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The Clebsch-Gordan coefficients for the quantum group $S_\mu U(2)$ and q-Hahn polynomials

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ABSTRACT

The tensor product of two unitary irreducible representations of the quantum group $S_\mu U(2)$ is decomposed in a direct sum of unitary irreducible representations with explicit realizations. The Clebsch-Gordan coefficients yield the orthogonality relations for q-Hahn and dual q-Hahn polynomials.

INTRODUCTION

There are many special functions of hypergeometric type which admit a group theoretic interpretation. (See Vilenkin [12].) For q-hypergeometric series only a few interpretations were known. (See [8], § 1 for an overview.) Nowadays, quantum groups, as introduced by Woronowicz [13] and Drinfeld [3], offer lots of possibilities for group theoretic interpretations of q-hypergeometric series.

For instance, the little q-Jacobi polynomials appear as matrix elements of irreducible representations of the quantum group $S_\mu U(2)$. Their orthogonality relations are implied by the Schur orthogonality relations for compact matrix quantum groups (cf. [13], theorem 5.7). See [8], [9] and [11]. For one of the q-analogues of the Krawtchouk polynomials there also exists a group theoretic interpretation. (See [8].)

In this paper we will show that the q-Hahn and dual q-Hahn polynomials admit a quantum group theoretic interpretation, quite analoguous to an interpretation of (dual) Hahn polynomials in terms of Clebsch-Gordan coefficients for $SU(2)$ (cf. Koornwinder [7]). Here we will closely follow the approach of [7], where the decomposition of the tensor product of two irreducible representations was realized in terms of a basis of homogeneous polynomials in four variables by the matrix elements of the irreducible representations.

In section 1 we will recall some facts from the theory of q-hypergeometric series. The reader will find some facts about the quantum group $S_\mu U(2)$ and its representations in section 2. The tensor product of two unitary irreducible representations will be decomposed in section 3, which furnishes a new proof of theorem 5.11 of [14] (i.e. our theorem 3.4). The Clebsch-Gordan coefficients will be defined in section 4. The essential symmetry relations for the Clebsch-Gordan coefficients will be derived in a simple algebraic way. However, in section 5 we have to do hard computational work in order to obtain the expression (5.3). The orthogonality relations for the q-Hahn and dual q-Hahn polynomials will be a relatively easy consequence of this expression.

After we completed this manuscript a preprint by Kirillov and Reshetkhin ([6]) reached us, where they also give (without proof) explicit expressions for the Clebsch-Gordan coefficients for the quantized universal enveloping algebra $U_q(\mathfrak{sl}(2))$ (cf. [5]). However, they do not express them as q-Hahn polynomials.

1. q-HYPERGEOMETRIC FUNCTIONS

In this section we state some definitions concerning q-hypergeometric functions. Some q-hypergeometric orthogonal polynomials are also discussed, as well as some identities for q-hypergeometric functions.

Let $1 \neq q \in \mathbb{C}$. For $a \in \mathbb{C}$, $k \in \mathbb{Z}_+$ the *q-shifted factorial* is defined by

$$(1.1) \qquad (a;q)_k = \prod_{i=0}^{k-1} (1-aq^i)$$

and if $|q|<1$ we also have

$$(1.2) \qquad (a;q)_\infty = \prod_{i=0}^{\infty} (1-aq^i).$$

The product of n q-shifted factorials is denoted by

$$(a_1,\dots,a_n;q)_k = (a_1;q)_k \cdots (a_n;q)_k.$$

We also have *q-combinatorial coefficients*. For n, $k \in \mathbb{Z}_+$, $n \geq k \geq 0$,

$$(1.3) \qquad \begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q;q)_n}{(q;q)_k (q;q)_{n-k}} = \begin{bmatrix} n \\ n-k \end{bmatrix}_q.$$

Then

$$\begin{bmatrix} n \\ 0 \end{bmatrix}_q = 1 = \begin{bmatrix} n \\ n \end{bmatrix}_q$$

and the q-combinatorial coefficients satisfy the following recurrence relation:

$$(1.4) \qquad \begin{bmatrix} n+1 \\ k \end{bmatrix}_q = q^k \begin{bmatrix} n \\ k \end{bmatrix}_q + \begin{bmatrix} n \\ k-1 \end{bmatrix}_q.$$

For $r \in \mathbb{Z}_+$ the *q-hypergeometric series* ${}_{r+1}\varphi_r$ is defined by

$$(1.5)\qquad {}_{r+1}\varphi_r\left(\begin{matrix} a_1, \dots, a_{r+1} \\ b_1, \dots, b_r \end{matrix}; q, z\right) = \sum_{k=0}^{\infty} \frac{(a_1, \dots, a_{r+1}; q)_k}{(b_1, \dots, b_r, q; q)_k} z^k.$$

The radius of convergence is 1 for generic values of the parameters. For us the most interesting case arises when $a_1 = q^{-n}$ ($n \in \mathbb{Z}_+$) while the parameters $b_1, \dots, b_r$ are not of the form $1, q^{-1}, \dots, q^{-n}$. Then (1.5) is a well defined terminating series with summation from 0 to n.

There is a q-analogue of the Chu-Vandermonde formula ([4], (1.5.3))

$$(1.6)\qquad {}_2\varphi_1\left(\begin{matrix} q^{-n}, b \\ c \end{matrix}; q, q\right) = \frac{(c/b; q)_n}{(c; q)_n} b^n.$$

The *little q-Jacobi polynomials* are also defined in terms of a ${}_2\varphi_1$ series:

$$(1.7)\qquad p_n(x; a, b|q) = {}_2\varphi_1\left(\begin{matrix} q^{-n}, abq^{n+1} \\ aq \end{matrix}; q, qx\right).$$

They are orthogonal polynomials. (See [1].)

The *q-Hahn polynomials* are defined in terms of a ${}_3\varphi_2$ series:

$$(1.8)\qquad \mathcal{Q}_n(x) = \mathcal{Q}_n(x; a, b, N|q) = {}_3\varphi_2\left(\begin{matrix} q^{-n}, abq^{n+1}, x \\ aq, q^{-N} \end{matrix}; q, q\right)$$

for $N \in \mathbb{Z}_+$ and $n \in \{0, 1, \dots, N\}$. They are orthogonal polynomials and the explicit orthogonality relation is

$$(1.9)\qquad \left\{\begin{aligned} &\sum_{x=0}^{N} \mathcal{Q}_m(q^{-x}) \mathcal{Q}_n(q^{-x}) \frac{(aq; q)_x (bq; q)_{N-x}}{(q, q)_x (q; q)_{N-x}} (aq)^{-x} \\ &= \delta_{mn} \frac{(abq^2; q)_N (aq)^{-N}}{(q; q)_N} \frac{(1-abq)(q, bq, abq^{N+2}; q)_n}{(1-abq^{2n+1})(aq, abq, q^{-N}; q)_n} \\ &\times (-aq)^n q^{\binom{n}{2} - Nn}. \end{aligned}\right.$$

(See [4], (7.2.22).)

The *dual q-Hahn polynomials* are also defined in terms of a terminating ${}_3\varphi_2$ series:

$$(1.10)\qquad \mathcal{R}_n(\mu(x)) = \mathcal{R}_n(\mu(x); a, b, N|q) = {}_3\varphi_2\left(\begin{matrix} q^{-n}, abq^{x+1}, q^{-x} \\ aq, q^{-N} \end{matrix}; q, q\right)$$

for $N \in \mathbb{Z}_+$ and $n \in \{0, 1, \dots, N\}$ and $\mu(x) = q^{-x} + q^{x+1}ab$. They are orthogonal polynomials and the explicit orthogonality relation can be derived from the orthogonality relations for the q-Racah polynomials. (See [2], (1.17)–(1.19) and p. 28, 29.) The relation is the following

$$
(1.11)\quad \begin{cases} \sum_{x=0}^{N} \mathscr{R}_m(\mu(x))\mathscr{R}_n(\mu(x)) \dfrac{(1-abq^{2x+1})(aq, abq, q^{-N}; q)_x}{(1-abq)(q, bq, abq^{N+2}; q)_x} \\ \times(-aq)^{-x} q^{Nx-\binom{x}{2}} \\ = \delta_{mn} \dfrac{(abq^2; q)_N (aq)^{-N}}{(bq; q)_N} \dfrac{(q, b^{-1}q^{-N}; q)_n}{(aq, q^{-N}; q)_n} (abq)^n. \end{cases}
$$

Note that for x, $n \in \mathbb{Z}_+$, $0 \le x$, $n \le N$

$$
(1.12)\quad \mathscr{R}_n(\mu(x); a, b; N|q) = \mathscr{Q}_x(q^{-n}; a, b; N|q)
$$

and that (1.11) is equivalent to (1.9).

We will also need a transformation for the ${}_3\varphi_2$ series. It is (see [4], (3.2.5))

$$
(1.13)\quad {}_3\varphi_2\left(\begin{matrix} q^{-n}, a, b \\ d, e \end{matrix}; q, \frac{deq^n}{ab}\right) = \frac{(e/a; q)_n}{(e; q)_n} {}_3\varphi_2\left(\begin{matrix} q^{-n}, a, d/b \\ d, aq^{1-n}/e \end{matrix}; q, q\right).
$$

2. THE QUANTUM GROUP $S_\mu U(2)$ AND ITS REPRESENTATION THEORY

The matrix elements of the representations of the quantum group $S_\mu U(2)$ are considered in this section.

Fix $\mu \in [-1, 1] \setminus \{0\}$. We are primarily interested in the case $|\mu| < 1$. Let A be the unital C^*-algebra generated by α and γ subject to the relations

$$
(2.1)\quad \begin{cases} \alpha^*\alpha + \gamma^*\gamma = I; \; \alpha\alpha^* + \mu^2\gamma\gamma^* = I \\ \gamma\gamma^* = \gamma^*\gamma; \; \alpha\gamma = \mu\gamma\alpha; \; \alpha\gamma^* = \mu\gamma^*\alpha. \end{cases}
$$

(For the construction of A see [14], § 1.)

Let

$$
(2.2)\quad u = \begin{pmatrix} \alpha & -\mu\gamma^* \\ \gamma & \alpha^* \end{pmatrix}.
$$

Woronowicz ([13], [14]) has proved that $S_\mu U(2) = (A, u)$ is a compact matrix quantum group (a quantum group for short). For $\mu = 1$ we can identify (A, u) with $SU(2)$.

The comultiplication is the unital C^*-algebra homomorphism $\Phi: A \to A \otimes A$ such that (see [14], (1.13))

$$
(2.3)\quad \begin{cases} \Phi(\alpha) = \alpha \otimes \alpha - \mu\gamma^* \otimes \gamma, \\ \Phi(\gamma) = \gamma \otimes \alpha + \alpha^* \otimes \gamma. \end{cases}
$$

Fix $l \in \frac{1}{2}\mathbb{Z}_+$ and let $n \in \{-l, -l+1, \ldots, l-1, l\}$. Then, using (2.3) and (2.1), we have

$$
(2.4)\quad \begin{cases} \Phi\left(\begin{bmatrix} 2l \\ l-n \end{bmatrix}_{\mu^{-2}}^{1/2} \alpha^{l-n}\gamma^{l+n}\right) \\ = \begin{bmatrix} 2l \\ l-n \end{bmatrix}_{\mu^{-2}}^{1/2} (\alpha\otimes\alpha-\mu\gamma^*\otimes\gamma)^{l-n}(\gamma\otimes\alpha+\alpha^*\otimes\gamma)^{l+n} \\ = \sum_{m=-l}^{l} t_{n,m}^{l,\mu}\otimes\begin{bmatrix} 2l \\ l-m \end{bmatrix}_{\mu^{-2}}^{1/2} \alpha^{l-m}\gamma^{l+m}. \end{cases}
$$

Evidently, $t_{n,m}^{l,\mu}\in\mathscr{A}$, where $\mathscr{A}$ denotes the *-subalgebra of A generated by the matrix elements of u defined in (2.2). Using the coassociativity ([13], (1.7)),

$$
(\Phi\otimes \mathrm{id})\circ\Phi = (\mathrm{id}\otimes\Phi)\circ\Phi,
$$

on (2.4) we obtain

$$
(2.5)\qquad \Phi(t_{n,m}^{l,\mu}) = \sum_{k=-l}^{l} t_{n,k}^{l,\mu}\otimes t_{k,m}^{l,\mu}.
$$

So the matrix $(t_{n,m}^{l,\mu})_{n,m=-l,\dots,l}$ defines a smooth representation $t^{l,\mu}$ of $S_\mu U(2)$ in $\mathbb{C}^{2l+1}$ (cf. [13]).

Define

$$
(2.6)\qquad \delta=\alpha^*;\ \beta=-\mu\gamma^*,
$$

then the relations (2.1) become

$$
(2.7)\quad \begin{cases} \alpha\beta=\mu\beta\alpha;\ \alpha\gamma=\mu\gamma\alpha;\ \gamma\delta=\mu\delta\gamma;\ \beta\delta=\mu\delta\beta; \\ \alpha\delta-\mu^2\delta\alpha=(1-\mu^2)I;\ \gamma\beta=\beta\gamma;\ \alpha\delta-\mu\beta\gamma=I. \end{cases}
$$

So we can identify $\mathscr{A}$ with $\mathscr{A}(\alpha,\beta,\gamma,\delta)$ the unital algebra of polynomials in non-commuting variables α, β, γ and δ with relations (2.7). Write $a(\alpha,\beta,\gamma,\delta)$ for a specific algebraic expression in the non-commuting variables α, β, γ and δ for some $a\in\mathscr{A}(\alpha,\beta,\gamma,\delta)$. Interchanging β and γ yields an isomorphism of $\mathscr{A}(\alpha,\beta,\gamma,\delta)$ which we denote by

$$
a(\alpha,\beta,\gamma,\delta)\mapsto a(\alpha,\gamma,\beta,\delta).
$$

Write $\tilde{a}(\alpha,\beta,\gamma,\delta)$ for the same algebraic expression as $a(\alpha,\beta,\gamma,\delta)$ with the order of the factors in each term inverted. Interchanging α and δ yields an anti-isomorphism which we denote by

$$
a(\alpha,\beta,\gamma,\delta)\mapsto \tilde{a}(\delta,\beta,\gamma,\alpha).
$$

We state some results on the representations $t^{l,\mu}$ and its matrix elements $t_{n,m}^{l,\mu}$. (See [8].)

PROPOSITION 2.1. ([8]) The matrix elements $t_{n,m}^{l,\mu}$ satisfy the following symmetry relations:

$$
(2.8)\qquad t_{n,m}^{l,\mu}(\alpha,\beta,\gamma,\delta) = t_{m,n}^{l,\mu}(\alpha,\gamma,\beta,\delta),
$$

(2.9) $\quad t^{l,\mu}_{n,n}(\alpha,\beta,\gamma,\delta)=(t^{l,\mu}_{-n,\,-m})^{\sim}(\delta,\gamma,\beta,\alpha),$

(2.10) $\quad t^{l,\mu}_{n,m}(\alpha,\beta,\gamma,\delta)=(t^{l,\mu}_{-m,\,-n})^{\sim}(\delta,\beta,\gamma,\alpha).$

THEOREM 2.2. ([8], [9], [11]) For $m\geq n\geq -m$ we have

(2.11) $\quad t^{l,\mu}_{n,m}(\alpha,\beta,\gamma,\delta)=c^{l,\mu}_{n,m}\delta^{m+n}p_{l-m}(-\mu^{-1}\beta\gamma;\ \mu^{2(m-n)};\ \mu^{2(m+n)}|\mu^2)\beta^{m-n},$

where p_{l-m} denotes a little q-Jacobi polynomial (see (1.7)) and

(2.12) $$c^{l,\mu}_{n,m}=\begin{bmatrix} l-n \\ m-n\end{bmatrix}^{1/2}_{\mu^2}\begin{bmatrix} l+m \\ m-n\end{bmatrix}^{1/2}_{\mu^2}\mu^{-(m-n)(l-m)}.$$

Note that we can obtain an expression for $t^{l,\mu}_{n,m}$ in the remaining cases by proposition 2.1.

THEOREM 2.3. ([8], see also [9], [14], § 5) The representations $t^{l,\mu}$ $(l\in\frac{1}{2}\mathbb{Z}_+)$ form a complete system of inequivalent irreducible unitary representations of the quantum group $S_\mu U(2)$.

See [13] for the meaning of this theorem.

3. DECOMPOSITION OF THE TENSOR PRODUCT $t^{l_1,\mu}ⓉT t^{l_2,\mu}$

In this section we decompose the unitary representation $t^{l_1,\mu}Ⓣt^{l_2,\mu}$ into a sum of unitary representations. We also give explicit realizations of these representations.

From [14], theorem 1.2 we know that $\alpha^k\gamma^m\beta^n$ and $\gamma^m\beta^n\delta^k$ form a basis for $\mathscr{A}$. Let $\mathscr{A}_d$ be the linear subspace of $\mathscr{A}$ spanned by all $\alpha^k\gamma^m\beta^n$ and $\gamma^m\beta^n\delta^k$ with $k+m+n=d$ for $d=0,1,\ldots$. Then $\dim(\mathscr{A}_d)=(d+1)^2$.

Let $\mathscr{A}^d$ be the linear subspace of $\mathscr{A}$ spanned by all $\alpha^{d_1}\gamma^{d_2}\beta^{d_3}\delta^{d_4}$ with d_1+ $+d_2+d_3+d_4=d$.

PROPOSITION 3.1.

$$\mathscr{A}^d=\sideset{^\oplus}{}\sum_{j=0,1,\ldots,[d/2]}\mathscr{A}_{d-2j}.$$

PROOF. From (see (2.7))

(3.1) $$\begin{cases}\alpha^{d_1}\gamma^{d_2}\beta^{d_3}\delta^{d_4}=\alpha^{d_1}\gamma^{d_2}(\alpha\delta-\mu\beta\gamma)\beta^{d_3}\delta^{d_4}\\ =\mu^{-(d_2+d_3)}\alpha^{d_1+1}\gamma^{d_2}\beta^{d_3}\delta^{d_4+1}-\mu\alpha^{d_1}\gamma^{d_2+1}\beta^{d_3+1}\delta^{d_4}\end{cases}$$

we see that $\mathscr{A}^{d-2j}\subset\mathscr{A}^d$, $j=0,1,\ldots,[d/2]$. Since $\mathscr{A}_{d-2j}\subset\mathscr{A}^{d-2j}$ we have

$$\sideset{^\oplus}{}\sum_{j=0,1,\ldots,[d/2]}\mathscr{A}_{d-2j}\subset\mathscr{A}^d.$$

Since the $\alpha^k\gamma^m\beta^n$ and $\gamma^m\beta^n\delta^k$ form a basis of $\mathscr{A}$ we see that

$$\dim\Big(\sideset{^\oplus}{}\sum_{j=0,1,\ldots,[d/2]}\mathscr{A}_{d-2j}\Big)=\sum_{j=0}^{[d/2]}(d-2j+1)^2=\binom{d+3}{3}\leq\dim(\mathscr{A}^d).$$

But the dimension of $\mathscr{A}^d$ is smaller than or equal to the dimension of the space of homogeneous polynomials of degree d in four variables, which is $\binom{d+3}{3}$. $\square$

COROLLARY 3.2. The monomials $\alpha^{d_1}\gamma^{d_2}\beta^{d_3}\delta^{d_4}$ $(d_1+d_2+d_3+d_4=d)$ constitute a basis for $\mathscr{A}^d$.

The equality (3.1) can be generalized into the following, which will be useful in the sequel.

LEMMA 3.3. For d_1, d_2, d_3, d_4, $k\in\mathbb{Z}_+$ we have

$$\alpha^{d_1}\gamma^{d_2}\beta^{d_3}\delta^{d_4}=\sum_{i=0}^{k}(-\mu)^i\mu^{-(d_2+d_3)(k-i)}\begin{bmatrix}k\\ i\end{bmatrix}_{\mu^{-2}}\alpha^{d_1+k-i}\gamma^{d_2+i}\beta^{d_3+i}\delta^{d_4+k-i}.$$

PROOF. By repeating (3.1) we see that we have an expression like

$$\alpha^{d_1}\gamma^{d_2}\beta^{d_3}\delta^{d_4}=\sum_{i=0}^{k}A_i^k\alpha^{d_1+k-i}\gamma^{d_2+i}\beta^{d_3+i}\delta^{d_4+k-i}.$$

To calculate A_i^k we apply (3.1) to every term of the sum on the right hand side. This yields the following recurrence relation for A_i^k:

$$A_i^{k+1}=\mu^{-(d_2+d_3)-2i}A_i^k-\mu A_{i-1}^k. \tag{3.2}$$

Now put $A_i^k=(-\mu)^i\mu^{-(d_2+d_3)(k-i)}B_i^k$, then (3.2) yields a recurrence relation for B_i^k:

$$B_i^{k+1}=(\mu^{-2})^iB_i^k+B_{i-1}^k. \tag{3.3}$$

Since $A_0^0=B_0^0=1$ we have from (1.4) the solution $B_i^k=\begin{bmatrix}n\\ i\end{bmatrix}_{\mu^{-2}}$ for the relation (3.3). $\square$

Now we consider $\mathscr{A}^{e,f}$, the linear span of the monomials $\alpha^{e_1}\gamma^{e_2}\beta^{f_1}\delta^{f_2}$ with $e_1+e_2=e$ and $f_1+f_2=f$. Note that (3.1) immediately yields

$$\mathscr{A}^{e,f}\subset\mathscr{A}^{e+1,f+1}. \tag{3.4}$$

We make $\mathscr{A}^{2l_1,2l_2}$ into a Hilbert space by declaring the following basis orthonormal

$$\psi_{n_1,n_2}^{l_1,l_2,\mu}=\begin{bmatrix}2l_1\\ l_1-n_1\end{bmatrix}_{\mu^{-2}}^{1/2}\begin{bmatrix}2l_2\\ l_2-n_2\end{bmatrix}_{\mu^{-2}}^{1/2}\alpha^{l_1-n_1}\gamma^{l_1+n_1}\beta^{l_2-n_2}\delta^{l_2+n_2}, \tag{3.5}$$

$n_1\in\{-l_1,\dots,l_1\}$, $n_2\in\{-l_2,\dots,l_2\}$. This is possible because of corollary 3.2.

From (2.3) and (2.6) it follows that

$$\Phi\left(\begin{bmatrix}2l\\ l-n\end{bmatrix}_{\mu^{-2}}^{1/2}\beta^{l-n}\delta^{l+n}\right)=\sum_{m=-l}^{l}t_{n,m}^{l,\mu}\otimes\begin{bmatrix}2l\\ l-m\end{bmatrix}_{\mu^{-2}}^{1/2}\beta^{l-m}\delta^{l+m}.$$

This and (2.4) imply

$$\Phi(\psi_{n_1,n_2}^{l_1,l_2,\mu})=\sum_{m_1=-l_1}^{l_1}\sum_{m_2=-l_2}^{l_2}t_{n_1,m_1}^{l_1,\mu}t_{n_2,m_2}^{l_2,\mu}\otimes\psi_{m_1,m_2}^{l_1,l_2,\mu}. \tag{3.6}$$

This proves that Φ acting on $\mathscr{A}^{2l_1,2l_2}$ gives a realization of the tensor product $t^{l_1,\mu}\textcircled{T}t^{l_2,\mu}$. Taking into account the Hilbert space structure of $\mathscr{A}^{2l_1,2l_2}$ we see that $t^{l_1,\mu}\textcircled{T}t^{l_2,\mu}$ is unitary.

From proposition 5.2 of [8] (or from theorem 2.2 and lemma 3.3) and (3.4) we know that

$$t^{l,\mu}_{n,m}\in\mathscr{A}^{l-m,l+m}\subset\mathscr{A}^{l-m+i,l+m+i}$$

for $i\in\mathbb{Z}_+$. In particular, if we take $m=l_2-l_1$ and $l=|l_1-l_2|,\ |l_1+l_2|+1,\dots,\ l_1+l_2$, we have

$$t^{l,\mu}_{n,l_2-l_1}\in\mathscr{A}^{l+l_1-l_2+i,l-l_1+l_2+i}=\mathscr{A}^{2l_1,2l_2} \tag{3.7}$$

for $i=l_1+l_2-l$.

Theorems 2.3 and theorem 5.7(i) of [13] imply that all $t^{l,\mu}_{n,l_2-l_1}$ are linearly independent and (2.5) yields

$$\Phi(t^{l,\mu}_{n,l_2-l_1})=\sum_{m=-l}^{l}t^{l,\mu}_{n,m}\otimes t^{l,\mu}_{m,l_2-l_1}. \tag{3.8}$$

If we define $\mathscr{A}^{2l_1,2l_2}_l$ to be the linear span of $t^{l,\mu}_{n,l_2-l_1}$, $n=-l,\dots,l$, then Φ acting on $\mathscr{A}^{2l_1,2l_2}_l$ gives a realization of the representation $t^{l,\mu}$.

THEOREM 3.4.

$$\mathscr{A}^{2l_1,2l_2}=\oplus\sum_{l=|l_1-l_2|}^{l_1+l_2}\mathscr{A}^{2l_1,2l_2}_l$$

and

$$t^{l_1,\mu}\textcircled{T}t^{l_2,\mu}\cong\oplus\sum_{l=|l_1-l_2|}^{l_1+l_2}t^{l,\mu}.$$

PROOF. We need only to prove the first statement in view of (3.6) and (3.8). This follows from (3.7) and

$$\sum_{l=|l_1-l_2|}^{l_1+l_2}(2l+1)=(2l_1+1)(2l_2+1). \qquad \square$$

REMARK. Theorem 3.4 is theorem 5.11 of [14], but the proof is new.

THEOREM 3.5. (i) For suitable complex constants $a^{l,\mu}_{l_1,l_2}\neq 0$

$$\varphi^{l_1,l_2,l,\mu}_j=a^{l,\mu}_{l_1,l_2}t^{l,\mu}_{j,l_2-l_1}, \tag{3.9}$$

$j=-l,\dots,l$, constitute an orthonormal basis for $\mathscr{A}^{2l_1,2l_2}_l$.

(ii) The constant $a^{l,\mu}_{l_1,l_2}$ is uniquely determined by the condition

$$(\varphi^{l_1,l_2,l,\mu}_l,\psi^{l_1,l_2,\mu}_{l_1,l-l_1})>0. \tag{3.10}$$

Then

$$a^{l,\mu}_{l_1,l_2}=(-\mu)^{l-l_1-l_2}\left\{\frac{(\mu^{-2};\mu^{-2})_{2l_1}(\mu^{-2};\mu^{-2})_{2l_2}(1-\mu^{-2(1+2l)})}{(\mu^{-2};\mu^{-2})_{l_1+l_2-l}(\mu^{-2};\mu^{-2})_{l_1+l_2+l+1}}\right\}^{1/2}. \tag{3.11}$$

PROOF. (i) This follows from theorem 3.4, (3.8) and from [11], theorem 5.8, propositions 2.2 and 2.3.

To prove (ii) we use

$$t^{l,\mu}_{l,l_2-l_1}(\alpha,\beta,\gamma,\delta)=\begin{bmatrix}2l\\ l+l_1-l_2\end{bmatrix}^{1/2}_{\mu^{-2}}\gamma^{l+l_1-l_2}\delta^{l-l_1+l_2}$$

by (2.5) and the q-binomial theorem (see [8], lemma 2.1). Hence, using lemma 3.3, we have

$$\varphi^{l_1,l_2,l,\mu}_{l}(\alpha,\beta,\gamma,\delta)=a^{l,\mu}_{l_1,l_2}\begin{bmatrix}2l\\ l+l_1-l_2\end{bmatrix}^{1/2}_{\mu^{-2}}\sum_{i=0}^{l_1+l_2-l}\mu^{-(l_1-l_2+l)(l_1+l_2-l-i)}$$

$$\times(-\mu)^i\begin{bmatrix}l_1+l_2-l\\ i\end{bmatrix}_{\mu^{-2}}\begin{bmatrix}2l_1\\ l_1+l_2-l-i\end{bmatrix}^{-1/2}_{\mu^{-2}}\begin{bmatrix}2l_2\\ i\end{bmatrix}^{-1/2}_{\mu^{-2}}\psi^{l_1,l_2,\mu}_{l-l_2+i,l_2-i}.$$

So (3.10) implies (take $i=l_1+l_2-l$) that

$$(3.12)\qquad a^{l,\mu}_{l_1,l_2}(-1)^{l_1+l_2-l}>0.$$

Now, $\varphi^{l_1,l_2,l,\mu}_{j}$ and $\psi^{l_1,l_2,\mu}_{n_1,n_2}$ are orthonormal bases, so

$$(3.13)\qquad \begin{cases} 1=|a^{l,\mu}_{l_1,l_2}|^2\begin{bmatrix}2l\\ l+l_1-l_2\end{bmatrix}_{\mu^{-2}}\sum_{j=0}^{l_1+l_2-l}\mu^{-2(l_1-l_2+l)j}\mu^{2(l_1+l_2-l-j)}\\ \times\begin{bmatrix}l_1+l_2-l\\ j\end{bmatrix}^2_{\mu^{-2}}\begin{bmatrix}2l_1\\ j\end{bmatrix}^{-1}_{\mu^{-2}}\begin{bmatrix}2l_2\\ l_1+l_2-l-j\end{bmatrix}^{-1}_{\mu^{-2}}\\ =|a^{l,\mu}_{l_1,l_2}|^2\mu^{2(l_1+l_2-l)}\dfrac{(\mu^{-2};\mu^{-2})_{2l}(\mu^{-2};\mu^{-2})_{l_1+l_2-l}}{(\mu^{-2};\mu^{-2})_{l_1-l_2+l}(\mu^{-2};\mu^{-2})_{2l_2}}\\ \times{}_2\varphi_1\left(\begin{matrix}\mu^{2(l_1+l_2-l)},\mu^{-2(l-l_1+l_2+1)}\\ \mu^{4l_1}\end{matrix};\mu^{-2},\mu^{-2}\right)\\ =|a^{l,\mu}_{l_1,l_2}|^2\mu^{2(l_1+l_2-l)}\dfrac{(\mu^{-2};\mu^{-2})_{l_1+l_2-l}(\mu^{-2};\mu^{-2})_{l_1+l_2+l+1}}{(\mu^{-2};\mu^{-2})_{2l_1}(\mu^{-2};\mu^{-2})_{2l_2}(1-\mu^{-2(1+2l)})}.\end{cases}$$

In the last step we used the Chu-Vandermonde formula (1.6). Finally, (3.12) and (3.13) imply (3.11). □

4. THE CLEBSCH-GORDAN COEFFICIENTS FOR $S_\mu U(2)$

The Clebsch-Gordan coefficients are defined in this section. Some of their properties will be derived.

Since we have two orthonormal bases in $\mathscr{A}^{2l_1,2l_2}$, we can consider the unitary matrix which maps one basis onto the other. Its matrix elements are called the *Clebsch-Gordan coefficients* $C^{l_1,l_2,l,\mu}_{j_1,j_2,j}$:

$$(4.1)\qquad \varphi^{l_1,l_2,l,\mu}_{j}=\sum_{j_1=-l_1}^{l_1}\sum_{j_2=-l_2}^{l_2}C^{l_1,l_2,l,\mu}_{j_1,j_2,j}\psi^{l_1,l_2,\mu}_{j_1,j_2}.$$

PROPOSITION 4.1. If $j \neq j_1 + j_2$, then

$$C^{l_1, l_2, l, \mu}_{j_1, j_2, j} = 0.$$

PROOF. We need the notion of a *quantum subgroup* of a quantum group $G = (A, u)$. This is a quantum group $K = (B, v)$ such that there exists a surjective unital C^*-algebra homomorphism $\pi : A \to B$ such that

$$\Phi_K \circ \pi = (\pi \otimes \pi) \circ \Phi_G, \tag{4.2}$$

where Φ_K and Φ_G denote the comultiplication of K and G.

Now let t^G be a matrix representation of G, then $t^K = \pi t^G = (\pi t^G_{ij})_{i,j}$ is a matrix representation of K because of (4.2).

Take $B = C(\mathbb{T})$, the unital commutative C^*-algebra of continuous functions on the unit circle $\mathbb{T}$. Pick $f \in C(\mathbb{T})$ defined by $f(z) = z$ for $z \in \mathbb{T}$ and put

$$v = \begin{pmatrix} f & 0 \\ 0 & f^* \end{pmatrix}.$$

It is easy to check that the unital C^*-homomorphism of the C^*-algebra A of $S_\mu U(2)$ into $C(\mathbb{T})$ generated by

$$\pi(\alpha) = f; \quad \pi(\gamma) = 0$$

makes $(C(\mathbb{T}), v)$ into a quantum subgroup of $S_\mu U(2)$. (See [8], [10].)

Apply $\pi \otimes \mathrm{id}$ on the last equality in (2.4) to obtain (cf. [8], § 4)

$$\pi(t^{l,\mu}_{n,m}) = \delta_{nm} f^{-2n}. \tag{4.3}$$

Apply $(\pi \otimes \mathrm{id}) \circ \Phi$ to (4.1) and use (3.6), (3.8) and (4.3) to obtain the following equality in $\mathscr{B} \otimes \mathscr{A}$, where $\mathscr{B}$ denotes the *-subalgebra of $C(\mathbb{T})$ generated by the elements of v,

$$f^{-2j} \otimes \varphi^{l_1, l_2, l, \mu}_j = \sum_{j_1 = -l_1}^{l_1} \sum_{j_2 = -l_2}^{l_2} f^{-2(j_1 + j_2)} \otimes C^{l_1, l_2, l, \mu}_{j_1, j_2, j} \psi^{l_1, l_2, \mu}_{j_1, j_2}.$$

This proves the proposition. $\square$

REMARK 4.2. If we define a quantum subgroup $K = (B, v)$ of $G = (A, u)$ with $u \in M_n(A)$ and $v \in M_m(B)$, if $m = n$ and if there exists a unital C^*-algebra homomorphism $\pi : A \to B$ with

$$\pi(u_{ij}) = v_{ij} \quad \forall i, j \in \{1, \ldots, n\},$$

then we have automatically (4.2) and the surjectivity of π. Also

$$\kappa_K \circ \pi = \pi \circ \kappa_G,$$

where κ_K and κ_G denote the coinverse (see [13], def. 1.1.) of K and G, and $\pi(\mathscr{A}) = \mathscr{B}$.

Because of proposition 4.1 it is sufficient to consider $C^{l_1, l_2, l, \mu}_{j_1, j_2, j}$ with $|l_1 - l_2| \leq$ $\leq l_1 + l_2$; $|j| \leq l$; $|j_1| \leq l_1$; $|j_2| \leq l_2$; $j = j_1 + j_2$. Here $|j| \leq l$ means $j \in \{-l, \ldots, l\}$. To narrow this area under consideration even more we will prove symmetry relations for the Clebsch-Gordan coefficients. We have the following relation in $\mathscr{A}(\alpha,\beta,\gamma,\delta)$:

$$
(4.4)\quad \left\{\begin{aligned}
&\frac{\phi_j^{l_1,l_2,l,\mu}(\alpha,\beta,\gamma,\delta)}{\{(\mu^{-2};\mu^{-2})_{2l_1}(\mu^{-2};\mu^{-2})_{2l_2}\}^{1/2}}=\\
&\sum_{\substack{j_1=-l_1\\ j_1+j_2=j}}^{l_1}\sum_{j_2=-l_2}^{l_2}\frac{C_{j_1,j_2,j}^{l_1,l_2,l,\mu}\alpha^{l_1-j_1}\gamma^{l_1+j_1}\beta^{l_2-j_2}\delta^{l_2+j_2}}{\{(\mu^{-2};\mu^{-2})_{l_1-j_1}(\mu^{-2};\mu^{-2})_{l_1+j_1}(\mu^{-2};\mu^{-2})_{l_2-j_2}(\mu^{-2};\mu^{-2})_{l_2+j_2}\}^{1/2}}.
\end{aligned}\right.
$$

Because $\mu\in[-1,1]\setminus\{0\}$ we see that the Clebsch-Gordan coefficients are real. Since $\varphi_j^{l_1,l_2,l,\mu}(\alpha,\beta,\gamma,\delta)=a_{l_1,l_2,j}^{l,\mu}t_{j,l_2-l_1}^{l,\mu}(\alpha,\beta,\gamma,\delta)$ the symmetry relations for $t_{n,m}^{l,\mu}(\alpha,\beta,\gamma,\delta)$ (see (2.8), (2.9) and (2.10)) yield symmetry relations for $\varphi_j^{l_1,l_2,l,\mu}(\alpha,\beta,\gamma,\delta)$.

$$
(4.5)\quad \frac{\phi_j^{l_1,l_2,l,\mu}(\alpha,\beta,\gamma,\delta)}{\{(\mu^{-2};\mu^{-2})_{2l_1}(\mu^{-2};\mu^{-2})_{2l_2}\}^{1/2}}=\frac{\phi_{l_2-l_1}^{1/2(l_1+l_2-j),1/2(l_1+l_2+j),l,\mu}(\alpha,\gamma,\beta,\delta)}{\{(\mu^{-2};\mu^{-2})_{l_1+l_2-j}(\mu^{-2};\mu^{-2})_{l_1+l_2+j}\}^{1/2}}
$$

$$
(4.6)\quad =\frac{(\phi_{-j}^{l_2,l_1,l,\mu})^{\sim}(\delta,\gamma,\beta,\alpha)}{\{(\mu^{-2};\mu^{-2})_{2l_2}(\mu^{-2};\mu^{-2})_{2l_1}\}^{1/2}}
$$

$$
(4.7)\quad =\frac{(\phi_{l_1-l_2}^{1/2(l_1+l_2+j),1/2(l_1+l_2-j),l,\mu})^{\sim}(\delta,\beta,\gamma,\alpha)}{\{(\mu^{-2};\mu^{-2})_{l_1+l_2+j}(\mu^{-2};\mu^{-2})_{l_1+l_2-j}\}^{1/2}}.
$$

Combination of (4.4) and (4.5), of (4.4) and (4.6) and of (4.4) and (4.7) gives the following symmetry relations for the Clebsch-Gordan coefficients. (Of course we use corollary 3.2 as well.)

PROPOSITION 4.3. The Clebsch-Gordan coefficients $C_{j_1,j_2,j}^{l_1,l_2,l,\mu}$ satisfy the following relations:

$$
\begin{aligned}
C_{j_1,j_2,j}^{l_1,l_2,l,\mu}&=C_{1/2(-l_1+l_2+j_1-j_2),1/2(-l_1+l_2-j_1+j_2),l_2-l_1}^{1/2(l_1+l_2-j),1/2(l_1+l_2+j),l,\mu}\\
&=C_{-j_2,-j_1,-j}^{l_1,l_2,l,\mu}\\
&=C_{1/2(l_1-l_2+j_1-j_2),1/2(l_1-l_2-j_1+j_2),l_1-l_2}^{1/2(l_1+l_2+j),1/2(l_1+l_2-j),l,\mu}.
\end{aligned}
$$

Thus we can restrict ourselves to one of the following four subsets in the (l_1,l_2,l,j_1,j_2,j)-space:

$$
(4.8)\quad \left\{\begin{aligned}
&\text{(i)}\ l_1-l_2\le j\le l_2-l_1\le l\le l_1+l_2;\ -l_1\le j_1\le l_1;\ j=j_1+j_2;\\
&\text{(ii)}\ l_2-l_1\le j\le l_1-l_2\le l\le l_1+l_2;\ -l_2\le j_2\le l_2;\ j=j_1+j_2;\\
&\text{(iii)}\ j\le l_1-l_2\le -j\le l\le l_1+l_2;\ -l_1\le j_1;\ -l_2\le j_2;\ j=j_1+j_2;\\
&\text{(iv)}\ -j\le l_1-l_2\le j\le l\le l_1+l_2;\ j_1\le l_1;\ j_2\le l_2;\ j=j_1+j_2.
\end{aligned}\right.
$$

5. EXPRESSION OF THE CLEBSCH-GORDAN COEFFICIENTS IN TERMS OF q-HAHN POLYNOMIALS

We derive an explicit expression for the Clebsch-Gordan coefficients in terms of q-Hahn polynomials. This will allow us to prove the orthogonality relations for the q-Hahn polynomials.

First of all we need a generating function for the Clebsch-Gordan coefficients. We restrict ourselves to case (i) of (4.8). Then it follows from (4.4), (3.11), theorem 2.2 and (1.7) that

$$
(5.1)\quad \begin{cases}
\displaystyle\sum_{\substack{j_1=-l_1\\ j=j_1+j_2}}^{l_1} \frac{C^{l_1,l_2,l,\mu}_{j_1,j_2,j}\alpha^{l_1-j_1}\gamma^{l_1+j_1}\beta^{l_2-j_2}\delta^{l_2+j_2}}{\{(\mu^{-1};\mu^{-2})_{l_1-j_1}(\mu^{-2};\mu^{-2})_{l_1+j_1}(\mu^{-2};\mu^{-2})_{l_2-j_2}(\mu^{-2};\mu^{-2})_{l_2+j_2}\}^{1/2}} \\
\displaystyle = \frac{(-\mu)^{-l_1-l_2+l}\mu^{-(l+l_1-l_2)(l_2-l_1-j)}(1-\mu^{-2(1+2l)})^{1/2}}{(\mu^2;\mu^2)_{l_2-l_1-j}((\mu^{-2};\mu^{-2})_{l_1+l_2-l}(\mu^{-2};\mu^{-2})_{l_1+l_2+l+1})^{1/2}} \\
\displaystyle \times\left\{\frac{(\mu^2;\mu^2)_{l-j}(\mu^2;\mu^2)_{l+l_2-l_1}}{(\mu^2;\mu^2)_{l+l_1-l_2}(\mu^2;\mu^2)_{l+j}}\right\}^{1/2}\delta^{l_2-l_1+j}\beta^{l_2-l_1-j} \\
\displaystyle \times\sum_{k=0}^{l+l_1-l_2}\frac{(\mu^{-2(l+l_1-l_2)};\mu^2)_k(\mu^{2(l-l_1+l_2+1)};\mu^2)_k}{(\mu^{2(l_2-l_1-j+1)};\mu^2)_k(\mu^2;\mu^2)_k}(-\mu\beta\gamma)^k.
\end{cases}
$$

In the right hand side we use

$$
\begin{aligned}
&\delta^{l_2-l_1+j}\beta^{l_2-l_1-j}(\beta\gamma)^k=\mu^{-(l_2-l_1-j+2k)(l_2-l_1+j)} \\
&\times\sum_{i=0}^{2l_1-k}(-\mu)^i\mu^{-(l_2-l_1-j+2k)(2l_1-k-i)}\begin{bmatrix}2l_1-k\\ i\end{bmatrix}_{\mu^{-2}} \\
&\times\alpha^{2l_1-k-i}\gamma^{k+i}\beta^{l_2-l_1-j+k+i}\delta^{l_2+l_1+j-k-i}
\end{aligned}
$$

by lemma 3.3 and the commutation relations (2.7). Put $j_1=k+i-l_1$ and change summation

$$
\sum_{k=0}^{l+l_1-l_2}\ \sum_{i=0}^{2l_1-k} = \sum_{j_1=-l_1}^{l_1}\ \sum_{k=0}^{(l+l_1-l_2)\wedge(l_1+j_1)}.
$$

After some manipulation with q-shifted factorials we obtain

$$
(5.2)\quad \begin{cases}
\displaystyle C^{l_1,l_2,l,\mu}_{j_1,j_2,j}=(-\mu)^{-l_2+l+j_1}\frac{(\mu^{-2};\mu^{-2})_{2l_1}}{(\mu^2;\mu^2)_{-l_1+l_2-j}} \\
\displaystyle \times\left\{\frac{(1-\mu^{-2(1+2l)})(\mu^{-2};\mu^{-2})_{l_2-j_2}(\mu^{-2};\mu^{-2})_{l_2+j_2}}{(\mu^{-2};\mu^{-2})_{l_1+l_2-l}(\mu^{-2};\mu^{-2})_{l_1+l_2+l+1}(\mu^{-2};\mu^{-2})_{l_1-j_1}(\mu^{-2};\mu^{-2})_{l_1+j_1}}\right\}^{1/2} \\
\displaystyle \times\left\{\frac{(\mu^2;\mu^2)_{l-l_1+l_2}(\mu^2;\mu^2)_{l-j}}{(\mu^2;\mu^2)_{l+l_1-l_2}(\mu^2;\mu^2)_{l+j}}\right\}^{1/2}\mu^{-(l_2-l_1-j)(l+l_1+j-j_1)} \\
\displaystyle \times{}_3\varphi_2\left(\begin{matrix}\mu^{-2(l_1-l_2+l)},\ \mu^{2(l-l_1+l_2+1)},\ \mu^{-2(l_1+j_1)}\\ \mu^{2(l_2-l_1-j+1)},\ \mu^{-4l_1}\end{matrix};\mu^2,\mu^{-2(l_2+j_2)}\right).
\end{cases}
$$

Transform (5.2) using

$$(\mu^{-2};\mu^{-2})_k=(-1)^k\mu^{-k(k+1)}(\mu^2;\mu^2)_k.$$

Next we introduce

$$x=l_1-j_1;\ n=l_1-l_2+l;\ N=2l_1,$$

$$a=-l_1+l_2+j;\ b=-l_1+l_2-j.$$

Then (4.8) (i) is equivalent to the condition that x, n, N, a and b are integers and

$$0\le x\le N;\ 0\le n\le N;\ a\ge 0;\ b\ge 0.$$

If we also use a transformation rule for the ${}_3\varphi_2$ (see (1.13) with $a=\mu^{2(n+a+b+1)}$, $b=\mu^{-2(N-x)}$, $d=\mu^{-2N}$ and $e=\mu^{2(b+1)}$), then we can recognize a q-Hahn polynomial $\mathcal{Q}_n$. The result is

$$(5.3)\quad \begin{cases} C^{1/2N,\,1/2N+a+b,\,n+1/2(a+b),\,\mu}_{1/2N-x,\,1/2(a-b-N)+x,\,1/2(a-b)}=(-1)^{N+n-x}\dfrac{(\mu^2;\mu^2)_N}{\mu^2;\mu^2)_b}\\ \times\left\{\dfrac{(\mu^2;\mu^2)_{N-x+b}(\mu^2;\mu^2)_{x+a}(\mu^2;\mu^2)_{n+b}(\mu^2;\mu^2)_{n+a+b}(1-\mu^{2(1+2n+a+b)})}{(\mu^2;\mu^2)_{N-n}(\mu^2;\mu^2)_{N+n+a+b+1}(\mu^2;\mu^2)_{N-x}(\mu^2;\mu^2)_x(\mu^2;\mu^2)_n(\mu^2;\mu^2)_{n+a}}\right\}^{1/2}\\ \times\mu^{(N-x)(a+1)+n(a+n)}\dfrac{(\mu^{-2(n+a)};\mu^2)_n}{(\mu^{2(b+1)};\mu^2)_n}\,\mathcal{Q}_n(\mu^{-2x};\mu^{2a},\mu^{2b};N\,|\,\mu^2). \end{cases}$$

Since the Clebsch-Gordan coefficients are matrix elements of a unitary matrix we have

$$(5.4)\quad \sum_{x=0}^{N} C^{1/2N,\,1/2N+a+b,\,n+1/2(a+b),\,\mu}_{1/2N-x,\,1/2(a-b-N)+x,\,1/2(a-b)}C^{1/2N,\,1/2N+a+b,\,m+1/2(a+b),\,\mu}_{1/2N-x,\,1/2(a-b-N)+x,\,1/2(a-b)}=\delta_{mn}.$$

Combination of (5.3) and (5.4) and $q=\mu^2$ yields

$$(5.5)\quad \sum_{x=0}^{N}\frac{(q;q)_{N-x+b}(q;q)_{x+a}}{(q;q)_x(q;q)_{N-x}}q^{-x(a+1)}\mathcal{Q}_n(q^{-x})\mathcal{Q}_m(q^{-x})=c_n\delta_{mn},$$

where $\mathcal{Q}_n(q^{-x})=\mathcal{Q}_n(q^{-x};q^a,q^b;N\,|\,q)$ and

$$(5.6)\quad \begin{cases} c_n=\dfrac{(q;q)_{N-n}(q;q)_{N+n+a+b+1}(q;q)_n(q;q)_{n+a}}{(q;q)_{n+b}(q;q)_{n+a+b}(1-q^{1+2n+a+b})}q^{-N(a+1)-n(a+n)}\\ \times\dfrac{(q;q)_b^2}{(q;q)_N^2}\dfrac{(q^{b+1};q)_n^2}{(q^{-(n+a)};q)_n^2}. \end{cases}$$

When we divide both sides of (5.5) by $(q;q)_a(q;q)_b$ we obtain the orthogonality relations (1.9) for the q-Hahn polynomials with a, b replaced by q^a, q^b. Since $q=\mu^2\in(0,1)$ we can use analytic continuation to obtain (1.9) for arbitrary a, b.

Of course we also have orthogonality relations dual to (5.4):

$$\sum_{n=0}^{N} C^{1/2N,\,1/2N+a+b,\,n+1/2(a+b),\,\mu}_{1/2N-x,\,1/2(a-b-N)+x,\,1/2(a-b)}\, C^{1/2N,\,1/2N+a+b,\,n+1/2(a+b),\,\mu}_{1/2N-y,\,1/2(a-b-N)+y,\,1/2(a-b)} = \delta_{xy}. \tag{5.7}$$

Substitution of (5.3) and (1.12) in (5.7) yields the orthogonality relations (1.11) for the dual q-Hahn polynomials with a, b replaced by q^a, q^b and n and x interchanged.

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