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A J -FUNCTION FOR INHOMOGENEOUS POINT PROCESSES

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Abstract

We propose new summary statistics for intensity-reweighted moment stationary point processes that generalise the well known J -, empty space, and nearest-neighbour distance distribution functions, represent them in terms of generating functionals and conditional intensities, and relate them to the inhomogeneous reduced second moment function. Extensions to space time and marked point processes are briefly discussed.

Keywords & Phrases: conditional intensity, empty space function, generating functional, J -function, nearest-neighbour distance distribution function, inhomogeneity, intensity-reweighted moment stationarity, marked point process, minus sampling estimator, product density, reduced second moment measure, spatial interaction.

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1 Introduction

The analysis of data in the form of a map of (marked) points often starts with the computation of summary statistics. Some statistics are based on inter-point distances, others on the average number of points in sample regions, or geometric information. For a survey of the state of the art and pointers to the literature, the reader is referred to the recent handbook of spatial statistics [10].

In the exploratory stage, it is usually assumed that the data constitute a realisation of a stationary point process and deviations from a homogeneous Poisson process are studied to suggest a suitable model. Although stationarity is a convenient assumption, especially if – as is often the case – only a single map is available, in many areas of application, though, heterogeneity *is* present. To account for possible non-stationarity, Baddeley *et al.* [3] defined a reduced second moment function by considering the random measure obtained from the mapped point pattern by weighting each observed point according to the (estimated) intensity at its location. Gabriel and Diggle [9] took this idea further into the domain of space time point processes.

In this paper, our aim is to define an extension of the J -function [18] that is able to accommodate spatial and/or temporal inhomogeneity. The idea underpinning the J -function is to compare the point pattern around a typical point in the map to that around an arbitrarily chosen origin in space in order to gain insight in the interaction structure of the point process that generated the data. The power of the J -function in hypothesis testing was assessed in [6] and [26]. Extensions to multivariate point processes were proposed by [19], and window based

J -functions suggested by [2] and [6]. For applications in agriculture, astronomy, forestry and geology, see [8, 13, 14, 15, 21, 24].

The plan of this paper is as follows. In Section 2 we fix notation and recall some basic concepts from stochastic geometry. In Section 3 we describe the most important summary statistics that are being used in exploratory analysis of point patterns under the assumption of stationarity. Section 4 introduces the new statistic J_{inhom} and gives representations of it in terms of generating functionals and conditional intensities. Section 5 is devoted to the explicit computation of J_{inhom} for some important classes of point process models. In Section 6 we develop a minus sampling estimator and apply it in Section 7 to simulated examples. The paper closes with suggestions for further extensions to space time and marked point processes.

2 Preliminaries and notation

Throughout this paper, let X be a simple point process on \mathbb{R}^d . Its *intensity measure* Λ is defined by

$$\Lambda(B) = \mathbb{E} \left[\sum_{x \in X} 1\{x \in B\} \right]$$

for Borel sets $B \subseteq \mathbb{R}^d$. We assume that Λ is locally finite, i.e. $\Lambda(B) < \infty$ whenever B is bounded, and absolutely continuous with respect to Lebesgue measure so that

$$\Lambda(B) = \int_B \lambda(x) dx$$

for some non-negative measurable function λ referred to as *intensity function*. Heuristically speaking $\lambda(x) dx$ is the probability of observing some point in the infinitesimal region dx and represents the heterogeneity of X .

Note that the intensity measure is also known as the *first order factorial moment measure* of X . Higher order factorial moment measures $\Lambda^{(n)}$, $n \in \mathbb{N}$, are defined by

$$\Lambda^{(n)}(B_1 \times \cdots \times B_n) = \mathbb{E} \left[\sum_{x_1, \dots, x_n \in X}^{\neq} 1\{x_1 \in B_1; \dots; x_n \in B_n\} \right],$$

where the superscript \neq indicates that the sum is taken over all n -tuples of distinct points and the B_i are Borel subsets of \mathbb{R}^d . As the intensity measure, $\Lambda^{(n)}$ is not necessarily locally finite, nor guaranteed to have a Radon–Nikodym derivative. If $\Lambda^{(n)}$ is absolutely continuous with respect to the n -fold product of Lebesgue measures,

$$\Lambda^{(n)}(B_1 \times \cdots \times B_n) = \int_{B_1} \cdots \int_{B_n} \rho^{(n)}(x_1, \dots, x_n) dx_1 \cdots dx_n$$

for some non-negative measurable function $\rho^{(n)}$ called *n -th order product density* of X . Note that $\rho^{(n)}$ is permutation invariant and satisfies the integral equation

$$\mathbb{E} \left[\sum_{x_1, \dots, x_n \in X}^{\neq} g(x_1, \dots, x_n) \right] = \int \cdots \int g(x_1, \dots, x_n) \rho^{(n)}(x_1, \dots, x_n) dx_1 \cdots dx_n$$

for all non-negative, measurable functions $g \geq 0$. Thus, $\rho^{(n)}(dx_1, \dots, dx_n)$ may be interpreted as the infinitesimal probability of finding points of X at each of dx_1, \dots, dx_n . For further details, see for example [10, 12, 16].

In the physics literature, *n-point correlation functions* tend to be used instead of product densities [22]. They are defined recursively by

$$\begin{aligned} \xi_1 &\equiv 1; \\ \frac{\rho^{(n)}(x_1, \dots, x_n)}{\lambda(x_1) \cdots \lambda(x_n)} &= \sum_{k=1}^n \sum_{D_1, \dots, D_k} \xi_{n(D_1)}(\mathbf{x}_{D_1}) \cdots \xi_{n(D_k)}(\mathbf{x}_{D_k}), \end{aligned}$$

where the last sum ranges over all partitions $\{D_1, \dots, D_k\}$ of $\{1, \dots, n\}$ in k non-empty, disjoint sets, and the $\mathbf{x}_{D_j} = \{x_i : i \in D_j\}$, $j = 1, \dots, k$ form the corresponding partition of points. Since for a Poisson point process $\xi_n \equiv 0$ for $n > 1$, heuristically speaking *n-point correlation functions* account for the excess due to *n-tuples* in comparison to a Poisson point process with the same intensity function.

3 Summary statistics

Summary statistics are used by spatial statisticians as tools for exploratory data analysis, testing, and model validation purposes. Popular examples include the *nearest neighbour distance distribution function* G , the *empty space function* F , the *reduced second moment function* K and the *J-function*. More specifically, for a stationary point process X with intensity $\lambda > 0$,

$$(1) \quad \begin{cases} F(t) &= \mathbb{P}(X \cap B(0, t) \neq \emptyset), \\ G(t) &= \mathbb{P}^{10}(X \cap B(0, t) \neq \emptyset), \\ K(t) &= \mathbb{E}^{10} [\sum_{x \in X} 1\{x \in B(0, t)\}] / \lambda, \\ J(t) &= (1 - G(t)) / (1 - F(t)), \end{cases}$$

where $B(0, t)$ is the closed ball of radius $t \geq 0$ centred at the origin and \mathbb{P}^{10} denotes the reduced Palm distribution of X . For further details about these and other summary statistics, see for example [12]. Note that the *J-function* is defined only for t such that $F(t) < 1$. Values larger than one indicate inhibition, whereas $J(t) < 1$ suggests clustering, but note the caveats against drawing too strong conclusions in [4].

All statistics defined in (1) can be expressed in terms of product densities. Indeed, if the second order factorial moment measure exists as a locally finite measure with Radon–Nikodym derivative $\rho^{(2)}(x_1, x_2) = \rho^{(2)}(\|x_1 - x_2\|)$,

$$K(t) = \int_{B(0, t)} \frac{\rho^{(2)}(\|x\|)}{\lambda^2} dx = \int_{B(0, t)} (1 + \xi_2(\|x\|)) dx.$$

Clearly, K depends only on product densities up to order two. In contrast, the empty space function depends on product densities of all orders [27],

$$F(t) = - \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \int_{B(0, t)} \cdots \int_{B(0, t)} \rho^{(n)}(x_1, \dots, x_n) dx_1 \cdots dx_n$$

provided all order product densities exist and the series is absolutely convergent, that is, $\limsup_{n \rightarrow \infty} \left(\frac{\Lambda^{(n)}(B(0,t)^n)}{n!} \right)^{1/n} < 1$. Similarly,

$$G(t) = - \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \int_{B(0,t)} \cdots \int_{B(0,t)} \frac{\rho^{(n+1)}(0, x_1, \dots, x_n)}{\lambda} dx_1 \cdots dx_n,$$

provided that the series is absolutely convergent. Thence [17],

$$J(t) = 1 + \sum_{n=1}^{\infty} \frac{(-\lambda)^n}{n!} J_n(t)$$

for all $t \geq 0$ for which $F(t) < 1$, where $J_n(t) = \int_{B(0,t)} \cdots \int_{B(0,t)} \xi_{n+1}(0, x_1, \dots, x_n) dx_1 \cdots dx_n$. If product densities of all orders do not exist, one may truncate the series. Indeed, using only product densities up to second order gives

$$J(t) - 1 \approx -\lambda(K(t) - |B(0,t)|)$$

so the K -function can be seen as a second order approximation to the J -function.

For non-stationary point processes, the definitions in (1) depend on the choice of origin and adaptations are called for. To this end, Baddeley *et al.* [3] introduced the notion of *second order intensity-reweighted stationarity*. A point process X possesses this property if the random measure

$$\Xi = \sum_{x \in X} \frac{\delta_x}{\lambda(x)}$$

is second-order stationary. Here, δ_x denotes the Dirac measure that places a single point at x . Clearly if Ξ is stationary, it is also second-order stationary but the converse does not hold. Examples of second order intensity-reweighted stationary point processes include Poisson point processes, the random thinning of a stationary point process, and log Gaussian Cox processes driven by a Gaussian random field with a translation invariant covariance function. Cluster processes, as well as more general superposition processes, typically are not second order intensity-reweighted stationary.

For a second order intensity-reweighted stationary point process, an inhomogeneous K -function [3] can be defined by

$$K_{\text{inhom}}(t) := \frac{1}{|B|} \mathbb{E} \left[\sum_{x,y \in X}^{\neq} \frac{1_B(x) 1_{\{y \in B(x,t)\}}}{\lambda(x) \lambda(y)} \right]$$

regardless of the choice of bounded Borel set $B \subset \mathbb{R}^d$ and using the convention $a/0 = 0$ for $a \geq 0$. Indeed, $K_{\text{inhom}}(t) = \mathcal{K}_{\Xi}(B(0,t) \setminus \{0\})$, where \mathcal{K}_{Ξ} is the reduced second moment measure of the random measure Ξ .

Gabriel and Diggle [9] restrict themselves to point processes X that are simple and have locally finite moment measures of first and second order. Additionally they assume that X has an intensity function λ that is bounded away from zero and a pair correlation function

$$g(x, y) = g(\|x - y\|) = \frac{\rho^{(2)}(x, y)}{\lambda(x) \lambda(y)}$$

that depends only on $\|x - y\|$. Clearly in this case

$$K_{\text{inhom}}(t) = \frac{1}{|B|} \int_B \int_{B(0,t)} g(\|z\|) dz dx = \int_{B(0,t)} g(\|z\|) dz,$$

which for any inhomogeneous planar Poisson process reduces to πt^2 .

Baddeley *et al.* [3] briefly discuss how to define empty space and nearest neighbour distance distribution functions for inhomogeneous point processes. First, for given $x \in \mathbb{R}^d$ and $t \geq 0$, they propose to determine $r(x, t)$ by solving

$$t = \int_{B(x, r(x, t))} \lambda(y) dy,$$

then set

$$\begin{aligned} F_x(t) &= \mathbb{P}(d(x, X) \leq r(x, t)) \\ G_x(t) &= \mathbb{P}^x(d(x, X) \leq r(x, t)), \end{aligned}$$

where $d(x, X)$ denotes the shortest distance from x to a point of X . For Poisson processes, the above definitions do not depend on x and are both equal to $1 - e^{-t}$. The obvious drawback of such an approach is that $r(x, t)$ may be hard to compute in practice. Moreover, the definitions depend on x as well as t . Our goal in the present paper is to give an alternative definition of F , G , and J for intensity-reweighted moment stationary point processes based on their representation in terms of product densities that does not depend on the choice of origin and is easy to use in practice.

4 Inhomogeneous J -function

Let X be a simple point process on \mathbb{R}^d whose intensity function λ exists and is bounded away from zero with $\inf_x \lambda(x) = \bar{\lambda} > 0$. Assume that for all $n \in \mathbb{N}$ the n^{th} order factorial moment measure exists as a locally finite measure and has a Radon–Nikodym derivative $\rho^{(n)}$ with respect to the n -fold product of Lebesgue measure ℓ with itself for which the corresponding n -point correlation function ξ_n is translation invariant, that is, $\xi_n(x_1 + a, \dots, x_n + a) = \xi_n(x_1, \dots, x_n)$ for almost all $a \in \mathbb{R}^d$. We shall call such a point process *intensity-reweighted moment stationary*. Note that a fortiori X is second order intensity-reweighted stationary. Moreover, a stationary point process is also intensity-reweighted moment stationary.

Definition 1. *Let X be an intensity-reweighted moment stationary point process. Set*

$$J_n(t) = \int_{B(0,t)} \cdots \int_{B(0,t)} \xi_{n+1}(0, x_1, \dots, x_n) dx_1 \cdots dx_n$$

and define

$$J_{\text{inhom}}(t) = 1 + \sum_{n=1}^{\infty} \frac{(-\bar{\lambda})^n}{n!} J_n(t),$$

for all $t \geq 0$ for which the series is absolutely convergent, that is, for which $\limsup_{n \rightarrow \infty} \left(\frac{\bar{\lambda}^n}{n!} |J_n(t)| \right)^{1/n} < 1$.

A few special cases deserve to be mentioned. For a Poisson point process with intensity function $\lambda(\cdot)$, as the n -point correlation functions vanish for $n > 1$, so do the J_n whence $J_{\text{inhom}}(t) \equiv 1$ for all $t \geq 0$. Furthermore, if X is stationary, $\bar{\lambda} = \lambda$ and by [17, Prop. 4.2], $J_{\text{inhom}} \equiv J$.

Like in the stationary case considered in Section 3, the series in Definition 1 may be truncated, for example when X is only second order intensity-reweighted stationary or not all n -point correlation functions exist. For $n = 1$, we obtain

$$J_{\text{inhom}}(t) - 1 \approx -\bar{\lambda} \int_{B(0,t)} \xi_2(0,x) dx = -\bar{\lambda} (K_{\text{inhom}}(t) - |B(0,t)|).$$

In the remainder of this section, we rewrite J_{inhom} in terms of generating functionals and conditional intensities. Recall that for any function $v : \mathbb{R}^d \rightarrow [0, 1]$ that is measurable and identically 1 except on some bounded subset of \mathbb{R}^d , the generating functional at v is defined as

$$G(v) = \mathbb{E} \left[\prod_{x \in X} v(x) \right],$$

where by convention an empty product is taken to be 1. The distribution of X is determined uniquely by its generating functional [7, Prop. 7.4.II]. The factorial moment measures, provided they exist as locally finite measures, can be derived from the generating functional using its Taylor expansion [7, Prop. 7.4.III]. Conversely, if product densities of all orders exist, let u be a measurable function with values in $[0, 1]$ that has bounded support. Then

$$G(v := 1 - u) = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \int \cdots \int u(x_1) \cdots u(x_n) \rho^{(n)}(x_1, \dots, x_n) dx_1 \cdots dx_n,$$

provided the series converges [25, p. 109].

Theorem 1. Write, for $t \geq 0$ and $a \in \mathbb{R}^d$,

$$u_t^a(x) = \frac{\bar{\lambda} \mathbf{1}\{x \in B(a,t)\}}{\lambda(x)}$$

and assume that $\limsup_{n \rightarrow \infty} \left(\frac{\bar{\lambda}^n}{n!} \int_{B(0,t)} \cdots \int_{B(0,t)} \frac{\rho^{(n)}(x_1, \dots, x_n)}{\lambda(x_1) \cdots \lambda(x_n)} dx_1 \cdots dx_n \right)^{1/n} < 1$. Under the assumptions of Definition 1, for almost all $a \in \mathbb{R}^d$,

$$J_{\text{inhom}}(t) = \frac{G^{\text{!}a}(1 - u_t^a)}{G(1 - u_t^0)}$$

for all $t \geq 0$ for which the denominator is non-zero, where $G^{\text{!}a}$ is the generating functional of the reduced Palm distribution $\mathbb{P}^{\text{!}a}$ at a , G that of \mathbb{P} itself.

Note that for a stationary point process, $u_t^a(x) = \mathbf{1}\{x \in B(a,t)\}$, hence

$$G(1 - u_t^a) = \mathbb{P}(X \cap B(a,t) = \emptyset) = 1 - F(t).$$

Therefore, the generating functional in the denominator can be interpreted as the inhomogeneous counterpart of the empty space function. A similar interpretation holds for the numerator in terms of the nearest neighbour distance distribution function and one retrieves the classic definition of the J -function given in Section 3. At this point it should be emphasised that the numerator and denominator in the definition of J_{inhom} generalise respectively the nearest neighbour distance distribution function and empty space function.

Proof: We begin by showing that

$$\mathbb{E}^{\!|x} \left[\sum_{x_1, \dots, x_n \in X}^{\neq} \prod_{i=1}^n \frac{1\{x_i \in B(x, t)\}}{\lambda(x_i)} \right] = \int_{B(0, t)} \cdots \int_{B(0, t)} \frac{\rho^{(n+1)}(0, x_1, \dots, x_n)}{\lambda(0) \lambda(x_1) \cdots \lambda(x_n)} dx_1 \cdots dx_n$$

for almost all $x \in \mathbb{R}^d$. To see this, consider the functions

$$g_A(x, X) = \frac{1\{x \in A\}}{\lambda(x)} \sum_{x_1, \dots, x_n \in X}^{\neq} \prod_{i=1}^n \frac{1\{x_i \in B(x, t)\}}{\lambda(x_i)}$$

defined for all bounded Borel sets $A \subset \mathbb{R}^d$. By the definition of Palm distributions and the Campbell–Mecke formula,

$$\int \int g(x, \varphi) \lambda(x) d\mathbb{P}^{\!|x}(\varphi) dx = \mathbb{E} \left[\sum_{x \in X} g(x, X \setminus \{x\}) \right].$$

Using Fubini’s theorem, for our choice of g the left hand side can be written as

$$\int_A \mathbb{E}^{\!|x} \left[\sum_{x_1, \dots, x_n \in X}^{\neq} \prod_{i=1}^n \frac{1\{x_i \in B(x, t)\}}{\lambda(x_i)} \right] dx$$

while the right hand side is equal to

$$\mathbb{E} \left[\sum_{x, x_1, \dots, x_n}^{\neq} \frac{1\{x \in A\}}{\lambda(x)} \prod_{i=1}^n \frac{1\{x_i \in B(x, t)\}}{\lambda(x_i)} \right].$$

The expectation can be computed in terms of $\rho^{(n+1)}$ and equals

$$\begin{aligned} \int_A \int_{B(x, t)} \cdots \int_{B(x, t)} \frac{\rho^{(n+1)}(x, x_1, \dots, x_n)}{\lambda(x) \lambda(x_1) \cdots \lambda(x_n)} dx dx_1 \cdots dx_n = \\ \int_A \int_{B(0, t)} \cdots \int_{B(0, t)} \frac{\rho^{(n+1)}(0, x_1, \dots, x_n)}{\lambda(0) \lambda(x_1) \cdots \lambda(x_n)} dx dx_1 \cdots dx_n \end{aligned}$$

by the translation invariance of the n -point correlation functions. Hence

$$\mathbb{E}^{\!|x} \left[\sum_{x_1, \dots, x_n \in X}^{\neq} \prod_{i=1}^n \frac{1\{x_i \in B(x, t)\}}{\lambda(x_i)} \right]$$

is constant for almost all $x \in \mathbb{R}^d$.

Next, note that

$$\prod_{x \in X} \left(1 - \frac{\bar{\lambda} 1\{x \in B(a, t)\}}{\lambda(x)} \right) = 1 + \sum_{n=1}^{\infty} \frac{(-\bar{\lambda})^n}{n!} \sum_{x_1, \dots, x_n \in X}^{\neq} \prod_{i=1}^n \frac{1\{x_i \in B(a, t)\}}{\lambda(x_i)}.$$

Since the number of points in $X \cap B(a, t)$ is almost surely finite, the expressions are well-defined under the convention that an empty product takes the value one. Consequently, for almost all a ,

$$(2) \quad G^{1a}(1 - u_t^a) = 1 + \sum_{n=1}^{\infty} \frac{(-\bar{\lambda})^n}{n!} \int_{B(0,t)} \cdots \int_{B(0,t)} \frac{\rho^{(n+1)}(0, x_1, \dots, x_n)}{\lambda(0) \lambda(x_1) \cdots \lambda(x_n)} dx_1 \cdots dx_n$$

provided the power series in the right hand side is absolutely convergent.

By the discussion preceding the statement of the theorem,

$$(3) \quad G(1 - u_t^0) = 1 + \sum_{n=1}^{\infty} \frac{(-\bar{\lambda})^n}{n!} \int_{B(0,t)} \cdots \int_{B(0,t)} \frac{\rho^{(n)}(x_1, \dots, x_n)}{\lambda(x_1) \cdots \lambda(x_n)} dx_1 \cdots dx_n,$$

since the power series in the right hand side is assumed to be absolutely convergent.

Upon recalling the definition of the n -point correlation functions and splitting into terms that do or do not contain the origin, one obtains that the right hand side of (2) is equal to

$$1 + \sum_{n=1}^{\infty} \frac{(-\bar{\lambda})^n}{n!} \sum_{D \subseteq \{1, \dots, n\}} J_{n(D)}(t) \sum_{k=1}^{n-n(D)} \sum_{\substack{D_1, \dots, D_k \neq \emptyset \\ \cup D_j = \{1, \dots, n\} \setminus D}}^{\text{disjoint}} I_{n(D_1)} \cdots I_{n(D_k)}$$

(with $\sum_{k=1}^0 = 1$) which in turn can be written as

$$(4) \quad \left[1 + \sum_{n=1}^{\infty} \frac{(-\bar{\lambda})^n}{n!} J_n(t) \right] \times \left[1 + \sum_{m=1}^{\infty} \frac{(-\bar{\lambda})^m}{m!} \sum_{k=1}^m \sum_{\substack{D_1, \dots, D_k \neq \emptyset \\ \cup D_j = \{1, \dots, m\}}}^{\text{disjoint}} I_{n(D_1)} \cdots I_{n(D_k)} \right]$$

where

$$I_n = \int_{B(0,t)} \cdots \int_{B(0,t)} \xi_n(x_1, \dots, x_n) dx_1 \cdots dx_n$$

and $n(D)$ denotes the cardinality of the set D . The sum over k in the rightmost term of (4) can be written as

$$\int_{B(0,t)} \cdots \int_{B(0,t)} \frac{\rho^{(m)}(x_1, \dots, x_m)}{\lambda(x_1) \cdots \lambda(x_m)} dx_1 \cdots dx_m,$$

hence the second term in (4) is equal to the right hand side of (3). Finally, since both sums in (4) are absolutely convergent, so is (2), an observation that completes the proof. \square

Next, we focus our attention on *conditional intensities* $\lambda(x; X)$, $x \in \mathbb{R}^d$. Assuming they exist, they are defined in integral terms by

$$\mathbb{E} \left[\sum_{x \in X} g(x, X \setminus \{x\}) \right] = \int \mathbb{E}^{!x} [g(x, X)] \lambda(x) dx = \int \mathbb{E} [g(x, X) \lambda(x; X)] dx$$

for any non-negative measurable function g .

Theorem 2. *Assume that X admits a conditional intensity and define the random variable $W_a(X) := \prod_{x \in X} (1 - u_t^a(x))$. Then, under the assumptions of Theorem 1, $\mathbb{E}[W_a(X)] = 0$ implies $\mathbb{E}[\lambda(a; X) W_a(X) / \lambda(a)] = 0$ for almost all $a \in \mathbb{R}^d$, and otherwise*

$$J_{\text{inhom}}(t) = \mathbb{E} \left[\frac{\lambda(a; X)}{\lambda(a)} W_a(X) \right] / \mathbb{E} W_a(X),$$

the W_a -weighted expectation of $\lambda(a; X) / \lambda(a)$.

Consequently, $J_{\text{inhom}}(t) \leq 1 \Leftrightarrow \text{Cov} \left(\frac{\lambda(a; X)}{\lambda(a)}, W_a(X) \right) \leq 0$ with a similar statement for the opposite inequality sign.

Proof: Consider the functions

$$g_A(x, X) = \frac{1_{\{x \in A\}}}{\lambda(x)} \prod_{y \in X} \left(1 - \frac{\bar{\lambda} 1_{\{y \in B(x, t)\}}}{\lambda(y)} \right)$$

defined for all bounded Borel sets $A \subset \mathbb{R}^d$. Arguing as in the proof of Theorem 1 and using the definition of conditional intensities, one obtains

$$\int_A \mathbb{E}^{!x} \left[\prod_{y \in X} \left(1 - \frac{\bar{\lambda} 1_{\{y \in B(x, t)\}}}{\lambda(y)} \right) \right] dx = \int_A \mathbb{E} \left[\frac{\lambda(x; X)}{\lambda(x)} \prod_{y \in X} \left(1 - \frac{\bar{\lambda} 1_{\{y \in B(x, t)\}}}{\lambda(y)} \right) \right] dx.$$

Hence,

$$\mathbb{E}^{!x} \left[\prod_{y \in X} \left(1 - \frac{\bar{\lambda} 1_{\{y \in B(x, t)\}}}{\lambda(y)} \right) \right] = \mathbb{E} \left[\frac{\lambda(x; X)}{\lambda(x)} \prod_{y \in X} \left(1 - \frac{\bar{\lambda} 1_{\{y \in B(x, t)\}}}{\lambda(y)} \right) \right]$$

for almost all $x \in \mathbb{R}^d$. An appeal to Theorem 1 completes the proof. \square

5 Theoretical examples

5.1 Poisson process

Let X be a Poisson point process with intensity function $\lambda : \mathbb{R}^d \rightarrow \mathbb{R}^+$ that is bounded away from zero. Since $\rho^{(n)}(x_1, \dots, x_n) = \prod_i \lambda(x_i)$, the n -point correlation functions vanish for $n > 1$, so $J_{\text{inhom}}(t) \equiv 1$ for all $t \geq 0$.

The generating functional of X is

$$G(1 - u) = \exp \left[- \int u(x) \lambda(x) dx \right].$$

In particular, for the function $u = u_t^0$ defined in Theorem 1, $G(1 - u_t^0) = \exp [-\bar{\lambda}|B(0, t)|]$. Also, since according to Slivnyak's theorem for a Poisson point process $\mathbb{P}^{!0} = \mathbb{P}$, $G^{!0}(1 - u_t^0) = G(1 - u_t^0)$. Finally, the conditional intensity $\lambda(\cdot, X)$ of X coincides almost everywhere with the intensity function $\lambda(\cdot)$.

5.2 Location dependent thinning

Let X be a simple, stationary point process on \mathbb{R}^d for which product densities $\rho^{(n)}$ of all orders exist. Let $p : \mathbb{R}^d \rightarrow (0, 1)$ be a measurable function that is bounded away from zero and consider the thinning of X with retention probability $p(x)$ as in Example 8.2 of [7]. Since the process is simple, the product densities $\rho_{\text{th}}^{(n)}$ of the thinned point process can be expressed in terms of those of X by $\rho_{\text{th}}^{(n)}(x_1, \dots, x_n) = \rho^{(n)}(x_1, \dots, x_n) \prod_{i=1}^n p(x_i)$. In particular, the intensity function of the thinned point process is $\lambda_{\text{th}}(x) = \lambda p(x)$, where $\lambda > 0$ is the intensity of X . Consequently,

$$\frac{\rho_{\text{th}}^{(n)}(x_1, \dots, x_n)}{\lambda_{\text{th}}(x_1) \cdots \lambda_{\text{th}}(x_n)} = \frac{\rho^{(n)}(x_1, \dots, x_n)}{\lambda^n}.$$

Therefore, the n -point correlation functions of the thinned point process coincide with those of the underlying stationary point process X , $\xi_n^{\text{th}}(x_1, \dots, x_n) = \xi_n(x_1, \dots, x_n)$, and inherit the property of translation invariance. Hence $J_n^{\text{th}}(t)$ is equal to the J_n -function of the underlying point process X . As the intensity function of the thinned point process is bounded from below by $\lambda \bar{p}$ where \bar{p} is the infimum of the retention probabilities,

$$J_{\text{inhom}}^{\text{th}}(t) = 1 + \sum_{n=1}^{\infty} \frac{(-\lambda \bar{p})^n}{n!} J_n(t)$$

for all $t \geq 0$ for which the series converges. Note that the power series coefficients are identical to those in the power series expansion of the J -function of X .

The generating functional of the thinned point process is $G_{\text{th}}(v) = G(vp + 1 - p)$, where G is the generating functional of X . Hence

$$G_{\text{th}} \left(1 - \frac{\bar{p}}{p(\cdot)} 1\{\cdot \in B(0, t)\} \right) = G(1 - \bar{p} 1\{\cdot \in B(0, t)\}) = \mathbb{E} \left[(1 - \bar{p})^{n(X \cap B(0, t))} \right],$$

the generating function of the number of points of X that fall in $B(0, t)$ evaluated at $1 - \bar{p}$.

As the reduced Palm distribution of the thinned point process coincides with a random location dependent thinning of the reduced Palm distribution of X with retention probabilities given by the function p ,

$$G_{\text{th}}^{!0}(1 - u_t^0) = \mathbb{E}^{!0} \left[(1 - \bar{p})^{n(X \cap B(0, t))} \right],$$

so that under the assumptions of Theorem 1

$$J_{\text{inhom}}^{\text{th}}(t) = \frac{\mathbb{E}^{\text{!0}} [(1 - \bar{p})^{n(X \cap B(0,t))}]}{\mathbb{E} [(1 - \bar{p})^{n(X \cap B(0,t))}]}$$

To conclude this example, note that the assumption of stationarity of the underlying point process X may be weakened to intensity-reweighted moment stationarity.

5.3 Scaling

Let X be a simple point process on \mathbb{R}^d for which product densities $\rho^{(n)}$ of all orders exist. Let $c > 0$ be a scalar constant and map the point pattern X to cX . Then all order product densities $\rho_{cX}^{(n)}$ of cX exist and are given by $\rho_{cX}^{(n)}(x_1, \dots, x_n) = c^{-dn} \rho^{(n)}(x_1/c, \dots, x_n/c)$. In particular for $n = 1$, $\lambda_{cX}(x) = c^{-d} \lambda(x/c)$. Therefore the n -point correlation functions

$$\xi_n^{cX}(x_1, \dots, x_n) = \xi_n(x_1/c, \dots, x_n/c)$$

of cX are invariant under translations if and only if the n -point correlation functions ξ_n of X are, in which case the J_n -functions J_n^{cX} of cX are scaled versions $J_n^{cX}(t) = c^{dn} J_n(t/c)$ of the corresponding functions of X . Furthermore, $\inf_{x \in \mathbb{R}^d} \lambda_{cX}(x) = \bar{\lambda} c^{-d}$, so the inhomogeneous J -function of cX is

$$J_{\text{inhom}}^{cX}(t) = 1 + \sum_{n=1}^{\infty} \frac{(-\bar{\lambda} c^{-d})^n}{n!} c^{dn} J_n(t/c) = 1 + \sum_{n=1}^{\infty} \frac{(-\bar{\lambda})^n}{n!} J_n(t/c) = J_{\text{inhom}}(t/c),$$

the inhomogeneous J -function of X evaluated at t/c provided the series converges. Note that in contrast to the thinning case, the power series coefficients are not identical to those of the underlying point process X .

The generating functional of the scaled process is given by $G_{cX}(v) = G(v(c \cdot))$, where G is the generating functional of X , whence

$$G_{cX} \left(1 - \frac{c^{-d} \bar{\lambda}}{c^{-d} \lambda(\cdot/c)} 1\{\cdot \in B(0, t)\} \right) = G \left(1 - \frac{\bar{\lambda}}{\lambda(\cdot)} 1\{\cdot \in B(0, t/c)\} \right).$$

Similarly, noting that $d\mathbb{P}_{cX}^{\text{!}x}(\varphi) = d\mathbb{P}^{\text{!}x/c}(\varphi/c)$,

$$G_{cX}^{\text{!}a} \left(1 - \frac{c^{-d} \bar{\lambda}}{c^{-d} \lambda(\cdot/c)} 1\{\cdot \in B(a, t)\} \right) = G^{\text{!}a/c} \left(1 - \frac{\bar{\lambda}}{\lambda(\cdot)} 1\{\cdot \in B(a/c, t/c)\} \right).$$

To conclude this example, a conditional intensity of cX is obtained by scaling that of X , i.e. $\lambda_{cX}(x, \varphi) = c^{-d} \lambda(x/c, \varphi/c)$ [11], from which we retrieve the formula $J_{\text{inhom}}^{cX}(t) = J_{\text{inhom}}(t/c)$ under the assumptions of Theorem 2.

5.4 Log Gaussian Cox process

Write Q for the distribution of a random measure defined in terms of its Radon–Nikodym derivative Λ with respect to Lebesgue measure. We assume that all moment measures of the random measure exist and are locally finite. Let X be the Cox process directed by the random intensity process Λ , that is, given a realisation $\Lambda = \lambda$, X is a Poisson point process with intensity function λ . It follows from [7, p. 262] that the factorial moment measures of X exist and are equal to the moment measures of the driving random measure. Hence X has product densities $\rho^{(n)}(x_1, \dots, x_n) = \mathbb{E} [\prod_{i=1}^n \Lambda(x_i)]$. Moreover, the reduced Palm distribution of X at x is the distribution of a Cox process with driving random measure distributed as Q^x , the Palm distribution of the driving measure of X at x [25, p. 141].

The class of log-Gaussian Cox processes [20] is especially convenient. For models in this class,

$$\Lambda(x) = \exp [Z(x)]$$

where Z is a Gaussian field. Such a field is defined fully by its mean and covariance function. Write $\mu(x)$ for the mean function, $\sigma^2(x)$ for the variance of $Z(x)$ and $r(x, y)$ for the correlation function. In other words, the covariance function of Z is given by $\sigma(x)\sigma(y)r(x, y)$. Conditions have to be imposed on these functions in order to make the resulting Cox process well-defined. In particular, the intensity function must be integrable almost surely, and $\Psi_\Lambda(B) = \int_B \Lambda(x) dx$ a finite random variable for all bounded Borel sets $B \subset \mathbb{R}^d$. Moreover, the distribution of the random measure Ψ_Λ must be uniquely determined by that of Z . Sufficient conditions are given in [1, Thm. 3.4.1] for zero mean Gaussian processes. Therefore, we additionally assume that the mean function μ is continuous and bounded. Now, since $\mathbb{E} [\prod_i \Lambda(x_i)] = \mathbb{E} [e^{\sum_i Z(x_i)}]$, the moment generating function of the normally distributed random variable $\sum_i Z(x_i)$ evaluated at 1,

$$\rho^{(n)}(x_1, \dots, x_n) = \exp \left[\sum_{i=1}^n \left(\mu(x_i) + \frac{\sigma^2(x_i)}{2} \right) + \sum_{i < j} \sigma(x_i)\sigma(x_j)r(x_i, x_j) \right].$$

Specialising to $n = 1$, it follows that the intensity function is $\log \lambda(x) = \mu(x) + \sigma^2(x)/2$ whence

$$\frac{\rho^{(n)}(x_1, \dots, x_n)}{\lambda(x_1) \cdots \lambda(x_n)} = \exp \left[\sum_{i < j} \sigma(x_i)\sigma(x_j)r(x_i, x_j) \right].$$

Thus, if $\sigma(\cdot) \equiv \sigma > 0$ and $r(x, y) = r(x - y)$, X is intensity-reweighted moment stationary and the intensity function is bounded away from zero with infimum $\exp [\sigma^2/2 + \inf_{x \in \mathbb{R}^d} \mu(x)]$.

In order to derive an explicit formula for J_{inhom} , we turn to the generating functional. Recall that a Cox process has a generating functional [7, Prop. 8.5.1] defined by $G(v) = \mathbb{E}_Q \exp [-\int (1 - v(x)) \Lambda(x) dx]$. Therefore, for the log-Gaussian Cox process

$$G(1 - u_t^0) = \mathbb{E}_Z \exp \left[-\bar{\mu} \int_{B(0,t)} e^{Z(x) - \mu(x)} dx \right].$$

where $\bar{\mu}$ denotes $\inf_{x \in \mathbb{R}^d} e^{\mu(x)}$.

The Palm distributions Q^x of a log Gaussian random measure are $\Lambda(x) = e^{Z(x)}$ -weighted. To see this, note that the Campbell measure evaluated at a bounded Borel set $B \subset \mathbb{R}^d$ and F in the smallest σ -algebra for which $\Psi_\Lambda(B) = \int_B \Lambda(x) dx$ is finite for all such B , can be calculated as

$$C(B \times F) = \mathbb{E}_Q [1_F(\Psi_\Lambda) \Psi_\Lambda(B)] = \int_B \lambda(x) \left[\int 1_F(\Psi_\Lambda) \frac{\Lambda(x)}{\lambda(x)} dQ(\Psi_\Lambda) \right] dx$$

by Fubini and the existence of a σ -finite intensity measure $\lambda(\cdot)$ that is bounded away from zero. Therefore,

$$G^{la}(1 - u_t^a) = \mathbb{E}_Z \left[\frac{e^{Z(a) - \mu(a)}}{e^{\sigma^2/2}} \exp \left[-\bar{\mu} \int_{B(a,t)} e^{Z(y) - \mu(y)} dy \right] \right].$$

Since $Y(x) := Z(x) - \mu(x)$, $x \in \mathbb{R}^d$, is a stationary Gaussian process, the above generating functional does not depend on the choice of a . Therefore, under the assumptions of Theorem 1,

$$J_{\text{inhom}}(t) = \frac{\mathbb{E}_Y e^{Y(0)} \left[\exp \left[-\bar{\mu} \int_{B(0,t)} e^{Y(x)} dx \right] \right]}{\mathbb{E}_Y \left[e^{Y(0)} \right] \mathbb{E}_Y \exp \left[-\bar{\mu} \int_{B(0,t)} e^{Y(x)} dx \right]}.$$

The mixed Poisson process considered in [18] is a special case.

Note that $J_{\text{inhom}}(t) < 1$ if and only if the random variables $e^{Y(0)}$ and $e^{-\bar{\mu} \int_{B(0,t)} e^Y}$ are negatively correlated. The geostatistical models used in practice, for example the one we shall use in Section 7, have a positive, continuously decreasing, correlation function. Therefore, by Pitt's theorem [23], the Gaussian fields defined by such correlation functions are associated. Under the further conditions of [1, Thm. 3.4.1.], the sample functions $Y(\cdot)$ are almost surely continuous and hence the integral of e^Y over $B(0,t)$ is uniquely defined and the limit of Riemann sums over ever finer partitions of $B(0,t)$. Since Y is associated, $\text{Cov} \left(e^{Y(0)}, e^{-c_i \sum_i e^{Y(x_i)}} \right) \leq 0$ for all finite sums with positive scalar multipliers $c_i > 0$. Upon taking the limit, it follows that $J_{\text{inhom}}(t) \leq 1$.

6 Estimation

The goal of this section is to develop an estimator for the inhomogeneous J -function of Definition 1. For this purpose, we shall use the representation in terms of generating functionals of Theorem 1 and apply the minus sampling principle outlined in [25, p. 127].

Specifically, let $W \subset \mathbb{R}^d$ be a compact set with non-empty interior and suppose the point process X is observed in W . For clarity of exposition, we assume that the intensity function λ is known. If it is not, it can be estimated (for instance using kernel estimation [5]) and plugged into the estimators outlined below.

In order to estimate $G(1 - u_t^0)$, let $L \subseteq W$ be a finite point grid. Set

$$(5) \quad \widehat{G(1 - u_t^0)} := \frac{\sum_{l_k \in L \cap W_{\ominus t}} \prod_{x \in X \cap B(l_k, t)} \left[1 - \frac{\bar{\lambda}}{\lambda(x)} \right]}{\#L \cap W_{\ominus t}},$$

where $W_{\ominus t}$ is the eroded set $\{x \in W : d(x, \partial W) \geq t\} = \{x \in W : x + B(0, t) \subseteq W\}$. Note that $G(\widehat{1 - u_t^0})$ is an estimator as for all grid points $l_k \in W_{\ominus t}$ the ball $B(l_k, t)$ is fully contained in W so that no points of $X \setminus W$ are needed for the computation of the product in the numerator of (5).

Similarly, set

$$(6) \quad G^{!a}(\widehat{1 - u_t^a}) = \frac{\sum_{x_k \in X \cap W_{\ominus t}} \prod_{x \in X \setminus \{x_k\} \cap B(x_k, t)} \left[1 - \frac{\bar{\lambda}}{\lambda(x)}\right]}{\#X \cap W_{\ominus t}}.$$

Compared to (5), the grid points l_k are replaced by the points x_k of $X \cap W_{\ominus t}$. Again, (6) is a function of $X \cap W$ only.

Proposition 1. *Under the assumptions of Theorem 1, the estimator (5) is unbiased, (6) is ratio-unbiased.*

Proof: We claim that

$$(7) \quad \mathbb{E} \left[\prod_{x \in X \cap B(l_k, t)} \left(1 - \frac{\bar{\lambda}}{\lambda(x)}\right) \right] = G(1 - u_t^0)$$

for all $l_k \in L \cap W_{\ominus t}$. To see this, note that

$$(8) \quad \prod_{x \in X \cap B(l_k, t)} \left(1 - \frac{\bar{\lambda}}{\lambda(x)}\right) = 1 + \sum_{n=1}^{\infty} \frac{(-\bar{\lambda})^n}{n!} \sum_{x_1, \dots, x_n \in X \cap W}^{\neq} \prod_{i=1}^n \frac{1_{\{x_i - l_k \in B(0, t)\}}}{\lambda(x_i)}.$$

Hence, under the assumptions made, the expectation of (7) can be expressed as

$$1 + \sum_{n=1}^{\infty} \frac{(-\bar{\lambda})^n}{n!} \int_{B(l_k, t)} \cdots \int_{B(l_k, t)} \frac{\rho^{(n)}(x_1, \dots, x_n)}{\prod_{i=1}^n \lambda(x_i)} dx_1 \cdots dx_n,$$

which, because of the translation invariance of the integrands, reduces to

$$1 + \sum_{n=1}^{\infty} \frac{(-\bar{\lambda})^n}{n!} \int_{B(0, t)} \cdots \int_{B(0, t)} \frac{\rho^{(n)}(x_1, \dots, x_n)}{\prod_{i=1}^n \lambda(x_i)} dx_1 \cdots dx_n = G(1 - u_t^0).$$

This proves the claim, from which the unbiasedness of (5) follows.

Next, turn to the numerator of (6). By the definition of Palm distributions and the reduced Campbell–Mecke theorem [25, p.107], its expectation can be expressed as

$$\int \int_{W_{\ominus t}} \lambda(x) \prod_{y \in \varphi} \left(1 - \frac{\bar{\lambda} 1_{\{y \in B(x, t)\}}}{\lambda(y)}\right) d\mathbb{P}^{!x}(\varphi) dx.$$

By (8), Fubini, the first equation in the proof of Theorem 1, and (2), the Palm expectation in the integrand equals $G^{!0}(1 - u_t^0)$ for almost all x , hence the numerator of (6) equals $G^{!0}(1 - u_t^0) \int_{W_{\ominus t}} \lambda(x) dx$. As the expectation of the denominator is equal to $\int_{W_{\ominus t}} \lambda(x) dx$, (6) is ratio-unbiased as claimed. \square

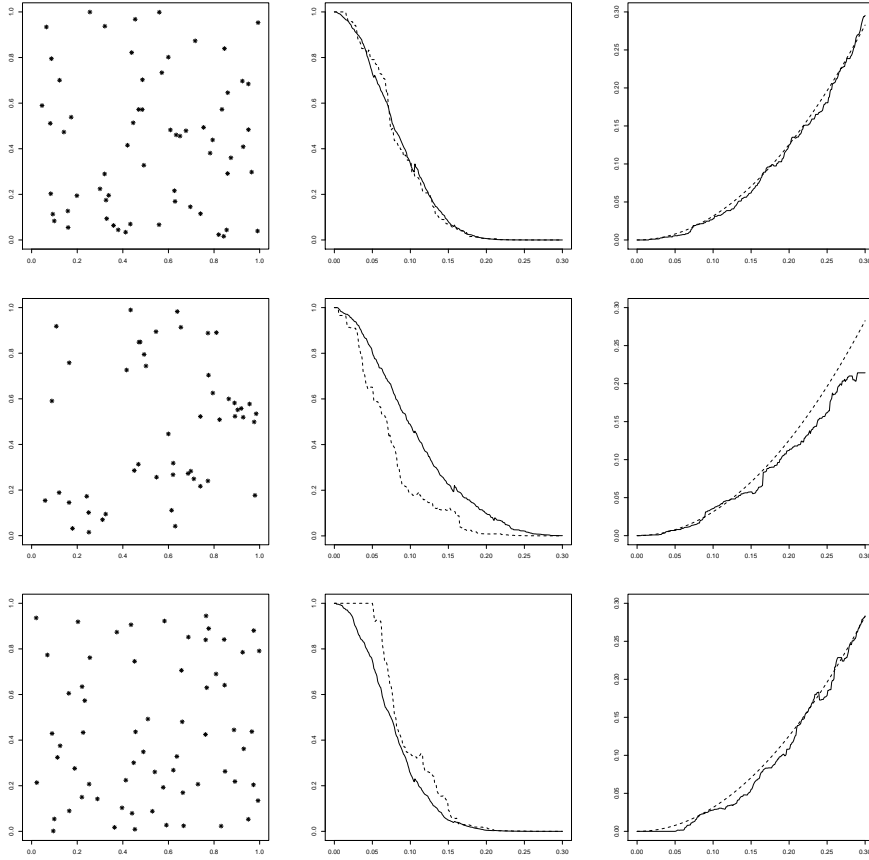


Figure 1: Each row contains a realisations of a point process in the leftmost frame, the graphs of (5) (solid line) and (6) (dashed line) in the middle frame, and the graph of $\widehat{K}_{\text{inhom}}(t)$ (solid line) compared to πt^2 (dashed line) in the rightmost frame. The models are a Poisson point process (top row), a log Gaussian Cox process (middle row), and a thinned hard core process (bottom row).

7 Examples

In order to see how $J_{\text{inhom}}(t)$ works in practice, we simulated realisations of three of the models presented in Section 5. Typical patterns are displayed in the leftmost column of Figure 1. In all three images a smooth intensity gradient can be observed: more points are located near the bottom of the square than near the top. However, the interaction structure seems different. For example, the middle picture contains groups of points that are close together, with large gaps in between the clusters. In the lower picture on the other hand, points seem to avoid being very close together and are more evenly spaced out. In the top picture, both very small and very large interpoint distances occur. In order to quantify the above qualitative remarks, we applied the ideas presented in this paper and compared the results to those obtained by a second order analysis. To simulate the patterns and calculate the estimators, the R packages `spatstat`¹ and `RandomFields`² were used.

Poisson point process The first example is a heterogeneous Poisson point process with intensity function $\lambda(x, y) = 100 e^{-y}$. Note that the mean number of points is $100(1 - e^{-1}) \approx 60$ per unit area. A realisation is shown in the top left frame in Figure 1. The top middle frame shows (5) (solid line) and (6) (dashed line). It can be seen that the graphs lie close together, in accordance with the fact that for any Poisson point process, $J_{\text{inhom}} \equiv 1$. For comparison, the plug in minus sampling estimator of K_{inhom} is shown as the solid line in the top right frame. Again, the graph is close to that of the theoretical value πt^2 (dashed line in the top right frame).

Log Gaussian Cox process The second example is a log Gaussian Cox process. The defining Gaussian random field has exponentially decaying correlation function, unit variance, and mean function μ satisfying $e^{\mu(x,y)} = 100 e^{-y-1/2}$. Note that the intensity function of the Cox process thus defined coincides with that of the Poisson point process discussed above. A realisation is shown in the middle row's leftmost frame in Figure 1. The middle frame in the same row show (5) (solid line) and (6) (dashed line). Note that the graph of (6) lies well below that of (5), indicative of attraction between points due to the positive correlation of Z after accounting for the inhomogeneity. For comparison, the plug in minus sampling estimator of K_{inhom} is shown as the solid line in the rightmost frame in the middle row. From about $t = 0.13$, the estimated value is smaller than πt^2 .

Thinned hard core process The third example is a thinned hard core (Strauss) process defined by its conditional intensity $\beta 1\{d(x, X \setminus \{x\}) > R\}$. A realisation for $\beta = 200$, $R = 0.05$ and retention probability $p(x, y) = e^{-y}$ is shown in the bottom left frame in Figure 1. The middle frame in the bottom row show (5) (solid line) and (6) (dashed line). Note that the hard core distance is clearly reflected in the flat initial segment in the graph of (6), which lies above the graph of (5) up to about $r = 0.2$, indicative of the inhibition between points due to that present in the underlying hard core process after accounting for the inhomogeneity. For

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comparison, the plug in minus sampling estimator of K_{inhom} is shown as the solid line in the bottom right frame. The estimated value is smaller than that of a Poisson point process up to about $t = 0.2$ confirming the picture painted by the J_{inhom} -function approach.

8 Summary and extensions

In this paper, we defined a J -function for intensity-reweighted moment stationary point processes in terms of their n -point correlation functions and gave representations in terms of the generating functional and conditional intensity. We calculated J_{inhom} explicitly for the three representative classes of intensity-reweighted moment stationary point processes presented in [3], derived an estimator, and presented simulation examples.

Although this paper focussed on point processes on \mathbb{R}^d , the approach may be extended to space time or marked point processes. First, assume that Y is a simple point process on the product space $\mathbb{R}^d \times \mathbb{R}$ equipped with the supremum distance whose intensity function $\lambda(\cdot)$ exists and $\inf_{(x,t)} \bar{\lambda}(x,t) > 0$. Furthermore assume all order factorial moment measures exist as locally finite measures that have Radon–Nikodym derivatives $\rho^{(n)}$ with respect to the n -fold product measure of ℓ with itself, $n \in \mathbb{N}$, and the corresponding n -point correlation functions are translation invariant in both components. Define J_n as in Definition 1, from which an inhomogeneous space time version of the J -function can be defined. If the series is truncated at $n = 1$, one obtains

$$J_{\text{inhom}}^{ST}(t) - 1 \approx -\bar{\lambda} \int_{-t}^t \int_{\|x\| \leq t} \xi_2((0,0), (x,s)) dx ds,$$

which corresponds to the K_{ST}^* -approach of Gabriel and Diggle [9]. If space and time are scaled differently, see Section 5.3, $J_{\text{inhom}}^{ST}(t,s)$ becomes a function of two variables, one for spatial distances, the other for time differences, which is more natural in many applications.

For marked point processes on \mathbb{R}^d with marks in some Polish space M equipped with a finite reference measure ν , make the same assumptions as above for space time point processes except that the n -point correlation functions are required to be translation invariant in the spatial component only. For any Borel set $B \subseteq M$ and $n \in \mathbb{N}$, set $J_n^B(t)$ equal to the common value of

$$\frac{1}{\nu(B)} \int_B \int_{B(0,t) \times M} \cdots \int_{B(0,t) \times M} \xi_{n+1}((a,b), y_1 + a, \dots, y_n + a) d\nu(b) d\ell \times \nu(y_1) \cdots d\ell \times \nu(y_n)$$

for almost all $a \in \mathbb{R}^d$ and define a family of inhomogeneous J -functions with respect to the mark set B as in Definition 1. Under suitable regularity conditions,

$$J_{\text{inhom}}^B(t) = \frac{G_B^0(1 - u_t^0)}{G(1 - u_t^0)},$$

where $u_t^a(y = (x, m)) = \bar{\lambda} 1\{x \in B(a, t)\} / \lambda(y)$ and

$$G_B^x(1 - u_t^x) = \frac{1}{\nu(B)} \int_B \int \left[\prod_{y \in Y} (1 - u_t^x(y)) \right] d\nu(b) d\mathbb{P}^{(x,b)}(Y),$$

which can be estimated using minus sampling ideas.

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