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A note on pooling of labels in random fields

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ABSTRACT

This paper studies the effect on the interaction structure arising from merging labels in certain classes of random field models.

1. Introduction

In image segmentation, often Markov random field models are employed in which neighbouring (blocks of) pixels have a similar texture (Winkler, 2003). Generally, the number of different textures is unknown. To overcome this problem, a number of Bayesian approaches have been suggested in which the unknown number is treated as a random variable. See for example Dryden et al. (2003) or Green and Richardson (2002).

A practical problem with such approaches is that most Markov random field densities are known only up to a normalising constant. When updating the number of labels in a Monte Carlo method, the normalising constants do not cancel out and have to be approximated. A more fundamental problem is that the interaction structure may change dramatically if two labels are pooled together, in other words, the class of Markov random fields is not closed under merging labels, making them unnatural models in an unsupervised image segmentation algorithm. In contrast, we show that the class of Markov connected component models (Møller and Waagepetersen, 1998) is closed under the above mentioned operation, and hence may provide more natural prior distributions for image segmentation with an unknown number of different textures.

The plan of this paper is as follows. First we review some random field theory in Section 2, while the main part Section 3 studies the effect on the interaction structure of changes in the number of labels and compares the results to their counterparts in a continuous point process set-up (Chin and Baddeley, 1999; Van Lieshout, 2000b). The paper closes with a short discussion.

2. Markov and connected component fields

Let $S = (s_1, \ldots, s_m)$ be a collection of sites, for example a rectangle of raster points in $\mathbb{Z}^2$, and assume there is a symmetric, reflexive relation $\sim$ on S. In a graph theoretical interpretation, the sites are the vertices, and an edge is drawn between s

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and $r, s \neq r$, if and only if $s \sim r$. Each site is assigned a label, or colour, from the set $\Lambda = \{1, \ldots, q\}, q \geq 2$, at random according to some probability distribution. The labels are nominal, so any convenient set of q distinct numbers may be used. The random field thus obtained is denoted by $X = (X_1, \ldots, X_m)$ with X_i indicating the label at s_i.

There is a close connection between the notion of a random field and the physics concept of a *Gibbs state*. Recall that X is a Gibbs state with interaction potentials $\{V_A : A \subseteq S\}$ if

$$P(X_1 = x_1, \ldots, X_m = x_m) = \frac{1}{Z} \exp\left[\sum_{A \subseteq S} V_A(x_1, \ldots, x_m)\right] \tag{1}$$

for $V_A : \Lambda^S \to \mathbb{R}$ such that $V_\emptyset(\cdot) \equiv 0$ and $V_A(\cdot)$ depends only on the labelling at sites in A. The potential V is normalised with respect to the label $\ell \in \Lambda$ if $x_a = \ell$ for some $s_a \in A$ implies $V_A(x_1, \ldots, x_m) = 0$. In fact, any random field with strictly positive probability mass function is a Gibbs state with respect to the ℓ-canonical potential

$$V_A(x_1, \ldots, x_m) = \sum_{B \subseteq A} (-1)^{|A \setminus B|} \log P(\mathbf{x}^B)$$

where $\mathbf{x}_i^B = x_i$ for $s_i \in B$ and a pre-fixed value $\ell \in \Lambda$ otherwise. The notation $|A \setminus B|$ is used for the cardinality of the set $A \setminus B$. Note that the above potential is the unique potential that is normalised with respect to label ℓ. See Geman (1990) for proofs and further details. From now on, we shall assume that $P(\cdot)$ is strictly positive so that conditional distributions are well-defined, and use the equivalent Gibbs state formulation ad libitum.

A random field X with probability mass function $P(\cdot)$ is said to be *Markov* with respect to $\sim$ if for all $i = 1, \ldots, m$ the conditional probability mass function

$$P(X_i = x_i \mid X_j = x_j, j \neq i) = P(X_i = x_i \mid X_j = x_j, s_j \sim s_i, j \neq i) \tag{2}$$

depends only on x_i and the labels at those sites $s_j, j \neq i$, that share an edge with s_i (Geman, 1990; Winkler, 2003). The collection $\partial(s_i)$ of such sites is called the *neighbourhood* of the ith site; the conditional distributions in (2) are known as the *local characteristics*.

To characterise Markov random fields, we need the following definition. A *clique* is a subset $C \subseteq S$ for which $s \sim t$ for all $s, t \in C$. Note that singletons and the empty set $\emptyset$ are cliques. Write $\mathcal{C}$ for the class of all cliques in S. Then, the Hammersley–Clifford theorem (Clifford, 1990) states that X is a Markov random field if and only if its probability mass function can be written as

$$P(X_1 = x_1, \ldots, X_m = x_m) = \prod_{C \in \mathcal{C}} \varphi_C(x_c, s_c \in C) \tag{3}$$

for some interaction functions $\varphi_C : \Lambda^C \to \mathbb{R}^+$ defined for each clique $C \in \mathcal{C}$. Eq. (3) amounts to saying that X is a Gibbs state with interaction potentials $V_C(\cdot) = \log \varphi_C(\cdot)$ for non-empty C.

Recall that $K \subset S$ is a *connected component* with respect to $\sim$ if it is non-empty and for every $s, r \in K$ there exists a path $s = s_1 \sim \cdots \sim s_n = r$ with $s_i \in K$. The *Markov connected component fields* proposed by Møller and Waagepetersen (1998) are defined by a factorisation of the form

$$P(X_1 = x_1, \ldots, X_m = x_m) = \frac{1}{Z} \prod_{K \in \mathcal{K}(\mathbf{x})} \Psi(K, l(\mathbf{x}_K)) \tag{4}$$

where the product ranges over the maximal $\sim$-connected components of identically labelled sites in $\mathbf{x} = \mathbf{x}^S = (x_1, \ldots, x_m)$, $l(\mathbf{x}_K)$ denotes the common label in the component $\mathbf{x}_K = (x_k, s_k \in K)$, and $\Psi(\cdot, \cdot)$ is a positive function on $\mathcal{K} \times \Lambda$, the product space of all $\sim$-connected components and the label set. Thus, the cliquewise interactions in (3) are replaced by connected componentwise interactions in (4). In general, a local dependence definition does not exist, except when one of the colours may be regarded as background.

It is important to observe that the two classes of Markov and Markov connected component fields are not comparable in the sense that neither class is contained in the other; see Møller and Waagepetersen (1998) for further details.

2.1. Example: the Potts model

Let $S = (s_1, \ldots, s_m)$ be a finite set of sites, $\sim$ a symmetric, reflexive relation on S and $\Lambda = \{1, \ldots, q\}, q \geq 2$, a finite set of labels. Then the *Potts model* is a Λ^S-valued random variable with joint probability mass function

$$\pi(x_1, \ldots, x_m) = \frac{1}{Z} \exp\left[-\beta \sum_{s_i \sim s_j, i < j} \mathbf{1}\left\{x_i \neq x_j\right\}\right]. \tag{5}$$

The parameter $\beta \in \mathbb{R}$ is known as the reciprocal temperature, Z is a normalising constant that ensures $\sum_{\mathbf{x} \in \Lambda^S} \pi(\mathbf{x}) = 1$. The special case $q = 2$ is known as the *Ising model*. Thus, (5) is of the form (1) with $V_C(x_i, x_j) = -\beta\, \mathbf{1}\left\{x_i \neq x_j\right\}$ for the two-point set $C = \{s_i, s_j\}$ with $s_i \sim s_j$, and $V_A \equiv 0$ otherwise.

The Potts model (5) has local characteristics satisfying

$$\frac{P(X_i=\ell \mid X_j=x_j, j\neq i)}{P(X_i=1 \mid X_j=x_j, j\neq i)} = \frac{\exp\left[-\beta \sum_{s_i\sim s_j} \mathbf{1}\left\{x_j\neq \ell\right\}\right]}{\exp\left[-\beta \sum_{s_i\sim s_j} \mathbf{1}\left\{x_j\neq 1\right\}\right]}$$

for any $\ell \in \Lambda$ and $x_j \in \Lambda, j \neq i \in \{1,\ldots,m\}$. We conclude that X is Markov with respect to $\sim$ for all q. If $\beta > 0$, majority voting amongst the neighbours of the site of interest determines which label has the highest conditional probability (ferromagnetic case); for $\beta < 0$, the label disagreeing with most of the neighbours' ones is most likely (antiferromagnetic case). The interaction functions are $\varphi_\emptyset = 1/Z$, $\varphi_C(x_i,x_j) = \exp\left[-\beta\,\mathbf{1}\left\{x_i\neq x_j\right\}\right]$ for cliques $C=\{s_i,s_j\}$ that consist of a pair of neighbours $s_i\sim s_j$, and $\varphi_C\equiv 1$ otherwise.

Since

$$\pi(x_1,\ldots,x_m) = \frac{1}{Z}\exp\left[-\frac{\beta}{2}\sum_{K\in\mathcal{K}(\mathbf{x})}\sum_{s_i\in K}|\{s_j\notin K : s_i\sim s_j\}|\right],$$

the Potts model is also a Markov connected component field with respect to $\sim$.

3. Merging of labels

Chin and Baddeley (1999) proved that the class of Markov connected component point processes (Baddeley and Møller, 1989) is closed under independent superposition. The purpose of the present paper is to show that the same is true for Markov connected component fields under merging of labels. Formally, if X is a random field with labels in $\{1,\ldots,q\}$, we replace the labels $q-1$ and q by a single label 0 (say) to obtain the field Y with label set $\{0,1,\ldots,q-2\}$. Note that as the labels are nominal values, we could have merged $q-1$ and q into the label "$q-1$" but chose not to in order to avoid confusion.

Theorem 1. *Let $S=(s_1,\ldots,s_m)$ be a finite set of sites, $\sim$ a symmetric, reflexive relation on S and $\Lambda=\{1,\ldots,q\}$, $q\geq 2$, a finite set of labels. Let X be a q-colour Markov connected component field (4) with respect to $\sim$ and define the random field Y with values in $\{0,\ldots,q-2\}$ by*

$$Y_i = X_i\,\mathbf{1}\{X_i\leq q-2\}$$

for $i=1,\ldots,m$. Then Y is a $(q-1)$-colour Markov connected component field with respect to $\sim$.

Proof. Fix $\mathbf{y}=(y_1,\ldots,y_m)\in\{0,\ldots,q-2\}^S$. For $\mathbf{x}\in\Omega_{\mathbf{y}} := \{\mathbf{x}\in\Lambda^S : y_i = x_i\,\mathbf{1}\{x_i\leq q-2\}, i=1,\ldots,m\}$ and $j\in\Lambda$, write $\mathcal{K}_j(\mathbf{x})$ for the set of maximal connected components in $\mathbf{x}$ labelled j. Note that the maximal j-labelled connected components in $\mathbf{x}$ and $\mathbf{y}$ are identical for $j=1,\ldots,q-2$, and that each $(q-1)$- or q-component is part of a single maximal 0-component in $\mathbf{y}$. Denote the family of these 0-components by $\mathcal{K}_0(\mathbf{y})$. Now, the probability mass function of Y is given by

$$\pi_Y(\mathbf{y}) = \sum_{\mathbf{x}\in\Omega_{\mathbf{y}}}\pi(\mathbf{x}) = \frac{1}{Z}\prod_{j=1}^{q-2}\prod_{K\in\mathcal{K}_j(\mathbf{y})}\Psi(K,j)\times\prod_{K\in\mathcal{K}_0(\mathbf{y})}\left\{\sum_{\mathbf{z}\in\{q-1,q\}^K}\prod_{j=q-1}^{q}\prod_{L\in K_j(\mathbf{z})}\Psi(L,j)\right\}.$$

Thus, π_Y admits a factorisation of the form (4) with $\Psi_Y(K,j) = \Psi(K,j)$ for $j\in\{1,\ldots,q-2\}$ and $\Psi_Y(K,0) = \sum_{\mathbf{z}\in\{q-1,q\}^K}\prod_{j=q-1}^{q}\prod_{L\in K_j(\mathbf{z})}\Psi(L,j)$. The claim is proved. □

Note that the normalisation constant Z involved in π_Y is the same as that in π, the probability mass function of X.

The factorisation in the proof above should be compared to that for point processes; see Chin and Baddeley (1999) or page 73 in Van Lieshout (2000a).

A similar result does not hold for Markov random fields. Here the situation is more complicated, reflecting the state of affairs for Markov point processes with respect to superposition (Van Lieshout, 2000a,b).

Example 1. Consider the Potts model X introduced in Section 2.1, and, for $i=1,\ldots,m$, set $Y_i = X_i\mathbf{1}\{X_i\leq q-2\}$. Then, using the notation introduced in the proof of Theorem 1, the probability mass function of Y equals

$$\pi_Y(\mathbf{y}) = \sum_{\mathbf{x}\in\Omega_{\mathbf{y}}}\frac{1}{Z}\exp\left[-\beta\sum_{s_i\sim s_j, i<j}\mathbf{1}\left\{x_i\neq x_j\right\}\right] = \frac{1}{Z}\exp\left[-\beta\sum_{s_i\sim s_j, i<j}\mathbf{1}\left\{y_i\neq y_j\right\}\right]$$
$$\times\prod_{K\in\mathcal{K}_0(\mathbf{y})}\left\{\sum_{\mathbf{z}\in\{q-1,q\}^K}\exp\left[-\beta\sum_{s_i\sim s_j\in K, i<j}\mathbf{1}\left\{z_i\neq z_j\right\}\right]\right\}. \quad (6)$$

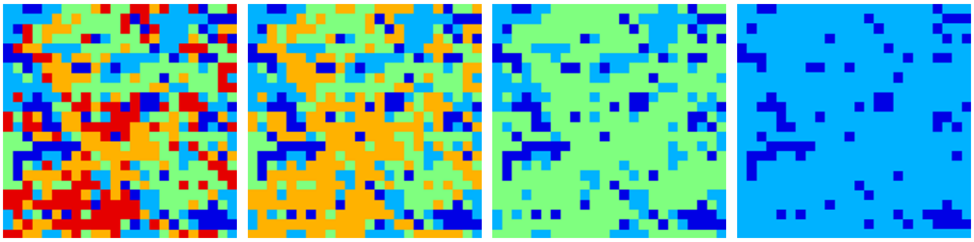

Fig. 1. Realisation of a Potts model with $q = 5$ labels, the first-order neighbour relation $s_i \sim s_j \Leftrightarrow \|s_i - s_j\| \leq 1$, and interaction parameter $\beta = 1$ (leftmost panel) after merging two labels successively (other panels).

Thus, (6) is proportional to the product of the probability mass functions of a Potts model with $q - 1$ colours and a Markov connected component field on the maximal 0-components collected in $\mathcal{K}_0(\mathbf{y})$. Realisations of X and Y with $q = 5$ are shown in the two leftmost panels of Fig. 1.

We conclude that merging two labels in a Potts model yields a Markov connected component field with respect to the underlying relation $\sim$. One might wonder whether Y is Markov with respect to $\sim$. This is not the case, as can be seen from the following counterexample.

Counterexample 1. Consider the following configuration with sites indexed in row major order and first-order neighbour relation $s_i \sim s_j \Leftrightarrow \|s_i - s_j\| \leq 1, i = 1, \ldots, 4$:

0	?
0/1	0

Let X be a Potts model with $q = 3$ and $\beta = 1$, and merge labels 2 and 3 into a single label 0 to obtain the random field Y. We are interested in the marginal distribution of Y at the top right site given that its neighbours are assigned value 0. Straightforward computations yield $P(Y_2 = 1; Y_1 = Y_3 = Y_4 = 0) = 2e^{-2}(1 + e^{-1})^2/Z = P(Y_1 = Y_2 = Y_4 = 0; Y_3 = 1)$. Also $P(Y_1 = Y_2 = Y_3 = Y_4 = 0) = (2 + 2e^{-4} + 12e^{-2})/Z$, and $P(Y_1 = Y_4 = 0; Y_2 = Y_3 = 1) = 4e^{-4}/Z$. We obtain respectively

$$P(Y_2 = 1 \mid Y_1 = Y_3 = Y_4 = 0) = \frac{e^{-2}(1 + e^{-1})^2}{1 + 7e^{-2} + 2e^{-3} + 2e^{-4}}$$

and

$$P(Y_2 = 1 \mid Y_1 = Y_4 = 0; Y_3 = 1) = \frac{2e^{-2}}{1 + 2e^{-1} + 3e^{-2}}.$$

Hence Y is not first-order Markov.

In the above counterexample, the neighbours of the pixel of interest were labelled 0. To see why, consider any q-colour Markov random field X with respect to some relation $\sim$, and focus (without loss of generality) on the site s_1. Let $(y_2, \ldots, y_m)$ be a configuration on the remaining sites such that the labels fall in $\{1, \ldots, q-2\}$ on the neighbourhood $\partial(s_1)$ of s_1. Define Y as in Theorem 1. Then, for $\ell \in \{1, \ldots, q-2\}$ and $\mathbf{y} = (\ell, y_2, \ldots, y_m)$,

$$\begin{aligned} P(Y_1 = \ell; Y_j = y_j, j \neq 1) &= \sum_{\mathbf{x} \in \Omega_{\mathbf{y}}} P(X_1 = l; X_j = x_j, j \neq 1) \\ &= \sum_{\mathbf{x} \in \Omega_{\mathbf{y}}} P(X_1 = \ell \mid X_j = x_j, s_j \in \partial(s_1)) P(X_j = x_j, j \neq 1) \\ &= P(Y_1 = \ell \mid Y_j = y_j, s_j \in \partial(s_1)) P(Y_j = y_j, j \neq 1) \end{aligned}$$

where as before $\Omega_{\mathbf{y}} = \{\mathbf{x} \in \Lambda^S : y_i = x_i \mathbf{1}\{x_i \leq q-2\}\}$. Hence

$$P(Y_1 = \ell \mid Y_j = y_j, j \neq 1) = P(Y_1 = \ell \mid Y_j = y_j, s_j \in \partial(s_1))$$

for any $\ell \neq 0$. Since probabilities add to unity, also

$$P(Y_1 = 0 \mid Y_j = y_j, j \neq 1) = P(Y_1 = 0 \mid Y_j = y_j, s_j \in \partial(s_1)).$$

For homogeneous models such as the Potts model in Counterexample 1, a single neighbour of the pixel of interest may be set to zero without affecting the relation (2) for Y. A similar phenomenon is observed for Markov point processes that are invariant under independent superposition up to second order (Van Lieshout, 2000a, p. 59–61).

Next, we turn our attention to the class of random fields that are both Markov and connected component Markov, characterised by Møller and Waagepetersen (1998) by a factorisation of the probability mass function as follows:

$$\pi(x_1, \ldots, x_m) = \frac{1}{Z} \prod_{K \in \mathcal{K}(\mathbf{x})} \prod_{\emptyset \neq C \in \mathcal{C}, C \subseteq K} \Phi(C, l(\mathbf{x}_K)). \tag{7}$$

For instance for the Potts model introduced in Section 2.1, $\Phi(C, l(\mathbf{x}_K)) = \exp[-\frac{\beta}{2}|\{s_j \notin K : c \sim s_j\}|]$ for $C = \{c\}$ with $c \in K$ and 1 otherwise.

We have the following corollary to Theorem 1.

Corollary 1. *Let $S = (s_1, \ldots, s_m)$ be a finite set of sites, $\sim$ a symmetric, reflexive relation on S and $\Lambda = \{1, \ldots, q\}$, $q \geq 2$, a finite set of labels. Let X be a q-colour Markov connected component field with respect to $\sim$ with a probability mass function given by (7) and define the random field Y with values in $\{0, \ldots, q-2\}$ by*

$$Y_i = X_i \mathbf{1}\{X_i \leq q-2\}$$

for $i = 1, \ldots, m$. Then Y is a $(q-1)$-colour Markov connected component field with respect to $\sim$ with

$$\Psi(K, 0) = \sum_{\mathbf{z} \in \{q-1,q\}^K} \prod_{j=q-1}^{q} \prod_{L \in K_j(\mathbf{z})} \prod_{\emptyset \neq C \in \mathcal{C}, C \subseteq L} \Phi(C, j).$$

In general, Y is not a Markov random field; see Counterexample 1.

For any Markov random field, the following result holds true.

Theorem 2. *Let $S = (s_1, \ldots, s_m)$ be a finite set of sites, $\sim$ a symmetric, reflexive relation on S and $\Lambda = \{1, \ldots, q\}$, $q > 2$, a finite set of labels. Let X be a Markov random field (3) with respect to $\sim$ and define the random field Y with values in $\{0, \ldots, q-2\}$ by*

$$Y_i = X_i \mathbf{1}\{X_i \leq q-2\}$$

for $i = 1, \ldots, m$. Then Y has probability mass function $\pi_Y(\mathbf{y})$ given by

$$\prod_{\mathcal{C} \ni C \subseteq (\cup \mathcal{K}_0(\mathbf{y}))^c} \varphi_C(y_c, s_c \in C) \prod_{K \in \mathcal{K}_0(\mathbf{y})} \left\{ \sum_{\mathbf{z} \in \{q-1,q\}^K} \prod_{\mathcal{C} \ni C \cap K \neq \emptyset} \varphi_C(z_c, s_c \in C \cap K; y_d, s_d \in C \setminus K) \right\}.$$

Proof. Fix $\mathbf{y} = (y_1, \ldots, y_m) \in \{0, \ldots, q-2\}^S$, and write $\Omega_{\mathbf{y}} := \{\mathbf{x} \in \Lambda^S : y_i = x_i \mathbf{1}\{x_i \leq q-2\}, i = 1, \ldots, m\}$. Then the probability mass function of Y is given by

$$\pi_Y(\mathbf{y}) = \sum_{\mathbf{x} \in \Omega_{\mathbf{y}}} \pi(\mathbf{x}) = \prod_{\mathcal{C} \ni C \subseteq (\cup \mathcal{K}_0)^c} \varphi_C(y_c, s_c \in C) \sum_{\mathbf{z} \in \{q-1,q\}^{\mathcal{K}_0(\mathbf{y})}} \prod_{\mathcal{C} \ni C \cap \mathcal{K}_0(\mathbf{y}) \neq \emptyset} \varphi_C(z_c, s_c \in C \cap \mathcal{K}_0(\mathbf{y}); y_d, s_d \in C \setminus \mathcal{K}_0(\mathbf{y})).$$

The key observation is that if $y_i = 0 = y_j$ and $s_i \sim s_j$ then s_i and s_j must belong to the same maximal 0-component in $\mathbf{y}$, so no clique can contain points from disjoint $K, L \in \mathcal{K}_0(\mathbf{y})$. Consequently, π_Y is as claimed. □

We conclude that Y is not necessarily a Markov connected component field. A counterexample is the Geman and Reynolds field (Geman and Reynolds, 1992). More precisely, let $S = \{s_1, s_2\}$ consist of two related sites, and take $q = 6$. Define a random field X by its probability mass function $\pi(x_1, x_2) \propto \exp[1/(|x_1 - x_2| + 1)]$. Then clearly X is a Markov random field with

$$\phi_S(x_1, x_2) = \exp[1/(|x_1 - x_2| + 1)],$$

$\phi_{s_1}(x_1) = \phi_{s_2}(x_2) = 1$, and the interaction function of the empty set equal to the normalising constant. However, by arguments similar to those in Example A2 of Møller and Waagepetersen (1998), the density of Y does not factorise over the maximal connected components.

4. Conclusion

In this paper, we considered the effect of merging labels on the interaction structure of random fields. It was shown that the class of Markov connected component fields is closed under the merge operation, and explicit expressions were derived for the component potentials. We then proved that the class of Markov random fields is not closed under the merge operation, and the resulting random field does not necessarily satisfy the Markov connected component condition.

Our results suggest that models such as those proposed by Møller and Waagepetersen (1998) are natural prior distributions for image segmentation with an unknown number of different textures. Except for trivial cases, Markov chain Monte Carlo methods will be needed in order to calculate an 'optimal' segmentation. As such a Markov chain in principle must be able to visit all possible labellings, transitions that assign label q or $q-1$ to the 0-components of $\mathbf{y}$, say, to obtain $\mathbf{x}$ must be considered. There are many valid options, for example picking the two labels independently and uniformly. Alternatively, and more naturally, one could sample from the conditional distribution $\pi(\mathbf{x})1\{\mathbf{x} \in \Omega_{\mathbf{y}}\}/\pi_Y(\mathbf{y})$. For a Markov connected component field, this conditional probability can be written as

$$\prod_{K \in \mathcal{K}_0(\mathbf{y})} \left\{ \frac{\prod_{j=q-1}^{q} \prod_{K \in K_j(\mathbf{x})} \Psi(K, j)}{\sum_{\mathbf{z} \in \{q-1,q\}^K} \prod_{j=q-1}^{q} \prod_{L \in K_j(\mathbf{z})} \Psi(L, j)} \right\} \tag{8}$$

by the proof of Theorem 1, so the **x**-labels in different maximal 0-components of **y** are assigned independently. Switching between labels in this way is amenable to Gibbs sampling (Geman, 1990; Winkler, 2003) and involves no ratio of normalising constants. Instead, a sample from (8) is required. Since single texture components tend be small compared to the image size, and only two labels have to be considered, such label assignment is quite feasible.

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References

Baddeley, A.J., Møller, J., 1989. Nearest-neighbour Markov point processes and random sets. Internat. Statist. Rev. 57, 89–121.
Chin, Y.C., Baddeley, A.J., 1999. On connected component Markov point processes. Adv. Appl. Probab. 31, 279–282.
Clifford, P., 1990. Markov random fields in statistics. In: Grimmett, G.R., Welsh, D.J.A. (Eds.), Disorder in Physical Systems. A Volume in Honour of J.M. Hammersley. Clarendon Press, Oxford, pp. 19–32.
Dryden, I.L., Scarr, M.R., Taylor, C.C., 2003. Bayesian texture segmentation of weed and crop images using reversible jump Markov chain Monte Carlo methods. J. Roy. Statist. Soc. Ser. C 52, 31–50.
Geman, D., 1990. Random fields and inverse problems in imaging. In: École d'Été de Probabilités de Saint-Flour XVIII–1988. In: Lecture Notes in Mathematics, vol. 1427. Springer-Verlag, Berlin, pp. 113–193.
Geman, D., Reynolds, G., 1992. Constrained restoration and the recovery of discontinuities. IEEE Trans. Pattern Anal. Mach. Intell. 14, 367–383.
Green, P.J., Richardson, S., 2002. Hidden Markov models and disease mapping. J. Amer. Statist. Assoc. 97, 1055–1070.
Van Lieshout, M.N.M., 2000a. Markov Point Processes and their Applications. Imperial College Press, World Scientific, London, Singapore.
Van Lieshout, M.N.M., 2000b. Propagation of spatial interaction under superposition. In: Heuvelink, G.B.M., Lemmens, M.J.P.M. (Eds.), Accuracy 2000. Proceedings of the 4th International Symposium on Spatial Accuracy Assessment in Natural Resources and Environmental Sciences. Delft University Press, Delft, pp. 687–694.
Møller, J., Waagepetersen, R.P., 1998. Markov connected component fields. Adv. Appl. Probab. 30, 1–35.
Winkler, G., 2003. Image Analysis, Random Fields, and Markov Chain Monte Carlo Methods: A Mathematical Introduction, second ed. Springer-Verlag, Berlin.