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Linear Groups and Distance-transitive Graphs

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A detailed treatment is given of the graphs on which a group with simple socle $PSL(n, q)$ acts primitively and distance-transitively.

1. Introduction

This paper may be viewed as a continuation of [5], in which all graphs are determined on which a group with socle $L(n, q)$ for some $n \geqslant 8$ acts distance transitively and primitively. Here we treat the case where the simple socle is isomorphic to $PSL(n, q)$ for some $n \in \mathbb{N}$ with $2 \leqslant n \leqslant 7$. This completes the determination of all graphs on which a group with simple socle isomorphic to some $L(n, q)$ acts distance transitively. We recall that a group G acting on a graph $\Gamma = (V\Gamma, E\Gamma)$ is said to be *distance-transitive* on Γ if its induced action on each of the sets

$$\{(x, y) \mid x, y \in V\Gamma, d(x, y) = i\}$$

is transitive, and that a graph is called *distance-transitive* if its automorphism group acts distance transitively on it. Here, d denotes the usual distance in Γ, and i runs through $\{0, \ldots, diam(\Gamma)\}$. For notation, standard terminology and facts concerning distance-transitive graphs, the reader is referred to [3] and [6].

Theorem 1.1. *Let G be a group with $PSL(n, q) \trianglelefteq G \leqslant \operatorname{aut} PSL(n, q)$, $n \geqslant 2$, and $(n, q) \neq (2, 2)$, $(2, 3)$. If Γ is a connected graph of diameter at least 2 on which G acts primitively and distance-transitively, then either Γ is a Grassmann graph or $K := N_{\operatorname{aut}\Gamma}(G^\infty)$, Γ, and the stabilizer H in K of a vertex are as listed in Table 1, with the understanding that, if $\operatorname{diam}(\Gamma) = 2$, only one of Γ and its complement is listed.*

For the precise definitions of the graphs listed, the reader is referred to [6]. In most cases, the group in the second column is the full automorphism group of Γ. But, for instance, $J(9, 2)$ has automorphism group Sym_9, whereas our group is $P\Gamma L(2, 8)$.

The results in [12], [6] and [4] imply that all imprimitive distance-transitive graphs whose primitive quotients are among those listed in Table 1 are known.

Proof. The proof is given in several steps. In view of Theorem 3.2 in [5] and known results on small valency (cf. [15]), we may (and shall) assume (without loss of generality) that $n \leqslant 7$ and $k \geqslant 14$, where k is the valency of Γ. Throughout the proof, we let $\gamma \in V\Gamma$, $v = |V\Gamma|$, $X = \operatorname{soc} G = PSL(n, q)$, $H = G_\gamma$, and $Y = H \cap X$. Then $H = N_G(Y)$. Finally, we set $q = p^a$, where p is a prime.

2. The Case $n = 2$

Since the graphs corresponding to Alt_5 are known (cf. [14] and [21]) and accord with the statement of the theorem, we may (and shall) take $q \geqslant 7$. Since G acts doubly transitively on the projective line $\Omega = \{\mathbb{F}_q v \mid v \in \mathbb{F}_q^2\}$ and the permutation character of G on (the cosets of) H is multiplicity-free, the group H has at most two orbits on Ω, and so is listed in an appendix ('Hering's Theorem') or the conclusion of the main theorem

TABLE 1.

(n, q)	K	H	Index	Array	Name
(2, 4)	Sym_5	$\mathrm{Sym}_3 \times 2$	10	$\{3, 2; 1, 1\}$	Petersen
(2, 7)	$PGL(2, 7)$	Sym_4	28	$\{3, 2, 2, 1; 1, 1, 1, 2\}$	Coxeter
(2, 8)	$P\Gamma L(2, 8)$	$Frob_{7.6}$	36	$\{14, 6; 1, 4\}$	$J(9, 2)$
(2, 9)	$P\Sigma L(2, 9)$	$L(2, 3) \times 2$	15	$\{6, 4; 1, 3\}$	Complement of $J(6, 2)$
(2, 9)	$P\Gamma L(2, 9)$	$AGL(1, 5) \times 2$	36	$\{5, 4, 2; 1, 1, 4\}$	$Inv(\mathrm{aut\ Sym}_6 \backslash \mathrm{Sym}_6)$
(2, 9)	$P\Gamma L(2, 9)$	[32]	45	$\{4, 2, 2, 2; 1, 1, 1, 2\}$	Gen. 8-gon (2, 1)
(2, 16)	$P\Gamma L(2, 16)$	$(2 \times L(2, 4)) \cdot 2$	68	$\{12, 10, 3; 1, 3, 8\}$	Doro
(2, 17)	$PSL(2, 17)$	Sym_4	102	$\{3, 2, 2, 2, 1, 1, 1; 1, 1, 1, 1, 1, 1, 3\}$	Biggs–Smith
(2, 19)	$PSL(2, 19)$	Alt_5	57	$\{6, 5, 2; 1, 1, 3\}$	Perkel
(2, 25)	$P\Sigma L(2, 25)$	$L(2, 5) \cdot 2 \times 2$	65	$\{10, 6, 4; 1, 2, 5\}$	Locally Petersen
$(3, q)$	aut $P\Gamma L(3, q)$	$Borel \cdot 2$	$(q^2 + q + 1)(q + 1)$	$\{2q, q, q; 1, 1, 2\}$	Gen. 6-gon $(q, 1)$
(3, 4)	aut $PSL(3, 4)$	$PSU(3, 2) \cdot Dih_{12}$	280	$\{9, 8, 6, 3; 1, 1, 3, 8\}$	Γ_3(Herm. forms (3, 4))
(3, 4)	$P\Sigma L(3, 4) \cdot \langle \mathrm{diag} \rangle$	$\mathrm{Alt}_6 \cdot 2^2$	56	$\{10, 9; 1, 2\}$	Gewirtz
(4, 2)	Sym_8	$\mathrm{Sym}_6 \times 2$	28	$\{15, 8; 1, 6\}$	Complement of $J(8, 2)$
(4, 2)	Sym_8	$\mathrm{Sym}_5 \times \mathrm{Sym}_3$	56	$\{15, 8, 3; 1, 4, 9\}$	$J(8, 3)$
(4, 3)	$PGO^+(6, 3)$	$PSp(4, 3){:}2 \times 2$	117	$\{36, 20; 1, 9\}$	Non-isotropics

of [20]. It is well known (cf. [23]) that $\mathrm{aut}\, X = P\Gamma L(2, q)$ has order $q(q^2 - 1)a$, and that the subgroups of $X = L(2, q)$ come in seven types, which we have labeled (ia), (ib), (ii), . . . , (vi) below.

(i) Y is a dihedral group, of order $|Y| = 2(q - \varepsilon)/(2, q - 1)$, where $\varepsilon \in \{\pm 1\}$. We show that Γ is the Johnson graph $J(9, 2)$ and $G = P\Gamma L(2, 8)$.

(ia) First, suppose $\varepsilon = 1$. Then as a G-set, $V\Gamma$ may be viewed as the set $\binom{\Omega}{2}$ of pairs of projective points. Furthermore, by Lemma 2.6 of [5], we may suppose that $G = P\Gamma L(2, q)$ or $diam\, \Gamma \leq 4$. We establish that the latter must hold. To this end, assume that $G = P\Gamma L(2, q)$.

Take $\gamma = \{0, \infty\}$ so that $H_1 = G_\gamma \cap PGL(2, q)$ is generated by the elements h, w with matrices

$$\begin{pmatrix} \zeta & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

where ζ is a generator of $\mathbb{F}_q^*$. Consider the H_1-orbits on $V\Gamma \backslash \{\gamma\}$. The element h acts on $\{\lambda, \mu\} \in \binom{\Omega}{2}$ by multiplication of its members by ζ and the element w by inversion and multiplication by -1. Clearly, the set X^γ of all vertices meeting γ in a singleton is a single orbit of size $2(q - 1)$. Each of the remaining $\binom{q}{2}$ vertices in $V\Gamma \backslash \{\gamma\}$ is H_1-conjugate to a vertex of shape $\{1, \zeta^j\}$ for some $j (1 \leq j \leq (q - 1)/2)$.

Now h^i fixes $\{1, \mu\}$ iff $\mu = -1 \neq 1$, and $h^i\{1, \mu\}$ coincides with $w\{1, \mu\}$ iff either $-1 = \zeta^i$ and $-\mu^{-1} = \zeta^i \mu$, or $-1 = \zeta^i \mu$ and $-\mu^{-1} = \zeta^i$. In the first case we have again $\mu = -1 \neq 1$, in the second case there is an i for each μ. This information determines the order of vertex stabilizers in H_1, and yields that on $V\Gamma \backslash (X^\gamma \cup \{\gamma\})$ we have $(q - 3)/2$ orbits of length $q - 1$ and a single orbit (with representative $\{1, -1\}$) of length $(q - 1)/2$ if q is odd, and $(q - 2)/2$ orbits of length $q - 1$ if q is even.

If $\{0, 1\}$ is adjacent to γ, then we must have $\Gamma = J(q + 1, 2)$, by definition of the Johnson graph $J(q + 1, 2)$ (cf. 1.2 of [5]), and so G must have a known rank 3 representation. Here $G = P\Sigma L(2, 8)$ appears with $H = Frob_{7.6}$.

More generally, let i be such that $X^\gamma = \Gamma_i(\gamma)$; then, since $J(q + 1, 2)$ has diameter 2, we have $diam\, \Gamma \leq 2i$. We fix a neighbour $\delta = \{1, \alpha\}$ of γ in Γ. Applying w and a suitable power of h to δ, we obtain $\{\eta, \eta\alpha^{-1}\} \in \Gamma_1(\gamma) \subseteq \Gamma_{\leq 2}(\delta)$. Transforming δ to γ by means of

$$\begin{pmatrix} -1 & \alpha \\ 1 & -1 \end{pmatrix}$$

we find

$$\left\{\frac{\alpha-\eta}{\eta-1}, \frac{\alpha-\eta\alpha^{-1}}{\eta\alpha^{-1}-1}\right\} \in \Gamma_{\leqslant 2}(\gamma).$$

Taking $\eta = \alpha^2$, we obtain $\{-\alpha/(\alpha+1), 0\} \in \Gamma_{\leqslant 2}(\gamma)$. If $\alpha \neq -1$, it follows that $X^\gamma = \Gamma_2(\gamma)$, and so, by the above remark, *diam* $\Gamma \leqslant 4$, as required. Therefore, suppose $\alpha = -1$ and p is odd. Taking $\eta \neq 1, -1$, we obtain

$$\left\{1, \left(\frac{\eta-1}{\eta+1}\right)^2\right\} \in \Gamma_{\leqslant 2}(\gamma) \subseteq \Gamma_{\leqslant 3}(\delta).$$

Taking $\eta = 2$, we see $\{1, 9\} \in \Gamma_{\leqslant 3}(\delta)$. If *diam* $\Gamma > 6$, this forces $9 \equiv -1 \bmod p$, whence $p = 5$. But then q is a non-trivial power of 5 and an $\eta \in \mathbb{F}_q \backslash \mathbb{F}_p$ can be found such that

$$\left(\frac{\eta-1}{\eta+1}\right)^2 \neq -1;$$

applying the same argument once more leads to a contradiction.

Consequently, *diam* $\Gamma \leqslant 6$. We show that q must be small. From the above, we see at least the H-orbits X^γ, the one containing $\{1, -1\}$, and at least $(q-3)/2a$ further orbits, so $2 + (q-3)/2a \leqslant$ *diam* $\Gamma \leqslant 6$. This shows that $a \leqslant 3$ if $p = 3$ and $a = 1$, $q \leqslant 11$ if $p \geqslant 5$. If $q = 9$, then soc G is an alternating group so Γ is known (cf. [5]) and if $q = 7, 11$, there are at least two suborbits of size at most 13, so $k \leqslant 13$ by Lemma 2.7 and Γ is known (cf. §1.5 of [5]). Since $q > 5$, only the case $q = 3^3$ remains. Then, there is a unique suborbit of size 13 and one of size 52, while the remaining four suborbits all have length 78. Since $k \neq 52$ (because Γ is not a Johnson graph) it follows that $k = 13$, contrary to the assumption $k \geqslant 14$.

This establishes that *diam* $\Gamma \leqslant 4$. Then, by the same argument as above, $2 + (q - 3)/2a \leqslant 4$ if p is odd, and $1 + (q-2)/2a \leqslant 4$ if $p = 2$. The only new cases to consider arise when $p = 2$, so let $q = 2^a$. Then $q \leqslant 32$. If $q = 32$, then all non-trivial suborbits distinct from X^γ have size 5×31, and so $k = k_2 = 155$, contradicting Lemma 2.7 [5]. If $q = 16$, then the suborbits have sizes 1, 15, 30, 30 and 60. Taking into account that $k_2 = 30$, we find that $k = 15$, $k_3 = 60$ and $k_4 = 30$. But it is readily seen that there is no corresponding feasible intersection array. We have seen above that for $q = 8$ we find the Johnson graph $J(9, 2)$. Since $q > 5$, this ends the proof of (ia).

(ib) Now let $\varepsilon = -1$. We shall view X as the group $PSU(2, q)$, so elements are (projectively) represented by matrices x with $x^{\mathrm{T}} = x^{-\sigma}$, where 'T' stands for transposed and σ for the standard Frobenius of order 2 of $\mathbb{F}_{q^2}$. The group X preserves the hermitean form $\langle(\alpha_1, \beta_1), (\alpha_2, \beta_2)\rangle = \alpha_1\alpha_2^q + \beta_1\beta_2^q$ on $\mathbb{F}_{q^2}^2$ (cf. [23] for details). Take ξ to be a generator of $\mathbb{F}_{q^2}^*$, and put $\zeta = \xi^{q-1}$. Then the elements h, w, described by the same matrices as in (ia), generate $H_1 := H \cap PGU(2, q)$. Denote by Ω the set of projective points over $\mathbb{F}_{q^2}$, and identify $\alpha \in \mathbb{F}_{q^2}$ with the 1-space containing $(\alpha, 1)$. Then G leaves invariant the subset Δ (of size $q+1$) of points represented by vectors (α, β) with $\langle(\alpha, \beta), (\alpha, \beta)\rangle = 0$, and for every point of $\Omega\backslash\Delta$ represented by (α, β), there is a unique orthogonal point $(\beta^q, -\alpha^q)$. Now H is the stabilizer of the orthogonal pair of points related to the standard basis, so $V\Gamma$ may be identified with the set of all orthogonal pairs $\{\alpha, -\alpha^{-1}\}$ with $\alpha \in \mathbb{F}_{q^2}$, $\alpha^{1+q} \neq -1$. Since h preserves α^{1+q} for $\alpha \in \mathbb{F}_{q^2}$, the 'double' value $\alpha^{\pm(1+q)} \in \mathbb{F}_q$ parametrizes $\langle h \rangle$-orbits. It readily follows from this description that on $V\Gamma$, the subgroup H_1 has $(q-2)/2$ orbits of length $q+1$ if q is even, and $(q-3)/2$ orbits of length $q+1$ and a single orbit of length $(q+1)/2$ (containing 1) if q is odd. The H-orbit structure will be completely determined if we know the Frobenius action; but this is also clear from the above picture. For instance, if q is odd, then, among the H_1-orbits of length $q+1$, there are precisely $(p-3)/2$

invariant under the Frobenius of order a. Then $a>1$ implies that there are orbits of length $>(q+1)$, so by Lemma 2.7 of [5] there are at most two orbits of length $q+1$. Thus $(p-3)/2 \leq 2$, i.e. $p \leq 7$. Let e be the number of divisors of a (including 1 and a). By Lemma 2.7 [5], and the orbit lengths, we must have $k_{e+1} \leq k_e$ if q is even and $k_{e+2} \leq k_{e+1}$ if q is odd, so $d \leq 3e$ if q is even and $d \leq 3e+3$ if q is odd. But H has at least $(q-2)/2a$ orbits if q is even and at least $1+(q-p)/2a+(p-3)/2$ if q is odd, so $2^a = q \leq 6ae+2$ if q is even and $p^a = q \leq 6ae+4a+3$ if q is odd. Using that $k \geq 14$, we also have $q \geq 13$, so that q is one of 16, 32, 64, 27, 81, 25 and 13. Inspection of the subdegrees in these specific cases shows that no feasible intersection array exists.

(ii) Y is a Borel subgroup of X. Then G acts doubly transitive on $V\Gamma$ and so Γ is a clique.

(iii) $\operatorname{soc} Y \cong \mathrm{Alt}_5$ and $p \neq 2, 5$. We may view $V\Gamma$ as the class of X-conjugates of Y. Thus $v = q(q^2-1)/120$ and $|H| = 120$ or 60 (as H is a maximal subgroup of G and there are precisely two conjugacy classes of Alt_5 in $L(2,q)$ which fuse in $PGL(2,q)$).

Let $x \in \operatorname{soc} Y$ be an element of order 5 and let $\varepsilon_5 \in \{\pm 1\}$ be such that 5 divides $q-\varepsilon_5$. There are $q(q+\varepsilon_5)/2$ groups of order 5 in X, all conjugate to $\langle x \rangle$. Hence, there are precisely $(q-\varepsilon_5)/10$ vertices of Γ containing x. Let a and b denote the number of vertices of Γ meeting $\operatorname{soc} Y$ in precisely $\langle x \rangle$, respectively, a dihedral of order 10 containing x. Then Y has b orbits of length 6 on the vertices of Γ and $a/2$ orbits of length 12. Moreover, $a+b+1=(q-\varepsilon_5)/10$. By the assumption $k>13$, so there is only one H-orbit of length at most 12.

Suppose first that q is a prime, so that $Y = H \cong \mathrm{Alt}_5$. If $q>31$, there are at least two H-orbits of size at most 12, a contradiction. Thus $q \leq 31$ and we are done by a straightforward check using the Atlas [7]. Next suppose q is not a prime. Then, by maximality of H, it must be the square of a prime, and by [20] $q=9$ or 49. Since in the first case the theorem is readily seen to hold, we may assume $q=49$. But then $\varepsilon_5=-1$ and, by the assumption $k>12$, there is a single H-orbit of size 24. At this point, it is straightforward to derive a contradiction. This ends the proof of the case where $Y \cong \mathrm{Alt}_5$ and $p \neq 2, 5$.

(iv) Let $p>3$, and $Y \cong \mathrm{Alt}_4$ (with $q \equiv 3$ or $5 \bmod 8$) or Sym_4 (with $q \equiv \pm 1 \bmod 8$). Then q is a prime number and, if $q \equiv \pm 1 \bmod 8$, there are two conjugacy classes of subgroups of X isomorphic to Sym_4 which fuse in $PGL(2,q)$ so $h=|H|=12$ or 24. But $h=12$ implies $k \leq 12$, in which case there is nothing left to prove. Thus $h=24$ and $\Gamma_1(\gamma)$ is a regular H-orbit.

If $d=2$, the complement of Γ is distance-transitive with the same group G, so we may assume $k_2=k$ so that $v=1+24+24=49$ and $v=q(q^2-1)/48$ or $q(q^2-1)/24$, contradicting that q is a prime. If $d>2$, we obtain $k=k_2=24$ and we are done by Lemma 2.7 [5].

(v) $Y = PSL(2,r)$, where $q=r^m$ and m is an odd prime number. There is a unique X-class, so $v=q(q^2-1)/(r(r^2-1))$. Recall that $q=p^a$ so that $r=p^{a/m}$. Now, by multiplicity freeness, H has at most two orbits on $\Omega = P(\mathbb{F}_q^2)$; but we see one H-orbit (which coincides with a Y-orbit) of length $r+1$. Other Y-orbits are regular of length $r(r^2-1)/(2,p-1)$, so we must have $q+1=(r+1)+br(r^2-1)/(2,p-1)$, where b divides $|G/X|$, so $b \mid (2,p-1)m$. It follows that $(r^{m-1}-1)/(r^2-1)=(q-r)/(r(r^2-1)) = b/(2,p-1) \leq m$. Consequently, either $m \leq 3$ or $r=2$ and $m=5$. In the latter case, H contains a torus and so is dealt with in (i).

Therefore, we have $m=3$ and $b=(2,p-1)$, so $H \geq PGL(2,q)$.

Now $v=r^2(r^4+r^2+1)$. Let $\varepsilon \in \{1,-1\}$. There are $r(r+\varepsilon)/2$ tori (i.e. abelian subgroups of consisting entirely of semi-simple elements) of order $r-\varepsilon$ in $H_1 = PGL(2,r)$ and similarly with q instead of r, whence each torus of $PGL(2,q)$ of order $r-\varepsilon$ is contained in $vr(r+\varepsilon)/(q(q+\varepsilon)) = r^2+\varepsilon r+1$ conjugates of H_1. Thus there are

$(r(r+\varepsilon)/2)(r^2+\varepsilon r) = r^2(r+\varepsilon)^2/2$ vertices of $V\Gamma$ meeting H in a torus of order $r-\varepsilon$. The same computation can be carried out for dihedral subgroups of order $2(r-\varepsilon)$; using that a dihedral of order $q-\varepsilon$ contains $(q-\varepsilon)/(r-\varepsilon)$ dihedral subgroups of order $r-\varepsilon$, it follows that, if D is a dihedral subgroup of H_1 of order $r-\varepsilon$, then H_1 is the unique member of its conjugacy class containing D. Hence any two conjugates of H_1 containing a torus of order $r-\varepsilon$ meet precisely in that torus.

Suppose *diam* $\Gamma \geq 5$. Then, by [5], Lemma 2.6, we may assume that $G = P\Gamma L(2, q)$. If e denotes the number of divisors of a (including 1 and a), then, since the H-orbit sizes of vertices meeting H in a torus of order $q-\varepsilon$ only depend on the order of the Galois automorphism, the number of H-orbits of vertices meeting H in a torus of order $r-\varepsilon$ is at least $r(r+\varepsilon)/2ea$. On the other hand, there are orbits of size larger than that, for instance those containing H^x, where x corresponds to the matrix

$$\begin{pmatrix} 1 & b \\ -b^{-1} & 0 \end{pmatrix}$$

where $b \in \mathbb{F}_q \backslash \mathbb{F}_r$. Thus, by [5], Lemma 2.7 we have $r(r+\varepsilon)/2ea \leq 2$. This implies that r is one of 2, 3, 4, 8 and 9. By straightforward analysis of the numbers involved, there must be strictly more than two H-orbits in the cases $r = 8, 9$. A straightforward check of subdegrees against feasible intersection arrays shows that the theorem holds for the remaining values $(2, 3, 4)$ of r.

Finally, suppose *diam* $\Gamma \leq 4$. Then the number of non-trivial H-orbits is four. One of these is accounted for by the same x as in the previous paragraph. Since both values of ε account for at least one, there is a value, say ε_0, of ε such that there is exactly one H-orbit of vertices meeting H in a torus of order $r-\varepsilon$. This implies $r(r+\varepsilon_0)/2a \leq 1$, whence $r = 2, 3, 4$. If $r = 2$, then H is not maximal, and we are done. If $r = 3$, the degrees of irreducible characters do not exceed $a \cdot 28 = 168$, so we would have $1 + 4 \cdot 168 \geq |V\Gamma| = 891$, a contradiction. If $r = 4$, then, by standard arithmetic, there must be more than two H-orbits meeting H in a torus of order $r-1=3$.

(vi) soc $Y \cong PSL(2, r)$, where $q = r^2$. By Lemma 2.6(i) of [5], we may assume $G \geq P\Sigma L(2, q)$. By maximality of H, and observing that if q is odd, there are two classes of subgroups isometric to $PSL(2, r)$, we have $G = P\Sigma L(2, q)$ and $H = P\Gamma L(2, r)\cdot\langle\gamma\rangle$, where γ is the standard Frobenius automorphism of $PSL(2, q)$ of order 2. Furthermore, as a G-set, $V\Gamma$ can be identified with the $L(2, q)$-class of γ. Thus, Proposition 2.5 of [5] applies. Clearly, cases (i) and (ii) of its conclusion do not hold.

Suppose q is odd. First consider the case where $\delta \in \Gamma$ is adjacent to γ if δ and γ commute. Then the product of any two non-commuting involutions in Y has the same order. But any element in a torus of Y order $(r \pm 1)/2$ arises as such a product, so (as r is odd) it follows that $(r-1)/2 = 2$ and $q = 25$. The resulting graph has been found by J. I. Hall [11] in his determination of locally Petersen graphs.

It remains to study the case where γ and $\delta \in \Gamma(\gamma)$ do not commute. Then case 2.5(iii) of [5] is at hand, so $\gamma\delta$ has order 2 iff $\delta \in \Gamma_d(\gamma)$. Also, no two involutions in $V\Gamma$ have a product of order 4, so (by consideration of involutions in Γ commuting with σ) $r \equiv 3, 5 \bmod 8$.

To finish, we shall use another interpretation of $V\Gamma$. Since $G = P\Sigma L(2, r^2) \cong P\Sigma O^-(4, r)$, we can view $H \cong P\Sigma O(3, r) \cdot 2$ as the stabilizer of a non-isotropic vector in elliptic projective 3-space. (The two choices of points according to square or non-square norm if q is odd correspond to the two classes of $PSL(2, r)$ in $PSL(2, q)$.) We can thus view $V\Gamma$ as the set of non-isotropic points with square norm.

Suppose q is odd. Then, from this picture it is readily seen that, if γ and δ are vertices of Γ, there is $g \in H = G_\gamma$ such that δ and δg are orthogonal (consider the projection of δ on the orthoplement of $\langle \gamma \rangle$). This yields that commuting involutions in the earlier picture occur at distance 2, whence $d \leq 2$, a contradiction.

Suppose q is even. Then, a direct computation (cf. [6], Ch. 12) shows that vertices corresponding to orthogonal points can be found at distance at most 3, regardless of the choice of adjacency, so $d \leq 3$, and $q = 16$, yielding the Doro graph. This ends the proof for $n = 2$.

3. Proof for $n \geq 3$; Structure Preserving Vertex Stabilizers

The following result is essentially due to Saxl [22], cf. the remark following [5], Lemma 2.1. Recall that, for $d \leq n/2$, the Grassmann graph $G(n, d, q)$ has vertex set $VG(n, d, q)$, the collection of d-dimensional subspaces of $\mathbb{F}_q^n$.

Lemma 3.1. *Let G, Γ and H be as above and suppose G acts multiplicity-freely on $V\Gamma$:*
(i) *If τ_n is the number of involutions in* Sym_n, *then*

$$|P\Gamma L(n, q) \cap H| \geq (1 + \tau_n)^{-1}[G : G \cap P\Gamma L(n, q)]^{-1} \prod_{i=2}^{n} \frac{q^i - 1}{q - 1}.$$

(ii) *If n is even, the group G acts multiplicity-freely on $VG(n, n/2, q)$ with rank $n/2 + 1$. Consequently, the number of H-orbits on $VG(n, n/2, q)$ is at most $n/2 + 1$.*

For dimension $n \leq 5$, the subgroups of $L(n, q)$ have been determined (cf. [17] for references and details). Nevertheless, we start with the same approach for finding all multiplicity-free permutation representations as used by Inglis, Liebeck and Saxl [13]; namely, to apply Aschbacher's division of cases for a skew-linear group $H_0 = P\Gamma L(V) \cap H$ (a normal subgroup of H of index at most 2) acting projectively on a module V over $\mathbb{F}_q$ of dimension n. Aschbacher [2] discerns eight cases (C1), . . . , (C8), in which H preserves a certain structure on V. We shall go over the various possibilities now. Denote by ϕ the natural projection map $\Gamma L(n, q) \to P\Gamma L(n, q)$.

(C1), (C2) *Y stabilizes a subspace.* We are as in one of (i), (ii), (iii) or (iv) of [13]. There are no changes with respect to [13] (i.e. this leads to the Grassmann graphs), except that for $n = 3$ generalized hexagons of order $(q, 1)$ occur (they are distance-transitive as polarities exist) and for $n = 3$ and $q = 2$, the Coxeter graph arises.

(C3) *There is an extension field of order $r = q^m$, for some prime $m \mid n$, and $\mathbb{F}_r H_0$-module W such that V is the module obtained from W by restriction of scalars to* $\mathbb{F}_q$. There is a torus, L say, in $SL(n, q)$ of order $q^{m-1} + q^{m-2} + \cdots + 1$ such that $H = N_G(\phi L)$. As all such tori are conjugate, we may take $V\Gamma$ to be the set of conjugates of L. Similarly to case (v) of the proof of Theorem 3.2 in [5], one can show that if L_1 is a conjugate of L which commutes with L, then $L_1 \in \Gamma_d(L)$. Let $N \in \Gamma(L)$. Then, according to Lemma 2.7(ii), (iii) of [5], $N_H(\phi N)$ is the unique one of maximal order among all $N_H(\phi M)$ for $M \in V\Gamma$ such that M and L do not commute. In other words, $k = [H : N_H(\phi N)]$ is minimal among all conjugates of L not commuting with L.

As here $n \leq 7$, we have either $m = n$ or one of $(m, n) = (2, 6), (3, 6), (2, 4)$. Consider the case $m = n$. In view of maximality of H, we have that n is a prime; in particular, $n \in \{3, 5, 7\}$. All non-trivial orbit sizes of $H_0 := \phi^{-1}H \cap \Gamma L(n, q)$ on $V\Gamma$ are multiples of $|L|/(n, |L|)$ (for the centralizer in L of a conjugate L_1 of L distinct from it is trivial and the normalizer interchanges the n distinct characters of L_1 on $V \otimes \mathbb{F}_{q^n}$). Thus, there are at most $e([H:N_G(\phi L_1)]) = e(2na(n, q^{n-1} + \cdots + 1))$ different non-trivial H-orbit sizes, where $e(x)$ stands for the number of divisors of x. By Lemma 2.7(vi) of [5], this yields $diam\, \Gamma \leqslant 3e(2na(n, q^{n-1} + \cdots + 1))$. On the other hand, we have $v \leqslant 1 + diam\, \Gamma \cdot |H|$, so

$$v = \frac{1}{m} q^{n^2(m-1)/2m}(q^n - 1)(q^{n-2} - 1)\cdots(q-1)/(q^n - 1)(q^{n-m} - 1)\cdots(q^m - 1)$$
$$\leqslant 1 + 6e(2na(n, q^{n-1} + \cdots + 1))an(q^n - 1)/(q - 1).$$

This gives that we have one of $(n, q) = (3, 2)$, $(3, 3)$, $(3, 4)$. In the first case, we find the projective line of order 7 on which $PGL(2, 7) \cong \text{aut}\, L(3, 2)$ acts doubly transitively, so $V\Gamma$ is a clique, a contradiction. In the cases $q = 3$ and $q = 4$, we obtain graphs on 144 and 960 vertices, respectively, which, by closer inspection of possible intersection arrays, are readily seen not to provide distance-transitive graphs.

From now on, we may assume that m is a proper divisor of n.

Suppose $m = 2$, so $n = 4$ or 6, and L is a torus of order $q + 1$. The case $n = 4$ can be done by geometry, using the isomorphisms $L(2, q^2) \cong P S\Omega^-(4, q)$ and $L(4, q) \cong P S\Omega^+(6, q)$. Thus, we can (and shall) view $V\Gamma$ as the set of elliptic lines in the hyperbolic geometry $O^+(6, q)$. Fix a line $l \in V\Gamma$. Any line $m \in V\Gamma$ belongs to one of the sets V_i $(1 \leqslant i \leqslant 6)$ given below:

V_i	$\lvert V_i \rvert$	Description of V_i
V_1	$(q^2-1)(q^2+1)$	$\langle l, m\rangle$ degenerate, $l \cap m = \varnothing$
V_2	$q(q^2+1)(q+1)(q-2)/2$	$\langle l, m\rangle$ non-degenerate, $l \cap m \neq \varnothing$
V_3	$q(q^2+1)(q^2-1)(q+1)(q-2)/2$	$\langle l, m\rangle$ degenerate, $l \cap m = \varnothing$
V_4	$q^3(q^2+1)(q^2-1)(q-1)/4$	$\langle l, m\rangle$ elliptic, $l \cap m = \varnothing$
V_5	$q^2(q^2+1)(q^2-1)(q-1)(q-2)/4$	$\langle l, m\rangle$ hyperbolic, $l \cap m = \varnothing$, $m \notin l^\perp$
V_6	$q^2(q^2+1)/2$	$\langle l, m\rangle$ hyperbolic, $l \cap m = \varnothing$, $m \in l^\perp$

If $q = 2$, then $V_i = \varnothing$ for $i = 2, 3, 5$, and the Johnson graph $J(8, 3)$ appears. Otherwise, $diam\, \Gamma \geqslant 6$, so, by Lemma 2.6 of [5], we may assume that H acts transitively on the set of non-isotropic points of $O^+(6, q)$. Now V_6 is a single orbit corresponding to L_1 (the commuting conjugate of L) so $\Gamma_d(l) = V_6$. On the other hand, a straightforward check shows that an H-orbit off V_6 of minimal length lies in V_2 (and has size $(q+1)(q^2+1)q/2$) if q is odd, and lies in V_1 (and has size $(q^2-1)(q^2+1)$) if q is even. In both cases, it is easily seen that there are members of $V_6 = \Gamma_d(l)$ in $\Gamma_{\leqslant 4}(l)$, contradicting that $d \geqslant 6$.

Suppose $n = 6$. Take l such that $L = \langle l \rangle$, and let $K = \langle k \rangle \in V\Gamma \backslash \Gamma_d(L)$ be such that $l^{-1}k$ has four-dimensional fixed space and $\langle l, k\rangle \cong SL(2, q)$ stabilizes a two-dimensional complement of this fixed space. The H-orbit size of K is certainly not maximal. So the number of such orbits is bounded by 2. Also $N_H(\phi L, \phi K) \leqslant C_G(\phi\langle L, K\rangle)$. Now the $n = 2$ case gives that the number of such orbits (varying K over the conjugates of L in $\langle L, K\rangle$) is at least $(p-3)/2$. Since this number is bounded by 2, we obtain $p \leqslant 7$. If p is odd, then H is centralized by an involution in $PGL(6, q)$ and so by Lemma 2.6 of [5], we may take $PGL(n, q) \leqslant G$ and H is the centralizer of an involution in $N_G(\phi L)$; but then there are pairs of involutions from this class with products of order 4 (from the $PGL(2, q)$ picture), so we are done by [5]. It remains to consider the case where $p = 2$.

Suppose $q = 2$. Then direct computation (we used CAYLEY) shows that the

H-orbits on $V\Gamma$ have sizes 1, 336, 5040, 201060, 25920, 315, 3780, in the respective cases where $\langle L, N\rangle$ is a group of type $\mathbb{Z}_3$, $\mathbb{Z}_3^2$, [36], Alt_5, $L(2, 8)$, Alt_4, [24]. Thus, $d = 6$, and $\Gamma(L)$ must be the H-orbit of size 315. But then, there is a subspace decomposition $V = V_1 \otimes V_2$ with $\dim V_i = 2i$ such that L and N coincide on V_1, and generate a subgroup isomorphic to Alt_4 in the subgroup A of G normalizing V_1 and V_2. Now A acts on L^A as $SL(V_2) \cong \mathrm{Alt}_8$ on its set of groups of order 3 fixing 5 points, and the above adjacency leads to an isomorphism of the subgraph of Γ induced on L^A with $J(8, 3)$. In particular, commuting pairs occurs at distance 3, so $d \leqslant 3$, a contradiction.

Now $q \geqslant 4$, q even. From the geometry it is readily seen that there are at least three H-orbits of the same length consisting of $\langle K\rangle$ such that $\langle L, K\rangle \cong SL(4, q)$, a contradiction.

Finally, suppose $m = 3$. Then $n = 6$. Now $|H_0| \leqslant (q^3 - 1)(q - 1)\,|PGL(2, q^3)| \cdot 3a$, so Lemma 3.1 yields $q \leqslant 4$. For $q = 3, 4$, direct check reveals that the number of H-orbits on the set of maximal flags in $\mathbb{F}_q^6$ exceeds $\tau_n + 1 = 76$, contradicting the remark after Lemma 2.1 of [5]. If $q = 2$, it can be verified that too many subdegrees are equal for the graph Γ to be distance-transitive.

(C4), (C7) There is a Y-invariant tensor product decomposition $V = V_1 \otimes \cdots \otimes V_j$ with $j > 1$ and $\dim V_i > 1$ for all i $(1 \leqslant i \leqslant j)$. Then, as $n \leqslant 7$, we have $j = 2$ and $(\dim V_1, \dim V_2) = (2, 2)$ or $(2, 3)$.

First consider $\dim V_1 = 2$, and $\dim V_2 = 3$, so $n = 6$. Then, by Lemma 3.1,

$$q(q^2-1)q^3(q^3-1)(q^2-1)(q-1)a \geqslant |H| \geqslant \tfrac{1}{2}\tfrac{1}{76}\prod_{i=1}^{6}\frac{q^i-1}{q-1}$$

implying $q^4 a \geqslant \frac{1}{152}(q^6-1)(q^5-1)(q^2+1)/(q-1)^2$, which is absurd.

Thus, assume $\dim V_1 = \dim V_2 = 2$. Then H is an orthogonal group and will be dealt with in (C8).

(C5) *There is a divisor m of a such that, with $q = r^m$, the subgroup H_0 is conjugate to a subgroup normalizing $PSL(n, r)$.*

Lemma *If σ is the standard Frobenius $\xi \mapsto \xi^r$ of order m. Then $H = C_G(\sigma)$, and the permutation character of G on H is multiplicity-free iff $m = 2$.*

If $m = 2$, the statement follows from [10].

Suppose for the remainder of the proof of this lemma that $m > 2$. Denote by P, S the set projective points of $\mathbb{F}_q^n$, $\mathbb{F}_r^n$, respectively. Then P partitions into the three H invariant sets S, $S_1 = \{p \in P\backslash S \mid pp^\sigma \cap S \neq \emptyset\}$, and $S_2 = \{p \in P\backslash S \mid pp^\sigma \cap S = \emptyset\}$, where pp^σ denotes the projective line of P on p and $p\sigma$. Since these three sets are non-empty and G is doubly transitive on P, we are done unless G contains a duality (i.e. graph automorphism) δ. Also, H cannot have four or more orbits on P. Consider $p \in S_1$ and denote by l the unique line pp^σ on p meeting S. Then $H_x \leqslant H_l$ and, as S_1 must be a single H-orbit, the group H_l acts transitively on the $r(r^{m-1} - 1)$ points of $l\backslash S$, so $r(r^{m-1} - 1) \mid r(r^2 - 1)m$. Hence either $m = 5$ and $r = 2$, or $m = 3$. In the first case, we obtain a contradiction with Lemma 3.1, so from now on we may assume $m = 3$.

Now consider the H-invariant sets of incident point, hyperplane pairs $\{s, t\}$, for $s \in S_i$, $t \in \delta S_j$ $(0 \leqslant i, j \leqslant 2)$. If $n > 3$, all six of them are non-empty and if $n = 3$, there are five non-empty sets among them. Since G acts multiplicity-freely on the set of all incident point, hyperplane pairs with rank 5 and 4 in the respective cases, this leads to a contradition with the multiplicity freeness of G on $V\Gamma$, and so finishes the proof of the lemma. □

Due to the lemma, we only need consider the case where $m = 2$. Then H is the centralizer of the involution σ and, in view of the proof of Theorem 3.2 Case (vii) [5], we may assume $\sigma \in G$, $V\Gamma = \sigma^G$, $\Gamma(\sigma) \leqslant H$, $H \cap \sigma^G$ is a class of s-transpositions for some prime s, and $n \leqslant 4$. According to [1] and [9], $n = 4$ and $r \in \{2, 3\}$.

If $r = 2$, then $\Gamma(\sigma)$ is isomorphic to the complement of the Johnson graph $J(8, 2)$, so Γ contains a quadrangle, $k = 28$, $a_1 = 6$, and, by [24], Γ has diameter at most 7, a contradiction as the permutation rank exceeds 8 (cf. Gow [10]).

If $r = 3$ then $\Gamma(\sigma)$ is the graph of non-isotropics in $O^+(6, 3)$, so Γ contains a quadrangle, $k = 117$ and $a_1 = 36$, leading to the same contradiction as for $r = 2$.

(C6) *There is a prime $r \neq p$ such that $r^m = n$ for some m, and an r-group R acting irreducibly on V and normalized by H_0, such that $R/Z(R)$ has order r^{2m} and $Z(R)$ has order at least* 3 (*and dividing* $q - 1$). *Furthermore, a is odd and equals the order of p in the group of units of the integers modulo* $|Z(R)|$. Now

$$|H \cap P\Gamma L(n, q)| \leqslant r^{2m} \, |Sp(2m, r)| \, a = r^{2m+m^2} \prod_{i=1}^{m} (r^{2i} - 1)a$$

so, by Lemma 3.1,

$$r^{2m+m^2} \prod_{i=1}^{m} (r^{2i} - 1) \geqslant \tfrac{1}{2}(1 + \tau_n)^{-1} \prod_{i=2}^{n} \frac{(q^i - 1)}{(q - 1)}.$$

Using that $|Z(R)|$ divides $(q - 1)$ and $2 < r^m = n \leqslant 7$, and that $|Z(R)|$ is either odd or divisible by 4, we see that the only possible values for the triple (r, m, q) are $(3, 1, 4)$, $(3, 1, 7)$, $(2, 2, 5)$. In the first case, we have the example on 280 vertices described in Table 1. In the second case, a look at the character table of aut $L(3, 7)$ (cf. the Atlas [7]) immediately gives a contradiction with multiplicity freeness. Finally, let $(m, r, q) = (2, 2, 5)$. Then, by use of the isomorphism $L(4, 5) \cong PS\Omega^+(6, 5)$, the vertex set $V\Gamma$ may be viewed as the stabilizer of an orthonormal frame (6 non-isotropic 1-spaces that are mutually orthogonal), say $\{\mathbb{F}_5 v_i\}_{1 \leqslant i \leqslant 6}$ in $O^+(6, 5)$. Now $v_1 + 2v_2$, $v_1 + v_2 + 2v_3 + 2v_4$, $v_1 + v_2 + v_3 + 2v_4 + 2v_5 + 2v_6$, $v_1 + v_2 + v_3 + v_4 + v_5$ are clearly representatives of distinct H-orbits, whose 1-spaces are isotropic, showing that H has at least four orbits of isotropic points. This implies that it cannot be multiplicity-free (cf. the remark following Lemma 2.1 of [5]).

(C8) *There is a non-degenerate H_0-invariant quadratic, symplectic, or hermitean form on V.* If the form is symplectic or hermitean, then H is the centralizer of an involution, and we proceed as in [5]. First, consider the case of a symplectic form. Then $m = 2$ in view of [5]. Using the isomorphisms $PSp(4, q) \cong PS\Omega(5, q)$ and $L(4, q) \cong PS\Omega^+(6, q)$, we can view $V\Gamma$ as the set of projective points $\langle x \rangle$ with $Q(x) = 1$, for $x \in W = \mathbb{F}_q^6$ and Q a fixed non-degenerate quadratic form on W of Witt index 3, and $G \cap L(4, q)$ as the simple socle of the group fixing Q. From this picture, it is straightforward that $V\Gamma$ cannot be distance-transitive, unless $q = 2$ or 3, in which cases there are distance-transitive graph structures on Γ as listed in Table 1 (on 28 and 117 vertices, respectively).

Now consider the case where H_0 fixes a hermitean form. Then, according to [5], there are involutions $x, y \in V\Gamma$ such that xy has order 4, so $\Gamma(x)$ coincides with a class of r-transpositions for some prime number r, and by [9] and [1], either $(n, q) = (4, 9)$ or $q = 4$. In the first case we obtain the result that Γ satisfies $k = 126$, $a_1 = 45$ and contains quadrangles, so that, by [24], $diam\,\Gamma \leqslant 5$, less than the number of H-orbits (cf. Gow [10]), a contradiction. Therefore assume $q = 4$. For $n = 3$, we obtain an example, the graph Γ from Table 1 on 280 vertices, so assume $n \geqslant 4$. Then the same argument as given at the end of the proof of Theorem 3.2 in [5] applies.

It remains to discuss the case where H_0 stabilizes a quadratic form. By maximality of H in G, we take q to be odd.

Suppose n is odd. If $G \leqslant P\Gamma L(n, q)$, then the permutation rank of G on $VG(n, 2, q)$ is 3 or 2 according as $n \geqslant 5$ or $n = 3$, whereas H has four, respectively three orbits on this set. Consequently, G is not multiplicity-free on $V\Gamma$, a contradiction. Hence G contains a graph automorphism. Now G has permutation rank 5 on the set of incident point, hyperplane pairs, whereas $H \cap P\Gamma L(n, q)$ has at least seven orbits on this set, again a contradiction with multiplicity freeness.

Thus $n = 2m$ is even. First, suppose the Witt index of the form is maximal (i.e. equal to m). Then G has permutation rank $m + 1$ on the set of m-spaces, but there are at least $m + 2$ H-orbits on this set (if $n = 4$, there are elliptic, hyperbolic, tangent and isotropic lines, and if $n = 6$, there are totally isotropic, degenerate with two-dimensional radical, degenerate with hyperbolic quotient, degenerate with elliptic quotient, non-degenerate).

Finally, let the Witt index be smaller than m. Then it is $m - 1$. If $G \leqslant P\Gamma L(n, q)$, then G has permutation rank 2 on the set of 1-spaces, and H has three orbits on this set (observe that if $n \geqslant 4$, no outer automorphsim can be realized in $P\Gamma L(n, q)$), so again G cannot be multiplicity-free on $V\Gamma$. Thus G contains a diagram automorphism. Now $H \cap P\Gamma L(n, q)$ has three orbits on the set of 1-spaces, and from this it readily follows that there are at least six orbits on the set of incident point, hyperplane pairs. Since G has permutation rank 5 on the latter set, we have a contradiction with multiplicity freeness, and we are done.

4. Proof for $n \geqslant 3$; Irreducible Groups with Simple Socle

We retain the notation $\phi : \Gamma L(n, q) \to P\Gamma L(n, q)$, $V = \mathbb{F}_q^n$, $H_0 = \phi^{-1}(H \cap P\Gamma L(n, q))$. In this section, we deal with the case where H_0 is not as described in one of (C1)–(C8). Then, according to [2], the socle Z of H is a non-abelian simple group acting absolutely irreducibly on $\mathbb{F}_q$. Moreover, we have $H = N_G(Z)$, and $C_G(\phi Z) = 1$, so H embeds in aut Z. The resulting upper bound $|\text{aut } Z|$ on H will be frequently applied in conjunction with Lemma 3.1. We further divide this case into four subcases, viz.: (i) Z is a simple Chevalley group of characteristic p; (ii) Z is a simple Chevalley group of characteristic $r \neq p$ and cannot be viewed as a simple Chevalley group of characteristic p; (iii) Z is an alternating group Alt_m with $m \geqslant 7$, $m \neq 8$; (iv) Z is a sporadic group.

(i) From known literature (e.g. [8, 17, 19]) we derive:

Lemma. *Let Z be a simple Chevalley group of characteristic p (including the derived groups $PSp(4, 2)'$, $G_2(2)'$, $G_2(3)'$, ${}^2F_4(2)'$) that is a subgroup of $L(n, q)$ for which (C1)–(C8) does not hold. Then either $Z \cong PSp(4, 2)'$ and $q = 4$, or $Z \cong L(2, r)$ for some power $r = p^m$ of p.*

The case $Z \cong PSp(4, 2)'$ leads to the graph on 56 vertices mentioned in Table 1. Therefore, we assume $Z \cong L(2, r)$. By a result of Donkin (cf. [19]), $n \geqslant 2^{m/(m,a)}$. As $n \leqslant 7$, we have $m/(m, a) \leqslant 2$. Suppose $m = (m, a)$. Then $m = a$, for otherwise (C5) would hold. By Lemma 3.1, we have

$$q(q^2 - 1)a \geqslant \tfrac{1}{2}(1 + \tau_n) \prod_{i=2}^{n} \frac{q^i - 1}{q - 1},$$

whence $n = 3$. But then $Z = PS\Omega(3, q)$ and belongs to (C8), a contradiction.

Therefore $x = (m, a)$ satisfies $m = 2x$ and there is an odd number k such that $a = kx$.

Set $s = p^x$. Then Lemma 3.1 gives

$$s^2(s^4 - 1)m \geq \tfrac{1}{2}(1 + \tau_n) \prod_{i=2}^{n} \frac{s^{ik} - 1}{s^k - 1},$$

leading to $k = 1$ (recall that $n \geq 2^2$), and either $n = 5$ and $q = 2$, or $n = 4$.

If $(n, q) = (5, 2)$, a look at the Atlas [7] shows that $H = N_G(Z)$ is non-maximal, again a contradiction. Consequently, $n = 4$, and we are in case (C3) (cf. [17]), a contradiction.

(ii) From known literature (e.g. [18]) we derive:

LEMMA *Let Z be a Chevalley group of characteristic $r \neq p$ acting projectively and irreducibly on the $\mathbb{F}_q$-vector space V of dimension at most 7. Denote by μ the minimal dimension of such a module. Then Z is isomorphic to one of $L(2, 4)$ ($\mu = 2$), $L(2, 8)$ ($\mu = 6$), $L(2, 7)$ ($\mu = 3$), $L(2, 9)$ ($\mu = 3$), $L(2, 11)$ ($\mu = 5$), $L(2, 13)$ ($\mu = 6$), $L(3, 4)$ ($\mu = 4$), $L(4, 2)$ ($\mu = 7$), $PSp(6, 2)$ ($\mu = 7$), $PSU(4, 2)$ ($\mu = 4$), $PSU(3, 3)$ ($\mu = 6$), $PSU(4, 3)$ ($\mu = 6$).*

Suppose $n = 3$. Then an absolutely irreducible embedding of each of the three groups listed in the table with $\mu \leq 3$ defies (C8).

So let $n \geq 4$. Each of $PSp(6, 2)$, $L(4, 2)$, $L(2, 13)$, $L(2, 8)$ fails in view of Lemma 3.1. We check the remaining possibilities for Z in their order of appearance in the lemma.

$Z \cong L(2, 4)$ or $L(2, 7)$: Lemma 3.1 yields $n = 4$ and $q \leq 3$, so $q = 3$. Now, in the former case, we obtain a contradiction with the maximality of H, and in the latter case is absurd as $L(2, 7)$ does not embed in $L(4, 3)$. Suppose $Z \cong L(2, 9)$. As $Z \cong PSp(4, 2)'$, we may also assume $p \neq 2$. But then Lemma 3.1 yields a contradiction.

Suppose $Z \cong L(2, 11)$. Then Lemma 3.1 (and $\mu \geq 5$) gives $n = 5$ and $q = 2$, which is absurd as 11 does not divide $|\text{aut}\, L(5, 2)|$.

Let $Z \cong L(3, 4)$. If $n \geq 6$, we obtain a contradiction with Lemma 3.1. By [17], we must have $n = 4$ and $q = 9$, in which case, X embeds via $PSU(4, 3)$, a contradiction with the maximality of H.

If $Z \cong PSU(4, 2)$, then we may assume $p \neq 2, 3$. Lemma 3.1 then yields $n = 4$ and $q = 5, 7$, whence, by the requirement $q \equiv 1 \bmod 3$ (cf. [17]) $q = 7$. In order to study the action of Z on V, we present Z as the group generated by the following matrices (they are given here as the matrices presented in [20] are in error):

$$\begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 1 & 1 & 2 \end{pmatrix},$$

$$\begin{pmatrix} 2 & 0 & 1 & 6 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 2 & 6 \\ 6 & 0 & 6 & 2 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}.$$

Straightforward computation shows that there are two orbits, say S and T, on the set of projective points (as stated in [20]) with length 40 and 360, respectively, and that there are 240 (projective lines) containing precisely 2 points from S, 90 lines having precisely 4 points of S, 1440 lines having precisely 1 point of S, and 1080 lines entirely contained in T. Consequently, the permutation rank of G on the set of lines (being 3) exceeds the number of H-orbits of lines, a contradiction with multiplicity freeness of G on H.

Suppose $Z \cong PSU(3, 3)$. Lemma 3.1 gives $n = 6$ and $q = 2$, but in view of $Z \cong G_2(2)'$, we may assume $p \neq 2$, and we are done.

Finally, suppose $Z \cong PSU(4, 3)$. Now either $n = 6$, and $q \in \{2, 4\}$ or $n = 7$ and $q = 2$. As the possibility $q = 2$ fails by Lagrange, we have $n = 6$ and $q = 4$. But then Z embeds in $PSU(6, 2)$ and hence H is not maximal in G. This ends the proof of case (ii).

(iii) By well known results, $Z \cong \mathrm{Alt}_m$ and $n = \dim V \leq 7$ gives $m \leq 9$.

Let $m = 7$. Then Lemma 3.1 gives that either $n = 5$ and $q = 2$, or $n \leq 4$. In the former case, H is non-maximal (cf. [7]), so assume $n \leq 4$.

If $n = 3$, then Lemma 3.1 gives $q \leq 25$. In view of [7], we must have $p \leq 7$, and by Lagrange and [7], $q = 25$ remains. But then Z is contained in $PSU(3, 5)$, yielding a contradiction with the maximality of H. Now suppose $n = 4$. If $p = 2$, then $q = 2$ and G is doubly transitive on $V\Gamma$, leading to a contradiction with *diam* $\Gamma > 1$, so $p \geq 3$. Lemma 3.1 gives $q = 3, 5$ contradicting Lagrange.

Finally, let $m = 9$. Then p divides $m!$ (as $n \leq 7$). If $p \neq 2$, then, by consideration of the subgroup Alt_8, $n = 7$, contradicting Lemma 3.1. So $p = 2$, forcing $n \geq 8$, a contradiction.

(iv) It is well known (cf. [20]) that the only sporadic groups having a projective representation of degree at most 7 are among M_{11}, M_{12}, M_{22}, J_1, J_2. If p does not divide $|Z|$, then by the Atlas [7] we have $Z = J_2$, $n = 6$, and $\phi^{-1}Z = 2 \cdot J_2$. Since p is odd, there is a symplectic form left invariant by Z, and so $H = N_G(Z)$ is non-maximal.

From now on, suppose p divides $|Z|$. We proceed with a case-by-case analysis.

Let $Z \cong M_{11}$. By [16], the only irreducible projective modular characters for Z of dimension at most 7 occurs for $p = 3$ and $n = 5$. If $G \leq PGL(5, 3)$, then Lemma 3.1 yields $|H| \geq 9680$. But $|H| = |Z| = M_{11} = 7920$, a contradiction. Hence G contains graph automorphisms, and by maximality of H, we have that there is a graph automorphism σ normalizing Z. Since out $M_{11} = 1$, we must have $H \leq C_Z(\sigma)$, a classical group, conflicting with maximality of H in G.

$Z \cong M_{12}$: if the representation has no multiplier, then, by [16], we have $n \geq 10$, which is absurd, so we may assume $\phi^{-1}H$ contains a subgroup $\hat{Z} \cong 2 \cdot M_{12}$. Now n must be even, and, in view of Lemma 3.1, either $n = 6$ and $q = 2$ or $n = 4$ and $q \leq 13$. But 11 must divide $|L(n, q)|$, whence $n = 4$ and $q = 11$. Since $|L(4, 11)|$ is not a multiple of 3^3, this is impossible.

$Z \cong M_{22}$: applying [17] gives $n \geq 6$. Lemma 3.1 then gives $q = 2$, contradicting Lagrange.

$Z \cong J_1$: consider a Frobenius subgroup F of order $7 \cdot 6$. Suppose $p \neq 7$. Then $n \geq 6$ for a faithful representation of F, and by Lemma 3.1 we obtain $q = 2$, again contradicting Lagrange. Thus $p = 7$. By [17], $n \geq 6$, contradicting Lemma 3.1.

$Z \cong J_2$: if $p = 3$, then $n = 4$ from Lemma 3.1. But consideration of the subgroup isomorphic to $5^2{:}D_{12}$ shows that $n \geq 6$. Then $q \leq 3$, contradicting Lagrange. This ends the proof of case (iv) and hence Theorem 1.1. □

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