JOURNAL OF ALGEBRA **125**, 348–366 (1989)

On the Local Recognition of Finite Metasymplectic Spaces

ARJEH M. COHEN

CWI, Kruislaan 413, 1098 SJ Amsterdam, The Netherlands

AND

BRUCE N. COOPERSTEIN

Mathematics Department, University of California, Santa Cruz, California 95064

Communicated by G. Stroth

Received October 23, 1986

This paper is concerned with the local recognition of certain graphs and geometries associated with exceptional groups of Lie type. The local approach to geometries is inspired by group theory. Finite simple groups are often characterized by local information, for example, the fusion pattern of involutions centralizing a given involution. The main results here, although of a geometric nature, are a contribution to obtaining a characterization of a group of exceptional Lie type by the fusion pattern of root subgroups centralizing a given root subgroup.

Let Δ be the shadow space of a (thin or thick) building of spherical type M_n (where n indicates the rank) with respect to a given node r of M (cf. Tits [13, 14]). We shall view Δ as a *space*, i.e., as a set of *points* together with a collection of subsets of size at least two of the point set, called *lines*. Thus, the points of Δ are the vertices of type r of the building in question and the lines are the residues of flags of cotype $\{r\}$. The local recognitions we intend to discuss are based on the fact that up to (nonspecial) isomorphisms, the building is uniquely determined by Δ. If p is a point of Δ, the set $\Delta_{\leqslant 1}(p)$ of points collinear with p (including p) constitutes a *subspace* in the sense that each line bearing two distinct points of $\Delta_{\leqslant 1}(p)$ is entirely contained in $\Delta_{\leqslant 1}(p)$. A space with this property is often called a *gamma space*. Since Δ affords a group of automorphisms which is transitive on the point set if $n \geqslant 2$ (cf. [13]), we can associate with Δ a space $\Delta_{\leqslant 1}$ such that for each point p of Δ the subspace $\Delta_{\leqslant 1}(p)$ of Δ is isomorphic to $\Delta_{\leqslant 1}$. (Here and elsewhere, a subspace X is regarded as a space by taking into account the lines of the ambient space completely contained in X.) A gamma space Γ will be called *locally isomorphic to* Δ if for each point p of Γ the subspace $\Gamma_{\leqslant 1}(p)$ is isomorphic to $\Delta_{\leqslant 1}$. The space Δ is called *locally recognizable* if every connected gamma space which is locally isomorphic to Δ is also isomorphic to Δ. Projective spaces are trivial examples of locally recognizable spaces because their points are pairwise collinear. In view of Johnson and Shult [9], polar spaces of rank at least three are locally recognizable as well. Thanks to known characterizations of the relevant Lie incidence systems, it is not hard to show that certain shadow spaces of thick buildings of type $M_n = E_6, E_7, E_8$ (among which the root group geometries) are locally recognizable. This is the content of Section 2 below.

If Δ does not possess subspaces which are projective planes, all lines of $\Delta_{\leqslant 1}$ meet in a unique point (the radical of $\Delta_{\leqslant 1}$); the "free construction" of a gamma space which is locally isomorphic to Γ has infinite diameter then, so that Δ is not locally recognizable. Examples of such shadow spaces are dual polar spaces. In order to capture these spaces in a local study as well, we observe that provided $n > 3$, for each natural number k there is a space $\Delta_{\leqslant k}$ such that for every point p of Δ the set $\Delta_{\leqslant k}(p)$ of points at distance at most k (measured in the collinearity graph of Δ) to p is a subspace isomorphic to $\Delta_{\leqslant k}$ (same argument as above). Thus the local recognition problem can be viewed as a particular case of the search for the minimal number k for which Δ is *k-recognizable*, i.e., satisfies the property that every space Γ such that for each point p the subset $\Gamma_{\leqslant k}(p)$ is a subspace isomorphic to $\Delta_{\leqslant k}$ is isomorphic to Δ. According to [1], thick finite dual polar spaces of rank $\geqslant 3$ are 3-recognizable and half dual polar spaces of rank $\geqslant 4$ are 2-recognizable. As a consequence of [6, 8], the spaces of the root group geometries of type F_4, E_6, E_7, E_8 are 2-recognizable. It has been mentioned above that this result has been improved for the latter three kinds of spaces. If the prevailing type M_n is F_4 and r corresponds to an end node, Δ is called a *metasymplectic space* (cf. [6, 7]). If Δ is the *thin* metasymplectic space (i.e., all lines of Δ have precisely two points and each line is in precisely three triangles), then Δ is the 24-cell associated with the Weyl group of type F_4 and $\Delta_{\leqslant 1}$ is the graph theoretic join of a single vertex graph and the cube, where the latter is viewed as a graph on 8 vertices and 12 edges. A. E. Brouwer and, independently, D. Buset [3] have shown that in this case each connected graph locally isomorphic to Δ is either isomorphic to Δ or to the complement E of the 3×5 grid.

If Δ is *thick*, i.e., if all lines of Δ have at least three points and each 4-circuit is collinear with at least three points, we do not know whether Δ is locally recognizable. However, the following approximation to local recognition of Δ will be shown to hold.

MAIN THEOREM. *Suppose Δ is a thick finite metasymplectic space. If Γ*

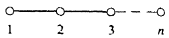
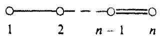
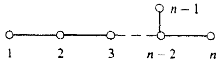
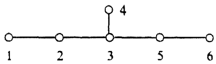
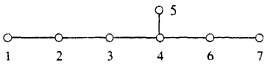
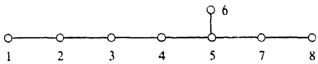
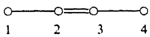
is a connected gamma space that is locally isomorphic to Δ, *then* Γ *is isomorphic to* Δ *provided the following condition is satisfied.*

(*) *If* x_1, x_2, x_3, x_4 *is a path of* Γ *with* x_1, x_3 *and* x_2, x_4 *pairs of non-collinear points such that* $\Gamma_{\leqslant 1}(x_1) \cap \Gamma_{\leqslant 1}(x_2) \cap \Gamma_{\leqslant 1}(x_3) \cap \Gamma_{\leqslant 1}(x_4)$ *contains at least two points, then each line of* $\Gamma_{\leqslant 1}(x_1) \cap \Gamma_{\leqslant 1}(x_2) \cap \Gamma_{\leqslant 1}(x_3)$ *contains a point collinear with* x_4.

Let us first give an alternative interpretation of (*). If p is a point of Δ, the space $\Delta_{\leqslant 1}(p)$ has radical $\{p\}$ (see [8] for terminology). Thus there is a natural quotient space Δ^p whose points are the lines on p on whose lines are the sets of lines on p contained in a plane on p. This quotient space Δ^p is isomorphic to a dual polar space of rank 3 and hence (cf. [12]) each pair of points of Δ^p at mutual distance 2 is contained in a *quad*, i.e., a geodesically closed subspace isomorphic to a generalized quadrangle. Condition (*) is equivalent to saying that if p and q are points of a line l of Γ, then every set of all planes containing l and determining the lines in a quad of Γ^p on the point l of Γ^p coincides with the set of all planes containing l and determining the lines in a quad of Γ^q on the point l of Γ^q. In the thin case, for any path x_1, x_2, x_3, where x_1, x_3 are noncollinear, the number of lines in $\Gamma_{\leqslant 1}(x_1) \cap \Gamma_{\leqslant 1}(x_2) \cap \Gamma_{\leqslant 1}(x_3)$ is at most 2, so that condition (*) is trivially satisfied. Thus, the above theorem can be viewed as an extension of the thin characterization in [3]. Although no "q-analogue" of E appears in the conclusion of the theorem, the heart of the proof consists of deriving the nonexistence of a local pattern resembling certain local patterns in E. To be more specific, there are points x, y in E at mutual distance 2 such that the subspace $E_{\leqslant 1}(x) \cap E_{\leqslant 1}(y)$ is a hexagon, whereas it takes some effort to establish that in the setting of the theorem there are no points x, y at mutual distance 1 in the space Γ such that $\Gamma_{\leqslant 1}(x) \cap \Gamma_{\leqslant 1}(y)$ is a generalized hexagon. By the results in Section 3, the possible generalized hexagons occurring as $\Gamma_{\leqslant 1}(x) \cap \Gamma_{\leqslant 1}(y)$ are the classical generalized hexagons of type G_2. The principal result of Section 3, Theorem (3.1), is a characterization of "locally quad" subspaces of a dual polar space of rank 3 which may be of interest in its own right. This proof of the Main Theorem is in Section 4. Apart from the thin example E, the existence of geometries of type F_4 related to geometries of type C_4, D_4 and of geometries with diagram $\circ\overset{c}{—}\circ=\circ—\circ$ related to the Monster simple group and Fischer's simple group Fi_{24} (cf. Buekenkout and Fischer [2], Ronan and Stroth [10]) seems to indicate that local recognition of metasymplectic spaces is less trivial than that of the root group geometries for E_6, E_7, E_8.

Although the infinite case is not covered by our methods, we have no reason to believe that the theorem would cease to hold if the finiteness restriction were removed.

TABLE I

M_n	Labeled diagram	Restriction
A_n		$n \geqslant 1$
$B_n = C_n$		$n \geqslant 2$
D_n		$n \geqslant 4$
E_6		
E_7		
E_8		
F_4		

The labeling of nodes of M_n shown in Table I will be used throughout in the sequel. A space is said to be a *space of type* $M_{n,r}$ if it is the shadow space of a building of type M_n with respect to node r as explained above. The labeling follows [11], where a survey of recognition theorems for spaces of type $M_{n,r}$ can be found.

2. Local Recognition of Spaces Related to E_6, E_7, E_8

This section is devoted to the proof of the following result.

(2.1) Theorem. *Every space of type* $D_{n,n}$ *(for* $4 \leqslant n \leqslant 7$*),* $E_{6,1}$, $E_{6,4}$, $E_{7,1}$, $E_{7,7}$, *or* $E_{8,1}$ *is locally recognizable.*

For notation, such as $\perp$ (for collinearity) and d (for distance), and terminology, such as "parapolar space," "symplecton," "singular subspace," and "rank," the reader is referred to [8]. In addition, for x a point of a gamma space Γ and X a subspace of Γ contained in $x^\perp$, we shall write X/x to denote the quotient space with respect to x, i.e., the space whose points are the lines containing x and a point of $X - \{x\}$ and whose lines are the sets of all lines on x meeting a given line entirely contained in $X - \{x\}$. Thus Γ^x and $x^\perp/x$ denote the same space.

Suppose $M_{n,r}$ is one of the types mentioned in the theorem. Let Δ be a space of type $M_{n,r}$. Let Γ be a connected gamma space which is locally isomorphic to Δ. If $M_{n,r} = D_{4,4}$, then Δ is a polar space so the result follows from Johnson and Shult [9]. Therefore, we (may) assume from now on that n is at least 5. Since maximal singular subspaces belong to $\Gamma_{\leqslant 1}(z)$ for any point z they contain, Γ is a partial linear space. Also for each point x of Γ there are subspaces S of Γ contained in $x^\perp$ such that S/x is a symplecton on Γ^x. (Conversely, if T' is a symplecton of Γ^x there is a unique subspace T of Γ containing x which is a union of lines of x such that $T/x = T'$.) The lemma below shows how to interpret such a space S at a point y collinear with x.

(2.2) LEMMA. *Retain the notation for Δ, $M_{n,r}$ of above. Let $n \geqslant 5$ and let Γ be a gamma space which is locally isomorphic to a space of type $M_{n,r}$. Let x, y be distinct collinear points of Γ and denote by l the line containing both x and y. If S is a subspace containing l which is a union of lines on x such that S/x is a symplecton of Γ^x then there is a subspace T which is a union of lines on y containing l such that T/y is a symplecton of Γ^y with the property that every plane on l is contained in S if and only if it is contained in T.*

Proof. Take two planes π_1, π_3 in S containing l such that $\pi_1 \not\subseteq \pi_3^\perp$. Since S/x is a polar space of rank $\geqslant 2$, there are planes π_2, π_4 in S containing l such that $\pi_2 \cup \pi_4 \subseteq \pi_1^\perp \cap \pi_3^\perp$ but $\pi_2 \not\subseteq \pi_4^\perp$. The set of planes on l contained in S now coincides with the set of planes π on l such that $\pi^\perp \cap \langle \pi_i, \pi_{i+1} \rangle$ contains a plane for each $i \in \{1, 2, 3, 4\}$, where indices are taken modulo 4. This description of the planes on l contained in S (independent of the choice of x in l) determines the set of lines on the point l of Γ^y of a symplecton T^1 of Γ^y. The inverse image of T^1 under the mapping $u \to uy$ from $y^\perp - \{y\}$ onto Γ^y is the desired subspace T. ∎

Before proceeding with the proof of Theorem (2.1), we list without proof some known properties of the spaces under study (see [8]).

(2.3) LEMMA. *Let Δ be a space of type $N_{u,v}$. Consider the following conditions concerning Δ.*

(xl) *If x is a point and l a line, then $d(x, l) \leqslant 2$.*

(xS) *If x is a point and S a symplecton, then $x^\perp \cap S \neq \emptyset$.*

(P) *If x_1, x_2, x_3, x_4, x_5 is a 5-circuit, then $\{x_1, x_3, x_5\}^\perp \neq \emptyset$.*

(SlS) *If S_1, S_2 are two symplecta, then there exists a line l such that $S_i \cap l \neq \emptyset$ for each $i \in \{1, 2\}$.*

(S) *If S is a subspace which is locally isomorphic to a symplecton, then S is a symplecton.*

(*Sy*) *If x, z are points at distance 2, then $\{x, z\}^{\perp}$ is a polar space of rank at least 2.*

The following hold:

(a) *If $N_{u,v} = A_{n,2}$ $(n \leqslant 7)$, $D_{5,5}$, or $E_{6,1}$, then Δ has diameter 2 and satisfies* (*SlS*) *and* (*S*).

(b) *If $N_{u,v} = A_{5,3}$, $D_{6,6}$, $C_{3,3}$, $E_{7,1}$, then Δ has diameter 3 and satisfies* (*xl*), (*xS*), *and* (*P*).

(c) *If $N_{u,v} = A_{n,2}$ $(n \geqslant 3)$, $D_{5,5}$, $A_{5,3}$, $E_{6,1}$, $D_{6,6}$, $E_{7,1}$, then Δ satisfies* (*Sy*).

The crux of the proof of Theorem (2.1) consists of identifying $\{x, z\}^{\perp}$, for two points x, z at mutual distance 2, with a symplecton. This is again done by local recognition, but now for subspaces.

A subspace X of a parapolar space is said to be *locally isomorphic to a symplecton* if for each point x of X there is a symplecton S such that $x^{\perp} \cap X = x^{\perp} \cap S$.

(2.4) LEMMA. *Let Δ, $M_{n,r}$, Γ be as above. For any two points x, z of Γ at mutual distance 2, the following hold.*

(i) *The mapping $\{x, z\}^{\perp} \ni u \to ux \in \Gamma^{x}$ is an isomorphism from $\{x, z\}^{\perp}$ onto the subspace $\{x, z\}^{\perp}/x$ of Γ^{x}; the connected components of this subspace are either singletons or locally isomorphic to a symplecton in Γ^{x} and hence to spaces of type $N_{u,v} = A_{3,2}$, $D_{4,1}$, $A_{3,2}$, $D_{5,1}$, $D_{4,1}$, $D_{6,1}$ in the respective cases $M_{n,r} = D_{n,n}$, $E_{6,1}$, $E_{6,4}$, $E_{7,1}$, $E_{7,7}$, $E_{8,1}$.*

(ii) *If a connected component of the space $\{x, z\}^{\perp}/x$ contains a line, it is a symplecton of Γ^{x}.*

(iii) *If $M_{n,r} = E_{6,4}$, $E_{7,7}$, or $E_{8,1}$, then $\{x, z\}^{\perp}/x$ is either a coclique or a symplecton.*

Proof. (i) Since Γ is a gamma space, it is readily seen that the mapping $u \to ux$ $(u \in \{x, z\}^{\perp})$ is an isomorphism from $\{x, z\}^{\perp}$ onto $\{x, z\}^{\perp}/x$. Let $u \in \{x, z\}^{\perp}$. Suppose there is a line on u in $\{x, z\}^{\perp}$. Then xu, zu have distance 2 in Γ^{u}. Now, by Lemma (2.3)(c), in Γ^{u} the points collinear with both xu and zu all belong to the unique subspace S on u, x, and z with the property that S is a union of lines on u and that S/u is a symplecton. In fact, each plane π on xu contains a line collinear with z if and only if it belongs to S. According to Lemma (2.2), there is a subspace T on x in $x^{\perp}$ such that T/x is a symplecton of Γ^{x} and each plane on xu contains a line collinear with z if and only if it belongs to T. This implies that the subspace of $\{x, z\}^{\perp}/x$ consisting of all points collinear with xu coincides with the subspace of the symplecton T/x consisting of all points collinear with xu.

Consequently, connected components of $\{x, y\}^{\perp}/x$ are locally isomorphic to a symplecton of the specified type or to singletons. Now (i) follows since the symplecta are of the types given in the statement.

(ii) In view of (i) it suffices to prove that any connected subspace S of a space Ψ of type $N_{u,v}$ that is locally isomorphic to a symplecton is itself a symplecton. As each point of S is contained in a line of S, the space S is locally polar in the sense of Johnson and Shult [9]. Since symplecta of Ψ have rank $\geqslant 4$, the rank of maximal singular subspaces is $\geqslant 3$, so that by [9], S is a polar space. It readily follows that S is a symplecton of Ψ (for instance since S is the geodesic closure of any 4-circuit that it contains.)

(iii) Suppose $T = \{x, z\}^{\perp}/x$ contains a connected component T^0 which is a symplecton in Γ^x. If $l \in T - T^0$, then by Lemma (2.3)(b) there is $m \in T^0$ collinear with l. But then l belongs to the connected component of T in m, i.e., to T^0, a contradiction. Hence $T = T^0$. ∎

(2.5) COROLLARY. *Let Γ be as in the previous lemma. Then Γ is a parapolar space.*

Proof. Clearly, Γ satisfies (*F*1), (*F*2) of [8]. Thus the proof comes down to showing

(F3) *If x, y are two points of Γ at mutual distance* 2, *the subspace $\{x, y\}^{\perp}$ is either a singleton or a polar space.*

In view of Lemma (2.4), we need only establish that $\{x, y\}^{\perp}$ is a connected space. Suppose that a, b are noncollinear points of $\{x, y\}^{\perp}$. We shall establish that $\{x, y\}^{\perp}$ is connected by distinguishing two cases:

(a) The diameter of Γ^x is 2, and hence $M_{n,r} = D_{n,n}, E_{6,1}, E_{7,1}$.

(b) The diameter of Γ^x is 3, and hence $M_{n,r} = E_{6,4}, E_{7,7}, E_{8,1}$.

(a) Since the diameter of Γ^a and Γ^b is 2, there are lines on a and on b inside $\{x, y\}^{\perp}$. Thus, by Lemma (2.4)(ii) the connected components of $\{x, y\}^{\perp}$ containing a and b respectively are symplecta. Since (*SlS*) in Lemma (2.3) holds, there is a line having a point in both symplecta, so that a and b are connected by a path in $\{x, y\}^{\perp}$.

(b) In view of (iii) of the above lemma, we are done if $\{x, y\}^{\perp}$ contains a line. Suppose, therefore, that $\{x, y\}^{\perp}$ is a coclique. If there is a point collinear with, but distinct from, three points from the 4 circuit x, a, y, b, then $\{x, a, y, b\}^{\perp} \neq \varnothing$. (For, if $z \in \{x, a, y\}^{\perp} - \{a\}$ then za and b both belong to $\{x, y\}^{\perp}$, so the latter space is a polar space in view of (iii) of the above lemma, whence $\varnothing \neq b^{\perp} \cap za \subseteq \{x, a, y, b\}^{\perp}$; and similarly for choices of other triples from x, a, y, b.) Therefore, we may restrict ourselves to the case where xa, ya have distance 3 in Γ^a, ay, by have distance 3 in Γ^y, and so on.

Take a plane π containing ay. By (xl) of Lemma (2.3) applied to Γ^a and Γ^y there exists a point c in π such that ca (resp. cy) has distance 2 in Γ^a (resp. Γ^y) to xa (resp. by). In particular $\{x, c\}^\perp$ and $\{b, c\}^\perp$ contain a line so that by (ii) of the above lemma $\{a, c\}^\perp/x$ and $\{b, c\}^\perp/b$ are symplecta of Γ^x and Γ^b, respectively. By (xS) of Lemma (2.3) applied to Γ^x, there is a point of $\{x, c\}^\perp/x$ collinear with xb. Denote by d the inverse image of this point in $\{x, c\}^\perp$ under the map $u \to ux$ ($u \in \{x, c\}^\perp$). Then d is collinear with b, c, x. Now y, $d \in \{b, c\}^\perp$ and a, $d \in \{x, c\}^\perp$, whence using that the symplecta $\{b, c\}^\perp/b$ and $\{x, c\}^\perp/x$ have diameter 2, there exist $e \in \{b, c, d, y\}^\perp$ and $f \in \{a, c, d, x\}^\perp$. Now by ($P$) of Lemma (2.3) applied to the 5-circuit a, y, e, d, f, there exists a line l on c inside $\{c, d, a, y\}^\perp$. Now $l \cup \{x\} \subseteq \{a, d\}^\perp$, so $\{a, d\}^\perp$ is a symplecton by Lemma (2.4) and there exists $g \in x^\perp \cap l$. Thus a and g are collinear and belong to $\{x, y\}^\perp$. Since a, g are distinct (otherwise xa, xd, xb would be a path of length 2 in Γ^x), this leads to a line inside $\{x, y\}^\perp$ contradicting that $\{x, y\}^\perp$ is a coclique. This ends the proof of the corollary. ∎

(2.6) *End of Proof of Theorem* (2.1). Now that the corollary has been established, it is straightforward to verify that Γ is a parapolar space and that axioms $(F3)_k$ and $(F4)_J$ hold for the relevant $k \in \mathbb{N}$, $J \subseteq \{-1, 0, 1\}$ in the distinguished cases for $M_{n,r}$. Application of Theorems 1 and 2 in [8] (and the "Added in Proof") shows that either Γ satisfies the conclusion of the theorem to be proved or Δ is of type $D_{n,n}$ and Γ is a quotient of Δ by a group A of automorphisms of Δ mapping each point to a point at distance at least 5 to it. However, in the latter case $n \leqslant 7$, so the diameter of Δ is at most 3, whence $A = 1$ and $\Gamma \cong \Delta$. Hence the theorem. ∎

3. Locally Quad Subspaces of Dual Polar Spaces

Let N be a dual polar space. A *quad* in N is a geodesically closed subspace that is isomorphic to a generalized quadrangle. A subspace X of a dual polar space N is said to be *locally quad* if for each point x of X there exists a quad Q of N such that $x^\perp \cap Q = x^\perp \cap X$. Examples of locally quad subspaces of N are quads. This section is devoted to the proof of the following result. Here, a dual polar space is called *thick* if each line has at least three points and if each point p has at least three lines inside every quad on p.

(3.1) Theorem. *Suppose N is a thick finite dual polar space of rank* 3. *If X is a locally quad subspace of N which is not a quad then N is the dual* *polar space $B_{3,3}(q)$ associated with a nondegenerate quadic in a 7-dimensional space over some finite field $\mathbb{F}_q$ and X is isomorphic to the classical generalized hexagon $G_2(q)$ of type G_2 over $\mathbb{F}_q$. Moreover, the map $x \to Q_x$ assigning to each point x of X the unique quad (=point of polar space) Q_x of N containing $x^\perp \cap X$ is the standard embedding of the generalized hexagon G_2 over $\mathbb{F}_q$ in the polar space $B_{3,1}(q)$ over $\mathbb{F}_q$.*

Here, the standard embedding of the classical generalized hexagon $G_2(q)$ in the polar space $B_{3,1}(q)$ refers to the construction of the generalized hexagon inside te polar space $B_{3,1}(q) \cong \Omega_7(q)$ as the absolute points and lines of a triality on the polar space $D_{4,1}(q) \cong \Omega_8^+(q)$. As a convenience to the reader, we list some known properties of dual polar spaces of rank 3. Proofs can be found in Cameron [4] or Shult and Yanushka [12].

(3.2) LEMMA. *Let N be a dual polar space of rank 3. The following hold.*

(i) *The space N is a gamma space whose lines are maximal cliques.*

(ii) *If $a_1 \perp a_2 \perp a_3 \perp a_4 \perp a_5 \perp a_1$ is a 5-circuit in N (i.e., a_i, a_{i+2} noncollinear, indices i modulo 5), then for each i the point a_i is collinear with a point on the line through a_{i+2} and a_{i+3}.*

(iii) *Each pair of points at mutual distance 2 is contained in a unique quad.*

(iv) *Each pair of quads has either empty intersection or meets in a line.*

(v) *If Q is a quad of N, then for each point $p \in N - Q$ the intersection $p^\perp \cap Q$ is a singleton $\{p_1\}$. Moreover, if $q \in Q$ the distance $d(p, q)$ from p to q satisfies $d(p, q) = 1 + d(p_1, q)$.*

(vi) *The diameter of N is 3. If x, y are points at mutual distance 3, each line containing x bears a unique point at distance 2 to y.*

In the proof of the proposition, we shall use a case by case argument to rule out all possible thick polar spaces of rank 3 except for those with parameters $(s, t, r) = (q, q^2 + q, q)$, i.e., $B_{3,1}(q) = \Omega_7(q)$ and $C_3(q) = Sp_6(q)$. Table II below lists the parameters of all thick dual polar spaces. For the duration of this section, let N and X be as in the hypothesis of the theorem.

(3.3) LEMMA. *The graph theoretical distance in X is the restriction to X of the distance in N. Moreover, X is a generalized hexagon of order (s, r).*

Proof. Suppose X contains a 4-circuit $a \perp b \perp c \perp d \perp a$. Let Q be the unique quad containing this 4-circuit. Since Q is the unique quad containing a, b, and d, we have $a^\perp \cap X = a^\perp \cap Q \subseteq Q$. Therefore, if x belongs to $a^\perp \cap X$, there is a point y on the line bc containing b and c such that

TABLE II

Thick Finite Dual Polar Spaces of Rank 3

Name of N	Parameters s, t, r	Number of points	Number of quads
$B_3(q) = \Omega_7(q)$			
	q, q^2+q, q	$(q+1)(q^2+1)(q^3+1)$	$\dfrac{q^6-1}{q-1}$
$C_3(q) = Sp_6(q)$			
${}^2D_4(q) = \Omega_8^-(q)$	$q^2, q^2 \pm q, q$	$(q^2+1)(q^3+1)(q^4+1)$	
${}^2A_5(q) = U_6(q)$	q, q^4+q^2, q^2	$(q+1)(q^3+1(q^5+1)$	$(q^5+1)(q^4+q^2+1)$
${}^2A_6(q) = U_7(q)$	q^3, q^4+q^2, q^2	$(q^3+1)(q^5+1)(q^7+1)$	

$x \perp y \perp b \perp a \perp x$ is a 4-circuit contained in Q. By the same argument as before, we get $x^\perp \cap X \subseteq Q$. By induction on the length of a path starting from a we find that the connected component of X containing a is contained in Q. On the other hand, each point of Q is collinear with a point of $a^\perp \cap Q$, so the connected component of X containing a coincides with Q. If $z \in X - Q$, then by Lemma (3.2)(v) there is a (unique) point in Q collinear with z. But then z is contained in the connected component of X containing a, a contradiction. It follows that $X = Q$, contradicting the hypothesis. Therefore, X does not contain 4-circuits, and hence, by Lemma (3.2)(i), (ii), no circuits for $m = 3, 4, 5$.

For $x, y \in X$, we denote by $d_X(x, y)$ the distance from x to y in X as opposed to the distance $d(x, y)$ in N. It is our goal to show that d and d_X coincide on X. Suppose $x, y \in X$ satisfy $d(x, y) = 2$. Let Q be the unique quad satisfying $x^\perp \cap Q = x^\perp \cap X$. By Lemma (3.2)(v) there is a point u in $x^\perp \cap y^\perp \cap Q$. Since $u \in x^\perp \cap Q \subseteq X$ we have a path from x to y inside X, whence $d_X(x, y) = 2$. Next suppose that $x, y \in X$ satisfy $d(x, y) > 2$. Then, again by Lemma (3.2)(v), there are $u \in y^\perp \cap Q$ and $v \in x^\perp \cap Q$ with $u \perp v$. Now $v \in X$ and $d_X(v, y) = 2$ by the preceding paragraph as $d(x, y) > 2$, so $d_X(x, y) = 1 + d_X(v, y) = 3$. This settles that d_X is the restriction of d to X. Since clearly by Lemma (3.2)(vi) for $x, y \in X$ with $d(x, y) = 3$ each line in X containing x contains a point of X at distance 2 to y, it follows that X is a generalized hexagon. As the line size of X is the line size $s+1$ of N and the number of lines on a point in X is the number of lines $r+1$ in a quad on a point in that quad, the order of X is (s, r). This proves the lemma. ∎

(3.4) LEMMA. (i) *If Q is a quad, then $Q \cap X$ is either empty, a line or of the form $x^\perp \cap X$ for some $x \in X$.*

(ii) *The mapping $x \to Q_x$ assigning to $x \in X$ the unique quad Q_x*

satisfying $x^\perp \cap X = x^\perp \cap Q_x$ is an injective mapping from X to the point set of the polar space underlying N.

Proof. (i) By Lemma (3.2), the intersection $Q \cap X$ contains two non-collinear points only if it has the form $x^\perp \cap X$ for some $x \in X$. Therefore we may and shall restrict ourselves to the case where $Q \cap X$ is a clique. Since $Q \cap X$ is a subspace, in view of Lemma (3.2)(i) it remains to show that it cannot be a point. Suppose $a \in Q \cap X$. Then there is a quad R such that $a^\perp \cap R = a^\perp \cap X$. Since $a \in Q \cap R$ and $Q \neq R$, Lemma (3.2)(iv) yields that $Q \cap R$ is a line containing a. Therefore $Q \cap X$ contains a line and we are done.

(ii) Obvious from (i) and the fact that $\{x\}$ is the radical in X of $Q_x \cap X$. ∎

(3.5) LEMMA. (i) *The parameters of N are $(s, t, r) = (q, q^2+q, q)$ for some prime power q.*

(ii) *The mapping $\pi: x \to Q_x$ of the preceding lemma is a bijection from X to the point set of the polar space P underlying N.*

(iii) *Each line of N has a point in common with X. In particular, each point of N is collinear with precisely q^2+q+1 points of x.*

(iv) *If $x, y \in X$, then x, y have distance $\leqslant 2$ if and only if $\pi(x)$ and $\pi(y)$ are collinear in the polar space P.*

Proof. (i) Thanks to (ii), the restriction of the mapping $x \to Q_x$ on X to a line l of X, is injective, too. Since the size of l is $s+1$, the number of quads containing l is $r+1$ and Q_x contains l for every $x \in l$, it follows that $s \leqslant r$. This rules out the possibility that $N \cong \Omega_8^-$ or $N \cong U_7$. Next, we count the number of quads Q with $Q \cap X \neq \varnothing$. By (ii) there are exactly $(s+1)(1+sr+s^2r^2)$ (=the cardinality of X) quads Q_x for x ranging over X. Since X has $(r+1)(1+sr+s^2r^2)$ lines and each line is contained in $r+1$ quads, $s+1$ of which have shape Q_x, there are $(1+sr+s^2r^2)((s+1)+(r+1)(r+1-(s+1))) = (1+sr+s^2r^2)(1+r^2+r-rs)$ quads that meet X nonemptily. On the other hand, the total number of quads is $((1+st)(r+1)+s^2t(t-r))\, t(t+1)/(1+r)(1+sr)$ as can be seen by a simple count using Lemma (3.2). It follows that there are

$$\alpha = \frac{((1+st)(r+1)+s^2t(t-r))\, t(t+1)}{(r+1)(sr+1)} - (1+sr+s^2r^2)(1+r+r^2-rs)$$

quads in N having an empty intersection with X. Since $\alpha < 0$ if N has the parameters of the type U_6 dual polar space, we conclude that N must have the parameters of the B_3 dual polar space (see Table II).

(ii) *Now* $s=r$, so the first paragraph of the proof of (i) yields that there are no quads Q such that $Q\cap X$ is a line. Since $\alpha=0$, the last paragraph of the same proof yields that there are no quads that have an empty intersection with X. The result therefore follows from Lemma (3.4).

(iii) Let l be a line of N. If Q is a quad containing l, then $Q=Q_x=x^\perp\cap Q$ for some $x\in X\cap Q$ by (ii), so there is a point y on l collinear with x by the definition of generalized quadrangle. Now $y\in x^\perp\cap Q$ so $y\in l\cap X$. If $z\in N-H$, then each line on z has at least one point in H by the above, and hence, since H is a subspace, precisely one point. Thus the size of $z^\perp\cap H$ is $t+1=q^2+q+1$.

(iv) Suppose x, y are points of X with $d(x,y)\leqslant 2$. Then $Q_x\cap Q_y$ contains a point of X (namely x), the line on x and y, the unique point in $x^\perp\cap y^\perp\cap X$ in the respective cases $d(x,y)=0, 1, 2$. Hence, by Lemma (3.2)(iv), the intersection $Q_x\cap Q_y$ contains a line. This means that $\pi(x)=Q_x$ and $\pi(y)=Q_y$ are collinear points of P. Conversely, let $x, y\in X$ be points such that $\pi(x)$, $\pi(y)$ are collinear. Then $Q_x\cap Q_y$ contains a line, l say. If $d(x,y)\leqslant 1$, there is nothing to show. Assume, therefore, $d(x,y)>1$. Then there are $u\in x^\perp\cap l$ and $v\in y^\perp\cap l$ as Q_x and Q_y are quads containing x, l and y, l, respectively. If $u\neq v$, then $l\subseteq Q_x\cap Q_y\cap X$ so $x, y\in l$, contradicting $d(x,y)>1$. Hence $u=v\in x^\perp\cap y^\perp$, so $d(x,y)=2$. ∎

We now prove Theorem (3.1). For $x\in X$, let $W_1(x)=\{\pi a\mid a\in x^\perp\cap X\}$ and $W_2(x)=\{\pi a\mid a\in X, d(a,x)\leqslant 2\}$. Because X is a generalized hexagon of order (q,q), we immediately see that $|W_1(x)|=(q^3-1)/(q-1)$ and $|W_2(x)|=(q^5-1)/(q-1)$. We next show that $W_1(x)$ is a maximal singular subspace of P. Suppose u, $v\in W_1(x)$ with $u\neq v$. Then there are $a, b\in x^\perp\cap X$ with $\pi(a)=u$, $\pi(b)=v$, and $d(a,b)\leqslant 2$. It follows from Lemma (3.5)(iv) that u, v are collinear in P. Thus $W_1(x)$ is a clique of P, so there exists a maximal clique W of P containing $W_1(x)$. Since P is the polar space $B_{3,1}(q)$ or $C_{3,1}(q)$, we have $|W|=(q^3-1)/(q-1)=|W_1(x)|$, whence $W=W_1(x)$.

Now consider $W_2(x)$. Let $\mathbb{P}$ be the standard projective space in which P is embedded. (Thus $\mathbb{P}$ has rank 6, 5 if P is $B_{3,1}(q)$, $C_{3,1}(q)$, respectively.) Denote by H the hyperplane of $\mathbb{P}$ that is orthogonal to $\pi(x)$. Clearly $W_2(x)\subseteq P\cap H$ by (iv) of the previous lemma. Also $|P\cap H|=(q^5-1)(q-1)=|W_2(x)|$, whence $W_2(x)=P\cap H$. It is now obvious that properties (a)–(g) of Section 3 in Cameron and Kantor [5] are satisfied for $P=\pi(X)$ with the graph structure of X. Therefore, from the result (3.2) in [5] it follows that if P is $B_{3,1}(q)$, then X is the classical generalized hexagon of type G_2 and the embedding of X in P is unique, and if P is $C_{3,1}(q)$, then q is a power of two and hence isomorphic to $B_{3,1}(q)$. ∎

(3.6) COROLLARY. *Let* N, X *be as in Theorem* (3.1). *Then*

(i) *If* $x, y \in X$ *and* $d(x, y) = 2$, *then there are a unique* $u \in \{x, y\}^\perp \cap X$ *and a unique set* l *of size* $q+1$ *containing* x, y, *called the ideal line on* x, y, *such that for any* $z \in (N - X) \cap \{x, y\}^\perp$ *we have* $z^\perp \cap u^\perp \cap X = l$.

(ii) *For* $x \in N - X$ *the set* $x^\perp \cap X$ *is an ideal plane of* X (*i.e., supplied with the structure of all ideal lines it contains, it is a projective plane*). *Thus* $|x^\perp \cap X| = q^2 + q + 1$.

(iii) *If* Y *is a subspace of* N *containing a connected component which is locally quad, then* Y *itself is locally quad.*

(iv) *For every pair* $z_1, z_2 \in N - X$ *we have* $d(z_1, z_2^\perp \cap X) \leqslant 2$.

Proof. Parts (i), (ii), and (iii) are immediate consequences of the above.

(iv) Take two distinct points a, b in $z_2^\perp \cap X$. Then $d(a, b) = 2$ so there is a quad Q containing a, b and z_2. Let c be the unique point in $\{a, b\}^\perp \cap X$. Then $c \in Q$ so $c^\perp \cap X = c^\perp \cap Q$ and each line of Q on c bears a point of $z_2^\perp$. Now Q is isomorphic to the classical quad $Sp_4(q)$ and $c^\perp \cap z_2^\perp \cap Q$ corresponds to an elliptic line under the isomorphism. If $z_1 \in Q$, then $d(z_1, a) \leqslant 2$ and we are done. Suppose, therefore, that $z_1 \in N - Q$. By Lemma (3.2)(v), there is $z_3 \in z_1^\perp \cap Q$. Since $c^\perp \cap z_2^\perp \cap Q$ corresponds to an elliptic line under the isomorphism $Q \to Sp_4(q)$, each point of Q is collinear with some point of $c^\perp \cap z_2^\perp \cap Q$. In particular, there is $u \in c^\perp \cap z_2^\perp \cap z_3^\perp \cap Q$. The conclusion is that z_1, z_3, u is a path of length 2 from z_1 to a member of $c^\perp \cap z_2^\perp \cap Q \subseteq z_2^\perp \cap X$, proving that $d(z_1, z_2^\perp \cap X) \leqslant 2$. ∎

4. Toward a Local Recognition of Thick Metasymplectic Spaces

Our goal is to prove the main theorem stated in Section 1. It is obvious that the theorem is a consequence of the following.

(4.1) Theorem. *Suppose* Γ *is a connected gamma space such that for each point* x *the maximal singular spaces containing* x *are thick projective spaces and* Γ^x *is isomorphic to a thick finite dual polar space* (*necessarily of rank* 3). *If condition* $(*)$ *of the main theorem is satisfied, then* Γ *is a metasymplectic space.*

The proof is comparable to the proof of Theorem (2.1) in the following sense. Condition $(*)$ enables us to obtain the analogue of the conclusion in Lemma (2.2) with quads taking over the role of symplecta. With this change from symplecta to quads, statement (i) of Lemma (2.4) also holds. However, due to the generalized hexagons appearing in Theorem (3.1), the argument of the proof of (ii) of that lemma does not suffice to obtain the same conclusion for type $M_{n,r}=F_{4,1}$. Therefore, further study of the subspace $\{x,y\}^{\perp}$ for x, y at mutual distance 2 is required. For x a point of Γ, we shall denote by d^x the distance function in Γ^x.

(4.2) LEMMA. *Let x, y be two points of Γ at mutual distance 2. If $z\in\{x,y\}^{\perp}$, then either $d^z(x,y)=3$ and $\{x,y\}^{\perp}$ is a coclique, or $d^z(x,y)=2$ and $\{x,y\}^{\perp}$ is a generalized quadrangle or a generalized hexagon of order (s,r). In the latter case $\{x,y\}^{\perp}$ induces a locally quad subspace in Γ^x, and hence satisfies all properties of the conclusion in Corollary* (3.6).

Proof. Set $X=\{x,y\}^{\perp}/x$. Observe that $X\cong\{x,y\}^{\perp}$ as $d(x,y)=2$. Assume X is not a coclique. Then there are distinct collinear points a, b in $\{x,y\}^{\perp}$. Since xa, ba, ya is a path of length 2 in $a^{\perp}/a$ we have $d^a(x,y)=2$ and there is a quad Q^a in $a^{\perp}/a$ containing xa, ba, and ya. Take $c\in a^{\perp}-\{a\}$ such that ac belongs to Q^a and is collinear with xa and ya but distinct from ba. Then there is a quad Q^x in $x^{\perp}/x$ containing xb, sa, and xc. Take $u\in x^{\perp}-\{x\}$ such that ux belongs to Q^x and is collinear with xc and xb but distinct from xa. Then u, x, a, y is a path with $d(u,a)=d(x,y)=2$ and b, $c\in\{u,x,a,y\}^{\perp}$, so by hypothesis $(*)$, the size of $\{u,x,a,y\}^{\perp}$ is $r+1$. On the other hand, the set is a coclique in $\{x,a,y\}^{\perp}$ as well as in $\{x,a,u\}^{\perp}$. Since the latter two sets induce $r+1$ lines of Q^a and Q^x on xa, respectively, it follows that $\{x,a,y\}^{\perp}$ induces the $r+1$ lines of Q^x on xa in Γ^x. Thus $\Gamma^x_{\leqslant 1}(ax)\cap X=\Gamma_{\leqslant 1}(ax)\cap Q^x$ and by induction with respect to the length of a path in X starting at a, we obtain that the connected component of X containing a is a locally quad subspace of $x^{\perp}/x$. By Lemma (3.2)(v) and Corollary (3.6)(ii), this implies that $\{x,y\}^{\perp}/x$ is a locally quad subspace of $x^{\perp}/x$, establishing the last statement of the lemma and also that $d^z(x,y)=2$ for any $z\in\{x,y\}^{\perp}$. Due to Theorem (3.1), X is a generalized quadrangle or a generalized hexagon of order (s,r). This settles the case where X is not a coclique. Finally, suppose X is a coclique. Then obviously, $d^z(x,y)>2$, whence $d^z(x,y)=3$ (since the dual polar space $\Delta_{\leqslant 1}$ of rank 3 has diameter 3). ∎

From now on, we shall say that two points x, y of Γ are a *coclique* (*quad, or hex*) *pair* if they have mutual distance 2 and $\{x,y\}^{\perp}$ is a coclique (quad, or hex, respectively).

(4.3) LEMMA. *Suppose x, y are a quad (hex) pair. Then for each point $y^1\in y^{\perp}-(\{y\}\cup x^{\perp})$ with $x^{\perp}\cap yy^1\neq\varnothing$, the pair x, y^1 is also a quad (hex) pair. Moreover, for $z\in x^{\perp}\cap yy^1$, we have $\Gamma^x_{\leqslant 2}(xz)\cap\{x,y\}^{\perp}/x=\Gamma^x_{\leqslant 2}(xz)\cap\{x,y^1\}^{\perp}/x$.*

Proof. Take $z\in x^{\perp}\cap yy^1$. Clearly $d(x,y^1)=2$ and $d^z(x,y^1)=2$, so by Lemma (4.2) the pair x,y^1 is either a quad or a hex pair. Let z, z_1, z_2 be a path in $x^{\perp} \cap y^{\perp}$ with $d(z, z_2) = 2$. Since zz_1, xz_1, z_2z_1, yz_1 is a 4-circuit in $z_1^{\perp}/z_1$ there is $z_2^1 \in (xz_2) \cap (y^1)^{\perp}$. Now x, y is a hex (quad) pair if and only if $\{x, y, z, z_2\}^{\perp}$ consists of exactly one (respectively $r+1$ points). Since $\{x, y, z, z_2\}^{\perp} = \{x, y^1, z, z_2^1\}^{\perp}$, it follows that x, y is a quad pair if and only if x, y^1 is a quad pair and similarly for hex pairs. Also, as $z_2x = z_2^1x$, the argument shows that $\Gamma^x_{\leqslant 2}(xz) \cap \{x, y\}^{\perp}/x = \Gamma^x_{\leqslant 2}(xz) \cap \{x, y^1\}^{\perp}/x$. ∎

(4.4) LEMMA. *Suppose x, y is either a quad or a hex pair. If $z \in x^{\perp} - y^{\perp}$* satisfies $xz \cap y^{\perp} = \varnothing$ *then z, y is a coclique pair. Moreover, if x, y is a quad pair, then $\{x, y, z\}^{\perp}$ is a singleton, and if x, y is a hex pair, then $\{x, y, z\}^{\perp}$ induces an ideal plane in $\{a, y\}^{\perp}$. (In particular its size is $t+1$.)*

Proof. Consider the point zx and the subspace $X = \{x, y\}^{\perp}/x$ in $x^{\perp}/x$. If x, y is a quad or hex pair, then by Lemma (3.2)(v) (resp. Corollary (3.5)(ii)) there is $a \in \{x, y\}^{\perp}$ such that ax is collinear to zx in $x^{\perp}/x$. Thus $a \in \{x, y, z\}^{\perp}$ and $d(z, y) = 2$. Suppose z, y is not a coclique pair. Then Lemma (4.2) yields the existence of a point b in $\{z, a, y\}^{\perp} - \{a\}$. On the other hand, since $d^a(x, y) = 2$, by Lemma (4.2), there is a quad Q^a in $a^{\perp}/a$ containing xa and ya. Now $d^a(z, y) = 2$ and z is collinear with xa in $a^{\perp}/a$. In view of Lemma (3.2)(v), this implies that za belongs to Q^a, so that $xz \cap y^{\perp} \neq \varnothing$, contradicting the hypothesis. The conclusion is that z, y is a coclique pair. Since $\{x, y, z\}^{\perp}/x$ is the set of points in X collinear to xz, the last statement of the lemma follows from Lemmas (2.4) and (3.2)(v) and Corollary (3.6)(ii). ∎

(4.5) LEMMA. *Suppose x, y is a hex pair. If $a, b \in x^{\perp} - \{x\}$ with $d^x(a, b) = 2$, then a, b is a hex pair.*

Proof. In light of $d^x(a, b) = 2$ and Lemma (4.2), either a, b is a hex pair or it is a quad pair. First assume that $a, b \in \{x, y\}^{\perp}$. Then due to $d^x(a, b) = 2$ and Corollary (3.6)(i) there exists $c \in \{a, b, x, y\}^{\perp}$. If a, b were a quad pair, then there would be $d \in \{a, b, x, y\}^{\perp} - \{c\}$ so that a, c, b, d is a quadrangle in $\{x, y\}^{\perp}$, contradicting that x, y is a hex pair. Thus a, b is a hex pair.

Next assume there are distinct points e, f in $\{a, b, x, y\}^{\perp}$. Then e, f are noncollinear, and form a hex pair by the first paragraph, so again by the first paragraph applied to $a, b \in \{e, f\}^{\perp}$ the pair a, b is also a hex pair.

Finally, let us deal with the general case. By Lemmas (3.2) and (3.5)(iii) each line in $x^{\perp}/x$ carries a point of $\{x, y\}^{\perp}/x$. In particular, if $u \in \{x, a, b\}^{\perp} - \{x\}$ there are points $v \in y^{\perp} \cap uxa$ and $w \in y^{\perp} \cap uxb$. If $v \neq w$, then clearly $d^x(v, w) = 2$ so v, w is a hex pair by what we have seen above: now $av \cap w^{\perp} \neq \varnothing$ and $a \notin w^{\perp}$ so a, w is a hex pair by Lemma (4.3). Furthermore, $bw \cap a^{\perp} \neq \varnothing$ and $b \notin a^{\perp}$, so by the same lemma a, b is a hex pair. Now if $v = w$ for all choices of $u \in \{x, a, b\}^{\perp} - \{x\}$, there are at least two points in $\{x, a, b, y\}^{\perp}$, so we can finish by appealing to the second paragraph. ▌

(4.6) LEMMA. *If Γ contains a hex pair, there are no quad pairs in Γ and for each point x there is a point y such that x, y is a hex pair.*

Proof. Let x_0, x_1 be distinct collinear points and assume x_0, y_0 is a hex pair. Take $y_1 \in x_0^{\perp} - \{x\}$ such that $d^{x_0}(x_1, y_1) = 2$. Then by the preceding lemma x_1, y_1 is a hex pair. In view of connectedness of Γ and by induction on the length of a path from an arbitrary point x to x_0, we obtain a point y such that x, y is a hex pair. Suppose that x^1, y^1 are points with $d(x^1, y^1) = 2$ which are not a coclique pair. By Lemma (4.2) there is $z \in \{x^1, y^1\}^{\perp}$ such that $d^z(x^1, y^1) = 2$, and by the first paragraph there is a point of w such that z, w is a hex pair. Then by Lemma (4.5) the pair x^1, y^1 is a hex pair, too. Thus there are no quad pairs, indeed. ▌

(4.7) LEMMA. *If Γ contains a hex pair, then Γ has diameter* 2.

Proof. Let z_1, z_2, z_3, z_4 be a path of length 4 with $d(z_1, z_2) = d(z_2, z_4) = 2$ and $d(z_1, z_4) \geqslant 2$. If z_1, z_3 is a hex pair then by Lemmas (4.3) and (4.4) it follows that $d(z_1, z_4) \leqslant 2$. Suppose therefore that z_1, z_3 is a coclique pair. Then, since $d^{z_2}(z_1, z_3) = 3$, there is $u \in \{z_2, z_3\}^{\perp}$ such that z_1, u is a hex pair (cf. Lemma (4.6)). Now, according to Lemma (4.4), $\{z_1, u, z_3\}^{\perp}$ induces an ideal plane of a generalized hexagon in $u^{\perp}/u$. We first show that $\{z_1, u, z_3\}^{\perp}$ also induces an ideal plane in $z_3^{\perp}/z_3$. Since $\{z_1, u, z_3\}^{\perp}$ is contained as an ideal plane in $\{u, z_1\}^{\perp}$ and the embedding of $\{z_1, u\}^{\perp}/z_1$ is $z_1^{\perp}/z_1$ is unique (see Theorem (3.1)) there exists $z_5 \in \{z_1\}^{\perp} - z_1$ such that $z_1 z_5$ is collinear in $z_1^{\perp}/z_1$ with each point of $\{z_1, u, z_3\}^{\perp}/z_1$. Thus $\{z_5, u, z_1\}^{\perp} = \{z_3, u, z_1\}^{\perp} = \{z_1, z_5, u, z_3\}^{\perp}$. Take $z_2^1 \in \{z_1, u, z_3\}^{\perp} - (z_2\}$. Then $d^{z_1}(z_2, z_2^1) = 2$ as $z_2 \perp z_5 \perp z_2^1$, so z_2, z_2^1 is a hex pair containing z_1, z_5, u, z_3. Since u, z_1 is a hex pair, we have $d^{z_2}(u, z_1) = 2$. By Corollary (3.6), this implies the existence of $v \in \{z_2, z_2^1, u, z_1\}^{\perp}$. Now z_3, u, v, z_1, z_5 is a path of length 5 in the generalized hexagon $\{z_2, z_2^1\}^{\perp}$. By the definition of generalized hexagon, there must be $z_5^1 \in z_1 z_5$ with $d^{z_2}(z_3, z_5^1) = 2$. By Lemma (4.2), z_3, z_5^1 is a hex pair, and by Lemma (4.3), u, z_5^1 is a coclique pair and $\{z_3, u, z_1\}^{\perp} = \{z_5, u, z_1, z_3)^{\perp} = \{z_5^1, u, z_1, z_3\}^{\perp} = \{z_5^1, u, z_3\}^{\perp}$. Thus $\{z_3, u, z_1\}^{\perp} = \{z_5^1, u, z_3\}^{\perp}$ induces an ideal plane of the generalized hexagon $\{z_3, z_5^1\}^{\perp}/z_3$ in $z^{\perp}/z_3$.

By Corollary (3.6)(iv), there is $w \in \{z_3, z_5', u\}^{\perp}$ with $d^{z_3}(w, z_4) \leqslant 2$. Observe that $w \in z_1^{\perp}$ as $\{z_5^1, u, z_3\}^{\perp} = \{z_1, u, z_3\}^{\perp}$. So if $d(w, z_4) = 1$, then $d(z_1, z_4) = 2$. Suppose $d^{z_3}(w, z_4) = 2$. Then, by Lemma (4.2), w, z_4 is a hex pair and $z_1 \in w^{\perp}$, so by the first paragraph of the proof applied to a path from z_1 to z_4 via w, $d(z_1, z_4) = 2$. The conclusion is that $d(z_1, z_4) = 2$ in all cases, so that, being connected, Γ has diameter 2. ▌

(4.8) LEMMA. *Suppose Γ contains a hex pair. If x, z is a coclique pair of Γ, then there is a quad of $x^{\perp}/x$ meeting $\{x, z\}^{\perp}/x$ in an elliptic line.*

Proof. Take $a \in \{x, z\}^{\perp}$. As $d^{a}(x, z) = 3$, there exists $y \in \{a, z\}^{\perp}$ such that x, y is a hex pair. By Lemma (4.6) the subspace $\{x, y, z\}^{\perp}/x$ of $y^{\perp}/y$ is an ideal plane in $\{x, y\}^{\perp}/y$, so there is $z^{1} \in x^{\perp} - \{x\}$ with $\{x, y, z^{1}\}^{\perp} = \{x, y, z\}^{\perp}$. Take $b \in \{x, y, z^{1}\}^{\perp} - \{a\}$. Then $d^{x}(a, b) = 2$, so there is a quad Q^{x} in $x^{\perp}/x$ containing both ax and bx. Let c be the unique common neighbor of a and b in $x^{\perp} \cap y^{\perp}$ (i.e., $\{x, y, a, b\}^{\perp} = \{c\}$). Now $L = \{c, z, x, y\}^{\perp}$ consists of $q+1$ points (a, b are among them) such that L/x is an elliptic line of Q^{x}. Therefore L/x is a maximal coclique in Q^{x}. Since $Q^{x} \cap \{x, y\}^{\perp}/x$ is a coclique of Q^{x} containing L/x, it follows that $L/x = Q^{x} \cap (\{z, x\}^{\perp}/x)$, proving the lemma. ∎

(4.9) LEMMA. *If x, y are two points of Γ at mutual distance 2, then they are either a quad pair or a coclique pair.*

Proof. Suppose there is a hex pair in Γ. Let x, z be a coclique pair. Let γ_1 (γ_x, γ_h) be the number of points y in $z^{\perp}$ such that $y \in X^{\perp}$ (x, y is a coclique pair; x, y is a hex pair, respectively). Counting $\#\{(u, y) \mid u \in \{x, z\}^{\perp}, y \in z^{\perp} \cap u^{\perp};\ x, y \text{ is a hex pair}\}$ in two ways we get $\gamma_h(t^2+q+1) = \gamma_1(q+1)\, q = \gamma_1(q^2+q+1)\, q$ so that

$$\gamma_h = q\gamma_1. \tag{1}$$

For u ranging over $\{x, z\}^{\perp}$, let α be the average of $\alpha_u = \#(x^{\perp} \cap z^{\perp} \cap \Gamma_2(u))$. Then counting $\#\{(u, v, y) \mid u, v \in \{x, z\}^{\perp}, u \neq v, y \in \{z, u, v\}^{\perp}, x, y \text{ is a hex pair}\}$ in two ways, we obtain $\gamma_1 \cdot \alpha \cdot (q+1) = \gamma_h(q^2+q+1)(q^2+q)$. So eliminating γ_h by use of (1) and dividing by $\gamma_1(q+1)$ we get

$$\alpha = q^2(q^2+q+1). \tag{2}$$

However, $x^{\perp} \cap z^{\perp} \cap \Gamma_2(u)$ induces a coclique in $x^{\perp}/x$ on the set of points in $x^{\perp}/x$ at distance two from ux. Since there are $q^2(q^2+q+1)$ lines in $x^{\perp}/x$ having exactly one point collinear with ux, it follows that $\alpha_u \leqslant q^2(q^2+q+1)$. In view of (2), we get $\alpha_u = q^2(q^2+q+1)$ for each $u \in \{x, z\}^{\perp}$. Hence, with respect to distance 2 in Γ the set $\{x, z\}^{\perp}$ is a graph of valency $q^2(q^2+q+1)$ and for each $u \in \{x, z\}^{\perp}$ each line of $x^{\perp}/x$ bearing a unique point collinear with ux contains a point vx where $v \in \{x, z\}^{\perp}$. Now let Q^{x} be a quad in $x^{\perp}/x$ meeting $\{z, x\}^{\perp}/x$ in an elliptic line (existence is guaranteed by Lemma (4.8)). Take $u \in \{x, z\}^{\perp}$ such that $ux \in Q^{x}$. Then by the above, each line of Q^{x} bearing a unique point at distance 1 to ux contains a point vx where $v \in \{x, z\}^{\perp}$, leading to q^2+1 points of $\{x, z\}^{\perp}/z$ inside Q^{x}. On the other hand, the intersection of Q^{x} and c is an elliptic line and hence of size $q+1$. Thus $q+1=q^2+1$, o the contradiction $q=1$. We conclude that Γ contains no hex Lemma (4.5), this ends the proof of the lemma. ▮

k. Other computations led to the question of existence of partial es PG(s, t, α) with parameters $s=q^2$, $t=q^2+q$, $\alpha=q+1$ leading ngly regular graphs with parameters $v=(1+q^3)(1+q^2)$, $^2+q+1)$, $\lambda=q^3+2q^2-1$, $\mu=(q^2+q+1)(q+1)$ (induced by x in $x^{\perp}/x$). Having excluded the hex pairs, we can resume the with the proof of Theorem (2.1). In fact, we have just established clusion of Lemma (2.4)(iii) for $M_{n,r}=F_{4,1}$. The next step is the e of Corollary (2.5).

COROLLARY. *The space Γ is a parapolar space.*

Copy the proof of Corollary (2.5), case (b). ▮

) *End of Proof of Theorem* 4.1. Since Γ is a parapolar space, there plecta (see [8]). A geometry of type F_4 can now be obtained by as objects of type 1, 2, 3, 4 the sets of points, lines, planes, and cta, respectively, and as incidence symmetrized containment. It is forward to verify that all residues of type B_3 or C_3 are polar spaces t conditions (O), (LL), (LH), and (HH) of Tits [14] are satisfied. of [14, Proposition 9], we conclude that this geometry is a build-type F_4 whose shadow space on the set of points is Γ, so that Γ is symplectic space. This ends the proof of Theorem (4.1). ▮

REFERENCES

. BROUWER AND A. M. COHEN, Local recognition of Tits geometries of classical type, *m. Dedicata* **20** (1986), 181–199.

UEKENHOUT AND B. FISCHER, A locally dual polar space for the monster, preprint.

BUSET, Graphs which are locally a cube, *Discrete Math.* **46** (1983), 221–226.

. CAMERON, Dual polar spaces, *Geom. Dedicata* **12** (1982), 75–85.

. CAMERON AND W. M. KANTOR, 2-Transitive and antiflag transitive collineation ups of finite projective spaces, *J. Algebra* **60** (1979), 384–422.

M. COHEN, An axiom system for metasymplectic spaces, *Geom. Dedicata* **12** (1982), –433.

M. COHEN, Points and lines in metasymplectic spaces, *Ann. Discrete Math.* **18** (1983), –196.

M. COHEN AND B. N. COOPERSTEIN, A characterization of some geometries of excep-al Lie type, *Geom. Dedicata* **15** (1983), 73–105.

JOHNSON AND E. E. SHULT, Local characterizations of polar spaces, *Geom. Dedicata* **28** 88), 127–151.

10. M. A. RONAN AND G. STROTH, Minimal parabolic geometries for the sporadic groups, *Europ. J. Combin.* **5** (1984), 59–91.
11. E. E. SHULT, Characterizations of the Lie incidence geometries, *in* "Surveys in Combinat," pp. 157–186. LMS Lecture Note Series 82 (E. K. Lloyd, Ed.), Cambridge Univ. Press, Cambridge, 1983.
12. E. E. SHULT AND A. YANUSHKA, Near n-gons and line systems, *Geom. Dedicata* **9** (1980), 1–72.
13. J. TITS, Buildings of spherical type and finite BN pairs, *in* "Lecture Notes in Mathematics, Vol. 386," Springer-Verlag, Berlin.
14. J. TITS, A local approach to buildings, *in* "The Geometric Vein, the Coxeter Festschrift," pp. 519–547, Springer-Verlag, Berlin, 1982.