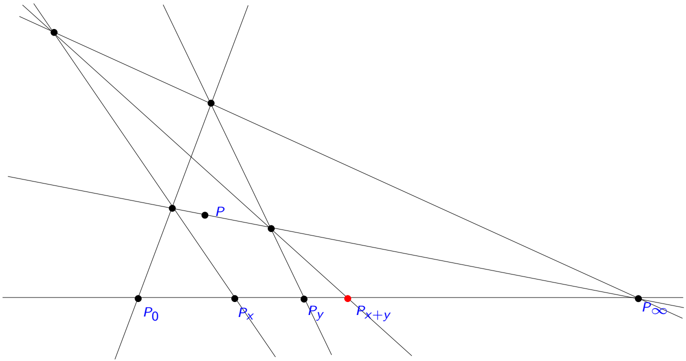
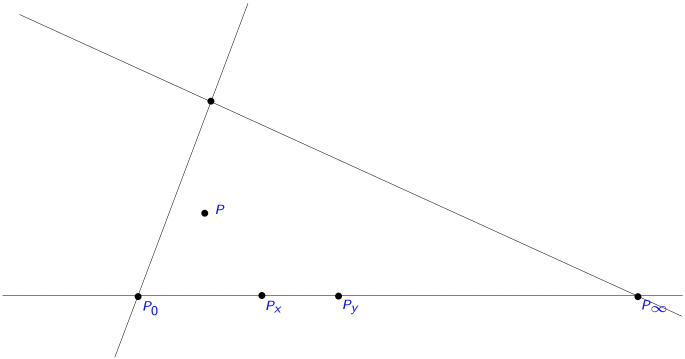


Von Staudt sequences – multiplication



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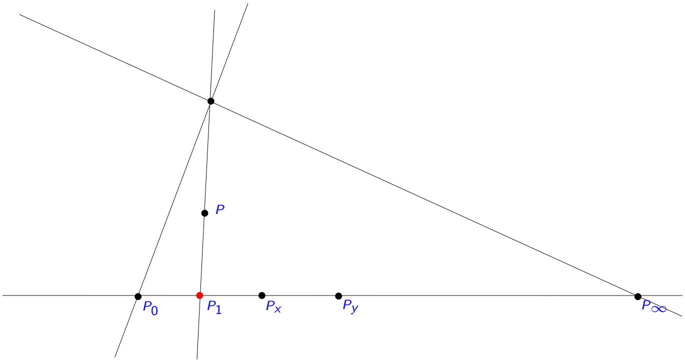

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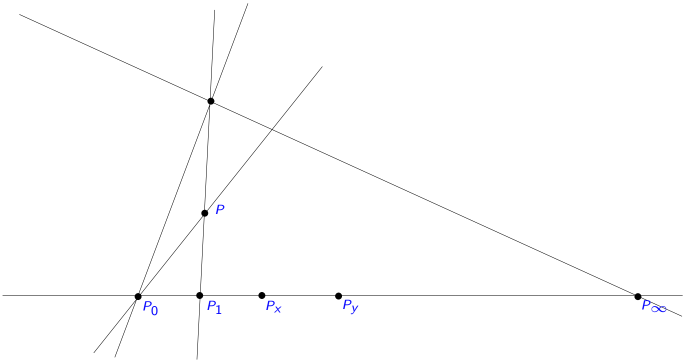

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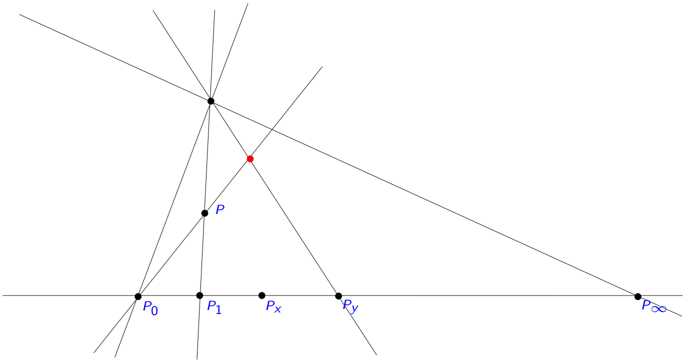

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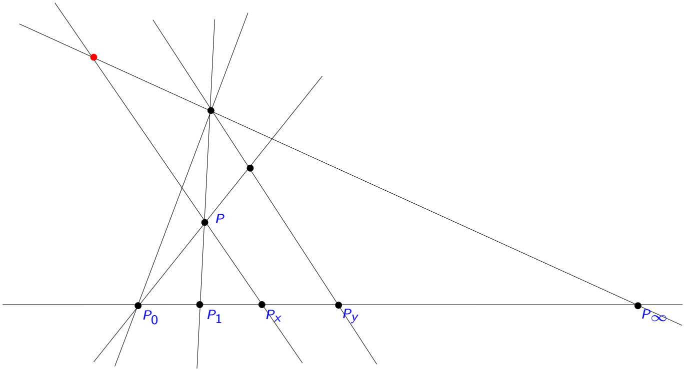

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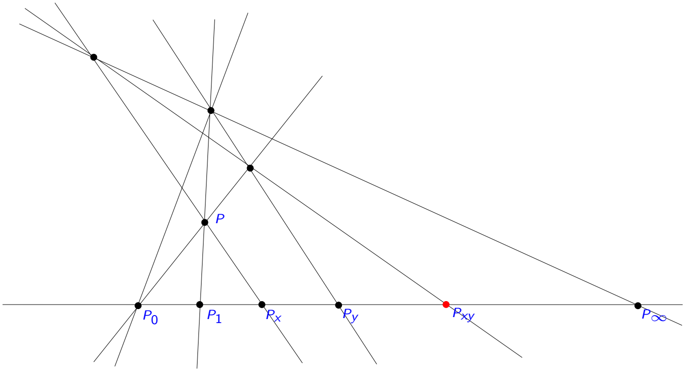


Remarks

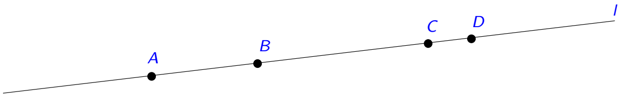
- There are also Von Staudt sequences for $x - y$ and x/y.
- Thus, we can "construct" P_q for any rational number $q \in \mathbb{Q}$.
- For instance, to get $3/2$ we first make P_1, then P_{1+1}, then P_{1+1+1} and finally $P_{(1+1+1)/(1+1)}$.

The cross ratio

For points A, B, C, D on a line ℓ the *cross ratio* is defined as

$$\mathrm{cr}(A, B, C, D) := \frac{\mathrm{length}(AB) \cdot \mathrm{length}(CD)}{\mathrm{length}(AC) \cdot \mathrm{length}(BD)}.$$

Here length(.) denotes the *signed length*, i.e. minus the length if the second point lies to the left of the first.



The cross ratio and Von Staudt

Theorem. [Von Staudt] Let $q \in \mathbb{Q}$ be a rational number, and consider a Von Staudt sequence for q. Then:

$$\mathrm{cr}(P_0, P_q, P_1, P_\infty) = q.$$

(Moreover, this property is "preserved under isomorphisms".)

The proof (and the precise definition of the Von Staudt constructions) is elementary, but not for 25 minute talks. So we skip it.

Two algorithmic decision problems

Existential theory of the reals:

INPUT:	A set of polynomial equalities and strict inequalities with integer coefficients.
OUTPUT:	"YES" if there is a simultaneous (real) solution, and "NO" otherwise.

Simple stretchability:

INPUT:	A combinatorial description $\mathcal{D}$ of a simple pseudoline arrangement.
OUTPUT:	"YES" if $\mathcal{D}$ is stretchable, and "NO" otherwise.

Mnëv's universality theorem

Using Von Staudt sequences, Mnëv proved a deep topological theorem on "realization spaces of line arrangements". As a corollary to this result he also obtained:

Theorem. [Mnëv, 1985] The existential theory of the reals is polynomially equivalent to simple stretchability.

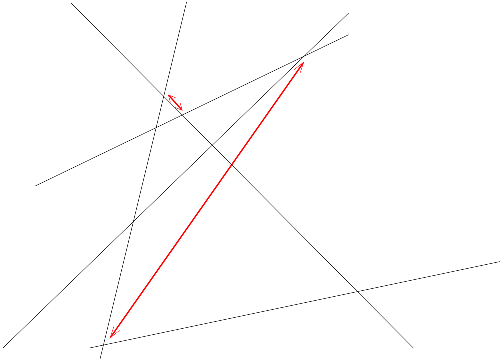
Corollary. Simple stretchability is NP-hard.

This last corollary was also obtained in a more direct way by Shor (1991). He reduced some SAT-variant using Pappus' and Desargues' theorems.

Part IV: Proof sketches

Another definition

Let $\text{span}(\mathcal{L})$ denote the ratio of the furthest distance between two intersection points of lines in $\mathcal{L}$ to the smallest distance between two (distinct) intersection points.



The lower bound on $f_{\mathcal{UDG}}$ – very brief proof sketch

Proof plan:

1. For $r \in \mathbb{N}$, we construct a set $S \subseteq \{-,+\}^{O(r)}$ with $|S| = O(r)$ such that whenever $S \subseteq \mathcal{D}(\mathcal{L})$ then $\text{span}(\mathcal{L}) \geq 2^{2^r}$.
2. We construct a unit disk graph G on $O(r)$ vertices, such that in any realization the lines $\ell_i := \{x : \| x - v_{2i-1} \| = \| x - v_{2i} \|\}$ induce a line arrangement with $S \subseteq \mathcal{D}(\mathcal{L})$. (here v_i is the center of the disk representing vertex i).
3. We apply some elementary computations to express $\text{span}(\mathcal{L})$ in terms of the coordinates of the v_i, and derive that at least one of them is $\geq 2^{2^r}$.

The lower bound on $f_{\mathcal{UDG}}$ – slightly more detail

Let $\mathcal{L}$ be a line arrangement that arises as a a Von Staudt sequence for $P_{2^{2^r+1}}$ with $O(r)$ lines in total.

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The rest of the construction in part 1 (to get the required $S \subseteq \mathcal{D}(\mathcal{L})$) is rather technical so we skip it.

We also skip steps 2, 3 which are technical as well.

Proof of the lower bounds on $f_{\mathcal{DG}}, f_{\mathcal{SEG}}, f_{d\text{-}\mathcal{DPG}}$

For disk, segment and dot-product graphs, we use the same $S \subseteq \{-,+\}^{O(r)}$ together with constructions for "embedding" it into a disk/segment/dot-product graph.

Proof of the upper bound on $f_{\mathcal{UDG}}$

Lemma. [Grigor'ev+Vorobjov,1985] For each $d, K \in \mathbb{N}$ there exists a constant $C = C(d, K)$ such that the following hold. Suppose that $h_1, \ldots, h_k$ are polynomials in n variables with integer coefficients, and degrees $\deg(h_i) < d$. Suppose all coefficients are less than K in absolute value. If there exists a solution $(x_1, \ldots, x_n) \in \mathbb{R}^n$ of the system $\{h_1 \geq 0, \ldots, h_k \geq 0\}$, then there also exists one with $|x_1|, \ldots, |x_n| \leq 2^{\log k \cdot C^n}$.

The proof of the upper bound on $f_{\mathfrak{UDG}}$, continued

Consider the set of inequalities:

$$
\begin{aligned}
&(x_i - x_j)^2 + (y_i - y_j)^2 \leq (r-3)^2, \quad \text{for all } ij \in E(G),\\
&(x_i - x_j)^2 + (y_i - y_j)^2 \geq (r+3)^2, \quad \text{for all } ij \notin E(G),\\
&r \geq 10.
\end{aligned}
$$

It has a solution (just inflate the coordinates and radii of some realization of G).

Hence, by G+V'85 there is also a real solution with all coordinates $\leq 2^{2^{cn}}$ for some constant c.

(we choose c such that $\left(\log \binom{n}{2}\right) \cdot C^n < 2^{cn}$, for all n).

The proof of the upper bound, continued

We now set $x_i' := \lfloor x_i \rfloor, y_i' := \lfloor y_i \rfloor, r' := \lfloor r \rfloor$.
Elementary computations (which we skip) give for $ij \in E(G)$:

$$(x_i' - x_j')^2 + (y_i' - y_j')^2 \leq (r')^2,$$

and for $ij \notin E(G)$ we find $(x_i' - x_j')^2 + (y_i' - y_j')^2 > (r')^2$. ■

Proof sketch for NP-hardness of k-unit ball graph recognition and dot product recognition

Given a combinatorial description $\mathcal{D}(\mathcal{L})$ of a simple pseudoline arrangement $\mathcal{L}$, we construct the adjacency matrix of a graph G on $O(|\mathcal{L}|^2)$ vertices (in polynomial time), such that G is a k-unit ball/k-dot product graph iff $\mathcal{L}$ is stretchable.
The construction is very similar to what we did in the lower bound of the integer representations proof.

Part V: dot product representations of planar graphs

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Theorem. [Reiterman et al '89, Fiduccia et al '98] Every forest is a 3-dot product graph.

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Theorem. [Kang+M, 10+] Every planar graph is a 4-dot product graph and there exist planar graphs that are not 3-product graphs.

The proof

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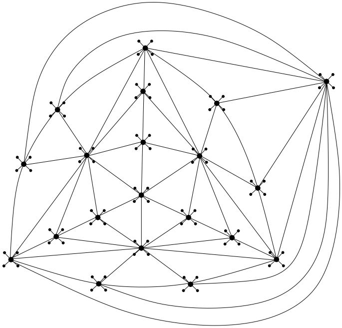

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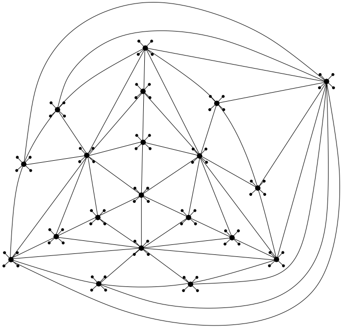

Then G_k is not 3-dot product for sufficiently large k.

We leave the proof as an exercise.
(Hint: use the spherical cosine rule and the Jordan curve theorem)

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- Dot-product dimension of other graph classes.