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RESEARCH ARTICLE

A HILLE-YOSIDA THEOREM FOR A CLASS OF WEAKLY * CONTINUOUS SEMIGROUPS

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0. Introduction.

In this paper we consider a class of weak * continuous semigroups of bounded linear operators on the dual of a Banach space X which are not necessarily the adjoints of C_0-semigroups on X. Such semigroups arise in a natural way as perturbations (in an appropriate sense) of adjoint C_0-semigroups: see Clément, Diekmann, Gyllenberg, Heijmans and Thieme [4-7]. There the perturbed semigroup is constructed by exploiting a variation-of-constants formula and duality arguments.

Here we shall introduce the notion of an integral weak * generator and use this to characterize the aforementioned class of weak * semigroups in a one-to-one manner.

Finally, we refer to Jefferies [12] for some related results.

1. Formal calculations with w^*-semigroups

A family $T^\times = \{T^\times(t): t \geq 0\}$ of bounded linear operators on a dual Banach space X^* such that

(i) $T^\times(0) = I$

(1.1) (ii) $T^\times(t+s) = T^\times(t)T^\times(s)$, $\quad t, s \geq 0$

(iii) $t \mapsto \langle x, T^\times(t)x^*\rangle$ is continuous for any given $x \in X$ and $X^* \in X^*$

is called a *weakly * continuous semigroup* or, in abbreviated form, a *w^*-semigroup.* The operator $A^\times$ defined by

$$A^\times x^* = w^* - \lim_{h\downarrow 0} \frac{1}{h}(T^\times(h)x^* - x^*) \tag{1.2}$$

with $\mathcal{D}(A^\times) = \{x^*: w^* - \lim_{h\downarrow 0} \frac{1}{h}(T^\times(h)x^* - x^*) \text{ exists}\}$ is called the *infinitesimal weak * generator* or, in abbreviated form, the *w^*-generator.*

The standard example of a w^*-semigroup is a dual semigroup, i.e.

$$T^\times(t) = T(t)^*$$

where $\{T(t)\}$ is a C_0-semigroup on X. In that case $A^\times = A^*$, where A is the infinitesimal generator of $T(t)$ and one can easily verify all the elegant and powerful relations between semigroup and generator which are familiar from C_0-semigroup theory provided one replaces strong differentiation and integration by the corresponding weak * analogs (see Butzer and Berens [3, §1.4]). In particular, a dual semigroup is uniquely determined by its w^*-generator. It is tempting to conjecture that this situation extends to w^*-semigroups in general.

However, an easy counterexample can be constructed as follows. Consider the C_0-semigroups $T(t)$ of translations on $X = C_0(\mathbf{R})$, the space of continuous functions defined on $\mathbf{R}$ which vanish at infinity. So $(T(t)x)(a) = x(t+a)$ and the dual semigroup T^* on X^* is defined by

$$\langle x, T^*(t)x^* \rangle = \langle T(t)x, x^* \rangle = \int_{\mathbf{R}} x(t+a)x^*(da).$$

It is well known that $X^\odot := \overline{\mathcal{D}(A^*)}$ is the maximal subspace of X^* on which $T^*(t)$ is strongly continuous in t. In this particular case $X^\odot$ is the subspace of measures which are Lebesgue absolutely continuous (so $X^\odot \simeq L_1(\mathbf{R})$) and one has the direct sum decomposition

$$X^* = X^\odot \oplus X^\perp$$

where $X^\perp$ denotes the subspace of measures which are singular with respect to the Lebesgue measure. We emphasize that both $X^\odot$ and $X^\perp$ are closed in X^* and invariant under $T^*(t)$. So for any $\alpha \in \mathbf{R}$ we can define a w^*-semigroup $T_\alpha^\times$ on X^* by

$$T_\alpha^\times(t)x^* = \begin{cases} T^*(t)x^* & \text{if } x^* \in X^\odot \\ T^*(\alpha t)x^* & \text{if } x^* \in X^\perp. \end{cases} \tag{1.3}$$

Obviously the maximal subspace of strong continuity does not depend on α and on this space $X^\odot$ the action does not depend on α either. So all these semigroups do have the same w^*-generator!

How can one distinguish the "bad" semigroups $T_\alpha^\times(t)$ with $\alpha \neq 1$ from the "good" semigroup $T^*(t)$ in a direct way, without invoking duality? The requirement that the semigroup operators are the solution operators corresponding to the Cauchy problem

$$\begin{aligned} &\frac{d^*}{dt}u(t) = A^\times u(t) \\ &u(0) = x^* \end{aligned} \tag{1.4}$$

is as such of not much help since in order to solve (1.4) one has to assume that $x^* \in \mathcal{D}(A^*)$ (and even that does not guarantee that a solution exists since

$\mathcal{D}(A^*)$ is not necessarily invariant under $T^\times(t)$). However, if we integrate (1.4) formally we obtain

$$u(t) - x^* = A^\times \int_0^t u(\tau)d\tau$$

and it seems reasonable to require that this should hold for $u(t) = T^\times(t)x^*$ and *all* $x^* \in X^*$. But with $T_\alpha^\times(t)$ defined by (1.3) we find

$$T_\alpha^\times(t)x^* - x^* = \begin{cases} A^\times \int_0^t T_\alpha^\times(\tau)x^*d\tau & \text{for } x^* \in X^\odot \\ \alpha A^\times \int_0^t T_\alpha^\times(\tau)x^*d\tau & \text{for } x^* \in X^\perp, \end{cases}$$

showing that the requirement is fulfilled iff $\alpha = 1$.

In order to rewrite the requirement in terms of semigroup operators only, we continue our *formal* calculations. If $x^* \in \mathcal{D}(A^\times)$ we write

$$A^\times \int_0^t T^\times(\tau)x^*d\tau = \int_0^t T^\times(\tau)A^\times x^*d\tau \tag{1.6}$$

even though a justification cannot be given. If we now consider the identity

$$T^\times(t)x^* = x^* + A^\times \int_0^t T^\times(\tau)x^*d\tau$$

and take x^* of the special form

$$x^* = \int_0^h T^\times(\sigma)y^*d\sigma \in \mathcal{D}(A^\times)$$

we obtain

$$\begin{aligned} T^\times(t)\int_0^h T^\times(\tau)y^*d\tau &= \int_0^h T^\times(\tau)y^*d\tau + \int_0^t T^\times(\tau)A^\times \int_0^h T^\times(\sigma)y^*d\sigma\ d\tau \\ &= \int_0^h T^\times(\tau)y^*d\tau + \int_0^h T^\times(\tau)\{T^\times(h)y^* - y^*\}d\tau \\ &= \int_0^h T^\times(t+\sigma)y^*d\sigma. \end{aligned}$$

This formal calculation motivates the introduction of property

$$T^\times(t)\int_0^h T^\times(\tau)x^*d\tau = \int_0^h T^\times(t+\tau)x^*d\tau \tag{$S1$}$$

$$\text{for all } x \in X^*,\ t \ge 0,\ h \ge 0.$$

We will call w^*-semigroups with property (S1) *integral w^*-semigroups*. A straightforward calculation shows that $T_\alpha^\times$ defined by (1.3) is an integral w^*-semigroup iff $\alpha = 1$.

Remark . Define
$$S^\times(t)x^* = \int_0^t T^\times(\tau)x^* d\tau.$$

Then $\{S^\times(t)\}$ is an *integrated semigroup* in the sense of Arendt [2], Kellermann and Hieber [13] and Neubrander [15] iff $\{T^\times(t)\}$ is an integral w^*-semigroup.

Up to now we are neither able to prove that (1.6) holds for all integral w^*-semigroups nor to find a counterexample within this class. So we are led to introduce the following concept of a generator.

Definition 1.1. $x^* \in \mathcal{D}(A_0^\times)$ and $y^* = A_0^\times x^*$ iff

$$T^\times(t)x^* - x^* = \int_0^t T^\times(\tau)y^* d\tau \ , \quad \text{for all } t \geq 0. \tag{1.7}$$

Note that, for $x^* \in \mathcal{D}(A_0^\times)$, y^* is uniquely determined by (1.7). We will call $A_0^\times$ the *integral generator* of $T^\times$. Observe that (1.7) is equivalent to

$$\frac{d^*}{dt} T^\times(t)x^* = T^\times(t)y^* \ , \qquad t \geq 0$$

and that automatically $\mathcal{D}(A_0^\times)$ is invariant under $T^\times(t)$ and $A_0^\times T^\times(t)x^* = T^\times(t)A_0^\times x^*$. Obviously $A^\times$ is an extension of $A_0^\times$.

One objective of this paper is to single out a large class of integral w^*-semigroups for which the two generators $A^\times$ and $A_0^\times$ are actually the same. The theory of dual semigroups suggests a way to achieve this end. For those we have [3, Corollary 2.1.5]

$$\mathcal{D}(A^*) = \mathrm{Fav}(T^*) = \{x^* \in X^* : t \mapsto T^*(t)x^* \text{ is Lipschitz on } [0,1]\}.$$

The fact that $A^\times$ extends $A_0^\times$ and the uniform boundedness principle imply that in general
$$\mathcal{D}(A_0^\times) \subset \mathcal{D}(A^\times) \subset \mathrm{Fav}(T^\times).$$

Therefore our strategy will be to forget about the w^*-generator for a while and to characterize those integral generators for which the domain coincides with the Favard class. The w^*-generator then coincides with the integral generator automatically.

2. The characterization theorem

Theorem 2.1. *Let* $A^\times$ *be a linear operator on* X^*. *The following sets* (G) *and* (S) *of properties are equivalent:*

(G_1) $(\lambda - A^\times)^{-1}$ *is an everywhere defined bounded operator such that for some* $M > 0$, $\omega \in \mathbf{R}$,

$$\|(\lambda - A^\times)^{-n}\| \le \frac{M}{(\lambda-\omega)^n} \quad \text{for } n \in \mathbf{N},\ \lambda > \omega.$$

(G_2) *If (i)* $x_n^* \in \mathcal{D}(A^\times)$, *(ii)* $\|x_n^* - x^*\| \to 0$ *as* $n \to \infty$ *and (iii)* $\|A^\times x_n^*\| \le C$ *for some* $C > 0$, *then* $x^* \in \mathcal{D}(A^\times)$ *and* $A^\times x_n^* \to A^\times x^*$ *weakly * as* $n \to \infty$.

(S) $A^\times$ *is the* w^**-generator of an integral* w^**-semigroup* $T^\times$ *which in addition to*

(S_1) $T^\times(t)\int_0^h T^\times(\tau)x^* d\tau = \int_0^h T^\times(t+\tau)x^* d\tau$, $x^* \in X^*$, $t, h \ge 0$, *satisfies*

(S_2) *If (i)* x_n^* *is a bounded sequence in* X^* *and (ii)* $S^\times(t)x_n^* = \int_0^t T^\times(\tau)x_n^* d\tau$ *converges strongly as* $n \to \infty$, *uniformly in* $t \ge 0$ *after scaling with a factor* $e^{-\lambda t}$ *with* $\operatorname{Re}\lambda$ *sufficiently large, then there exists* $x^* \in X^*$ *such that* $x_n^* \to x^*$ *weakly* as* $n \to \infty$ *and* $\|S^\times(t)x_n^* - S^\times(t)x^*\| \to 0$ *as* $n \to \infty$.

In the following we shall abbreviate the sentence "Let $A^\times$ be the w^*-generator of an integral w^*-semigroup such that (G) or, equivalently, (S) in Theorem 2.1 is satisfied" to "Assume G/S".

Theorem 2.2. *Assume* G/S. *Then*

a) $A^\times$ *is the integral generator of* $T^\times$. *Hence* $\mathcal{D}(A^\times)$ *is invariant under* $T^\times(t)$ *and* $\frac{d^*}{dt}T^\times(t)x^* = A^\times T^\times(t)x^* = T^\times(t)A^\times x^*$ *for* $x^* \in \mathcal{D}(A^\times)$ *and* $t > 0$.

b) $\|T^\times(t)\| \le Me^{\omega t}$ *and* $(\lambda - A^\times)^{-1}x^* = \int_0^\infty e^{-\lambda\tau}T^\times(\tau)x^* d\tau$ *for* $\lambda > \omega$.

c) $X^\odot := \overline{\mathcal{D}(A^\times)}$ *is the maximal subspace of strong continuity of* $T^\times$.

d) $\mathcal{D}(A^\times) = \operatorname{Fav}(T^\times) = \{x^*: \|T^\times(t)x^* - x^*\| \le Ct \text{ for } 0 \le t \le 1\}$ $= \{x^*: t \mapsto T^\times(t)x^*$ *is locally Lipschitz on* $[0,\infty)\}$.

e) *For* $x^* \in X^*$, $\int_0^t T^\times(\tau)x^* d\tau \in \mathcal{D}(A^\times)$ *and* $A^\times(\int_0^t T^\times(\tau)x^* d\tau) = T^\times(t)x^* - x^*$. *In particular* $\mathcal{D}(A^\times)$ *is* w^**-dense in* X^*.

f) $T^\times(t)x^* = w^* - \lim_{n\to\infty}(I - \frac{t}{n}A^\times)^{-n}x^*$.

Proof. Let $A^\odot$ denote the part of $A^\times$ in $X^\odot = \overline{\mathcal{D}(A^\times)}$. Assume ($G_1$). The Hille-Yosida theorem shows that $A^\odot$ generates a C_0-semigroup $T^\odot(t)$ on $X^\odot$.

We claim that $\mathcal{D}(A^\times) \subset \operatorname{Fav}(T^\odot) = \{x^\odot \in X^\odot: \limsup_{t\downarrow 0} \frac{1}{t}\|T^\odot(t)x^\odot - x^\odot\| < \infty\} = \{x^\odot \in X^\odot: t \mapsto T^\odot(t)x^\odot$ is locally Lipschitz on $[0,\infty)\}$. Take any $t \geq s \geq 0$ and $x^{\odot} \in \mathcal{D}(A^{\times})$; then

$$\begin{aligned}T^{\odot}(t)x^{\odot} - T^{\odot}(s)x^{\odot} &= \lim_{\lambda\to\infty}(T^{\odot}(t) - T^{\odot}(s))\lambda(\lambda - A^{\odot})^{-1}x^{\odot} \\ &= \lim_{\lambda\to\infty}\int_s^t T^{\odot}(\tau)A^{\odot}\lambda(\lambda - A^{\odot})^{-1}x^{\odot}d\tau.\end{aligned}$$

Since $x^{\odot} \in \mathcal{D}(A^{\times})$ we have $A^{\odot}\lambda(\lambda - A^{\odot})^{-1}x^{\odot} = \lambda(\lambda - A^{\times})^{-1}A^{\times}x^{\odot}$ and this remains bounded for $\lambda \to \infty$. Hence $\|T^{\odot}(t)x^{\odot} - T^{\odot}(s)x^{\odot}\| \leq C|t-s|$ and the claim is proved.

Any $x^{\odot} \in X^{\odot}$ can be strongly approximated by elements $\frac{1}{t}\int_0^t T^{\odot}(s)x^{\odot}ds \in \mathcal{D}(A^{\odot})$. If $x^{\odot} \in \mathrm{Fav}(T^{\odot})$, then $A^{\odot}\frac{1}{t}\int_0^t T^{\odot}(s)x^{\odot}ds = \frac{1}{t}(T^{\odot}(t)x^{\odot} - x^{\odot})$ remains bounded as $t \downarrow 0$. Assume (G_2). It follows that any $x^{\odot} \in \mathrm{Fav}(T^{\odot})$ necessarily belongs to $\mathcal{D}(A^{\times})$. Hence $\mathcal{D}(A^{\times}) = \mathrm{Fav}(T^{\odot})$.

Obviously $\mathrm{Fav}(T^{\odot})$ is invariant under $T^{\odot}$ and so the following definition makes sense:

$$T^{\times}(t)x^* = (\lambda - A^{\times})T^{\odot}(t)(\lambda - A^{\times})^{-1}x^* \tag{2.1}$$

for $\lambda \in \rho(A^{\times})$. The resolvent identity shows that this definition does not depend on the choice of λ. Clearly $\{T^{\times}(t)\}$ is a semigroup. Because of (G_1), $\lambda T^{\odot}(t)(\lambda - A^{\times})^{-1}x^*$ remains bounded as $\lambda \to \infty$. Since $T^{\times}(t)x^*$ is independent of λ, $A^{\times}T^{\odot}(t)(\lambda - A^{\times})^{-1}x^*$ has to remain bounded as well. (G_1) implies that $T^{\odot}(t)(\lambda - A^{\times})^{-1}x^*$ tends to zero strongly as $\lambda \to \infty$. It then follows from (G_2) that $A^{\times}T^{\odot}(t)(\lambda - A^{\times})^{-1}x^*$ tends to zero in the weak* topology. We conclude that

$$T^{\times}(t)x^* = w^* - \lim_{\lambda\to\infty}\lambda T^{\odot}(t)(\lambda - A^{\times})^{-1}x^*. \tag{2.2}$$

Using (G_1) once more we obtain the estimate

$$\|T^{\times}(t)x^*\| \leq \|T^{\odot}(t)\|M\|x^*\| \tag{2.3}$$

which shows that $\|T^{\times}(t)\|$ is exponentially bounded. Since $t \mapsto T^{\odot}(t)(\lambda - A^{\times})^{-1}x^*$ is norm continuous we deduce from (G_2) that $t \mapsto T^{\times}(t)x^*$ is weak* continuous. We now know that $\{T^{\times}(t)\}$ is a w^*-semigroup. In order to verify (S_1) we need a lemma.

Lemma 2.3. *Let $A^{\times}$ satisfy (G_2). Let $x^*: [t_1, t_2] \to X^*$ be continuous with values in $\mathcal{D}(A^{\times})$ and such that $\|A^{\times}x^*(t)\| \leq C$ for some $C > 0$ and $t_1 \leq t \leq t_2$. Then $t \mapsto A^{\times}x^*(t)$ is w^*-continuous on $[t_1, t_2]$, $\int_{t_1}^{t_2} x^*(\tau)d\tau \in \mathcal{D}(A^{\times})$ and $A^{\times}\int_{t_1}^{t_2} x^*(\tau)d\tau = \int_{t_1}^{t_2} A^{\times}x^*(\tau)d\tau$.*

Proof. The w^*-continuity of $A^{\times}x^*(t)$ is an immediate consequence of (G_2). As $x^*(t)$ is strongly continuous the integral $\int_{t_1}^{t_2} x^*(\tau)d\tau$ is strongly approximated by Riemann sums $\sum x^*(t_j)(t_{j+1} - t_j) \in \mathcal{D}(A^{\times})$. Similarly $\sum A^{\times}x^*(t_j)(t_{j+1} - t_j)$ approximates $\int_{t_1}^{t_2} A^{\times}x^*(\tau)d\tau$ in the weak* sense since $A^{\times}x^*(t)$ is weakly* continuous. The assertion now follows from (G_2). ∎

Armed with this lemma we can write

$$\begin{aligned}
T^{\times}(t)\int_0^h T^{\times}(\tau)x^*d\tau &= T^{\times}(t)(\lambda - A^{\times})\int_0^h T^{\odot}(\tau)(\lambda - A^{\times})^{-1}x^*d\tau \\
&= (\lambda - A^{\times})T^{\odot}(t)\int_0^h T^{\odot}(\tau)(\lambda - A^{\times})^{-1}x^*d\tau \\
&= (\lambda - A^{\times})\int_0^h T^{\odot}(t+\tau)(\lambda - A^{\times})^{-1}x^*d\tau \\
&= \int_0^h (\lambda - A^{\times})T^{\odot}(t+\tau)(\lambda - A^{\times})^{-1}x^*d\tau \\
&= \int_0^h T^{\times}(t+\tau)x^*d\tau
\end{aligned}$$

which is exactly (S_1). It remains to verify (S_2).

The definition (2.1) implies that

$$\int_0^t e^{-\lambda\tau}T^{\times}(\tau)d\tau = (\lambda - A^{\odot})\int_0^t e^{-\lambda\tau}T^{\odot}(\tau)d\tau(\lambda - A^{\times})^{-1}. \tag{2.4}$$

Hence, for $\operatorname{Re}\lambda$ sufficiently large,

$$(\lambda - A^{\times})^{-1} = \int_0^\infty e^{-\lambda\tau}T^{\times}(\tau)d\tau = \lambda\int_0^\infty e^{-\lambda\tau}S^{\times}(\tau)d\tau. \tag{2.5}$$

Consider any bounded sequence x_n^* in X^* such that $e^{-\lambda\tau}S^{\times}(t)x_n^*$ converges strongly as $n \to \infty$, uniformly in $t \geq 0$. Put $y_n^* = (\lambda - A^{\times})^{-1}x_n^*$. Then y_n^* converges strongly to a limit, say y^*. Moreover, $A^{\times}y_n^*$ is bounded since x_n^* is bounded. So (G_2) implies that $y^* \in \mathcal{D}(A^{\times})$ and $A^{\times}y_n^* \to A^{\times}y^*$ weakly *. Hence $x_n^* = (\lambda - A^{\times})y_n^* = \lambda y_n^* - A^{\times}y_n^* \to \lambda y^* - A^{\times}y^*$ weakly *. Put $x^* = \lambda y^* - A^{\times}y^*$; then $y^* = (\lambda - A^{\times})^{-1}x^*$. From (2.1) we deduce $S^{\times}(t) = (\lambda - A^{\odot})S^{\odot}(t)(\lambda - A^{\times})^{-1} = (\lambda S^{\odot}(t) - T^{\odot}(t) + I)(\lambda - A^{\times})^{-1}$ and consequently $S^{\times}(t)x_n^* \to (\lambda S^{\odot}(t) - T^{\odot}(t) + I)y^* = (\lambda S^{\odot}(t) - T^{\odot}(t) + I)(\lambda - A^{\times})^{-1}x^* = S^{\times}(t)x^*$. Hence ($S_2$) holds. This concludes the (G) implies (S) part of the proof of Theorem 2.1.

Let $T^{\times}$ be a w^*-semigroup with *integral* generator $A_0^{\times}$. Applying the uniform boundedness theorem twice we deduce that $\|T^{\times}(t)\|$ is bounded on $[0,1]$. The semigroup property then implies that $\|T^{\times}(t)\|$ is exponentially bounded. Assume (S_1). We claim that $S^{\times}(t)x^* \in \mathcal{D}(A_0^{\times})$ and $A_0^{\times}S^{\times}(t)x^* = T^{\times}(t)x^* - x^*$. In order to prove this claim we first note that $S^{\times}(t+h) = S^{\times}(t)T^{\times}(h) + S^{\times}(h)$. Hence ($S_1$) can be rewritten as

$$T^{\times}(t)S^{\times}(h) = S^{\times}(t+h) - S^{\times}(t) = S^{\times}(t)T^{\times}(h) + S^{\times}(h) - S^{\times}(t).$$

Therefore $T^{\times}(t)S^{\times}(h) - S^{\times}(h) = S^{\times}(t)(T^{\times}(h) - I)$, which, by the very definition of an integral generator, proves the claim.

Define $X^{\odot} = \overline{\mathcal{D}(A_0^\times)}$. If $x^* \in \mathcal{D}(A_0^\times)$, then $T^\times(t)x^* - x^* = S^\times(t)A_0^\times x^*$ and consequently $t \mapsto T^\times(t)x^*$ is norm continuous. As $T^\times(t)$ is exponentially bounded, this property extends to the closure $\overline{\mathcal{D}(A_0^\times)}$. Assume, conversely, that $\|T^\times(t)x^* - x^*\| \to 0$ as $t \downarrow 0$. Then $\|\frac{1}{t}S^\times(t)x^* - x^*\| \to 0$ as $t \downarrow 0$ as well. Since $S^\times(t)x^* \in \mathcal{D}(A_0^\times)$ we conclude that $x^* \in \overline{\mathcal{D}(A_0^\times)}$. So $X^{\odot}$ is the maximal subspace of strong continuity for $T^\times$. If we restrict $T^\times$ to the invariant subspace $X^{\odot}$ we obtain a C_0-semigroup which we call $T^{\odot}$. The definition of integral generator is such that it immediately follows that $A^{\odot}$ is the part of $A_0^\times$ in $X^{\odot}$. We now want to use the Hille-Yosida estimates for $A^{\odot}$ to prove (G_1).

We show that $\lambda \in \rho(A_0^\times)$ if $\mathrm{Re}\,\lambda > \omega$. Define, for $\mathrm{Re}\,\lambda > \omega$ and $x^* \in X^*$,

$$R_\lambda^\times x^* = \int_0^\infty e^{-\lambda s} T^\times(s)x^* ds.$$

We note that, by an approximation argument,

$$T^\times(t)\int_0^s T^\times(r)f^\times(r)dr = \int_0^s T^\times(t+r)f^\times(r)dr, \;\; s,t \geq 0,$$

for every strongly continuous X^*-valued function f. In particular,

$$\begin{aligned} T^\times(t)\int_0^\infty e^{-\lambda s}T^\times(s)x^*ds &= \int_0^\infty e^{-\lambda s}T^\times(t+s)x^*ds \\ &= \int_t^\infty e^{-\lambda(s-t)}T^\times(s)x^*ds, \end{aligned}$$

which is weakly * differentiable with weak * derivative $\lambda T^\times(t)R_\lambda^\times x^* - T^\times(t)x^*$. Therefore $R_\lambda^\times x^* \in \mathcal{D}(A_0^\times)$ and $A_0^\times R_\lambda^\times x^* = \lambda R_\lambda^\times x^* - x^*$, which yields that $(\lambda - A_0^\times)R_\lambda^\times = I$. On the other hand, if $T^\times(t)$ is a weakly * continuous semigroup satisfying (S_1), then $e^{-\lambda t}T^\times(t)$ is a weakly * continuous semigroup satisfying (S_1) and its integral weak * generator is $A_0^\times - \lambda$ with domain $\mathcal{D}(A_0^\times)$. Thus

$$e^{-\lambda t}T^\times(t)x^* - x^* = \int_0^t e^{-\lambda s}T^\times(s)(A_0^\times - \lambda)x^*ds,$$

for $x^* \in \mathcal{D}(A_0^\times)$. If $\mathrm{Re}\,\lambda > \omega$ we can take $t \to \infty$ and get that $x^* = R_\lambda^\times(\lambda - A_0^\times)x^*$. This shows that for $\mathrm{Re}\,\lambda > \omega$ we have $\lambda \in \rho(A_0^\times)$ and

$$R(\lambda, A_0^\times)x^* = R_\lambda^\times x^* = \int_0^\infty e^{-\lambda s}T^\times(s)x^*ds.$$

Now note that for $\mu \in \rho(A_0^\times)$ we have

$$(\lambda - A_0^\times)^{-1} = (\mu - A^{\odot})(\lambda - A^{\odot})^{-1}(\mu - A_0^\times)^{-1}.$$

We want to control the term $A^{\odot}(\lambda - A^{\odot})^{-1}(\mu - A_0^{\times})^{-1}$. Since

$$\begin{aligned}A^{\odot}(\lambda - A^{\odot})^{-1}x^{\odot} = \lambda(\lambda - A^{\odot})^{-1}x^{\odot} - x^{\odot} &= \lambda \int_0^{\infty} e^{-\lambda\tau}T^{\odot}(\tau)x^{\odot}d\tau - x^{\odot} \\ &= \lim_{h\downarrow 0}\int_0^{\infty} \frac{1}{h}(e^{-\lambda(t-h)} - e^{-\lambda t})T^{\odot}(t)x^{\odot}dt - x^{\odot} \\ &= \lim_{h\downarrow 0}\int_0^{\infty} e^{-\lambda t}\frac{1}{h}(T^{\odot}(t+h) - T^{\odot}(t))x^{\odot}dt \\ &= \lim_{h\downarrow 0}\int_0^{\infty} e^{-\lambda t}T^{\odot}(t)\frac{1}{h}(T^{\odot}(h) - I)x^{\odot}dt\end{aligned}$$

we obtain $\|A^{\odot}(\lambda - A^{\odot})^{-1}x^{\odot}\| \leq \frac{C}{\lambda-\omega}\|x^{\odot}\|$ provided $T^{\odot}(t)x^{\odot}$ is Lipschitz. The definition of integral generator implies at once that $T^{\times}(t)x^{\odot}$ is Lipschitz for $x^{\odot} \in \mathcal{D}(A_0^{\times})$. Hence ($G_1$) is a corollary of the Hille-Yosida estimates for $A^{\odot}$

Assume (S_2). Consider $x_n^* \in \mathcal{D}(A_0^{\times})$ such that $x_n^* \to x^*$ strongly while $\|A_0^{\times}x_n^*\|$ is bounded. The identity

$$T^{\times}(t)x_n^* - x_n^* = S^{\times}(t)A_0^{\times}x_n^*$$

and (S_2) imply that $A_0^{\times}x_n^*$ converges weakly * to a limit, say y^*, and that

$$T^{\times}(t)x^* - x^* = S^{\times}(t)y^*.$$

By the definition of integral generator this implies that $x^* \in \mathcal{D}(A_0^{\times})$ and $y^* = A_0^{\times}x^*$. Hence (G_2) holds.

Finally we claim that $\mathcal{D}(A_0^{\times}) = \mathrm{Fav}(T^{\odot})$. We know already that $\mathcal{D}(A_0^{\times}) \subset \mathrm{Fav}(T^{\odot})$. The fact that $x^{\odot} \in \mathrm{Fav}(T^{\odot})$ implies $x^{\odot} \in \mathcal{D}(A_0^{\times})$ follows from (G_2) exactly as before. Let $A^{\times}$ be the w^*-generator of $T^{\times}$; then $\mathcal{D}(A_0^{\times}) \subset \mathcal{D}(A^{\times}) \subset \mathrm{Fav}(T^{\times}) = \mathrm{Fav}(T^{\odot})$. We conclude that $A_0^{\times} = A^{\times}$.

We have now proved Theorem 2.1 but during the proof we have also shown that Theorem 2.2 a,b,c,d,e are true. It remains to prove Theorem 2.2 f. From the theory of C_0-semigroups we know that

$$(I - \frac{t}{n}A^{\odot})^{-n}(\lambda - A^{\times})^{-1}x^* \to T^{\odot}(t)(\lambda - A^{\times})^{-1}x^*$$

strongly for $n \to \infty$. By (G_1)

$$(\lambda - A^{\times})(I - \frac{t}{n}A^{\odot})^{-n}(\lambda - A^{\times})^{-1}x^* = (I - \frac{t}{n}A^{\times})^{-n}x^*$$

remains bounded as $n \to \infty$. The assertion now follows from (G_2) and the intertwining formula (2.1). ∎

Remark . (i) If T is a C_0-semigroup on X with generator A, then T^* satisfies (S_1)- (S_2) and A^* satisfies (G_1)- (G_2).

(ii) If $A^{\times}$ satisfies (G_1)-(G_2) and $B^{\times}: X^{\odot} \to X^*$ is a bounded linear operator, then $A^{\times} + B^{\times}$ satisfies (G_1)- (G_2) as well.

3. Duality

Throughout this section we assume that (G_1) is satisfied. Let $A^{\odot}$ be the part of $A^{\times}$ in $X^{\odot}$. Then $A^{\odot}$ is a densely defined operator on $X^{\odot}$ (even more, $A^{\odot}$ is the generator of a C_0-semigroup $T^{\odot}$) and so we can define its adjoint $A^{\odot *}$. Let $X^{\odot\odot} = \overline{\mathcal{D}(A^{\odot *})}$ and define $A^{\odot\odot}$ to be the part of $A^{\odot *}$ in $X^{\odot\odot}$. Then $A^{\odot\odot}$ satisfies the Hille-Yosida conditions and therefore is the generator of a C_0-semigroup $T^{\odot\odot}$ on $X^{\odot\odot}$.

In this section we show that $X^{\odot\odot}$ can be continuously embedded in X^{**} if (G_1) is satisfied and that $T^{\times}$ is the restricted dual of $T^{\odot\odot}$ if G/S is satisfied. To begin, let us assume (G_1) and define a pairing between $X^{\odot\odot}$ and X^* in the following way. Choose $\mu \in \rho(A^{\times})$. For $x^* \in X^*$ and $x^{\odot\odot} \in \mathcal{D}(A^{\odot\odot})$ we define

$$[x^{\odot\odot}, x^*] = \langle (\mu - A^{\odot\odot})x^{\odot\odot},\ (\mu - A^{\times})^{-1}x^*\rangle \tag{3.1}$$

(note that $(\mu - A^{\times})^{-1}x^* \in \mathcal{D}(A^{\times}) \subset X^{\odot}$). Our first result implies, among other thing, that this expression is independent of μ.

Lemma 3.1. *For every* $x^* \in X^*$ *and* $x^{\odot\odot} \in \mathcal{D}(A^{\odot\odot})$,

$$[x^{\odot\odot}, x^*] = \lim_{\lambda\to\infty} \langle x^{\odot\odot},\ \lambda(\lambda - A^{\times})^{-1}x^*\rangle.$$

Proof.

$$\begin{aligned}[x^{\odot\odot},\ x^*] &= \langle (\mu - A^{\odot\odot})x^{\odot\odot},\ (\mu - A^{\times})^{-1}x^*\rangle = \\ &\lim_{\lambda\to\infty} \langle (\mu - A^{\odot\odot})x^{\odot\odot},\ \lambda(\lambda - A^{\times})^{-1}(\mu - A^{\times})^{-1}x^*\rangle = \\ &\lim_{\lambda\to\infty} \langle (\mu - A^{\odot\odot})x^{\odot\odot},\ (\mu - A^{\odot})^{-1}\lambda(\lambda - A^{\times})^{-1}x^*\rangle = \\ &\lim_{\lambda\to\infty} \langle x^{\odot\odot},\ \lambda(\lambda - A^{\times})^{-1}x^*\rangle.\end{aligned}$$

■

Using this characterization the following estimate is easily derived:

$$|[x^{\odot\odot},\ x^*]| \le M\|x^{\odot\odot}\|\ \|x^*\| \tag{3.2}$$

for $x^* \in X^*$ and $x^{\odot\odot} \in \mathcal{D}(A^{\odot\odot})$. Since $\mathcal{D}(A^{\odot\odot})$ is dense in $X^{\odot\odot}$ we can extend the continuous linear functional $x^{\odot\odot} \to [x^{\odot\odot},\ x^*]$ to the whole space $X^{\odot\odot}$. Using the same notation for this extension we find that for every $x^{\odot\odot} \in X^{\odot\odot}$ and $x^* \in X^*$,

$$[x^{\odot\odot},\ x^*] = \lim_{\lambda\to\infty} \langle x^{\odot\odot},\ \lambda(\lambda - A^{\times})^{-1}x^*\rangle \tag{3.3}$$

and (3.2) holds. Furthermore,

$$[x^{\odot\odot},\ x^{\odot}] = \langle x^{\odot\odot},\ x^{\odot}\rangle \tag{3.4}$$

if $x^{\odot} \in X^{\odot}$ and $x^{\odot\odot} \in X^{\odot\odot}$. Let k be the embedding of $X^{\odot\odot}$ into X^{**} given by

$$kx^{\odot\odot}(x^*) = [x^{\odot\odot},\ x^*], \tag{3.5}$$

then, by (3.2), $\|kx^{\odot\odot}\| \leq M\|x^{\odot\odot}\|$. Furthermore,

$$\|kx^{\odot\odot}\| \geq \sup_{\|x^{\odot}\|\leq 1} |[x^{\odot\odot},\ x^{\odot}]| = \|x^{\odot\odot}\|. \tag{3.6}$$

Theorem 3.2. *Assume* (G_1). *Then*
a) $\langle A^{\odot *}x^{\odot\odot},\ x^{\odot}\rangle = [x^{\odot\odot},\ A^{\times}x^{\odot}],\ x^{\odot\odot} \in \mathcal{D}(A^{\odot *}),\ x^{\odot} \in \mathcal{D}(A^{\times})$.
b) $[(\lambda - A^{\odot *})^{-1}x^{\odot *},\ x^*] = \langle x^{\odot *},\ (\lambda - A^*)^{-1}x^*\rangle,\ x^{\odot *} \in X^{\odot *},\ x^* \in X^*$.

Proof. We only prove a).

Let $x^{\odot\odot} \in \mathcal{D}(A^{\odot *})$ and $x^{\odot} \in \mathcal{D}(A^{\times})$. Then

$$\langle A^{\odot *}x^{\odot\odot}, x^{\odot}\rangle = \lim_{\lambda\to\infty} \langle A^{\odot *}x^{\odot\odot}, \lambda(\lambda - A^{\odot})^{-1}x^{\odot}\rangle$$

$$= \lim_{\lambda\to\infty} \langle x^{\odot\odot}, \lambda(\lambda - A^{\times})^{-1}A^{\times}x^{\odot}\rangle = [x^{\odot\odot}, A^{\times}x^{\odot}].$$ ∎

Our next result gives a rather useful characterization of $A^{\times}$.

Theorem 3.3. *Assume* (G_1). *Let* $\widehat{X}$ *be a closed subspace of* $X^{\odot\odot}$ *which is invariant under* $T^{\odot\odot}$ *and separates point in* X^*. *Let* $x^*, y^* \in X^*$ *be such that*

$$[A^{\odot\odot}\hat{x}, x^*] = [\hat{x}, y^*]$$

for all $\hat{x} \in \widehat{X} \cap \mathcal{D}(A^{\odot\odot})$. *Then* $x^* \in \mathcal{D}(A^{\times})$ *and* $A^{\times}x^* = y^*$.

Proof. Let $\widehat{T}$ be the restriction of $T^{\odot\odot}$ to $\widehat{X}$ and let $\widehat{A}$ be the generator of $\widehat{T}$. Then $\mathcal{D}(\widehat{A}) = \widehat{X} \cap \mathcal{D}(A^{\odot\odot})$. Assume that $x^*, y^* \in X^*$ are such that $[\widehat{A}\hat{x}, x^*] = [\hat{x}, y^*]$ for all $\hat{x} \in \mathcal{D}(\widehat{A})$. From Theorem 3.2.b we get that

$$\langle \hat{x},\ (\lambda - A^{\times})^{-1}y^*\rangle = [(\lambda - \widehat{A})^{-1}\hat{x}, y^*] =$$

$$[\widehat{A}(\lambda - \widehat{A})^{-1}\hat{x},\ x^*] = [\lambda(\lambda - \widehat{A})^{-1}\hat{x} - \hat{x}, y^*] =$$

$$[\hat{x},\ \lambda(\lambda - A^{\times})^{-1}x^* - x^*]$$

for all $\hat{x} \in \widehat{X}$. Since $\widehat{X}$ separates points in X^* this yields

$$(\lambda - A^{\times})^{-1}y^* = \lambda(\lambda - A^{\times})^{-1}x^* - x^*,$$

hence $x^* \in \mathcal{D}(A^{\times})$ and $y^* = \lambda x^* - (\lambda - A^{\times})x^* = A^{\times}x^*$. ∎

From this point on we assume that G/S is satisfied. Let $T^{\times}$ be the w^*-continuous semigroup generated by $A^{\times}$.

Theorem 3.4. *If G/S is satisfied, then*

$$[T^{\odot\odot}(t)x^{\odot\odot}, x^*] = [x^{\odot\odot}, T^\times(t)x^*], \tag{3.7}$$

for all $x^{\odot\odot} \in X^{\odot\odot}$ *and* $x^* \in X^*$.

Proof. $[T^{\odot\odot}(t)x^{\odot\odot}, x^*] = \lim_{\lambda\to\infty} \langle T^{\odot\odot}(t)x^{\odot\odot}, \lambda(\lambda - A^\times)^{-1}x^*\rangle =$

$$\lim_{\lambda\to\infty} \langle x^{\odot\odot}, T^{\odot}(t)\lambda(\lambda - A^\times)^{-1}x^*\rangle =$$

$$\lim_{\lambda\to\infty} \langle x^{\odot\odot}, \lambda(\lambda - A^\times)^{-1}T^\times(t)x^*\rangle = [x^{\odot\odot}, T^\times(t)x^*].$$

Here we have used the intertwining formula (2.1). ■

In Sections 1 and 2 we have seen two different characterizations of $A^\times$, namely as the w^*-generator of $T^\times$ and as the integral generator of $T^\times$. The next theorem gives a third characterization, namely as the derivative of $T^\times(t)$ with respect to the $\sigma(X^*, X^{\odot\odot})$-topology at $t = 0$.

Theorem 3.5. *Assume G/S and let* $x^*, y^* \in X^*$. *Then* $x^* \in \mathcal{D}(A^\times)$ *and* $A^\times x^* = y^*$ *if and only if*

$$[x^{\odot\odot}, \frac{1}{h}(T^\times(h)x^* - x^*)] \to [x^{\odot\odot}, y^*] \quad \textit{as } h \downarrow 0, \tag{3.8}$$

for every $x^{\odot\odot} \in X^{\odot\odot}$.

Proof. "if". Suppose (3.8) is satisfied. If $x^{\odot\odot} \in \mathcal{D}(A^{\odot\odot})$, then

$$\begin{aligned}[x^{\odot\odot}, \frac{1}{h}(T^\times(h)x^* - x^*)] &= [\frac{1}{h}(T^{\odot\odot}(h)x^{\odot\odot} - x^{\odot\odot}),\ x^*] \\ &\to [A^{\odot\odot}x^{\odot\odot},\ x^*], \quad h \downarrow 0.\end{aligned}$$

Hence $[A^{\odot\odot}x^{\odot\odot}, x^*] = [x^{\odot\odot}, y^*]$ for $x^{\odot\odot} \in \mathcal{D}(A^{\odot\odot})$. Thus by Theorem 3.3 with $\widehat{X} = X^{\odot\odot}$, we get that $x^* \in \mathcal{D}(A^\times)$ and $A^\times x^* = y^*$.

"only if". Assume that $x^* \in \mathcal{D}(A^\times)$ and $A^\times x^* = y^*$, and let $x^{\odot\odot} \in \mathcal{D}(A^{\odot\odot})$. Then

$$\begin{aligned}[x^{\odot\odot},\ \frac{1}{h}(T^\times(h)x^* - x^*)] &= [\frac{1}{h}(T^{\odot\odot}(h)x^{\odot\odot} - x^{\odot\odot}), x^*] \\ &\to [A^{\odot\odot}x^{\odot\odot}, x^*] = [x^{\odot\odot}, A^\times x^*]\end{aligned}$$

as $h \downarrow 0$. Since $\mathcal{D}(A^{\odot\odot})$ is dense in $X^{\odot\odot}$ and $\{h^{-1}(T^\times(h)x^* - x^*): 0 < h < 1\}$ is bounded (recall that $\mathcal{D}(A^\times) = \mathrm{Fav}(T^\times)$) this result holds for every $x^{\odot\odot} \in X^{\odot\odot}$ which proves the "only if" part. ■

Theorem 3.6. *Assume G/S. Then*

$$[x^{\odot\odot},\ \int_0^t T^\times(s)x^* ds] = \int_0^t [x^{\odot\odot},\ T^\times(s)x^*]ds, \tag{3.9}$$

for every $x^{\odot\odot} \in X^{\odot\odot}$ *and* $x^* \in X^*$.

Proof. Let $x^* \in X^*$, $x^{\odot\odot} \in X^{\odot\odot}$, and $\lambda \in \rho(A^\times)$. Define $y^\odot = (\lambda - A^\times)^{-1}x^*$. Then $y^\odot \in \mathcal{D}(A^\times)$. The characterization of $A^\times$ as the integral generator of $T^\times$ yields that

$$T^\odot(t)y^\odot - y^\odot = \int_0^t T^\times(s)A^\times y^\odot ds =$$

$$\int_0^t T^\times(s)(\lambda y^\odot - x^*)ds = \lambda\int_0^t T^\odot(s)y^\odot ds - \int_0^t T^\times(s)x^* ds.$$

This yields that

$$[x^{\odot\odot}, \int_0^t T^\times(s)x^* ds] =$$

$$[x^{\odot\odot},\ \lambda\int_0^t T^\odot(s)y^\odot ds] - [x^{\odot\odot}, T^\odot(t)y^\odot - y^\odot] =$$

$$\int_0^t [x^{\odot\odot}, \lambda T^\odot(s)y^\odot]ds - [A^{\odot\odot}\int_0^t T^{\odot\odot}(s)x^{\odot\odot}ds,\ y^\odot] =$$

$$\int_0^t [x^{\odot\odot}, \lambda T^\odot(s)y^\odot]ds - [\int_0^t T^{\odot\odot}(s)x^{\odot\odot}ds,\ A^\times y^\odot] =$$

$$\int_0^t [x^{\odot\odot}, \lambda T^\odot(s)y^\odot]ds - \int_0^t [T^{\odot\odot}(s)x^{\odot\odot},\ A^\times y^\odot]ds =$$

$$\int_0^t [T^{\odot\odot}(s)x^{\odot\odot},\ (\lambda - A^\times)y^\odot]ds = \int_0^t [x^{\odot\odot},\ T^\times(s)x^*]ds.$$

∎

An immediate consequence of this result is the following characterization of the pairing $[\cdot,\cdot]$:

$$[x^{\odot\odot}, x^*] = \lim_{t\downarrow 0}\langle x^{\odot\odot}, \frac{1}{t}\int_0^t T^\times(s)x^* ds\rangle, \tag{3.10}$$

for every $x^{\odot\odot} \in X^{\odot\odot}$ and $x^* \in X^*$.

In the practically important case that $A^\times$ is the adjoint of a generator of a C_0-semigroup on X (or a bounded perturbation of it: see Clément et al [5]), this space X is continuously embedded in $X^{\odot\odot}$. Below we present two assumptions, one on $A^\times$ and one on $T^\times$, both of which guarantee that X lies embedded in $X^{\odot\odot}$.

Let $j\colon X \to X^{\odot *}$ be the embedding $jx(x^\odot) = \langle x, x^\odot\rangle$, for $x \in X$, $x^\odot \in X^\odot$. If we give X the new but equivalent norm

$$\|x\|' = \sup\{|\langle x, x^\odot\rangle| : x^\odot \in X^\odot,\ \|x^\odot\| \le 1\}$$

then j is an isometry from X onto $j(X)$ (see Hille and Phillips [11, Chapter XIV]). We introduce the following assumptions.

(G_0) For each $x \in X, \langle x, \lambda(\lambda - A^\times)^{-1}x^* - x^*\rangle \to 0,\ \lambda \to \infty$, uniformly in $\|x^*\| \le 1$.

(S_0) For each $x \in X, \langle x, T^\times(t)x^* - x^*\rangle \to 0,\ t \downarrow 0$, uniformly in $\|x^*\| \le 1$.

Note that both (G_0) and (S_0) are trivially satisfied if $T^\times$ is the adjoint of a C_0-semigroup on X.

Lemma 3.7. *Assume G/S. For every* $x \in X$ *and* $x^* \in X^*$,

$$\lim_{\lambda\to\infty} \langle x, \lambda(\lambda - A^\times)^{-1}x^* - x^*\rangle = 0.$$

Proof. Take $x^* \in X^*$. Then $x^* = (\lambda - A^\times)x^*_\lambda$, where $x^*_\lambda = (\lambda - A^\times)^{-1}x^*$. Then $\mu(\mu - A^\times)^{-1}x^*_\lambda = x^*_\lambda + (\mu - A^\times)A^\times x^*_\lambda \to x^*_\lambda$, $\mu \to \infty$, in norm. Furthermore, $A^\times\mu(\mu - A^\times)^{-1}x^*_\lambda = \mu(\mu - A^\times)^{-1}A^\times x^*_\lambda$ is bounded for $\mu \to \infty$. Thus, by (G_2), $x^*_\lambda \in \mathcal{D}(A^\times)$ and

$$A^\times\mu(\mu - A^\times)^{-1}x^*_\lambda \to A^\times x^*_\lambda, \quad \mu \to \infty,$$

with respect to the weak * topology. We already saw that

$$\lambda\mu(\mu - A^\times)^{-1}x^*_\lambda \to \lambda x^*_\lambda, \quad \mu \to \infty,$$

in norm. By subtraction we get,

$$(\lambda - A^\times)\mu(\mu - A^\times)^{-1}x^*_\lambda \to (\lambda - A^\times)x^*, \quad \mu \to \infty$$

in the weak * sense. Thus

$$\mu(\mu - A^\times)^{-1}x^* \to x^*, \quad \mu \to \infty$$

in the weak * sense. ∎

Theorem 3.8. *Assume G/S. Then* (G_0) *and* (S_0) *are equivalent. Moreover, if one (hence both) of these assumptions is satisfied, then* $j(X) \subseteq X^{\odot\odot}$ *and* $[jx, x^*] = \langle x, x^*\rangle$ *for* $x \in X$ *and* $x^* \in X^*$.

Proof. Assume (G_0). We first show that $j(X) \subseteq X^{\odot\odot}$. For $x \in X$,

$$\|\lambda(\lambda - A^{\odot *})^{-1}jx - jx\| = \sup_{\|x^\odot\|\le 1} |\langle\lambda(\lambda - A^{\odot *})^{-1}jx - x,\ x^\odot\rangle| =$$
$$\sup_{\|x^\odot\|\le 1} |\langle x, \lambda(\lambda - A^\odot)^{-1}x^\odot - x^\odot\rangle| \to 0, \quad \lambda \to \infty$$

by (G_0), hence $jx \in X^{\odot\odot}$. Furthermore,

$$\begin{aligned}[jx, x^*] &= \lim_{\lambda\to\infty} \langle jx, \lambda(\lambda - A^\times)^{-1}x^*\rangle \\ &= \lim_{\lambda\to\infty} \langle x, \lambda(\lambda - A^\times)^{-1}x^*\rangle = \langle x, x^*\rangle\end{aligned}$$

by Lemma 3.7.

We show that (S_0) is satisfied.

$$|\langle x, T^\times(t)x^* - x^*\rangle| = |[jx, T^\times(t)x^* - x^*]| =$$
$$|[T^{\odot\odot}(t)jx - jx, x^*]| \le \|T^{\odot\odot}(t)jx - jx\|\|x^*\| \to 0, \ t \downarrow 0,$$

uniformly for $\|x^*\| \le 1$. Thus (S_0) is satisfied.

Assume (S_0). We first show that $j(X) \subseteq X^{\odot\odot}$ and that $[jx, x^*] = \langle x, x^* \rangle$

$$\|T^{\odot *}(t)jx - jx\| = \sup_{\|x^\odot\| \le 1} |\langle T^{\odot *}(t)jx - jx,\ x^\odot \rangle| =$$

$$\sup_{\|x^\odot\| \le 1} |\langle x, T^\odot(t)x^\odot - x^\odot \rangle| \to 0, \quad t \downarrow 0,$$

by (S_0), hence $jx \in X^{\odot\odot}$. Furthermore, by (3.10),

$$[jx, x^*] = \lim_{t \downarrow 0} \langle x, \frac{1}{t} \int_0^t T^\times(s) x^* dx \rangle =$$

$$\lim_{t \downarrow 0} \frac{1}{t} \int_0^t \langle x, T^\times(s)x^* \rangle ds = \langle x, x^* \rangle.$$

Finally we prove (G_0).

$$|\langle x, \lambda(\lambda - A^\times)^{-1} x^* - x^* \rangle| = |[\lambda(\lambda - A^{\odot\odot})^{-1} jx - jx,\ x^*]| \le$$

$$\|\lambda(\lambda - A^{\odot\odot})^{-1} jx - jx\| \|x^*\| \to 0, \quad \lambda \to \infty$$

uniformly for $\|x^*\| \le 1$. ∎

4. An alternative characterization of $X^{\odot\odot}$

In the previous section we have seen that $X^{\odot\odot}$ lies continuously embedded in X^{**}, the embedding operator being denoted by k. In this section we give a direct definition of $k(X^{\odot\odot})$ in terms of the adjoint of $(\lambda - A^\times)^{-1}$. Throughout this section we assume that (G_1) is satisfied.

We define

$$X^{*\odot} = \{x^{**} \in X^{**} : \|\lambda(\lambda - A^\times)^{-1*} x^{**} - x^{**}\| \to 0 \text{ as } \lambda \to \infty\}. \tag{4.1}$$

From (G_1) one easily derives that $X^{*\odot}$ is a closed subspace of X^{**} which is invariant under $(\lambda - A^\times)^{-1*}$. For future use we prove the following lemma.

Lemma 4.1. *Let* $x^{**} \in X^{*\odot}$ *satisfy* $\langle x^{**}, x^* \rangle = 0$ *for every* $x^* \in \mathcal{D}(A^\times)$. *Then* $x^{**} = 0$.

Proof. From the assumption it follows that $\langle x^{**}, (\lambda - A^\times)^{-1} x^* \rangle = \langle (\lambda - A^\times)^{-1*} x^{**}, x^* \rangle = 0$ for every $x^* \in X^*$. Taking the supremum over all $x^* \in X^*$ we get that $\|\lambda(\lambda - A^\times)^{-1} x^{**}\| = 0$. Now letting $\lambda \to \infty$ and using that $x^{**} \in X^{*\odot}$ we find that $x^{**} = 0$. ∎

Let $p: X^{**} \to X^{\odot *}$ be the projection operator given by

$$px^{**}(x^\odot) = \langle x^{**}, x^\odot \rangle. \tag{4.2}$$

For a Banach space Y we denote by I_Y the identity operator on Y. We are ready to state the main theorem of this section.

Theorem 4.2.
a) $k(X^{\odot\odot}) \subseteq X^{*\odot}$ *and* $\langle kx^{\odot\odot}, x^* \rangle = [x^{\odot\odot}, x^*]$.
b) $p(X^{*\odot}) \subseteq X^{\odot\odot}$ *and* $[px^{**}, x^*] = \langle x^{**}, x^* \rangle$.
c) $k \circ p = I_{X^{*\odot}}$.
d) $p \circ k = I_{X^{\odot\odot}}$.

Proof. a) Let $x^{\odot\odot} \in X^{\odot\odot}$. Then

$$\begin{aligned}
&\|\lambda(\lambda - A^\times)^{-1*} kx^{\odot\odot}\| = \\
&\sup_{\|x^*\| \le 1} |\langle \lambda(\lambda - A^\times)^{-1*} kx^{\odot\odot} - kx^{\odot\odot}, x^* \rangle| = \\
&\sup_{\|x^*\| \le 1} |\langle kx^{\odot\odot}, \lambda(\lambda - A^\times)^{-1} x^* - x^* \rangle| = \\
&\sup_{\|x^*\| \le 1} |[x^{\odot\odot}, \lambda(\lambda - A^\times)^{-1} x^* - x^*]| = \\
&\sup_{\|x^*\| \le 1} |[\lambda(\lambda - A^{\odot\odot})^{-1} x^{\odot\odot} - x^{\odot\odot}, x^*]| \le \\
&\|\lambda(\lambda - A^{\odot\odot})^{-1} x^{\odot\odot} - x^{\odot\odot}\| \to 0, \quad \lambda \to \infty,
\end{aligned}$$

which proves the first assertion. The second assertion follows from definition (3.5).
b) Let $x^{*\odot} \in X^{\odot *}$. Then

$$\begin{aligned}
&\|\lambda(\lambda - A^{\odot *})^{-1} px^{*\odot} - px^{*\odot}\| = \\
&\sup_{\|x^\odot\| \le 1} |\langle \lambda(\lambda - A^{*\odot}) px^{*\odot} - px^{*\odot}, x^\odot \rangle| = \\
&\sup_{\|x^\odot\| \le 1} |\langle x^{*\odot}, \lambda(\lambda - A^\odot)^{-1} x^\odot - x^\odot \rangle| = \\
&\sup_{\|x^\odot\| \le 1} |\langle \lambda(\lambda - A^\times)^{-1*} x^{*\odot} - x^{*\odot}, x^\odot \rangle| \le \\
&\|\lambda(\lambda - A^\times)^{-1*} x^{*\odot} - x^{*\odot}\| \to 0, \quad \lambda \to \infty,
\end{aligned}$$

which proves the first part of b). The second part is proved by

$$\begin{aligned}
&[px^{*\odot}, x^*] = \lim_{\lambda\to\infty} \langle px^{*\odot}, \lambda(\lambda - A^\times)^{-1} x^* \rangle = \\
&\lim_{\lambda\to\infty} \langle x^{*\odot}, \lambda(\lambda - A^\times)^{-1} x^* \rangle = \lim_{\lambda\to\infty} \langle \lambda(\lambda - A^\times)^{-1*} x^{*\odot}, x^* \rangle = \\
&\langle x^{*\odot}, x^* \rangle.
\end{aligned}$$

c) For every $x^{*\odot} \in X^{*\odot}$ and $x^* \in X^*$,

$$\langle k \cdot px^{*\odot}, x^* \rangle = [px^{*\odot}, x^*] = \langle x^{*\odot}, x^* \rangle.$$

Here we have used a) and b).
d) For every $x^{\odot\odot} \in X^{\odot\odot}$ and $x^* \in X^*$,

$$[p \cdot kx^{\odot\odot}, x^*] = \langle kx^{\odot\odot}, x^* \rangle = [x^{\odot\odot}, x^*].$$

and d) is proved. ∎

This theorem says among other things that $k: X^{\odot\odot} \to X^{*\odot}$ is an isomorphism and that $k^{-1} = p$.

Now suppose that G/S is satisfied, and define $T^{\times *}(t) = T^{\times}(t)^*$, $t > 0$. One might suspect that

$$X^{*\odot} = \{x^{**} \in X^{**} : \|T^{\times *}(t)x^{**} - x^{**}\| \to 0, \quad t \downarrow 0\}.$$

And indeed, the inclusion $\subset$ is proved as follows. By Theorem 4.2b,

$$\|T^{\times *}(t)x^{*\odot} - x^{*\odot}\| = \sup_{\|x^*\| \leq 1} |\langle T^{\times *}(t)x^{*\odot} - x^{*\odot}, x^*\rangle| =$$
$$\sup_{\|x^*\| \leq 1} |\langle x^{*\odot}, T^{\times}(t)x^* - x^*\rangle| = \sup_{\|x^*\| \leq 1} |[px^{*\odot}, T^{\times}(t)x^* - x^*]| =$$
$$\sup_{\|x^*\| \leq 1} |[T^{\odot\odot}(t)px^{*\odot} - px^{*\odot}, x^*]| \leq \|T^{\odot\odot}(t)px^{*\odot} - px^{*\odot}\| \to 0, \quad t \downarrow 0.$$

But the reverse inclusion in general does not hold as the example below shows.

Example . Let S^1 be the one-dimensional circle group with $+$ being the addition modulo 2π. For a function $y: S^1 \to \mathbf{R}$ we define its translate y_t as: $y_t(\theta) = y(t+\theta)$, $0 \leq \theta \leq 2\pi$. Let Y be some vector space of bounded functions on S^1 such that

i) Y contains the constant functions,

ii) $y \in Y$ implies $y_t \in Y$, $t \in \mathbf{R}$.

For example, $Y = L^\infty(S^1)$ or $Y = C(S^1)$. (In what follows we mean by $C(S^1)$ the embedding of the space of continuous functions into $L^\infty(S^1)$.) A linear functional y^* on Y is called an *invariant mean* if

1. $y^*(y_t) = y^*(y)$, $y \in Y$, $t \in \mathbf{R}$,
2. $y^*(\mathbf{1}) = 1$,
3. $|y^*(y)| \leq \sup_{\theta \in S^1} |y(\theta)|$.

Here $\mathbf{1}$ stands for the element of Y which is identically one. On $C(S^1)$ the only invariant mean is given by the Haar integral. There is also an invariant mean on $L^\infty(S^1)$, but on this latter space there are many others; see Rudin [16].

Now let $X = L^1(S^1)$ and let T be the C_0-group of translations on X, i.e.

$$T(t)x = x_t, \quad t \in \mathbf{R}.$$

Then $X^* = L^\infty(S^1)$, $X^\odot = C(S^1)$ and $X^{**} = L^\infty(S^1)^*$. By the result of Rudin [16] mentioned before there exist at least two different invariant means x_1^{**}, $x_2^{**} \in X^{**}$ on X^*.

The restrictions of x_1^{**} and x_2^{**} to $X^\odot$ coincide and correspond to the Haar integral. Let $v^{**} = x_1^{**} - x_2^{**}$. Then $v^{**} \in X^{**}$ and for every $x^* \in X^*$,

$$\langle T^{**}(t)v^{**} - v^{**},\ x^*\rangle = \langle v^{**},\ T^*(t)x^* - x^*\rangle =$$
$$\langle v^{**},\ x^*_{-t} - x^*\rangle = 0$$

by property 1 of an invariant mean. Thus $T^{**}(t)v^{**} = v^{**}$. Suppose $v^{**} \in X^{*\odot}$. Since $\langle v^{**}, x^{\odot}\rangle = 0$ for every $x^{\odot} \in X^{\odot}$, Lemma 4.1 now implies that $v^{**} = 0$, a contradiction. Thus $v^{**} \notin X^{*\odot}$.

We conclude this section with an alternative characterization of $A^{\odot\odot}$. Let the operator $A^{\times\odot}$ on $X^{*\odot}$ be defined as follows: if $x^{*\odot}$, $y^{*\odot} \in X^{*\odot}$ and $\langle x^{*\odot}, A^{\times}x^{*}\rangle = \langle y^{*\odot}, x^{*}\rangle$ for every $x^{*} \in \mathcal{D}(A^{\times})$, then $x^{*\odot} \in \mathcal{D}(A^{\times\odot})$ and $A^{\times\odot}x^{*\odot} = y^{*\odot}$. Lemma 4.1 guarantees that this is a good definition.

Theorem 4.3. *$\mathcal{D}(A^{\times\odot}) = k(\mathcal{D}(A^{\odot\odot}))$ and $A^{\times\odot} \circ k = k \circ A^{\odot\odot}$ on $\mathcal{D}(A^{\odot\odot})$.*

Proof. "$\supset$": Let $x^{\odot\odot} \in \mathcal{D}(A^{\odot\odot})$ and $x^{*} \in \mathcal{D}(A^{\times})$. From Theorem 3.2.a we get that

$$\langle kx^{\odot\odot}, A^{\times}x^{*}\rangle = [x^{\odot\odot}, A^{\times}x^{*}] =$$

$$[A^{\odot\odot}x^{\odot\odot}, x^{*}] = \langle kA^{\odot\odot}x^{\odot\odot}, x^{*}\rangle,$$

whence it follows that $kx^{\odot\odot} \in \mathcal{D}(A^{\times\odot})$ and $A^{\times\odot}kx^{\odot\odot} = kA^{\odot\odot}x^{\odot\odot}$.

"$\subset$" is proved analogously. ∎

5. Generators with non-dense domain

The class of generators $A^{\times}$ on X^{*} satisfying $(\mathrm{G}_1) - (\mathrm{G}_2)$ is nothing but a special case of a class of generators with non-dense domain on an arbitrary Banach space.

Let $(X, \|\cdot\|)$ be an arbitrary Banach space and let $A: \mathcal{D}(A) \to X$ be a linear operator satisfying (G_1). By setting $\widetilde{A} = A - \omega I$ and renormalizing X by the equivalent norm

$$\|x\|' = \sup_{h>0} \sup_{n\geq 0} \|(I - h\widetilde{A})^{-n}x\|, \quad x \in X,$$

we may replace this assumption by

(H_1) $\quad A$ is m-dissipative on $(X, \|\cdot\|)$.

Following Amann [1], DaPrato and Grisvard [9], Nagel [14] and Walther [17], we define

$$|||x||| = \|(I - A)^{-1}x\|, \quad x \in X$$

to get a new norm on X. By (H_1)

$$|||x||| \leq \|x\|, \quad x \in X.$$

In general X is not complete with respect to $|||\cdot|||$ (it is if and only if A is bounded), and we define $\widehat{X}$ as the completion of X. Obviously, X is densely and continuously embedded in $\widehat{X}$.

Let $X_0 = \overline{\mathcal{D}(A)}$ and let A_0 be the part of A in X_0. Then A_0 is densely defined and m-dissipative in X_0. Let T_0 be the C_0-contraction semigroup on X_0 generated by A_0. If $\mathcal{D}(A)$ is invariant under T_0, we can define

$$T(t) = (I - A)T_0(t)(I - A)^{-1}, \quad t \geq 0. \tag{5.1}$$

Then T is a semigroup of bounded linear operators which is not necessarily strongly continuous. Clearly

$$|||T(t)x||| = \|T_0(t)(I - A)^{-1}x\| \leq \|(I - A)^{-1}x\| = |||x|||, \ x \in X$$

and

$$|||T(t)x - T(s)x||| = \|T_0(t)(I - A)^{-1}x - T_0(s)(I - A)^{-1}x\| \to 0 \text{ as } |t - s| \to 0,$$

which yields that T is a C_0-contraction semigroup on X with respect to $|||\cdot|||$. Let $\widehat{T}$ be the extension of T to $\widehat{X}$. Then $\widehat{T}$ is a C_0-contraction semigroup on the Banach space $\widehat{X}$. We denote its infinitesimal generator by $\widehat{A}$. If $\mathcal{D}(A)$ is *not* invariant under T_0, then definition (5.1) makes no sense. However, as the theorem below shows, we still have an extension $\widehat{T}(t): \widehat{X} \to \widehat{X}$ of T_0.

Theorem 5.1. *Assume* (H_1). *Then*
i) X_0 *is dense in* $(\widehat{X}, |||\cdot|||.)$
ii) T_0 *has a unique continuous extension* $\widehat{T}$ *on* $(\widehat{X}, |||\cdot|||)$.
iii) $\widehat{T}$ *is a* C_0*-contraction semigroup on* $\widehat{X}$.
iv) $\mathcal{D}(\widehat{A}) = X_0$
v) A *is the part of* $\widehat{A}$ *in* X.
vi) $\widehat{T}(t) = (I - \widehat{A})T_0(t)(I - \widehat{A})^{-1}, \quad t \geq 0.$
vii) $\lim\limits_{h \downarrow 0} |||\widehat{T}(t)\hat{x} - T_0(t)(I - h\widehat{A})^{-1}\hat{x}||| = 0, \ t \geq 0, \ \hat{x} \in \widehat{X}.$
viii) $\hat{x} \in \mathcal{D}(\widehat{A})$ *and* $\widehat{A}\hat{x} = \hat{y}$ *iff* $\widehat{T}(h)\hat{x} - \hat{x} = \int_0^h \widehat{T}(s)\hat{x}ds, h > 0.$
ix) X *is invariant under* $\widehat{T}$ *iff* $\mathcal{D}(A)$ *is invariant under* T_0.
From (viii) it follows that for every $\hat{x} \in \widehat{X}$ *and* $t \geq 0$,

$$\widehat{S}(t)\hat{x} := \int_0^t \widehat{T}(s)\hat{x} \ ds \in \mathcal{D}(\widehat{A}) = X_0$$

and

$$\widehat{A}\widehat{S}(t)\hat{x} = \widehat{T}(t)\hat{x} - \hat{x}.$$

Let $S(t)$ *be the restriction of* $\widehat{S}(t)$ *to* X. *Then* $S(t)$ *is the integrated semigroup associated with* A.

We assume

$$\{x \in X : \|x\| \leq 1\} \text{ is closed in } (\widehat{X}, \ |||\cdot|||). \tag{H_2}$$

Remark . One can easily show that (H_2) is equivalent to the following. $x_n \in \mathcal{D}(A)$, $n \geq 1$, $x_n \to x$, $n \to \infty$, and $\|Ax_n\|$ bounded implies that $x \in \mathcal{D}(A)$ and

$$\|(I - A)x\| \leq \liminf_{n \to \infty} \|(I - A)x_n\|.$$

Theorem 5.2. *Assume* $(\mathrm{H}_1)-(\mathrm{H}_2)$. *Then*

i) $\mathcal{D}(A)=\mathrm{Fav}(T_0)$.
So in particular, $\mathcal{D}(A)$ *is invariant under* T_0 *and* X *is invariant under* $\widehat{T}$. *Let* T *be the restriction of* $\widehat{T}$ *to* X.

ii) $\|T(t)x\|\leq\|x\|, \quad t\geq 0,\ x\in X$.

iii) $T(t)S(h)x=S(h)T(t)x$.

iv) $x\in\mathcal{D}(A)$ *and* $y=Ax$ *iff* $T(h)x-x=S(h)y,\ h>0$.

v) If $\{x_n\}$ *is a bounded sequence in* X *such that* $\{e^{-t}S(t)x_n\}$ *converges uniformly as* $n\to\infty$, *then there exists an* $x\in X$ *such that* $|||x_n-x|||\to 0$ *and* $\|S(h)x_n-S(h)x\|\to 0,\ h>0$.

Weakly * continuous semigroups satisfying $(\mathrm{S}_1)-(\mathrm{S}_2)$ fit into this framework surprisingly well. Let $A^\times$ be a linear operator on the dual Banach space X^* satisfying $(\mathrm{G}_1)-(\mathrm{G}_2)$ (with $M=1$, and $\omega=0$). Then (H_1) holds. Let $\widehat{X}^*$ be the completion of X^* with respect to the norm $|||\cdot|||$.

Lemma 5.3. *Let* $y_n^*\in X^*$, $\|y_n^*\|\leq M$ *and* $|||y_n^*-\hat{y}|||\to 0$ *as* $n\to\infty$ *for some* $\hat{y}\in\widehat{X}^*$. *Then* $\hat{y}\in X^*$ *and* $y_n^*\to\hat{y}$ *weakly * as* $n\to\infty$.

Proof. Define $x_n^*\in\mathcal{D}(A^\times)$ by $x_n^*=(I-A^\times)^{-1}y_n^*$. By (G_1), $\|x_n^*\|\leq\|y_n^*\|\leq M$, and $\|A^\times x_n^*\|=\|-y_n^*+x_n^*\|\leq 2M$. Since $\{y_n^*\}$ is a Cauchy sequence with respect to $|||\cdot|||$, $\{x_n^*\}$ is a Cauchy sequence with respect to $\|\cdot\|$, hence there exists an $x^*\in X^*$ such that $\|x_n^*-x^*\|\to 0$ as $n\to\infty$. Now (G_2) implies that $x^*\in\mathcal{D}(A^\times)$ and $A^\times x_n^*\to A^\times x^*$ weakly * as $n\to\infty$. Thus $y_n^*\to(I-A^\times)x^*$ weakly * as $n\to\infty$. From $\|x_n^*-x^*\|\to 0$ we also deduce that $|||y_n^*-(I-A^\times)x^*|||\to 0$ as $n\to\infty$, hence $\hat{y}=(I-A^\times)x^*$. ∎

This lemma shows in particular that (H_2) is satisfied. Thus from Theorems 5.1 and 5.2 it follows that $A^\times$ generates a semigroup $T^\times$ on X^* which is continuous with respect to $|||\cdot|||$, hence weakly * continuous by Lemma 5.3. Furthermore, (S_1) follows from Theorem 5.2(iii) and (S_2) from Theorem 5.2(v).

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