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Graphs Related to Held's Simple Group

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We analyze the permutation representations of low degree of Held's simple group *He*. We also determine its primitive multiplicity free permutation representations and show that there is no graph on which it or its automorphism group acts as a distance transitive group of automorphisms. In doing so, we supply a computer-free construction of *He*. © 1989 Academic Press, Inc.

1. STATEMENT OF RESULTS

Let *He* be the finite simple group of order $2^{10} \cdot 3^3 \cdot 5^2 \cdot 7^3 \cdot 17$ discovered by Held [4, 5] and *H* its automorphism group Aut *He*, which is of twice the order of *He*. The maximal subgroups of *He* and *H* are determined by Butler [2] and Wilson [7]. In Section 3, we give a construction of *H* which differs from the known computer-free construction in that it does not depend on the construction of the Monster. We subsequently derive some properties of its permutation representations. In particular, we determine the distance distribution diagrams of graphs underlying the two primitive permutation representations of lowest degrees (viz. 2058 and 8330, cf. Sections 3 and 4) and prove

1.1. THEOREM. *Let G be H or He and let K be a maximal subgroup of G. Then*:

(i) *The permutation rank of G on K is strictly smaller than* 6 *if and only if K has index* 2058 *in G and* $[K, K]$ *is isomorphic to* $PSp(4, 4)$. (*In this case, the rank is* 4 *if* $G = H$ *and* 5 *if* $G = He$.)

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(ii) *The permutation rank of G on K lies between* 6 *and* 7 *if and only if K has index* 8330 *in G and* $K \cap He$ *is isomorphic to* $2^2 \cdot L(3,4) \cdot \mathrm{Sym}_3$. (*In this case, the rank is* 6 *if* $G = H$ *and* 7 *if* $G = He$.)

(iii) *The permutation character of G on K is multiplicity free if and only if K is one of the groups described in* (i) *and* (ii).

We heavily employ the information on H as given in the "Atlas" [3], notably its character table. Study of the graph on 8330 vertices (cf. Section 4) has led to the permutation character $1_a + 51_a + 51_b + 680_a + 1275_a + 1920_a + 4352_a$ for He on $K \cap He$ as in (ii), which differs from the character given in the "Atlas". (The character given there cannot be a permutation character as the value on an element of type 14A has value -1.) The paper was motivated by the quest for the following consequence of the above theorem and the results of Sections 3 and 4. We recall (cf. Biggs [1]) that a group G is said to act distance transitively on a graph if, for each distance i realized in the graph, G is transitive on the set of (ordered) pairs of vertices at distance i.

1.2. COROLLARY. *Let G be He or H. There is no graph on which G acts distance transitively.*

In order to prove parts (i) and (ii) of the theorem, we need some properties of $PSp(4,4)$; these are derived in Section 2. Part (iii) of the theorem requires analysis of several possible subgroups K; this is done in Section 5. The final Section 6 is devoted to the proof of the corollary.

2. LEMMAS ON $PSp(4,4)$

Held's simple group He has a subgroup isomorphic to $PSp(4,4){:}2$. In this section we describe some properties of the group $PSp(4,4)$ relevant to the construction of He in the next section.

Consider a nondegenerate symplectic geometry in $PG(3,4)$. Then the isotropic points and lines form a self-dual generalized quadrangle, denoted by GQ, of order $(4,4)$. The automorphism group S of GQ is the extension of $PSp(4,4)$ by a field automorphism τ of order 2. There is a duality between points and lines established by an outer automorphism δ of S of order 4 squaring to τ such that $S \cdot \langle \delta \rangle$ is transitive on the Levi graph of GQ (i.e., the graph whose vertex set consists of the points and lines of GQ and in which adjacency is incidence). Inside GQ, the following substructures can be found: points; lines; flags, i.e., incident point line pairs; subquadrangles of order $(4,1)$, called grids; subquadrangles of order $(1,4)$, called dual grids; subquadrangles of order $(2,2)$, called quads. We denote by P the set of points, by L the set of lines, by F the set of flags, by G the set of grids, by DG the set of dual grids, and by Q the set of quads in GQ. The first lemma collects some well-known facts.

2.1. LEMMA. *Let S_x be the stabilizer in S of x ($x \in P, L, F, G, DG$, or Q). Then S is transitive on each of the sets P, L, F, G, DG, and Q. Moreover, $|P| = 85$ and $S_p \cong 2^6{:}(3 \times A_5){:}2$ if $p \in P$; $|L| = 85$ and $S_l \cong 2^6{:}(3 \times A_5){:}2$ if $l \in L$; $|F| = 425$ and $S_f \cong (2^2 \times 2^{2+4}{:}3){:}6$ if $f \in F$; $|G| = 136$ and $S_g \cong (A_5 \times A_5){:}2^2$ if $g \in G$; $|DG| = 136$ and $S_d \cong (A_5 \times A_5){:}2^2$ if $d \in DG$; $|Q| = 1360$ and $S_q \cong S_6 \times 2$ if $q \in Q$.*

Let g be a grid. We shall investigate the orbits of S_g on F, G, DG, and Q. First we consider the action of S_g on G. We need the following definitions. Let D be a set of 5 pairwise noncollinear points of g. If there is a point $p \in P$ such that p is collinear with all points in D, then D is called a grid diagonal and otherwise it is called a quad diagonal.

2.2. LEMMA. *The subgroup S_g is transitive on the set of 60 grid diagonals and on the set of 60 quad diagonals of g.*

Proof. For any 3 pairwise noncollinear points of GQ, either there is a unique point of GQ colinear with all 3 or the 3 points are on a hyperbolic line of GQ. Clearly no 3 pairwise noncollinear points of a grid are on a hyperbolic line. As there are 60 points outside g, we find there are 60 grid diagonals in a grid. Thus there are also 60 quad diagonals. Finally, S_g acts transitively on both sets as it is transitive on the set of 3 pairwise non-collinear points in g. ∎

2.3. LEMMA. *There are precisely 3 S_g-orbits on G: $\{g\}$, $\{h \in G \mid h \cap g$ is a grid diagonal$\}$ (of size 60), and $\{h \in G \mid h \cap g$ is the union of 2 intersecting lines$\}$ (of size 75).*

Proof. Let g be a grid and p a point of GQ not in g. Then p is collinear with 5 points of g forming a grid diagonal of g. Let q_1 and q_2 be two points in $p^\perp \cap g$ and suppose q_3 and q_4 are the two points in $q_1^\perp \cap q_2^\perp$ different from p and not in g. Let g' be the grid spanned by q_1, q_2, q_3, and q_4. We show that $g \cap g' = p^\perp \cap g$. Let r be a point in $p^\perp \cap g \setminus \{q_1, q_2\}$. There is a unique line l through r that intersects the line through q_1 and q_3 and a unique line l' through r that intersects the line through q_1 and q_4. But since there are only 5 lines through a point, the line l also intersects the line through q_2 and q_4 and l' also intersects the line through q_2 and q_3. But that means $r \in g'$ and therefore $g \cap g' = p^\perp \cap g$. So for every grid diagonal of g there is a unique grid g' that intersects g in that grid diagonal, and, by the above lemma, S_g is transitive on the set of grids intersecting g in a grid diagonal and this orbit of S_g has size 60.

Now suppose l_1 and l_2 are two intersecting lines of g and let p_1, p_2 be two noncollinear points on l_1 and l_2, respectively. Then every point in $p_1^\perp \cap p_2^\perp$ that is not in g determines a unique grid intersecting g in l_1 and l_2. Hence there are $25 \cdot 3 = 75$ grids intersecting g in two intersecting lines. Since S has rank 3 on G (see the "Atlas" [3]) these 75 grids form an S_g-orbit. ∎

2.4. LEMMA. *The S_g-orbits on DG are $\{x \in DG \mid x \cap g$ is a quadrangle$\}$ (of size 100), and $\{x \in DG \mid x \cap g = \emptyset\}$ (of size 36).*

Proof. Let g be a grid and x a dual grid that intersects g nontrivially. Since x consists of two perpendicular hyperbolic lines of GQ, this intersection is a quadrangle. Conversely, every quadrangle is in a unique dual grid. Hence there are 100 dual grids that intersect g in a quadrangle and, as S_g is transitive on the set of quadrangles in g, these 100 dual grids form an S_g-orbit. The remaining 36 dual grids have empty intersection with g and, by consideration of rank (see the "Atlas" [3]), form also an S_g-orbit. ∎

2.5. LEMMA. *The S_g-orbits on F are $\{f \in F \mid f \subseteq g\}$ of size 50, $\{f \in F \mid f$ contains a point of g but no line of $g\}$ (of size 75) and $\{f \in F \mid f \cap g = \emptyset\}$ (of size 300).*

Proof. Clearly, S_g has two orbits of size 50 and 75 as stated. It remains to check that S_g is transitive on $\{f \in F \mid f \cap g = \emptyset\}$. Let $f = (p, l)$ where p is a point not in g and l a line through p. Then p determines the grid diagonal, $d = p^\perp \cap g$ of g, and $l \cap g$ is a point of d, say p'. Clearly, f is uniquely determined by the pair (p', d) and, since S_g is transitive on the set of all such pairs, it is also transitive on $\{f \in F \mid f \cap g = \emptyset\}$. ∎

2.6. LEMMA. *The S_g-orbits on Q are $\{q \in Q \mid q \cap g$ is a 3×3 grid$\}$ (of size 100), $\{q \in Q \mid q \cap g$ is a set of 5 points contained in two intersecting lines of $g\}$ (of size 900), $\{q \in Q \mid q \cap g$ is a point$\}$ (of size 300), and $\{q \in Q \mid q \cap g$ is a quad diagonal$\}$ (of size 60).*

Proof. Let γ be a 3×3 grid, then there is a unique grid that contains γ. A quad contains 3×3 grids, so there are quads that intersect a grid in a 3×3 grid. So fix a 3×3 grid γ in g. Since any 3 pairwise noncollinear points in a grid determine a unique point collinear with all three, there is at most one quad that intersects g in γ. But as S_g is transitive on the 100 3×3 grids contained in g, there is a unique quad that intersects g in γ, and S_g is transitive on the 100 quads intersecting g in a 3×3 grid.

Now let l_1 and l_2 be two intersecting lines of GQ and choose two points on l_1 and two on l_2 different from $l_1 \cap l_2$. This can be done in $6 \cdot 6 = 36$ ways and there are $85 \cdot 20/2 = 850$ ways to choose l_1, l_2. Since there are 1360 quads, there are $1360 \cdot 15 \cdot 3/850 \cdot 36 = 2$ quads that contain the above 4 points and $l_1 \cap l_2$. Now suppose l_1 and l_2 are in g. Then one of these two quads intersects g in a 3×3 grid containing the above 5 points, and the other quad intersects g in precisely this set of 5 points. Hence given such a set of 5 points in g, there is a unique quad that intersects g in those points. Since S_g transitively permutes the sets of 5 points chosen in the above way, it has an orbit $\{q \in Q \mid q \cap g$ is a set of 5 points contained in two intersecting lines of $g\}$ on Q of length $25 \cdot 6 \cdot 6 = 900$.

Suppose q is a quad and p_1, p_2 are two noncollinear points in q. Denote by p_3 and p_4 the two points in $p_1^\perp \cap p_2^\perp \backslash q$ and consider the grid spanned by p_1, p_2, p_3, and p_4. This grid intersects q in 5 points that form a quad diagonal of the grid. Furthermore every quad diagonal of a grid is in at most one quad. So there is an S_g-orbit of length 60 in Q, consisting of quads that intersect g in a quad diagonal.

Fix a quad q and a point p of q. There are two lines l_1, l_2 on p not in q. Furthermore, there are 4 grids on $l_1 \cup l_2$, two of which meet q in a quad diagonal. The other two meet q precisely in p. Counting pairs, we find $1360 \cdot 15 \cdot 2/136 = 300$ quads intersecting g in a point. As the two grids meeting q in precisely p are easily seen to be interchanged by a "field" automorphism (fixing q pointwise), we get that S_g is transitive on the above set of 300 quads. ∎

There are dual versions of the above lemmas for the dual grids.

We now concentrate on flags and quads. We say a line is in a quad if and only if at least two points of the line are in the quad. In this case of course precisely 3 points of the line are in the quad.

Fix a quad $q \in Q$.

2.7. Lemma. *The subgroup S_q has the following orbits on F: the 45 flags in q; the 30 flags that intersect q in a point; the 30 flags that intersect q in a line; the 120 flags that have an empty intersection with q and such that the point of the flag is on a line of q; the 120 flags that have an empty intersection with q and such that the line of the flag contains a point of q; the remaining 80 flags (these are disjoint from q).*

Proof. Straightforward. ∎

2.8. Lemma. *There are 120 quads that intersect q in a $K_{2,3}$ (a complete bipartite graph with parts of sizes 2 and 3), 120 in the dual (a 2×3 grid), 15 in $p^\perp \cap q$ for some $p \in q$, 15 in the dual, 360 in a $K_{2,1}$, 360 in the dual, 45 in* *the union of three points on a line and three lines on a point*, 180 *in a flag, and* 144 *are disjoint from* q.

Proof. First observe that for every $K_{3,3}$ there is a unique quad containing it. Now suppose q' is a quad such that $q \cap q'$ contains a quadrangle ζ. This quadrangle is contained in a unique dual grid, h say. There are 4 quads q'' on ζ that intersect q in 5 points of h and 4 that intersect q in 4 points of h. But then $q \cap q''$ is a $K_{2,3}$ or the dual, and there are 120 possibilities for q'' in both cases.

Let p be a point of q, then there is a unique quad q' (distinct from q) that intersects q in $p^\perp \cap q$. So there are 15 quads that intersect q in $p^\perp \cap q$ for some p and 15 that intersect q in the dual. Let g be a grid intersecting q in 5 points that form a quad diagonal, D say, and suppose q' is another quad that intersects g in a quad diagonal that is disjoint from D. Then $q \cap q' = \emptyset$. Suppose g' is another grid that intersects q in a quad diagonal. Then clearly these two grids intersect in two intersecting lines. So $g' \cap q'$ is not a quad diagonal. Note that q intersects 6 grids in a quad diagonal and, given a fixed quad diagonal in a grid, there are 24 quad diagonals disjoint from that fixed one. Therefore, there are at least $144 = 6 \cdot 24$ quads q' that are disjoint from q.

Let p be a point of q, and q_1, q_2 two noncollinear points in $p^\perp \cap q$. Then there is a unique dual grid that contains p, q_1, and q_2. In this dual grid we can choose a $K_{3,3}$ that intersects q in p, q_1, and q_2 in two different ways. Each of these $K_{3,3}$'s is contained in a unique quad that intersects q in p, q_1, and q_2. Hence there are $15 \cdot 6 \cdot 4 \cdot 2/2 = 360$ quads that intersect q in a $K_{1,2}$ and also 360 that intersect q in the dual.

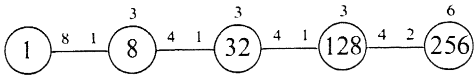
Now suppose p_0 is a point in q and fix a line through p_0 in q. Let p_1, p_2, p_3, and p_4 be the points of the two other lines of q on p_0 that are not in q. There are two quads containing $p_0, p_1, \ldots, p_4$; one intersects q in the dual of $p^\perp \cap q$ for some p and the other in the union of 3 points on a line and 3 lines on a point. So there are $15 \cdot 3$ quads that intersect q in the latter union.

Finally, fix a point p and a line l in q containing p. Then there are two lines l_1 and l_2 on p that are not in q. If we fix a point on one of these lines, then there are two quads that contain that point, p, the points of l not in q, and l_1, l_2. These quads intersect q in p and l, so there are $15 \cdot 3 \cdot 8 \cdot 2/4 = 180$ quads intersecting q in a flag.

Since there are 1360 quads, we have analyzed all situations. ∎

To end this section, we recall—without proof—some facts about the flag graph associated with GQ.

2.9. LEMMA. *The distribution diagram of the flag graph of* GQ, *i.e., the* *graph whose vertex set is F and in which two flags are adjacent if and only if they have a point or a line in common, is*



Moreover, $S \cdot \langle \delta \rangle$ *acts distance transitively on this graph.*

3. The Graph Γ on 2058 Vertices and a Construction of Aut He

Let Γ be the graph with vertex set $\{\infty\} \cup G \cup DG \cup Q \cup F$ and the minimal symmetric adjacency relation satisfying the following conditions:

(i) ∞ is adjacent to all grids and dual grids;

(ii) if $g \in G$, then g is adjacent to all dual grids that have an empty intersection with g, to all flags that intersect g in a point, and to all quads that intersect g in a quad diagonal or a 3×3 grid;

(iii) if $d \in DG$, then d is adjacent to all flags that intersect d in a line (i.e., the line of the flag contains two points of it, but the point of the flag is not in d), and all quads that intersect d in 5 pairwise disjoint lines or in 6 points (a $K_{3,3}$);

(iv) if $q \in Q$, then q is adjacent to all quads that intersect q in a flag, and to all flags that intersect q in a point or a line;

(v) two flags are adjacent if and only if they are at distance two in the flag graph of GQ.

From the above lemmas it follows that Γ has distribution diagram (seen from the point ∞) given in Fig. 1, and that the stabilizer in Aut Γ of ∞ is isomorphic to $Sp(4, 4): 4$.

Let g be a grid, then with the help of the above lemmas it is

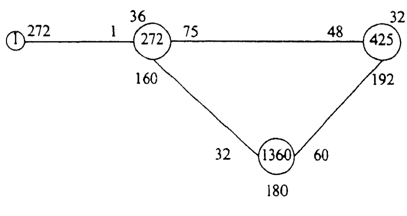


Figure 1

straightforward to check that the S_g-orbits on the vertex set of Δ are the following 13 sets:

Γ_0: g;

Γ_1: the 36 dual grids that are disjoint from g;

Γ_2: the 60 grids that intersect g in a grid diagonal;

Γ_3: the 60 quads that intersect g in a quad diagonal;

Γ_4: the 100 dual grids that intersect g in a 2×2 grid;

Γ_5: the 100 quads that intersect g in a 3×3 grid;

Γ_6: the 75 grids that intersect g in two intersecting lines;

Γ_7: the 75 flags that have their point in g but not their line;

Γ_8: the 50 flags inside g;

Γ_9: the 300 flags outside g;

Γ_{10}: the 300 quads that intersect g in a point;

Γ_{11}: the 900 quads that intersect g in the set of all points in a quad on two intersecting lines;

Γ_∞: ∞.

Even more detailed information on Γ is provided by the distribution matrix in Table I whose ij-entry (where $i, j = 0, 1, \ldots, 11, \infty$) contains the number of elements of Γ_i that are adjacent to a fixed element of Γ_j. Again, proofs are quite straightforward from the above.

In a diagram, this information can be displayed as shown in Fig. 2 (for reasons of legibility, the numbers that can be determined from symmetry are not always given).

TABLE I

	0	1	2	3	4	5	6	7	8	9	10	11	∞
0	—	1	—	1	—	1	—	1	—	—	—	—	1
1	36	—	6	6	—	—	12	12	—	6	6	4	36
2	—	10	—	10	18	6	—	4	12	12	10	6	60
3	60	10	10	—	6	18	4	—	12	10	12	6	—
4	—	—	30	10	—	16	24	12	24	18	10	12	100
5	100	—	10	30	16	—	12	24	24	10	18	12	—
6	—	25	—	5	18	9	—	19	12	12	6	10	75
7	75	25	5	—	9	18	19	—	12	6	12	10	—
8	—	—	10	10	12	12	8	8	8	2	2	8	—
9	—	50	60	50	54	30	48	24	12	24	46	42	—
10	—	50	50	60	30	54	24	48	12	46	24	42	—
11	—	100	90	90	108	108	120	120	144	126	126	120	—
∞	1	1	1	—	1	—	1	—	—	—	—	—	—

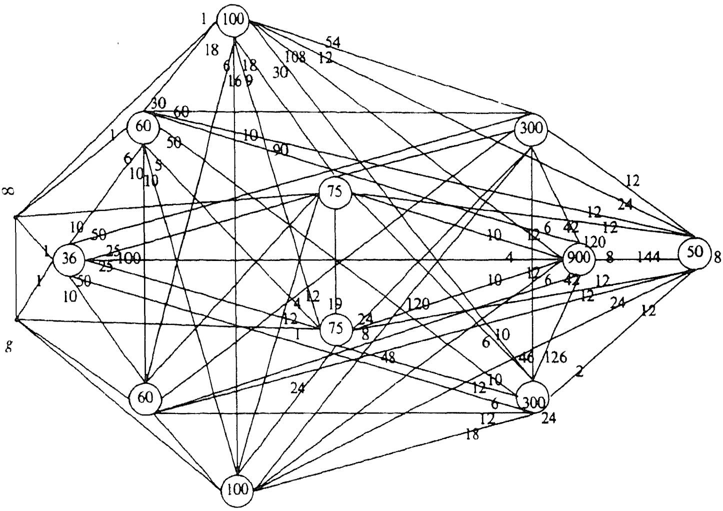


FIGURE 2

3.1. THEOREM. *The group* Aut Γ *is isomorphic to* $He \cdot 2$ *and acts flag transitively on* Γ. *Moreover, the distribution diagram of* Γ *is as given in Fig.* 1.

Proof. The bulk of the proof consists of establishing the existence of a symmetry of the graph which interchanges ∞ and a grid. Fix a grid $g \in \Gamma$, and adopt the above notation $\Gamma_0, \Gamma_1, ..., \Gamma_{11}, \Gamma_\infty$ with respect to g. The elements of the sets $\Gamma_1, ..., \Gamma_{11}$ will be identified with certain subconfigurations of g as follows.

Those of $\Gamma_2, \Gamma_3, \Gamma_4, \Gamma_5, \Gamma_8$, and Γ_{11} are identified with their intersections with g. Suppose $f = (p, l) \in \Gamma_9$, then we identify f with the point, grid diagonal pair $(l \cap g, p^\perp \cap g)$ in g. Let $q \in \Gamma_{10}$, and denote by p the unique point in $q \cap g$. As we saw before, there is a unique quad $q' \in \Gamma_3$ that intersects q in $p^\perp \cap q$. Now q will be identified with the point, quad diagonal pair $(q \cap g, q' \cap g)$.

Let $f = (p, l) \in \Gamma_7$. We identify f with the 4-tuple of grid diagonals $\{(p')^\perp \cap g \mid p' \in l \setminus \{p\}\}$ in g; such a 4-tuple will be called a grid quartet.

Suppose $h \in \Gamma_6$ and let p be the point of intersection of the two lines in $h \cap g$. Let p' be a point of $h \setminus h \cap g$. Then there is a unique point $p'' \in g$ that is on the hyperbolic line $\{p, p'\}^{\perp\perp}$, and there are three points, p_1, p_2, and p_3 in $((p')^\perp \cap g) \setminus (g \cap h)$. The points p, p'', p_1, p_2, and p_3 form a quad diagonal. The 16 points of $h \setminus g$ determine 4 quad diagonals in the above way, and we can identify h with the 4-tuple of these quad diagonals. Such a 4-tuple will be called a quad quartet.

Finally we have to identify the elements of Γ_1 with subconfigurations of g. This will be done in two ways, one in terms of grid diagonals of g and the other in terms of quad diagonals of g. Let d be a dual grid that has empty intersection with g. Then every point p of d determines the grid diagonal $p^\perp \cap g$ in g, and the 5 points of a hyperbolic line in d determine a partition of the point set of g by 5 grid diagonals. Such a partition will be called a grid partition corresponding to the two hyperbolic lines of d. Let C_1, C_2 be the grid partitions coming from d.

For all $d_1 \in C_1$ and $d_2 \in C_2$ we have $|d_1 \cap d_2| = 1$. (In fact, the elements of C_2 are the only grid diagonals that intersect all 5 elements of C_1 in a point. This follows from the observation that the 25 points and 60 grid diagonals of g form the point set of a generalized quadrangle isomorphic to GQ if adjacency is defined in the following way:

two points are adjacent iff they are collinear in g; a point and a grid diagonal are adjacent iff the point is in the diagonal;

two diagonals are adjacent iff they intersect in a unique point.

So C_1 and C_2 are two perpendicular hyperbolic lines in this generalized quadrangle. The same observation also holds for quad diagonals.) The pair (C_1, C_2) is called a double grid partition of g. Suppose C is a set of 5 quad diagonals partitioning the 25 points of g. Then the corresponding 5 quads have empty intersections. So there are $60 - 5 \cdot (15 - 5) = 10$ points that are in neither of these quads. Clearly these 10 points form a dual grid, d say, that does not intersect g. Let p be one of these 10 points and let l be a line through p. Exactly one of the 5 quads intersects l in 3 points which are not in d. Call this quad q. Now by Lemma 2.8 there is a unique quad q' that intersects q in these 3 points. Then $q' \cap g$ is a diagonal that intersects all diagonals of C in one point. In this way we obtain a set of 5 diagonals such that q is not one of the quads corresponding to these diagonals. By the same observation as above these 5 quad diagonals form the unique partition of the points of g with the property that every diagonal intersects all elements of C nontrivially. A partition by quad diagonals of g will be called a quad partition of g, and a pair of quad partitions (C_1, C_2) with the property that for all $d_1 \in C_1$ and $d_2 \in C_2$, $|d_1 \cap d_2| = 1$, will be called a double quad partition. Thus we can identify the elements of Γ with the double quad partitions of g. A double grid partition (C_1, C_2) of g and a double quad partition (C'_1, C'_2) of g determine the same dual grid in Γ if and only if, for all $d \in C_1 \cup C_2$ and $d' \in C'_1 \cup C'_2$, $|d \cap d'| = 1$.

It is possible to describe the adjacency relation between the vertices of the graph in terms of these subsets of g.

Let $\sigma = (ij) \times Id \in \mathrm{Sym}_5 \times \mathrm{Sym}_5$ in $\mathrm{Aut}(g)$ $(1 \leqslant i < j \leqslant 5)$. We define an involutory permutation $\bar{\sigma}$ on the vertex set of Γ:

on $\Gamma_0 \cup \Gamma_\infty : \infty^{\bar{\sigma}} = g$ and $g^{\bar{\sigma}} = \infty$;

on $\Gamma_2 \cup \Gamma_3$: $d^{\bar{\sigma}} = d^{\sigma}$ for all diagonals d of g;

on Γ_1: $(C_1, C_2)^{\bar{\sigma}} = (C_1^{\sigma}, C_2^{\sigma})$ for all double grid and quad partitions (C_1, C_2) of g;

on $\Gamma_9 \cup \Gamma_{10}$: $(p, d)^{\bar{\sigma}} = (p^{\sigma}, d^{\sigma})$ for all point, grid diagonal and point, quad diagonal pairs (p, d) of g;

on $\Gamma_6 \cup \Gamma_7$: $(d_1, ..., d_4)^{\bar{\sigma}} = (d_1^{\sigma}, ..., d_4^{\sigma})$ for all grid and quad diagonal quartets $(d_1, ..., d_4)$ of g;

on Γ_8: $(p, l)^{\bar{\sigma}} = (p^{\sigma}, (l^{\sigma})')$ for all flags (p, l), where $(l^{\sigma})'$ is the line through p^{σ} distinct from l^{σ};

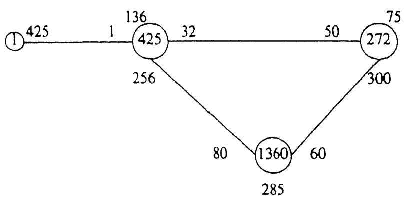
on Γ_{11}: $(\{p, a_1, a_2, b_1, b_2\})^{\bar{\sigma}} = \{p, a_3, a_4, b_3, b_4\}^{\sigma}$ if $\{p, a_1, a_2, a_3, a_4\}$ and $\{p, b_1, b_2, b_3, b_4\}$ are intersecting lines of g;

on $\Gamma_4 \cup \Gamma_5$: $(s)^{\bar{\sigma}} = (s^{\sigma})'$ for any $m \times m$ grid s of g, where $m = 2$ or 3 and $(s^{\sigma})'$ is the complement of the span of s^{σ} (a $(5-m) \times (5-m)$ grid).

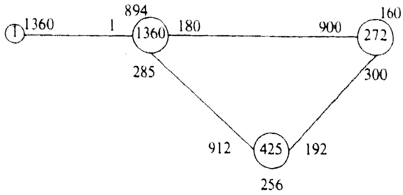
Using the above description of the graph Γ it is straightforward (the arguments involved being similar to those of Section 2), but tedious, to check that $\bar{\sigma} \in \mathrm{Aut}\, \Gamma$. So far, we have established that Aut Γ contains $S \cdot \langle \delta \rangle$ and $\bar{\sigma}$. It follows that Aut Γ is transitive on the points of Γ and that the vertex stabilizer is isomorphic to $PSp(4, 4){:}4$.

We shall now derive that Aut Γ is isomorphic to the group H. First of all, Aut Γ is transitive on the vertex set of Γ, so, by the fact that its vertex stabilizer is $S \cdot \langle \delta \rangle$, its order is $|H|$. Let N be a nontrivial minimal normal subgroup of Aut Γ. As the latter is primitive on the vertex set of Γ, the group N is transitive on the vertex set as well. By standard arguments, N cannot be regular, and so it contains the normal subgroup of any vertex stabilizer isomorphic to $PSp(4, 4)$. In particular, N has index at most 4 in Aut Γ and N is simple. Since $\bar{\sigma}$ is an odd permutation of the vertices of Γ, we cannot have $N = \mathrm{Aut}\, \Gamma$. Now by the classification of finite simple groups, there is no simple group of order $|He|/2$, and a unique one of order $|He|$, whence N is isomorphic to He, has index 2 in Aut Γ, and Aut $\Gamma \cong H$. This proves the theorem. ∎

3.2. *Remarks.* (i) If adjacency is defined with respect to the 425-orbit, the distribution diagram of the resulting graph is



and with respect to the 1360-orbit it is

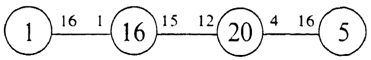


We shall refer to these graphs by Γ_{425} and Γ_{1360}, respectively.

(ii) The permutation representation on the vertex set of Γ has character $1_a + 51_a + 51_b + 680_a + 1275_a$.

4. Description of the Graph on 8330 Vertices

Let Γ be the graph on 2058 vertices of valency 272 of the previous section. Thus $H = \text{Aut}\,\Gamma \cong \text{Aut}\,He$, and $He = [H, H]$ has index 2 in H. Fix a vertex ∞ of Γ. We retain the notation of Section 2. Take a point p of GQ. The subgraph $\Pi(p)$ of Γ induced on the set consisting of ∞, the 16 dual grids on p, and the 25 flags $\{q, l\}$ with $q = p$ or $p \in l$ has distribution diagram



.

We define a double plane to be a graph whose vertex set is the union of the point and the line set of the projective plane of order 4 and whose adjacency is the union of collinearity, incidence, and intersecting. Thus it has 42 vertices, 2 maximal cliques of size 21, and 42 maximal cliques of size 6. The above subgraph $\Pi(p)$ of Γ_{425} is a double plane.

There are $85 = |P|$ double planes of shape $\Pi(p)$ ($p \in P$) on ∞, and, similarly, there are 85 double planes of shape $\Pi(l)$ on ∞ constructed "dually" with a line l of GQ instead of a point. A look at the construction of Γ shows that there are no other double planes in Γ containing ∞.

4.1. Lemma.

(i) *The group H is transitive on each of the sets M_6 and M_{21} of maximal Γ_{425}-cliques of sizes* 6 *and* 21, *respectively*;

(ii) *each member of M_{21} lies is a unique double plane in Γ_{425}; in particular, there are* 8330 *double planes and H permutes them transitively.*

The proof, being straightforward from transitivity of H on Γ, is left to the reader.

We recall that a hyperoval in the projective plane of order 4 is a set of 6 points with the property that every line meets the set in 0 or 2 points. Let Π be a double plane, with point set R_1 and line set R_2. A double hyperoval is defined to be the union of a hyperoval K_1 in R_1 and the set K_2 of 6 lines not containing a point of R_1. (Observe, that K_2 is a hyperoval in the dual plane so that the choice of R_1 as point set does not lead to ambiguity.) A double hyperoval in Γ_{425} will be understood to be a double hyperoval in a double plane occurring in Γ_{425}.

4.2. LEMMA.

(i) *Each double hyperoval is contained in a unique double plane; furthermore, the group H is transitive on the set of double hyperovals*;

(ii) *the map* $K \mapsto \{\gamma, \delta\}$, *where* γ *and* δ *are the two vertices* Γ*-adjacent to all members of K, establishes a bijective correspondence between the double hyperovals K of* Γ *and the unordered* Γ_{1360}*-adjacent pairs* γ, δ;

(iii) *for each* $\gamma \in \Gamma$, *the assignment* $\delta \mapsto \Pi(\gamma, \delta)$, *where* $\Pi(\gamma, \delta)$ *is the unique double plane containing the double hyperoval corresponding to* $\{\gamma, \delta\}$ *defines an injective map on* $\Gamma_{1360}(\gamma)$.

Proof. (i) The first statement follows by consideration of the graph from ∞; the second from the action of the stabilizer in H of a double plane Π (isomorphic to $2^2 \cdot L(3, 4) \cdot Dih_{12}$).

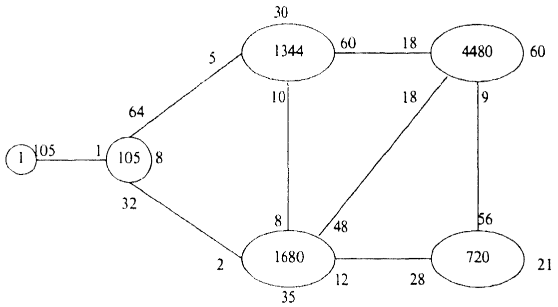
(ii) Let K be a double hyperoval in Γ_{425}. By (i), we may take K to be the union of 6 grids pairwise intersecting in two intersecting lines and 6 dual grids (uniquely determined by the grids). The 15 points of intersection of the 15 pairs of intersecting lines thus obtained are the point set of a member $\delta \in Q$. From the definition of Γ, it is clear that $\gamma = \infty$ and δ are only two vertices of Γ adjacent to every vertex of K.

Conversely, let $\gamma, \delta \in \Gamma$ satisfy $\delta \in \Gamma_{1360}(\gamma)$. Taking (without loss of generality) $\gamma = \infty$ and $\delta \in Q$, we see that the subgraph of Γ induced on the 32 common neighbors of these two vertices has precisely two 6-cliques, say K_1 and K_2. Setting $K = K_1 \cup K_2$, we obtain the required correspondence.

(iii) Without loss of generality, we take $\gamma = \infty$. Then, again, from the definition of Γ, it is clear that, by the correspondence of (ii), distinct double hyperovals in $G \cup DG$ lead to distinct quads δ. ∎

We denote by Δ the graph induced on the set of all double planes by letting two members Π_1 and Π_2 be adjacent if and only if they have precisely 10 vertices of Γ in common.

4.3. THEOREM. *The graph Δ has* 8330 *vertices. The permutation character of He on its vertex set is* $1_a + 51_a + 51_b + 680_a + 1275_a + 1920_a + 4352_a$. *The distance distribution diagram of Δ with respect to a vertex is*:



Proof. First, we determine the He_∞-orbits on the vertex set of Δ. The sets $\{\Pi(x) \mid x \text{ a line or point of } GQ\}$ and $\{\Pi(\infty, \delta) \mid \delta \in \Gamma_{1360}(\infty)\}$ are H_∞-orbits of respective sizes 170 and 1360 (cf. Lemma 4.2). The first one splits into two He_∞-orbits, the latter is a single He_∞-orbit.

Fixing a grid g and a point p of g, the 21-set consisting of g, the 8 flags in g with point collinear to but distinct from p and line not containing p, and the 12 quads on p intersecting g in the singleton $\{p\}$, form a maximal 21-clique in Γ_{425}, which in turn determines a unique double plane (cf. Lemma 4.1). Varying (g, p) over all incident grid, point pairs, we obtain an He_∞-orbit of 3400 double planes $X(g, p)$, each consisting of 4 grids (g is one of them), 10 flags, and 28 quads. Dualizing yields another He_∞-orbit of 3400 double planes $X(d, l)$ determined by a dual grid d and a line l in d (i.e., having two points in d), containing 4 dual grids and the same number of flags and quads. These two fuse under H_∞ to the third and (by counting) final H_∞-orbit, of size 6800.

Fix a line l of GQ. Exploiting the viewpoint from ∞, the structure of Δ on the vertices Π with $\Pi \cap \Pi(l)$ is readily determined. As a consequence, we get that each $\Pi \in \Delta$ is as described in one of the lines in Table II (where m is a line in the dual grid d, p is a point of the grid g, and q is a quad). Also, it is not hard to derive that each line represents a single $He_{\infty, \Pi(l)}$-orbit.

Let $\Pi \in \Delta$. From the Table II, there are 105 vertices adjacent to Π, 1344 meeting Π in an edge of Γ, and 1680 meeting Π in an edge of Γ_{425}. Using transitivity of H_Π on Π, we find three H_Π-orbits of the indicated sizes.

Next, we determine the permutation character π of He on the vertex set Δ. By the above, the rank of the permutation representation lies between 5 and 9. Using that He_∞ and H_∞ have 5, 3 orbits on the vertex set of Δ, respectively, we see that the low degree characters 51_a and 51_b occur with

TABLE II

Π	$\Pi \cap \Pi(l)$	Restriction	Size
$\Pi(l)$	Π		1
$\Pi(m)$	10 Vertices	m meets l	20
$\Pi(m)$	Edge	m disjoint from l	64
$\Pi(p)$	10 Vertices	$p \in l$	5
$\Pi(p)$	Γ_{425}-edge	$p \in P \setminus l$	80
$\Pi(x, q)$	$\varnothing$	$\lvert l \cap q \rvert = 3$	240
$\Pi(x, q)$	Γ_{425}-edge	$\lvert l \cap q \rvert = 1$	480
$\Pi(x, q)$	$\varnothing$	$\lvert l \cap q \rvert = 0$	640
$X(g, p)$	10 Vertices	l a line of g, $p \in l$	80
$X(g, p)$	Γ_{425}-edge	l a line of g, $p \notin l$	320
$X(g, p)$	Γ_{425}-edge	l a line not in g, $\{p\} = g \cap l$	120
$X(g, p)$	Edge	l a line not in g, $p \perp (l \cap g)$	960
$X(g, p)$	$\varnothing$	l a line not in g, $p \not\perp (l \cap g)$	1920
$X(d, m)$	Γ_{425}-edge	l a line of d, $m = l$	40
$X(d, m)$	Edge	l a line of d, m meets l	320
$X(d, m)$	Γ_{425}-edge	l a line of d, m disjoint from l	640
$X(d, m)$	$\varnothing$	l a line not in d, m meets l	480
$X(d, m)$	$\varnothing$	l a line not in d, m disjoint from l	1920

small multiplicities. It is then straightforward from the character table in the "Atlas" [3] that π is the character as described in the statement of the theorem. In particular, H_Π has 6 orbits on the vertex set of Δ. Thus, the set of 5200 vertices Π' with $\Pi \cap \Pi' = \varnothing$ partitions into two orbits. By Lagrange and Table II, there is only one possibility for the two orbits sizes, namely $720 = 240 + 480$ and $4480 = 640 + 1920 + 1920$. By Δ_x, for $x \in \{1344, 1680, 4480, 720\}$, we denote the graph structure on the vertex set of Δ induced by the suborbit of size x.

We next exhibit an edge between a vertex in $\Delta_{4480}(\Pi)$ and $\Delta_{720}(\Pi)$. Let d be a dual grid. Take two intersecting lines l_1 and l_2 in d. Denote by l a line of GQ not in d and meeting l_1 in a point (outside d). Now $X(d, l_1) \in \Delta_{720}(\Pi(l))$ and $X(d, l_2) \in \Delta_{4480}(\Pi(l))$; moreover the 3 flags (p, m) with p a point of d and m a line of GQ incident with p and the point of $l_1 \cap l_2$, are vertices of Γ in $X(d, l_i)$ for both i, whence there is an edge from $X(d, l_1)$ to $X(d, l_2)$.

Taking suitable representatives of vertices in the orbits of sizes 1344 and 1680, one deduces the remaining numbers given in the distribution diagram. ∎

5. THE REMAINING PERMUTATION REPRESENTATIONS OF *He* AND *H* ARE NOT MULTIPLICITY FREE

In order to establish Theorem 1.1 it remains to show that permutation representations distinct from those in Sections 3 and 4 are not multiplicity free. The lemma below bounds the index of the subgroups to be taken into consideration.

5.1. LEMMA. *Let $G = He$ or H be multiplicity free on a subgroup K. Then $|G/K| \leqslant 268{,}618$ if $G = He$, and $|G/K| \leqslant 478{,}976$ if $G = H$.*

Proof. Let π be the permutation character of G on K, and denote by $\mathrm{Irr}(G)$ the set of irreducible characters of G. Then, as π is multiplicity free, $|G/K| = \pi(1) \leqslant \sum_{\chi \in \mathrm{Irr}(G)} \chi(1)$, and the lemma follows from a direct computation (using the character table) of the latter sum. ∎

As the degree of an irreducible character of G is at most 23,324, we conclude from knowledge of maximal subgroups (cf. the "Atlas" [3])

5.2. COROLLARY. *Let X be a set of size v, and let G, with $G = He$ or H, act primitively on X. If this action is multiplicity free or of rank at most 7, then G_γ, for $\gamma \in X$, is one of the subgroups of G indicated in Table* III *(up to conjugacy in G, cf. the "Atlas"* [3]*).*

The third and sixth column of the table contain references to sections where the proof for G_γ of the indicated isomorphism type can be found.

5.3. LEMMA (maximal 6-cliques in Γ_{425} with stabilizer $2^6{:}3 \cdot \mathrm{Sym}_6$). *The permutation character of He on a subgroup isomorphic to $2^6{:}3 \cdot \mathrm{Sym}_6$ is not multiplicity free and of rank at least 8.*

TABLE III

$G \cong He$			$G \cong H$		
G_γ	v	Section	G_γ	v	Section
$PSp(4,4){:}2$	2,058	3	$PSp(4,4){:}4$	2,058	3
$2^2 \cdot L(3,4) \cdot \mathrm{Sym}\ 3$	8,330	4	$2^2 \cdot L(3,4) \cdot Dih_{12}$	8,330	4
$2^6{:}3 \cdot \mathrm{Sym}_6$	29,155	5.3	$\mathrm{Sym}_5 \times \mathrm{Sym}_5{:}2$	279,888	5.4
$2^6{:}3 \cdot \mathrm{Sym}_6$	29,155	5.3	$2^{4+4} \cdot \mathrm{Sym}_3 \times \mathrm{Sym}_3 \cdot 2$	437,325	5.4
$2_+^{1+6} \cdot L(3,2)$	187,425	5.5	$2_+^{1+6} \cdot L(3,2) \cdot 2$	187,425	5.6
$7^2{:}SL(2,7)$	244,800	5.6	$7^2{:}SL(2,7) \cdot 2$	244,800	5.6
$3 \cdot \mathrm{Sym}_7$	266,560	5.5	$3 \cdot \mathrm{Sym}_7 \times 2$	266,560	5.5

Proof. *He* contains two classes of maximal subgroups of index 29,155 interchanged by an outer automorphism of *He*. Any such maximal subgroup K can be obtained as the stabilizer of a maximal 6-clique in Γ_{425}. We exhibit 6 maximal 6-cliques in Γ_{425} which are from distinct H_∞-orbits: 5 flags on a common line and ∞ (of size 85); a dual grid d and 5 flags in d whose lines are on a common point p and whose points differ from p (of size 1360); 2 flags (p, l), (p, m) and 4 grids any two of which intersect each other in $l \cup n$ (of size 850); a grid g and 5 quads intersecting g in a point such that the 5 points so obtained form the quad diagonal corresponding to each quad (of size 8160); 2 grids intersecting each other in the union of two lines l, m and 4 quads intersecting the grids only in $l \cap n$ such that in either grid their quad diagonals cover the set of 16 points of the grids not on l or n (of size 5100); 2 flags (p, l), (p', m) $(p \neq p', l \cap n = r, r \neq p, p')$ and 4 quads having the following property:

> for each grid g containing the two flags let r' be the unique other point collinear with p and p'. Then three of the quads intersect g in r'; their quad diagonals contain r and r' and the union of the quad diagonals form, after deletion of r and r', a 3×3 grid which is the intersection of g with the remaining quad (of size 13,600).

Thus, there are at least 6 He_∞-orbits of maximal 6-cliques in Γ_{425}. Moreover, up to substitution of the representatives just given by their duals, we may take them to be in the same *He*-orbit. From Frobenius reciprocity it follows that the permutation character of *He* on K is not multiplicity free and that its permutation rank is at least 8. ∎

5.4. Lemma (Novelties). *If* $K \cong \mathrm{Sym}_5 \times \mathrm{Sym}_5 : 2$ *or* $2^{4+4} \cdot \mathrm{Sym}_3 \times \mathrm{Sym}_3 \cdot 2$, *then the permutation character of* H *on* K *is not multiplicity free and the permutation rank is at least* 10.

Proof. Recall that Γ (resp. Γ_{425}) is the graph on 2058 vertices with valency 272 (resp. 425). This graph has 279,888 (resp. 437,325) edges, and K is the stabilizer in H of one of them. (In the first case cf. Lemma 2.1; in the second case by the results of Section 4 every edge is on precisely two maximal 6-cliques, one of each *He*-orbit, so $K \cap He$ is contained in a subgroup $2^6 : 3 \cdot \mathrm{Sym}_6$ with index 15). Fixing a vertex $\gamma \in \Gamma$, we see from the distance distribution diagram in Section 3 that H_γ has at least 7 orbits on the edge set. By Frobenius reciprocity, this implies that the inner product of the permutation characters of H on K and on $S \cdot \langle \delta \rangle$ is at least 7. Since the character of H on $S \cdot \langle \delta \rangle$ has 4 irreducible constituents (cf. Remark 3.1(ii)), it follows that the permutation character on K has a multiplicity > 1 and that the permutation rank is at least 10. ∎

5.5. LEMMA.

(i) *The permutation character of He on a subgroup K isomorphic to* $2_+^{1+6} \cdot L(3, 2)$ *is not multiplicity free*;

(ii) *the permutation character of H on a subgroup K isomorphic to* $3 \cdot \mathrm{Sym}_7 \times 2$ *is not multiplicity free. Consequently, neither is the permutation character of He on a subgroup isomorphic to* $3 \cdot \mathrm{Sym}_7$. *In this case, the permutation rank is at least* 14.

Proof. (i) Since K is the centralizer of an involution, say b of type $2B$, the set He/K can be identified with the class B of involutions of this type, with He-action given by conjugation. Now b fixes a member x of B if and only if xb has order at most 2. A standard computation using the character table shows that there are 84 elements $x \in B$ with xb an involution of type $2A$ and 364 with $xb \in B$. Thus, b has $1+84+364=449$ fixed points in Γ. As $\sum_{\chi(b)>0} \chi(b)=315$, an irreducible character with $\chi(b)>0$ must occur with multiplicity >1.

(ii) As in (i), we can identity the H-set H/K with the class, C say, of involutions of type $2C$. For $c \in C$, there are 168 $x \in C$ with xc an involution of type $2A$ and 315 with xc an involution of type $2B$, whence b has $1+168+315=484$ fixed points on Γ. As $\sum_{\chi(b)>0} \chi(b)=442$, we again encounter an irreducible character with multiplicity >1. It follows immediately that the restriction to He is also not multiplicity free. The statement about the permutation rank is a consequence of the observation that the quotient of the index by the maximum degree of an irreducible character is strictly bigger than 11. ∎

5.6. *Computation of two permutation characters.* For the two classes of maximal subgroups below, we have no better way of proving the main theorem than to explicitly determine the entire permutation character as a sum of irreducible characters.

5.6.1. LEMMA. *Denote by K a maximal subgroup of H.*

(i) *If K is isomorphic to* $7^2 : SL(2, 7) \cdot 2$, *then the permutation character of He on* $K \cap He$ *is*

$$\begin{aligned}\pi|_{K\cap He} = {} & 1_a + 153_a + 153_b + 1920_a + 4080_a + 4352_a + 2\times 6272_a \\ & + 6528_a + 2\times 7650_a + 2\times 10{,}880_a + 11{,}475_a + 11{,}475_b + 2\times 13{,}720_a \\ & + 2\times 14{,}400_a + 2\times 17{,}493_a + 20{,}825_a + 21{,}504_a + 21{,}504_b.\end{aligned}$$

(ii) *If K is isomorphic to* $2_+^{1+6} \cdot L(3, 2) \cdot 2$, *then the permutation character of He on* $K \cap He$ *is*

$$\pi|_{K\cap He} = 1_a + 680_a + 2\times 1275_a + 1920_a + 4080_a + 4352_a + 6528_a + 2\times 7650_a + 10{,}880_a + 3\times 11{,}900_a + 2\times 13{,}720_a + 2\times 17{,}493_a + 21{,}504_a + 21{,}504_b.$$

Proof. (i) The conjugacy classes of $K\cap He \cong 7^2{:}SL(2,7)$ are as indicated in Table IV, where ux denotes a representative with $u\in O_7(K\cap He)$ and x an element of a fixed complement isomorphic to $SL(2,7)$. Here $\{\{Y, Y^*\}, \{X, X^*\}, \{Z, Z^*\}\} = \{\{A, B\}, \{C\}, \{D, E\}\}$, and z is the nontrivial central element, c, d are nonconjugate transvections, a is a diagonal element of order 6, and b is an element of order 8 in $SL(2,7)$. Now $(\pi|_{K\cap He}, \chi)$, for every irreducible χ of He, can be computed as $(1_{K\cap He}, \chi|_{K\cap He})$ due to Frobenius reciprocity, yielding the assertion of the lemma.

(ii) Let x_T, for T the type of an element in He, denote the size of the class to which it belongs. From Wilson [7] we deduce that $x_{2A} = 126$ and $x_{2B} = 449$. By the fixed point free action of an element of order 7 in K on $O_2(K)/Z(K)\cong 2^6$ (on which $K/O_2(K)$ acts as the sum of the natural $L(3,2)$-module and its contragredient), there are precisely two classes of elements of order 7 (and they are algebraically conjugate), which must be of type $7D$ and $7E$ in He; both have size 1536. Therefore, $x_{7D} = x_{7E} = 1536$, and from this it follows directly that $x_{14C} = x_{15D} = 1536$ as well, whereas $x_T = 0$ for all other type of elements of orders 7 and 14. Since, from the power map in the

TABLE IV

x	u	Size	He-class
1	(0, 0)	1	$1A$
1	(1, 0)	48	$7C$
z	(0, 0)	49	$2B$
c	(0, 0)	168	$7D$
c	(1, 0)	336	$7X$
c	(2, 0)	336	$7Y$
c	(3, 0)	336	$7Z$
d	(0, 0)	168	$7E$
d	(1, 0)	336	$7Y^*$
d	(2, 0)	336	$7Z^*$
d	(3, 0)	336	$7X^*$
zc	(0, 0)	1176	$14C$
zd	(0, 0)	1176	$14C$
a	(0, 0)	2744	$6B$
a^2	(0, 0)	2744	$3B$
b	(0, 0)	2058	$8A$
b^2	(0, 0)	2058	$4C$
b^3	(0, 0)	2058	$8A$

"Atlas" [3], there is no element of order 6 whose square has type $3A$ and whose cube has type $2B$, we must have $x_{3A}=0$, and hence also $x_{6A}=0$. We have

If $x_T \neq 0$, then
$T \in \{1A, 2A, 2B, 3B, 4A, 4B, 4C, 6B, 7D, 7E, 8A, 12B, 14C, 14D\}$.

Counting the number of elements of $K \cap He$ mapping onto an element of order 3 in $K \cap He/O_2(K \cap He)$, we find

$$x_{3B} + x_{6B} + x_{12B} = 7168.$$

The innerproducts of the permutation character of He on $K \cap He$ with the characters 1_a, 680_a, 51_a, and the permutation character of 3.2(ii) give the following restrictions on x_{4A}, x_{4B}, x_{4C}, and x_{8A}:

$$x_{4A} + x_{4B} + x_{4C} + x_{4A} = 7616,$$

$$(10{,}304 + 8x_{4B})/21504 \in \mathbf{Z}_{\geqslant 0},$$

$$(-3360 + 3x_{4A} + 3x_{4B} - x_{4C} + x_{8A})/21{,}504 \in \mathbf{Z}_{\geqslant 0},$$

$$(40{,}320 + 14x_{4A} + 18x_{4B} + 2x_{4C} + 2x_{8A})/21{,}504 \in \mathbf{Z}_{\geqslant 0},$$

leading to the unique solution $x_{4B}=1400$, $x_{4A}=672$, $x_{4C}=4200$, and $x_{8A}=1344$. We now examine elements of order a multiple of 3. The inner product with $22{,}050_a$ yields $(14{,}336 - 6x_{3B} - 2x_{6B})/21{,}504 \in \mathbf{Z}_{\geqslant 0}$, and the inner product with 4352_a yields $(14{,}336 + 8x_{3B})/21{,}504 \in \mathbf{Z}_{\geqslant 0}$. This forces $x_{3B}=896$ and $x_{6B}=4480$, whence $x_{12B}=1792$, and all x_T are determined. It is now straightforward to compute the permutation character. ∎

5.6.2. COROLLARY. *Theorem* 1.1 *holds for pairs* H, K *as in the above lemma.*

Proof. Obviously, the permutation rank of G on K exceeds 7. Also in case (ii), the multiplicity 3 of $11{,}900_a$ in $\pi|_{K \cap He}$ implies that H is not multiplicity free on K, so let K be as in (i), and suppose that π is multiplicity free. Then, the lemma implies that we must have

$$\pi = 1_a + \delta_1 1920_a + \delta_2 4080_a + \delta_3 4352_a + \delta_4 6528_a$$
$$+ \delta_5 20{,}825_a + \delta_6 21{,}504_a + \delta_7 21{,}504_b + \psi,$$

where δ_i is the identity or the sign character with kernel He, the characters x_m refer to the "Atlas" [3], and $\psi(g)=0$ if $g \in H \backslash He$. Now consider $x, y \in H \backslash He$ of type $8B$ and $6D$, respectively. Then x^2 and y^2 are of type $4B$

and $3A$, resp., so $\pi(x)=\pi(y)=0$. Therefore, we get $\delta_5(x)=1$ and $1+3\delta_1(y)+2\delta_3(y)-3\delta_4(y)-\delta_5(y)=0$, a contradiction with $\delta_m(x)=\delta_m(y)\in\{1,-1\}$. ∎

6. Proof of Corollary 1.2

If $G=He$, H acts distance transitively on an imprimitive distance regular graph Γ, then, by a result of D. Smith [6], there is also a primitive distance regular graph Γ' on which G acts distance transitively. So suppose Γ is a primitive distance regular graph and G acts distance transitively on Γ. Then the stabilizer K in G of a vertex is a maximal subgroup, the permutation character of this permutation representation is multiplicity free, and all irreducibles occurring in that character are real. By Theorem 1.1, we must have $K\cap He\cong Sp(4,4):2$ or $K\cap He\cong 2^2\cdot L(3,4)\cdot Dih_{12}$. From the results in Section 3 it follows that in the first case the corresponding graph is not distance transitive. So suppose we are in the second case. Then the subdegrees are 1, 105, 720, 1344, 1680, and 4480, and $\Gamma=\Delta_x$ with x being one of these subdegrees and notation as in Section 4. Furthermore, it is well known that in a distance regular graph of diameter d and subdegrees $1, k_1, ..., k_d$ we have the following inequalities.

$$k_1\leqslant k_i \qquad \text{for all} \quad i\in\{2, ..., d-1\}. \tag{$*$}$$

Hence $k_1=105$ or 720. The case $k_1=105$ leads to a contradiction as the corresponding graph Δ is not distance regular (see Section 4). So assume $k_1=720$. From the distribution diagram of Δ given in Section 4 it follows that there are 3 vertices x, y, and z such that x, z and x, y are Δ_{720}-edges and x, y is a Δ-edge. So if $k_1=720$, then $k_2=105$, a contradiction with $(*)$. Hence there is no graph Γ on which He or H acts distance transitively.

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