A generalization of Talagrand's variance bound in terms of influences

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July 5, 2010

Abstract

Consider a random variable of the form $f(X_1, \ldots, X_n)$, where f is a deterministic function, and where $X_1, \ldots, X_n$ are i.i.d random variables. For the case where X_1 has a Bernoulli distribution, Talagrand (in [15]) give an upper bound for the variance of f in terms of the individual influences of the variables X_i for $i = 1, \ldots, n$. We generalize this result to the case where X_1 takes finitely many vales.

Keywords and phrases: influences, concentration inequalities, sharp threshold
AMS 2010 classifications: 42B05, 60B15, 60B11

1 Introduction and statement of the results

1.1 Statement of results

Let $(\Omega, \mathcal{F}, \mu^1)$ be an arbitrary probability space. We denote its n-fold product by itself by $(\Omega^n, \mathcal{F}^n, \mu^n)$. Let $f : \Omega^n \to \mathbb{C}$ be a function with finite second moment, that is $\int_{\Omega^n} |f|^2 d\mu^n < \infty$. The influence of the ith variable on the function f is defined as

$$\Delta_i f(x_1, \ldots, x_n) = f(x_1, \ldots, x_n) - \int_\Omega f(x_1, \ldots, x_{i-1}, \xi, x_{i+1}, \ldots, x_n) \mu^1(d\xi)$$

for $x = (x_1, \ldots, x_n) \in \Omega^n$ and $i = 1, \ldots, n$. We will use the notation $\|f\|_q$ for the $\mathfrak{L}_q$ norm $q \in [1, \infty)$ of f, that is $\|f\|_q = \sqrt[q]{\int_{\Omega^n} |f|^q d\mu^n}$.

Using Jensen's inequality, Efron and Stein gave an upper bound on the variance of f (see [9]):

$$Var(f) \leq \sum_{i=1}^n \|\Delta_i f\|_2^2. \tag{1.1}$$

In some cases (1.1) has been improved. We write $\mathcal{P}(S)$ for the power set of a set S. For the case when Ω has two elements, say 0 and 1, and $\mu(\{1\}) = 1 - \mu(\{0\}) = p$, Talagrand showed the following result:

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Theorem 1.1. *(Theorem 1.5 of [15]) There exists a universal constant K such that for every $p \in (0,1)$, $n \in \mathbb{N}$ and for every real valued function f on $(\Omega^n, \mathcal{F}^n, \mu^n) = (\{0,1\}^n, \mathcal{P}(\{0,1\}^n), \mu_p)$,*

$$Var(f) \leq K \log\left(\frac{2}{p(1-p)}\right) \sum_{i=1}^{n} \frac{\|\Delta_i f\|_2^2}{\log\left(e \|\Delta_i f\|_2 / \|\Delta_i f\|_1\right)}, \tag{1.2}$$

where $\mu_p(\{x\}) = p^{\sum_{i=1}^n x_i} (1-p)^{n-\sum_{i=1}^n x_i}$ for $x \in \{0,1\}^n$.

Remark 1.2. An alternative proof of Theorem 1.1 for the case $p = 1/2$ can be found in [4].

We will generalize Theorem 1.1 for all finite sets Ω in the following way.

Theorem 1.3. *There is a universal constant $K > 0$ such that for each finite set Ω and each measure μ^1 on Ω with $p_{min} = \min_{j \in \Omega} \mu^1(\{j\}) > 0$, and for all complex valued functions f on $(\Omega^n, \mathcal{P}(\Omega^n), \mu^n)$,*

$$Var(f) \leq K \log(1/p_{min}) \sum_{i=1}^{n} \frac{\|\Delta_i f\|_2^2}{\log\left(e \|\Delta_i f\|_2 / \|\Delta_i f\|_1\right)}. \tag{1.3}$$

Remark 1.4. The special case of Theorem 1.3 where μ^n is the uniform measure on Ω^n has been proved in [8].

1.2 Background and discussion

Inequality (1.2) gives a bound on $Var(f)$ in terms of the influences. These and related inequalities, for example the widely used KKL lower bound for influences (see [12]) and various so called sharp-threshold results (see e.g. [11]), are useful when the function f is complicated, but its influences are still tractable. Such situations occur for example in percolation theory (see e.g. [4, 6, 7, 16]). In fact the above mentioned KKL bound is, in some sense a consequence of (1.2) (see Corollary 1.4 in [15]). (We write 'in some sense' because the paper [12] appeared significantly earlier.) This also holds for certain sharp-threshold results (see Corollary 1.3 in [15]).

Inequality (1.3) is the most literal extension of (1.2) to the case where Ω has cardinality k, $k \geq 2$. It will be explicitly used in [17].

Falik and Samarodnotsky (see [10]) used logarithmic Sobolev inequalities to derive edge isoperimetric inequalities. Rossignol used this method to derive sharp threshold results (see [13, 14]). Furthermore, Rossignol with Benaïm extended the results of [4] (where Talagrand's Theorem 1.1 above is applied to first-passage percolation), in [3], again with the use of logarithmic Sobolev inequalities. These similar applications suggest a deeper connection between logarithmic Sobolev inequalities and (1.2). Indeed, Bobkov and Houdré in [5], proved that a version of (1.2) actually implies a logarithmic Sobolev inequality in a continuous setup.

We finish this introduction with some remarks on the proof of Theorem 1.3. The proof of Theorem 1.5 of [15] uses a hypercontractive result (Bonami-Beckner inequality, see [2]) followed by a subtle symmetrization procedure (see Step 2 and 3 of the proof of Lemma 2.1 in [15]). In the proof our more general Theorem 1.3 above, we use a consequence of the extended Bonami-Beckner inequality (for an extension of the Bonami-Beckner inequality see Claim 3.1 in [1]) from [8] and then modify Talagrand's symmetrization procedure. This generalization of Talagrand's symmetrization argument, which covers Sections 2.2 and 2.3 is the main part of our proof.

2 Proof of Theorem 1.3

Without loss of generality, we assume that $\Omega = \mathbb{Z}_k$ (the integers modulo k) for some $k \in \mathbb{N}$.

Let η be an arbitrary measure on $\mathbb{Z}_k^n$. For each η, we will write $\mathfrak{L}_\eta\left(\mathbb{Z}_k^n\right)$ for the Hilbert space of complex valued functions on $\mathbb{Z}_k^n$, with the inner product

$$\langle f, g\rangle_\eta = \int_{\mathbb{Z}_k^n} f\overline{g}d\eta \text{ for } f, g \in \mathfrak{L}_\eta\left(\mathbb{Z}_k^n\right).$$

We will write $\|f\|_{\mathfrak{L}_q(\eta)}$ for the q-norm, $q \in [1, \infty)$, of a function $f : \mathbb{Z}_k^n \to \mathbb{C}$ with respect to the measure η, that is

$$\|f\|_{\mathfrak{L}_q(\eta)} = \left(\int |f|^q \, d\eta\right)^{1/q}.$$

When it is clear from the context which measure we are working with, we will simply write $\|f\|_q$.

2.1 A hypercontractive inequality

Let ν denote the uniform measure on $\mathbb{Z}_k^n$. Then the inner product on $\mathfrak{L}_\nu\left(\mathbb{Z}_k^n\right)$ is

$$\langle f, g\rangle_\nu = \int_{\mathbb{Z}_k^n} f\overline{g}d\nu = \frac{1}{k^n}\sum_{x\in\mathbb{Z}_k^n} f(x)\overline{g(x)} \text{ for } f, g \in \mathfrak{L}_\nu\left(\mathbb{Z}_k^n\right).$$

Define the 'scalar product' on $\mathbb{Z}_k^n$ by

$$\langle x, y\rangle = \sum_{i=1}^n x_i y_i, \text{ for } x, y \in \mathbb{Z}_k^n.$$

Let $\varepsilon = e^{2\pi i/k}$. For every $y \in \mathbb{Z}_k^n$, define the functions

$$w_y(x) = \varepsilon^{\langle x,y\rangle} \text{ for } x \in \mathbb{Z}_k^n.$$

It is easy to check the following lemma.

Lemma 2.1. $\{w_y\}_{y\in\mathbb{Z}_k^n}$ *form an orthonormal basis in* $\mathfrak{L}_\nu\left(\mathbb{Z}_k^n\right).$

Let us denote the number of non-zero coordinates of $\xi\in\mathbb{Z}_k^n$ by $[\xi]$. We will use the following hypercontractive inequality:

Lemma 2.2. *(Lemma 1 of [8]) There are positive constants C,γ such such that for any $k,n\in\mathbb{N}$, $m\in\{0,1,\dots,n\}$ and complex numbers a_y, for $y\in\mathbb{Z}_k^n$, we have*

$$\left\|\sum_{[y]=m}a_yw_y\right\|_{\mathfrak{L}_4(\nu)}\leq\left(Ck^\gamma\right)^m\left(\sum_{[y]=m}\left|a_y\right|^2\right)^{1/2}.$$

Remark 2.3. The proof (in [8]) of Lemma 2.2 is based on Claim 3.1 of [1]. Claim 3.1 of [1] is a generalization of the so called Bonami-Beckner inequality (see Lemma 1 of [2]). That inequality played an important role in [15] in the original proof of Theorem 1.1.

2.2 Finding a suitable basis

We assume that $\mu^1(\{j\})>0$ for all $j\in\mathbb{Z}_k$. Moreover, to ease the notation, we write μ in stead of μ^n.

Let $\mathfrak{L}_{\mu^1}(\mathbb{Z}_k)$ be the Hilbert space of functions from $\mathbb{Z}_k$ to $\mathbb{C}$, with the inner product

$$\langle a,b\rangle_{\mu^1}=\sum_{j\in\mathbb{Z}_k}a(j)\overline{b(j)}\mu^1(\{j\})\text{ for }a,b\in\mathfrak{L}_{\mu^1}(\mathbb{Z}_k).$$

Let $c_0\in\mathfrak{L}_{\mu^1}(\mathbb{Z}_k)$ be the constant 1 function. By Gram-Schmidt ortogonalization, there exist functions $c_l\in\mathfrak{L}_{\mu^1}(\mathbb{Z}_k)$ for $l\in\mathbb{Z}_k\setminus\{0\}$, such that c_j, $j\in\mathbb{Z}_k$ form an orthonormal basis in $\mathfrak{L}_{\mu^1}(\mathbb{Z}_k)$.

Using the functions c_j, $j\in\mathbb{Z}_k$ we define an orthonormal basis in $\mathfrak{L}_\mu\left(\mathbb{Z}_k^n\right)$ analogous to the basis w_y, $y\in\mathbb{Z}_k^n$.

Lemma 2.4. *The functions u_y, for $y\in\mathbb{Z}_k^n$, defined by*

$$u_y(x)=\prod_{i=1}^n c_{y_i}(x_i)\ \textit{for}\ x\in\mathbb{Z}_k^n,\tag{2.1}$$

form an orthonormal basis in $\mathfrak{L}_\mu\left(\mathbb{Z}_k^n\right).$

Proof. Since $\mu=\mu^n=\mu^1\otimes\mu^1\otimes\ldots\otimes\mu^1$, we have

$$\begin{aligned}\langle u_y,u_z\rangle_\mu &= \int\prod_{i=1}^n c_{y_i}(x_i)\overline{c_{z_i}(x_i)}\mu(dx)\\ &= \prod_{i=1}^n\langle c_{y_i},c_{z_i}\rangle_p\\ &= \begin{cases}1 & \text{if } y=z\\ 0 & \text{if } y\neq z.\end{cases}\end{aligned}$$

□

2.3 Extension of Lemma 2.2

The key ingredient in the proof of Theorem 1.3 is the following generalization of Lemma 2.2. It can also be seen as an extension of Lemma 2.1 of [15].

Lemma 2.5. *With the constants of Lemma 2.2, we have for every $k, n \in \mathbb{N}$, $m \in \{0, 1, \ldots, n\}$ and complex numbers a_y, $y \in \mathbb{Z}_k^n$,*

$$\left\| \sum_{[y]=m} a_y u_y \right\|_{\mathfrak{L}_4(\mu)} \leq (C\theta k^\gamma)^m \left(\sum_{[y]=m} |a_y|^2 \right)^{1/2}$$

holds, where $\theta = k \max_{i,,j} |c_i(j)|$.

Proof. The proof generalizes the symmetrization technique of the proof of Lemma 2.1 of [15]. Recall the definitions of ε and w_y for $y \in \mathbb{Z}_k^n$ of Section 2.1. Let n, k, m and the numbers a_y $y \in \mathbb{Z}_k^n$ as in the statement of Lemma 2.2.

Step 1 Define the product space $G = (\mathbb{Z}_k^n)^k$ with the probability measure $\mu_k = \bigotimes_{i=1}^k \mu$. Note that the measures μ_k and μ^k are different, the first is a measure on G, while the second is a measure on $\mathbb{Z}_k^k$. For $y, z \in \mathbb{Z}_k^n$ define the functions $g_y, g_{y,z}$ on G as follows. For $X = (X^0, \ldots, X^{k-1}) \in (\mathbb{Z}_k^n)^k$ and $z \in \mathbb{Z}_k^n$, let

$$g_y(X) = \prod_{1 \leq i \leq n,\, y_i \neq 0} \sum_{l=0}^{k-1} c_{y_i}(X_i^l) \varepsilon^{l y_i}, \tag{2.2}$$

$$g_{y,z}(X) = \prod_{1 \leq i \leq n,\, y_i \neq 0} \varepsilon^{z_i y_i} \sum_{l=0}^{k-1} c_{y_i}(X_i^l) \varepsilon^{l y_i} = g_y(X) w_y(z). \tag{2.3}$$

Recall that ν is the uniform measure on $\mathbb{Z}_k^n$, and define the set $H = G \times \mathbb{Z}_k^n$ and the product measure $\kappa = \mu_k \otimes \nu$ on H. We also define, for $y \in \mathbb{Z}_k^n$ the functions h_y on H by $h_y(X, z) = g_{y,z}(X) = g_y(X) w_y(z)$.

Step 2 For X as before and for $z \in \mathbb{Z}_k^n$ define X_z as

$$(X_z)_i^l = X_i^{l+z_i \bmod k}.$$

Then

$$\begin{aligned} g_{y,z}(X_z) &= \prod_{1 \leq i \leq n,\, y_i \neq 0} \sum_{l=0}^{k-1} c_{y_i}(X_i^{l+z_i \bmod k}) \varepsilon^{(l+z_i) y_i} \\ &= \prod_{1 \leq i \leq n,\, y_i \neq 0} \sum_{l=0}^{k-1} c_{y_i}(X_i^l) \varepsilon^{l y_i} \\ &= g_y(X). \end{aligned}$$

Hence for each fixed $z \in \mathbb{Z}_k^n$, we have

$$\left\| \sum_{[y]=m} a_y g_y \right\|_{\mathfrak{L}_4(\mu_k)} = \left\| \sum_{[y]=m} a_y g_{y,z} \right\|_{\mathfrak{L}_4(\mu_k)}. \tag{2.4}$$

Integrating over the variable z with respect to ν, Fubini's theorem gives that

$$\left\| \sum_{[y]=m} a_y g_y \right\|_{\mathfrak{L}_4(\mu_k)} = \left\| \sum_{[y]=m} a_y h_y \right\|_{\mathfrak{L}_4(\kappa)}. \tag{2.5}$$

Step 3 For fixed X, use Lemma 2.2 for the numbers $a_y g_y(X)$, and get

$$\int \left| \sum_{[y]=m} a_y g_y(X) w_y(z) \right|^4 d\nu(z) \le (Ck^\gamma)^{4m} \left(\sum_{[y]=m} |a_y g_y(X)|^2 \right)^2. \tag{2.6}$$

Since $\theta = k \max_{i,j} |c_i(j)|$, we have that $|g_y(X)| \le \theta^m$, which together with (2.6) gives

$$\int \left| \sum_{[y]=m} a_y g_y(X) w_y(z) \right|^4 d\nu(z) \le (C\theta k^\gamma)^{4m} \left(\sum_{[y]=m} |a_y|^2 \right)^2.$$

Integrating with respect to $d\mu_k(X)$ and taking the 4th root gives

$$\left\| \sum_{[y]=m} a_y h_y \right\|_{\mathfrak{L}_4(\kappa)} \le (C\theta k^\gamma)^m \left(\sum_{[y]=m} |a_y|^2 \right)^{1/2}. \tag{2.7}$$

By (2.7) and (2.5) we only have to show that

$$\left\| \sum_{[y]=m} a_y u_y \right\|_{\mathfrak{L}_4(\mu)} \le \left\| \sum_{[y]=m} a_y g_y \right\|_{\mathfrak{L}_4(\mu_k)}. \tag{2.8}$$

Step 4 Now we prove an alternative form of the function g_y. Recall the definition (2.2) of g_y. Expand the product, and get

$$\begin{aligned} g_y(X) &= \prod_{1\le i\le n,\, y_i\ne 0} \sum_{l=0}^{k-1} c_{y_i}(X_i^l)\varepsilon^{l y_i} \\ &= \sum_{\alpha:(*)} \prod_{1\le i\le n,\, y_i\ne 0} c_{y_i}(X_i^{\alpha(i)})\varepsilon^{\alpha(i) y_i}, \end{aligned} \tag{2.9}$$

where $(*)$ denotes the sum over all functions $\alpha : \{i \,|\, y_i \ne 0\} \to \mathbb{Z}_k$.

We will use the following trivial observation:
Observation: $c_{y_i}(X_i^l)\varepsilon^{ly_i} = 1$ whenever $y_i = 0$.
With the Observation we rewrite (2.9) as follows.

$$\begin{aligned} g_y(X) &= \sum_{\alpha\in\mathcal{A}_y}\prod_{i=1}^{n} c_{y_i}(X_i^{\alpha(i)})\varepsilon^{\alpha(i)y_i} \\ &= \sum_{\alpha\in\mathcal{A}_y}\prod_{t\in\mathbb{Z}_k}\prod_{1\le i\le n,\,\alpha(i)=t} c_{y_i}(X_i^t)\varepsilon^{ty_i}, \end{aligned} \tag{2.10}$$

where $\mathcal{A}_y$ is the set of functions $\alpha : \{1,2,\ldots,n\} \to \mathbb{Z}_k$ with the property that $\alpha(i) = 0$ if $y_i = 0$. For a function $\alpha \in \mathcal{A}_y$ we can define the vectors $v^t = v^t(\alpha) \in \mathbb{Z}_k^n$ for $t \in \mathbb{Z}_k$ by

$$v_i^t = v_i^t(\alpha) = \begin{cases} y_i & \text{if } \alpha(i) = t \\ 0 & \text{otherwise.} \end{cases}$$

The map $\alpha \mapsto (v^t(\alpha))_{t\in\mathbb{Z}_k}$ is one-to-one, furthermore the image of $\mathcal{A}_y$ under this map is

$$\mathcal{V}_y = \left\{ v = \left(v^t\right)_{t\in\mathbb{Z}_k} \,\middle|\, \sum_{t\in\mathbb{Z}_k} v^t = y, \text{ and } \forall i\, v_i^t \ne 0 \text{ for at most one } t\in\mathbb{Z}_k \right\}.$$

Using the properties of the map $\alpha \mapsto (v^t(\alpha))_{t\in\mathbb{Z}_k}$ together with the Observation and the definition of u, we can conclude from (2.10) that

$$\begin{aligned} g_y(X) &= \sum_{v\in\mathcal{V}_y}\prod_{t\in\mathbb{Z}_k}\prod_{i=1}^{n} c_{v_i^t}(X_i^t)\varepsilon^{tv_i^t} \\ &= \sum_{v\in\mathcal{V}_y}\prod_{t\in\mathbb{Z}_k} u_{v^t}(X^t)\varepsilon^{t\langle v^t,\mathbf{1}\rangle} \end{aligned} \tag{2.11}$$

where $\mathbf{1}$ is vector in $\mathbb{Z}_k^n$ with all coordinates equal to 1.

Step 5 Now we prove (2.8). Jensen's inequality gives that

$$\begin{aligned} &\int \left| \sum_{[y]=m} a_y g_y(X) \right|^4 d\mu_k(X) \\ &\quad \ge \int \left| \int \sum_{[y]=m} a_y g_y(X)\, d\mu_{k-1}\left(X^1,\ldots,X^{k-1}\right) \right|^4 d\mu\left(X^0\right). \\ &\quad = \int \left| \sum_{[y]=m} a_y \int g_y(X)\, d\mu_{k-1}\left(X^1,\ldots,X^{k-1}\right) \right|^4 d\mu\left(X^0\right). \end{aligned} \tag{2.12}$$

By (2.11), the inner integral of the left hand side of (2.12) is

$$\begin{aligned}
\int g_y\left(X\right)d\mu_{k-1}\left(X^1,\ldots,X^{k-1}\right)& \\
&=\int\sum_{v\in\mathcal{V}_y}\prod_{t\in\mathbb{Z}_k}u_{v^t}(X^t)\varepsilon^{t\left\langle v^t,\mathbf{1}\right\rangle}d\mu_{k-1}\left(X^1,\ldots,X^{k-1}\right)\\
&=\sum_{v\in\mathcal{V}_y}\left(\prod_{t\in\mathbb{Z}_k}\varepsilon^{t\left\langle v^t,\mathbf{1}\right\rangle}\right)u_{v^0}(X^0)\prod_{l=1}^{k-1}\int u_{v^l}(X^l)d\mu\left(X^l\right). \qquad (2.13)
\end{aligned}$$

Since u_0 is the constant 1 function on $\mathbb{Z}_k^n$, and by Lemma 2.4 (u_w, $w\in\mathbb{Z}_k^n$) is an orthonormal basis of $\mathfrak{L}_\mu\left(\mathbb{Z}_k^n\right)$, we have

$$\int u_w d\mu \quad =\int u_w u_0 d\mu=\begin{cases}1 & \text{if } w=0\\ 0 & \text{otherwise.}\end{cases}$$

By this and the definition of $\mathcal{V}_y$ we conclude from (2.13) that

$$\begin{aligned}
\int g_y\left(X\right)d\mu_{k-1}\left(X^1,\ldots,X^{k-1}\right)& \\
=\quad\sum_{v\in\mathcal{V}_y,\,v^1=\ldots=v^{k-1}=0}\left(\prod_{t\in\mathbb{Z}_k}\varepsilon^{t\left\langle v^t,\mathbf{1}\right\rangle}\right)u_{v^0}(X^0)\quad&=\quad u_y(X^0). \qquad (2.14)
\end{aligned}$$

(2.14) together with (2.12) gives that

$$\int\left|\sum_{[y]=m}a_y g_y\left(X\right)\right|^4 d\mu_k\left(X\right)\geq\int\left|\sum_{[y]=m}a_y u_y(X^0)\right|^4 d\mu\left(X^0\right),$$

from which by taking the 4th root, we get (2.8). This completes the proof of Lemma (2.5). □

From Lemma (2.5) and duality, we conclude the following lemma.

Lemma 2.6. *With the constants of Lemma 2.2, for any function $g\in\mathfrak{L}_\mu\left(\mathbb{Z}_k^n\right)$ we have*

$$\sum_{[y]=l}\left|\hat{g}\left(y\right)\right|^2\leq\left(C\theta k^\gamma\right)^{2l}\left\|g\right\|^2_{\mathfrak{L}_{4/3}(\mu)}.$$

2.4 Completion of the proof of Theorem 1.3

Notice that

$$\begin{aligned}\int_\Omega u_y\left(x_1,\ldots,x_{i-1},\xi,x_{i+1},\ldots,x_n\right)\mu^1(d\xi) &= \sum_{j\in\mathbb{Z}_k} c_{y_i}(j)p_j \prod_{1\le l\le n\, l\ne i} c_{y_l}(x_l)\\ &= \langle c_{y_i},c_0\rangle_p \prod_{1\le l\le n\, l\ne i} c_{y_l}(x_l)\\ &= \begin{cases} u_y(x) & \text{if } y_i=0\\ 0 & \text{if } y_i\ne 0.\end{cases}\end{aligned}$$

Hence

$$\int_\Omega f\left(x_1,\ldots,x_{i-1},\xi,x_{i+1},\ldots,x_n\right)\mu^1(d\xi) = \sum_{y\in\mathbb{Z}_k^n,\, y_i=0} \hat{f}(y)u_y$$

where $f=\sum_y \hat{f}(y)u_y$, i.e $\hat{f}(y)=\langle f,u_y\rangle_\mu$.

By the definition of $\Delta_i f$ we have

$$\Delta_i f = \sum_{y\in\mathbb{Z}_k^n,\, y_i\ne 0} \hat{f}(y)u_y. \tag{2.15}$$

Recall that $[y]$ was the number of non-zero coordinates of a vector $y\in\mathbb{Z}_k$. Define $M(g)$ by

$$M(g)^2 = \sum_{y\in\mathbb{Z}_k^n,\, y\ne 0} \frac{\hat{g}(y)^2}{[y]} \text{ for } g\in\mathfrak{L}_\mu\left(\mathbb{Z}_k^n\right).$$

Take a function $f\in\mathfrak{L}_\mu\left(\mathbb{Z}_k^n\right)$ with $\int f d\mu=0$ (which is equivalent to $\hat{f}(0)=0$). Then Parseval's formula and (2.15) gives that

$$\|f\|^2_{\mathfrak{L}_2(\mu)} = \sum_{y\ne 0}\hat{f}(y)^2 = \sum_{i=1}^n M(\Delta_i f)^2. \tag{2.16}$$

Since $1=\sum_{j=0}^{k-1}\left|c_i(j)\right|^2 p_j$, we can conclude that $\theta\le k/\min_j\sqrt{p_j}$. Hence Theorem 1.3 follows from the following Proposition 2.7 and (2.16).

Proposition 2.7. *There is a positive constant K, such that if $\int g d\mu=0$, we have*

$$M(g)^2\le K\log\left(C\theta k^\gamma\right)\frac{\|g\|_2^2}{\log\left(e\,\|g\|_2/\|g\|_1\right)},$$

where $\theta=k\max_{i=1,\ldots,n\, j\in\mathbb{Z}_k}\left|c_i(j)\right|$, and the constants C,γ are the same as in Lemma 2.2.

Proof. The proof of Proposition (2.7) is the same as the proof of Proposition 2.3 in [15] with the following modifications. Take $q = 4$ instead of $q = 3$, and use Lemma 2.6 instead of Proposition 2.2 of [15]. The only difference will be in the constants. First we get the term $2\log(C\theta k^{\gamma})$ in stead of $\log\left(2\theta^2\right)$. Furthermore we have to replace the estimate

$$\frac{\|g\|_2}{\|g\|_1} \leq \left(\frac{\|g\|_2}{\|g\|_{3/2}}\right)^3$$

by

$$\frac{\|g\|_2}{\|g\|_1} \leq \left(\frac{\|g\|_2}{\|g\|_{4/3}}\right)^2,$$

which is a consequence of the Cauchy-Schwartz inequality. This substitution only affects the constant K.

This completes the proof of Proposition (2.7) and the proof of Theorem 1.3. □

Acknowledgment.

I thank Rob van den Berg, for introducing me the subject, and for the comments on drafts of this paper. in preparation

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