The Clebsch-Gordan coefficients for the quantum group $\mathfrak{su}(2)$ and $q$-Hahn polynomials

by H.T. Koelink$^1$ and T.H. Koornwinder$^2$

$^1$ Mathematical Institute, University of Leiden, P.O. Box 9512, 2300 RA Leiden, the Netherlands
$^2$ Centre for Mathematics and Computer Science, P.O. Box 4079, 1009 AB Amsterdam, the Netherlands

Communicated by Prof. T.A. Springer at the meeting of December 19, 1988

ABSTRACT
The tensor product of two unitary irreducible representations of the quantum group $\mathfrak{su}(2)$ is decomposed in a direct sum of unitary irreducible representations with explicit realizations. The Clebsch-Gordan coefficients yield the orthogonality relations for $q$-Hahn and dual $q$-Hahn polynomials.

INTRODUCTION
There are many special functions of hypergeometric type which admit a group theoretic interpretation. (See Vilenkin [12].) For $q$-hypergeometric series only a few interpretations were known. (See [8], § 1 for an overview.) Nowadays, quantum groups, as introduced by Woronowicz [13] and Drinfeld [3], offer lots of possibilities for group theoretic interpretations of $q$-hypergeometric series.

For instance, the little $q$-Jacobi polynomials appear as matrix elements of irreducible representations of the quantum group $\mathfrak{su}(2)$. Their orthogonality relations are implied by the Schur orthogonality relations for compact matrix quantum groups (cf. [13], theorem 5.7). See [8], [9] and [11]. For one of the $q$-analogues of the Krawtchouk polynomials there also exists a group theoretic interpretation. (See [8].)

In this paper we will show that the $q$-Hahn and dual $q$-Hahn polynomials admit a quantum group theoretic interpretation, quite analogous to an interpretation of (dual) Hahn polynomials in terms of Clebsch-Gordan coefficients.

443
for $SU(2)$ (cf. Koornwinder [7]). Here we will closely follow the approach of [7], where the decomposition of the tensor product of two irreducible representations was realized in terms of a basis of homogeneous polynomials in four variables by the matrix elements of the irreducible representations.

In section 1 we will recall some facts from the theory of $q$-hypergeometric series. The reader will find some facts about the quantum group $S_{\mu}U(2)$ and its representations in section 2. The tensor product of two unitary irreducible representations will be decomposed in section 3, which furnishes a new proof of theorem 5.11 of [14] (i.e. our theorem 3.4). The Clebsch-Gordan coefficients will be defined in section 4. The essential symmetry relations for the Clebsch-Gordan coefficients will be derived in a simple algebraic way. However, in section 5 we have to do hard computational work in order to obtain the expression (5.3). The orthogonality relations for the $q$-Hahn and dual $q$-Hahn polynomials will be a relatively easy consequence of this expression.

After we completed this manuscript a preprint by Kirillov and Reshetikhin ([6]) reached us, where they also give (without proof) explicit expressions for the Clebsch-Gordan coefficients for the quantized universal enveloping algebra $U_q(\mathfrak{sl}(2))$ (cf. [5]). However, they do not express them as $q$-Hahn polynomials.

1. $q$-HYPERGEOMETRIC FUNCTIONS

In this section we state some definitions concerning $q$-hypergeometric functions. Some $q$-hypergeometric orthogonal polynomials are also discussed, as well as some identities for $q$-hypergeometric functions.

Let $1 \neq q \in \mathbb{C}$. For $a \in \mathbb{C}, k \in \mathbb{Z}^+$ the $q$-shifted factorial is defined by

$$
(a; q)_k = \prod_{i=0}^{k-1} (1 - aq^i)
$$

and if $|q| < 1$ we also have

$$
(a; q)_\infty = \prod_{i=0}^{\infty} (1 - aq^i).
$$

The product of $n$ $q$-shifted factorials is denoted by

$$(a_1, \ldots, a_n; q)_k = (a_1; q)_k \cdots (a_n; q)_k.
$$

We also have $q$-combinatorial coefficients. For $n, k \in \mathbb{Z}^+, n \geq k \geq 0$,

$$
\begin{bmatrix} n \end{bmatrix}_q = \frac{(q; q)_n}{(q; q)_k(q; q)_{n-k}} = \begin{bmatrix} n \\ n - k \end{bmatrix}_q.
$$

Then

$$
\begin{bmatrix} n \\ 0 \end{bmatrix}_q = 1 = \begin{bmatrix} n \\ n \end{bmatrix}_q,
$$

and the $q$-combinatorial coefficients satisfy the following recurrence relation:

$$
\begin{bmatrix} n+1 \\ k \end{bmatrix}_q = q^k \begin{bmatrix} n \\ k \end{bmatrix}_q + \begin{bmatrix} n \\ k-1 \end{bmatrix}_q.
$$

444
For $r \in \mathbb{Z}_+$ the $q$-hypergeometric series $r+1 \phi_r$ is defined by

\begin{equation}
(1.5) \quad r+1 \phi_r \left( \frac{a_1, \ldots, a_{r+1}}{b_1, \ldots, b_r}; q, z \right) = \sum_{k=0}^{\infty} \frac{(a_1, \ldots, a_{r+1}; q)_k}{(b_1, \ldots, b_r, q; q)_k} z^k.
\end{equation}

The radius of convergence is 1 for generic values of the parameters. For us the most interesting case arises when $a_1 = q^{-n}$ ($n \in \mathbb{Z}_+$) while the parameters $b_1, \ldots, b_r$ are not of the form $1, q^{-1}, \ldots, q^{-n}$. Then (1.5) is a well defined terminating series with summation from 0 to $n$.

There is a $q$-analogue of the Chu-Vandermonde formula ([4], (1.5.3))

\begin{equation}
(1.6) \quad _2\phi_1 \left( \frac{q^{-n}, b}{c}; q, q^n \right) = \frac{(c/b; q)_n}{(c; q)_n} b^n.
\end{equation}

The little $q$-Jacobi polynomials are also defined in terms of a $2 \phi_1$ series:

\begin{equation}
(1.7) \quad p_n(x; a, b | q) = _2\phi_1 \left( \frac{q^{-n}, abq^{n+1}}{aq, q}; q, qx \right).
\end{equation}

They are orthogonal polynomials. (See [1].)

The $q$-Hahn polynomials are defined in terms of a $3 \phi_2$ series:

\begin{equation}
(1.8) \quad \varphi_n(x; a, b, N | q) = _3\phi_2 \left( \frac{q^{-n}, abq^{n+1}, x}{aq, q^{-N}}, q, q \right)
\end{equation}

for $N \in \mathbb{Z}_+$ and $n \in \{0, 1, \ldots, N\}$. They are orthogonal polynomials and the explicit orthogonality relation is

\begin{equation}
(1.9) \quad \sum_{x=0}^{N} \varphi_n(q^{-x}) \varphi_n(q^{-x}) \frac{(aq; q)_x(bq; q)_{N-x}}{(q, q)_x(q; q)_{N-x}} (aq)^{-x} = \delta_{mn} \frac{(abq^2; q)_N(aq)^{-N}}{(q; q)_N} \frac{(1 - abq)(q, bq, abq^{N+2}; q)_n}{(1 - abq^{2n+2})(aq, abq, q^{-N}; q)_n} \times (-aq)^n q^{(5)-Nn}.
\end{equation}

(See [4], (7.2.22).)

The dual $q$-Hahn polynomials are also defined in terms of a terminating $3 \phi_2$

\begin{equation}
(1.10) \quad \mathcal{R}_n(\mu(x)) = \mathcal{R}_n(\mu(x); a, b, N | q) = _3\phi_2 \left( \frac{q^{-n}, abq^{x+1}, q^{-x}}{aq, q^{-N}}; q, q \right)
\end{equation}

for $N \in \mathbb{Z}_+$ and $n \in \{0, 1, \ldots, N\}$ and $\mu(x) = q^{-x} + q^{x+1}ab$. They are orthogonal polynomials and the explicit orthogonality relation can be derived from the orthogonality relations for the $q$-Racah polynomials. (See [2], (1.17)-(1.19) and p. 28, 29.) The relation is the following

\begin{equation}
(1.11) \quad \sum_{x=0}^{N} \mathcal{R}_n(q^{-x}) \mathcal{R}_n(q^{-x}) \frac{(aq; q)_x(bq; q)_{N-x}}{(q, q)_x(q; q)_{N-x}} (aq)^{-x} = \delta_{mn} \frac{(abq^2; q)_N(aq)^{-N}}{(q; q)_N} \frac{(1 - abq)(q, bq, abq^{N+2}; q)_n}{(1 - abq^{2n+2})(aq, abq, q^{-N}; q)_n} \times (-aq)^n q^{(5)-Nn}.
\end{equation}
Note that for $x, n \in \mathbb{Z}_+, 0 \leq x, n \leq N$

\begin{equation}
\sum_{x=0}^{N} \mathcal{R}_m(\mu(x)) \mathcal{R}_n(\mu(x)) \frac{(1 - abq^{2x+1})(aq, abq, q^{-N}; q)_x}{(1 - abq)(q, bq, abq^{N+2}; q)_x} \\
\times (-aq)^{-1} q^{N - (1 - x)} \\
= \delta_{mn} \frac{(abq^2; q)_N(aq)_N}{(bq; q)_N} \frac{(q, b^{-1}q^{-N}; q)_n}{(aq, q^{-N}; q)_n} (abq)_n.
\end{equation}

(1.11)

and that (1.11) is equivalent to (1.9).

We will also need a transformation for the $\varphi_2$ series. It is (see [4], (3.2.5))

\begin{equation}
3\varphi_2\left( q^{-n}, a, b; d, e, \frac{deq^n}{ab} \right) = \left(\frac{e/a}{e; q}_n \right) \frac{\varphi_2\left( q^{-n}, a, d/b \right)}{\varphi_2\left( d, aq^{1-n}/e \right; q, q)}.
\end{equation}

(1.12)

\begin{equation}
3\varphi_2\left( q^{-n}, a, b; N|q \right) = \varphi_2\left( q^{-n}; a, b; N|q \right)
\end{equation}

2. THE QUANTUM GROUP $S_\mu U(2)$ AND ITS REPRESENTATION THEORY

The matrix elements of the representations of the quantum group $S_\mu U(2)$ are considered in this section.

Fix $\mu \in [-1, 1] \setminus \{0\}$. We are primarily interested in the case $|\mu| < 1$. Let $A$ be the unital $C^*$-algebra generated by $a$ and $y$ subject to the relations

\begin{equation}
\begin{cases}
\alpha^*\alpha + \gamma^*\gamma = I; \\
a\alpha^* + \mu^2\gamma\gamma^* = I \\
\gamma\gamma^* = \gamma^*\gamma; \\
\alpha\gamma = \mu\gamma\alpha; \\
\gamma\alpha^* = \mu\gamma^*\alpha.
\end{cases}
\end{equation}

(2.1)

For the construction of $A$ see [14], § 1.)

Let

\begin{equation}
u = \begin{pmatrix} \alpha & -\mu\gamma^* \\ \gamma & \alpha^* \end{pmatrix}.
\end{equation}

(2.2)

Woronowicz ([13], [14]) has proved that $S_\mu U(2) = (A, u)$ is a compact matrix quantum group (a quantum group for short). For $\mu = 1$ we can identify $(A, u)$ with $SU(2)$.

The comultiplication is the unital $C^*$-algebra homomorphism $\Phi : A \to A \otimes A$ such that (see [14], (1.13))

\begin{equation}
\begin{cases}
\Phi(\alpha) = \alpha \otimes \alpha - \mu\gamma^* \otimes \gamma, \\
\Phi(\gamma) = \gamma \otimes \alpha + \alpha^* \otimes \gamma.
\end{cases}
\end{equation}

(2.3)

Fix $l \in \frac{1}{2} \mathbb{Z}_+$ and let $n \in \{-l, -l + 1, \ldots, l - 1, l\}$. Then, using (2.3) and (2.1), we have

446
Evidently, \( t^l_{n,m} \in \mathcal{A} \), where \( \mathcal{A} \) denotes the \(*\)-subalgebra of \( A \) generated by the matrix elements of \( u \) defined in (2.2). Using the coassociativity ([13], (1.7)),

\[
(\Phi \otimes \text{id}) \circ \Phi = (\text{id} \otimes \Phi) \circ \Phi,
\]
on (2.4) we obtain

\[
\Phi \left( \left[ \begin{array}{c} 2l \\ l-n \end{array} \right]^{1/2}_{\mu} \alpha^{l-n} y^{l+n} \right) = \left[ \begin{array}{c} 2l \\ l-n \end{array} \right]^{1/2}_{\mu} (\alpha \otimes \alpha - \mu y \otimes y)^{l-n} (y \otimes \alpha + \alpha \otimes y)^{l+n}
\]

\[
= \sum_{m=-l}^{l} t^l_{n,m} \otimes \left[ \begin{array}{c} 2l \\ l-m \end{array} \right]^{1/2}_{\mu} \alpha^{l-m} y^{l+m}.
\]

So the matrix \( (t^l_{n,m})_{n,m=-l,\ldots,l} \) defines a smooth representation \( t^l_{\mu} \) of \( S_{\mu} U(2) \) in \( \mathbb{C}^{2l+1} \) (cf. [13]).

Define

\[
\delta = \alpha^*; \quad \beta = -\mu y^*;
\]
then the relations (2.1) become

\[
\begin{cases}
\alpha \beta = \mu \beta \alpha; \quad \alpha y = \mu y \alpha; \quad \gamma \delta = \mu \delta \gamma; \quad \beta \delta = \mu \delta \beta; \\
\alpha \delta - \mu^2 \delta \alpha = (1 - \mu^2) I; \quad \gamma \beta = \beta \gamma; \quad \alpha \delta - \mu \beta \gamma = I.
\end{cases}
\]

So we can identify \( \mathcal{A} \) with \( \mathcal{A}(\alpha, \beta, \gamma, \delta) \) the unital algebra of polynomials in non-commuting variables \( \alpha, \beta, \gamma \) and \( \delta \) with relations (2.7). Write \( a(\alpha, \beta, \gamma, \delta) \) for a specific algebraic expression in the non-commuting variables \( \alpha, \beta, \gamma \) and \( \delta \) for some \( a \in \mathcal{A}(\alpha, \beta, \gamma, \delta) \). Interchanging \( \beta \) and \( \gamma \) yields an isomorphism of \( \mathcal{A}(\alpha, \beta, \gamma, \delta) \) which we denote by

\[
a(\alpha, \beta, \gamma, \delta) \rightarrow a(\gamma, \beta, \delta, \alpha).
\]

Write \( \bar{a}(\alpha, \beta, \gamma, \delta) \) for the same algebraic expression as \( a(\alpha, \beta, \gamma, \delta) \) with the order of the factors in each term inverted. Interchanging \( \alpha \) and \( \delta \) yields an anti-isomorphism which we denote by

\[
a(\alpha, \beta, \gamma, \delta) \rightarrow \bar{a}(\delta, \beta, \gamma, \alpha).
\]

We state some results on the representations \( t^l_{\mu} \) and its matrix elements \( t^l_{n,m} \).

(See [8].)

PROPOSITION 2.1. ([8]) The matrix elements \( t^l_{n,m} \) satisfy the following symmetry relations:

\[
t^l_{n,m}(\alpha, \beta, \gamma, \delta) = t^l_{m,n}(\alpha, \gamma, \beta, \delta),
\]
(2.9) \( t_{n,n}^{l,a,b,c}(\alpha, \beta, \gamma, \delta) = (t_{n,n}^{l,a,b,c})^{-1}(\delta, \gamma, \beta, \alpha) \),

(2.10) \( t_{n,m}^{l,a,b,c}(\alpha, \beta, \gamma, \delta) = (t_{n,m}^{l,a,b,c})^{-1}(\delta, \beta, \gamma, \alpha) \).

**THEOREM 2.2.** ([8], [9], [11]) For \( m \geq n \geq -m \) we have

(2.11) \( t_{n,m}^{l,a,b,c}(\alpha, \beta, \gamma, \delta) = c_{n,m}^{l,a,b,c} \delta^{n+m} + p_{l,m}(-\mu^{-1} \beta \gamma; \mu^{2(m-n)}, \mu^{2(m+n)}) \beta^{m-n} \),

where \( p_{l,m} \) denotes a little \( \varphi \)-Jacobi polynomial (see (1.7)) and

(2.12) \( c_{n,m}^{l,a,b,c} = \left[ \frac{l-n}{m-n} \right]^{1/2} \left[ \frac{l+m}{m-n} \right]^{1/2} \mu^{-(m-n)(l-m)} \).

Note that we can obtain an expression for \( t_{n,m}^{l,a,b,c} \) in the remaining cases by Proposition 2.1.

**THEOREM 2.3.** ([8], see also [9], [14], § 5) The representations \( t_{l,a}^{i} \) (\( l \in \frac{1}{2} \mathbb{Z}_+ \)) form a complete system of inequivalent irreducible unitary representations of the quantum group \( S_{l}(U(2)) \).

See [13] for the meaning of this theorem.

3. **DECOMPOSITION OF THE TENSOR PRODUCT** \( t_{l,a}^{i} \otimes t_{l,b}^{j} \)

In this section we decompose the unitary representation \( t_{l,a}^{i} \otimes t_{l,b}^{j} \) into a sum of unitary representations. We also give explicit realizations of these representations.

From [14], theorem 1.2 we know that \( \alpha^{k} \gamma^{m} \beta^{n} \) and \( \gamma^{m} \beta^{n} \delta^{k} \) form a basis for \( \mathcal{A} \). Let \( \mathcal{A}_{d} \) be the linear subspace of \( \mathcal{A} \) spanned by all \( \alpha^{k} \gamma^{m} \beta^{n} \) and \( \gamma^{m} \beta^{n} \delta^{k} \) with \( k + m + n = d \) for \( d = 0, 1, \ldots \). Then \( \dim(\mathcal{A}_{d}) = (d+1)^{2} \).

Let \( \mathcal{A}_{d}^{+} \) be the linear subspace of \( \mathcal{A} \) spanned by all \( \alpha^{d_{1}} \gamma^{d_{2}} \beta^{d_{3}} \delta^{d_{4}} \) with \( d_{1} + d_{2} + d_{3} + d_{4} = d \).

**PROPOSITION 3.1.**

\[ \mathcal{A}^{d} = \bigoplus_{j = 0, 1, \ldots, [d/2]} \mathcal{A}_{d-2j}. \]

**PROOF.** From (see (2.7))

(3.1) \[ \begin{align*}
\alpha^{d_{1}} \gamma^{d_{2}} \beta^{d_{3}} \delta^{d_{4}} &= \alpha^{d_{1}} \gamma^{d_{2}}(\alpha \delta - \mu \beta \gamma) \beta^{d_{3}} \delta^{d_{4}} \\
&= \mu^{-(d_{2} + d_{3})} \alpha^{d_{1} + 1} \gamma^{d_{2}} \beta^{d_{3}} \delta^{d_{4} + 1} - \mu \alpha^{d_{1}} \gamma^{d_{2}} \beta^{d_{3} + 1} \delta^{d_{4}}
\end{align*} \]

we see that \( \mathcal{A}^{d-2j} \subset \mathcal{A}_{d}^{+} \), \( j = 0, 1, \ldots, [d/2] \). Since \( \mathcal{A}_{d-2j} \subset \mathcal{A}_{d-2j}^{+} \) we have

\[ \bigoplus_{j = 0, 1, \ldots, [d/2]} \mathcal{A}_{d-2j} \subset \mathcal{A}^{d}. \]

Since the \( \alpha^{k} \gamma^{m} \beta^{n} \) and \( \gamma^{m} \beta^{n} \delta^{k} \) form a basis of \( \mathcal{A} \) we see that

\[ \dim \left( \bigoplus_{j = 0, 1, \ldots, [d/2]} \mathcal{A}_{d-2j} \right) = \sum_{j = 0}^{[d/2]} (d - 2j + 1)^{2} = \left( \frac{d+3}{3} \right)^{2} = \dim(\mathcal{A}^{d}). \]
But the dimension of $\mathcal{A}^d$ is smaller than or equal to the dimension of the space of homogeneous polynomials of degree $d$ in four variables, which is $\binom{d+3}{3}$.

**Corollary 3.2.** The monomials $\alpha^{d_1} \gamma^{d_2} \beta^{d_3} \delta^{d_4}$ $(d_1 + d_2 + d_3 + d_4 = d)$ constitute a basis for $\mathcal{A}^d$.

The equality (3.1) can be generalized into the following, which will be useful in the sequel.

**Lemma 3.3.** For $d_1, d_2, d_3, d_4, k \in \mathbb{Z}_+$ we have

$$\alpha^{d_1} \gamma^{d_2} \beta^{d_3} \delta^{d_4} = \sum_{i=0}^{k} (-\mu)^i \mu^{- (d_3 + d_4)(k-i)} \left[ \begin{array}{c} k \\ i \end{array} \right] \alpha^{d_1 + k-i} \gamma^{d_2 + i} \beta^{d_3 + i} \delta^{d_4 + k-i}.$$ 

**Proof.** By repeating (3.1) we see that we have an expression like

$$\alpha^{d_1} \gamma^{d_2} \beta^{d_3} \delta^{d_4} = \sum_{i=0}^{k} A^k_i \alpha^{d_1 + k-i} \gamma^{d_2 + i} \beta^{d_3 + i} \delta^{d_4 + k-i}.$$ 

To calculate $A^k_i$ we apply (3.1) to every term of the sum on the right hand side. This yields the following recurrence relation for $A^k_i$:

$$A^k_{i+1} = \mu^{- (d_3 + d_4)} A^k_{i} - 2i A^k_{i-1}.$$ 

Now put $A^k_i = (-\mu)^i \mu^{- (d_3 + d_4)(k-i)} B^k_i$, then (3.2) yields a recurrence relation for $B^k_i$:

$$B^k_{i+1} = \mu^{-2i} B^k_{i} + B^k_{i-1}.$$ 

Since $A^0_0 = B^0_0 = 1$ we have from (1.4) the solution $B^k_i = \left[ \begin{array}{c} n \\ i \end{array} \right]$ for the relation (3.3).

Now we consider $\mathcal{A}^{e,f}$, the linear span of the monomials $\alpha^{e_1} \gamma^{e_2} \beta^{f_1} \delta^{f_2}$ with $e_1 + e_2 = e$ and $f_1 + f_2 = f$. Note that (3.1) immediately yields

$$\mathcal{A}^{e,f} \subset \mathcal{A}^{e+1,f+1}.$$ 

We make $\mathcal{A}^{2l_1,2l_2}$ into a Hilbert space by declaring the following basis orthonormal

$$\psi_{n_1,n_2}^{l_1,l_2,\mu} = \left[ \begin{array}{c} 2l_1 \\ l_1 - n_1 \end{array} \right]^{1/2} \left[ \begin{array}{c} 2l_2 \\ l_2 - n_2 \end{array} \right]^{1/2} \alpha^{l_1 - n_1} \beta^{l_2 - n_2} \delta^{l_1 + n_1} \delta^{l_2 + n_2},$$ 

$n_1 \in \{-l_1, \ldots, l_1\}$, $n_2 \in \{-l_2, \ldots, l_2\}$. This is possible because of corollary 3.2.

From (2.3) and (2.6) it follows that

$$\Phi\left( \left[ \begin{array}{c} 2l \\ l - n \end{array} \right]^{1/2} \beta^{- n} \delta^{l + n}, \right) = \sum_{m=-l}^{l} t_{n,m}^{l,\mu} \left[ \begin{array}{c} 2l \\ l - m \end{array} \right]^{1/2} \beta^{- m} \delta^{l + m}.$$ 

This and (2.4) imply

$$\Phi(\psi_{n_1,n_2}^{l_1,l_2,\mu}) = \sum_{m_1 = -l_1}^{l_1} \sum_{m_2 = -l_2}^{l_2} t_{n_1,n_2}^{l_1,\mu} t_{m_1,m_2}^{l_2,\mu} \psi_{m_1,m_2}^{l_1,l_2,\mu}.$$ 

449
This proves that \( \Phi \) acting on \( \mathcal{A}^{2l_1,2l_2} \) gives a realization of the tensor product \( t^{l_1,\mu} \otimes t^{l_2,\mu} \). Taking into account the Hilbert space structure of \( \mathcal{A}^{2l_1,2l_2} \) we see that \( t^{l_1,\mu} \otimes t^{l_2,\mu} \) is unitary.

From proposition 5.2 of [8] (or from theorem 2.2 and lemma 3.3) and (3.4) we know that

\[
t^{l_1,\mu}_{n,m} \in \mathcal{A}^{l-m, l+m} \subseteq \mathcal{A}^{l+m, l+m}
\]

for \( i \in \mathbb{Z}_+ \). In particular, if we take \( m = l_2 - l_1 \) and \( l = |l_1 - l_2|, |l_1 + l_2| + 1, \ldots, l_1 + l_2 \), we have

\[
t^{l_1,\mu}_{n,l_2 - l_1} \in \mathcal{A}^{l_2 - l_1 + l_1, l_2 - l_1} = \mathcal{A}^{2l_1,2l_2}
\]

for \( i = l_1 + l_2 - l \).

Theorems 2.3 and theorem 5.7(i) of [13] imply that all \( t^{l_1,\mu}_{n,l_2 - l_1} \) are linearly independent and (2.5) yields

\[
\Phi(t^{l_1,\mu}_{n,l_2 - l_1}) = \sum_{m=-l}^{l} t^{l,\mu}_{n,m} \otimes t^{l_2,\mu}_{m,l_2 - l_1}.
\]

If we define \( \mathcal{A}^{2l_1,2l_2} \) to be the linear span of \( t^{l_1,\mu}_{n,l_2 - l_1} \), \( n = -l, \ldots, l \), then \( \Phi \) acting on \( \mathcal{A}^{2l_1,2l_2} \) gives a realization of the representation \( t^{l,\mu} \).

**THEOREM 3.4.**

\[
\mathcal{A}^{2l_1,2l_2} = \bigoplus_{l = |l_1 - l_2|}^{l_1 + l_2} \mathcal{A}^{2l_1,2l_2}
\]

and

\[
t^{l_1,\mu} \otimes t^{l_2,\mu} = \bigoplus_{l = |l_1 - l_2|}^{l_1 + l_2} t^{l,\mu}.
\]

**PROOF.** We need only to prove the first statement in view of (3.6) and (3.8). This follows from (3.7) and

\[
\sum_{l = |l_1 - l_2|}^{l_1 + l_2} (2l + 1) = (2l_1 + 1)(2l_2 + 1).
\]

**REMARK.** Theorem 3.4 is theorem 5.11 of [14], but the proof is new.

**THEOREM 3.5.**

(i) For suitable complex constants \( a^{l_1,\mu}_{l_1,l_2} \neq 0 \)

\[
\phi^{l_1,\mu}_{j,l_2} = a^{l_1,\mu}_{l_2,l_1} t^{l_1,\mu}_{l_2,l_2 - l_1}
\]

\( j = -l, \ldots, l \), constitute an orthonormal basis for \( \mathcal{A}^{2l_1,2l_2} \).

(ii) The constant \( a^{l_1,\mu}_{l_1,l_1} \) is uniquely determined by the condition

\[
\left(\phi^{l_1,\mu}_{j,l_2}, \psi^{l_1,\mu}_{l_1,l_1, l_2, l_1-j}\right) > 0.
\]

Then

\[
a^{l_1,\mu}_{l_1,l_2} = (-\mu)^{l_1 - l_2} \left\{ \frac{(\mu^{-2}; \mu^{-2})_{2l_1} (\mu^{-2}; \mu^{-2})_{2l_2}}{(\mu^{-2}; \mu^{-2})_{l_1 + l_2} (\mu^{-2}; \mu^{-2})_{l_1 - l_2}} \right\}^{1/2}.
\]
PROOF. (i) This follows from theorem 3.4, (3.8) and from [11], theorem 5.8, propositions 2.2 and 2.3.

To prove (ii) we use

\[
\ell_{l_1+1, l_2-l_1}(\alpha, \beta, \gamma, \delta) = \left[ \frac{2l}{l + l_1 - l_2} \right]^{1/2} \phi^{l_1+1, l_2-l_1, l_1+1, l_2-l_2} \]

by (2.5) and the \( q \)-binomial theorem (see [8], lemma 2.1). Hence, using lemma 3.3, we have

\[
\phi^{l_1+1, l_2-l_1}(\alpha, \beta, \gamma, \delta) = a^{l_1+1, l_2-l_1} \left[ \frac{2l}{l + l_1 - l_2} \right]^{1/2} \sum_{i=0}^{l_1+1, l_2-l_1} \mu^{-i(l_1-l_1+1)(l_1+1-l_1-1)}
\]

\[
\times (-\mu)^{i} \left[ \frac{2l}{l + l_1 - l_2} \right]^{1/2} \phi^{l_1+1, l_2-l_1, l_1+1, l_2-l_2} \psi^{l_1+1, l_2-l_1, l_1+1, l_2-l_2}.
\]

So (3.10) implies (take \( i = l_1+1, l_2-l_1 \) that

\[
(3.12) \quad a^{l_1+1, l_2-l_1} = (1)^{l_1+1, l_2-l_1} > 0.
\]

Now, \( \phi^{l_1+1, l_2-l_1} \) and \( \psi^{l_1+1, l_2-l_1} \) are orthonormal bases, so

\[
(3.13) \quad \begin{cases}
1 = \left| a^{l_1+1, l_2-l_1} \right|^2 \left[ \frac{2l}{l + l_1 - l_2} \right]^{1/2} \sum_{j=0}^{l_1+1, l_2-l_1} \mu^{-2(l_1-l_1+1)(l_1+1-l_1-1)} \\
\times \left[ \frac{2l}{l + l_1 - l_2} \right]^{1/2} \left[ \frac{2l_2}{l_1 + l_2 - l_1} \right] \left[ \frac{2l_1}{l_1 + l_2 - l_1} \right]^{-1} \\
\times \phi^{l_1+1, l_2-l_1, l_1+1, l_2-l_2} \\
= \left| a^{l_1+1, l_2-l_1} \right|^2 \mu^{2(l_1+l_2)} \\
\times (\mu^{-2}; \mu^{-2})_{l_1, l_1} (\mu^{-2}; \mu^{-2})_{l_1, l_2} \\
\times 2\phi^{l_1+1, l_2-l_1, l_1+1, l_2-l_2} \\
= \left| a^{l_1+1, l_2-l_1} \right|^2 \mu^{2(l_1+l_2)} \\
\times (\mu^{-2}; \mu^{-2})_{l_1, l_1} (\mu^{-2}; \mu^{-2})_{l_1, l_2} \\
\times (1 - \mu^{-2(l_1+1+l_2)}).
\end{cases}
\]

In the last step we used the Chu-Vandermonde formula (1.6). Finally, (3.12) and (3.13) imply (3.11).

4. THE CLEBSCH-GORDAN COEFFICIENTS FOR \( S_\mu U(2) \)

The Clebsch-Gordan coefficients are defined in this section. Some of their properties will be derived.

Since we have two orthonormal bases in \( \mathcal{A}^{2l_1, 2l_2} \), we can consider the unitary matrix which maps one basis onto the other. Its matrix elements are called the Clebsch-Gordan coefficients \( C^{l_1, l_2, l_1, l_2}_{j_1, j_2, j} \):

\[
(4.1) \quad \phi^{l_1, l_2, l_1, l_2} = \sum_{j_1=-l_1}^{l_1} \sum_{j_2=-l_2}^{l_2} C^{l_1, l_2, l_1, l_2}_{j_1, j_2, j} \psi^{l_1, l_2, l_1, l_2}.
\]

451
PROPOSITION 4.1. If \( j \neq j_1 + j_2 \), then
\[
C_{j_1, j_2, \mu}^{l_1, l_2, \mu} = 0.
\]

PROOF. We need the notion of a quantum subgroup of a quantum group \( G = (A, u) \). This is a quantum group \( K = (B, v) \) such that there exists a surjective unital C*-algebra homomorphism \( \pi : A \to B \) such that
\[
\Phi_K \circ \pi = (\pi \otimes \pi) \circ \Phi_G,
\]
where \( \Phi_K \) and \( \Phi_G \) denote the comultiplication of \( K \) and \( G \).

Now let \( t^G \) be a matrix representation of \( G \), then \( t^K = \pi t^G = (\pi t^G_j)_{i,j} \) is a matrix representation of \( K \) because of (4.2).

Take \( B = C(\mathbb{T}) \), the unital commutative C*-algebra of continuous functions on the unit circle \( \mathbb{T} \). Pick \( f \in C(\mathbb{T}) \) defined by \( f(z) = z \) for \( z \in \mathbb{T} \) and put
\[
u = \begin{pmatrix} f & 0 \\ 0 & f^* \end{pmatrix}.
\]

It is easy to check that the unital C*-homomorphism of the C*-algebra \( A \) of \( S_\mu U(2) \) into \( C(\mathbb{T}) \) generated by
\[
\pi(a) = f; \quad \pi(y) = 0
\]
makes \( (C(\mathbb{T}), \nu) \) into a quantum subgroup of \( S_\mu U(2) \). (See [8], [10].)

Apply \( \pi \otimes \text{id} \) on the last equality in (2.4) to obtain (cf. [8], § 4)
\[
(4.3) \quad \pi(t_{n,m}^{l_1, l_2}) = \delta_{nm} f^{-2n}.
\]

Apply \( (\pi \otimes \text{id}) \circ \Phi \) to (4.1) and use (3.6), (3.8) and (4.3) to obtain the following equality in \( \mathcal{B} \otimes \mathcal{A} \), where \( \mathcal{B} \) denotes the *-subalgebra of \( C(\mathbb{T}) \) generated by the elements of \( \nu \),
\[
f^{-2j} \otimes \theta_j^{l_1, l_2, \mu} = \sum_{j_1 = -l_1}^{l_1} \sum_{j_2 = -l_2}^{l_2} f^{-2(j_1 + j_2)} \otimes C_{j_1, j_2, \mu}^{l_1, l_2, \mu} \psi_j \otimes \theta_j^{l_1, l_2, \mu}.
\]

This proves the proposition. \( \square \)

REMARK 4.2. If we define a quantum subgroup \( K = (B, v) \) of \( G = (A, u) \) with \( u \in M_n(A) \) and \( v \in M_m(B) \), if \( m = n \) and if there exists a unital C*-algebra homomorphism \( \pi : A \to B \) with
\[
\pi(u_{ij}) = \nu_{ij} \quad \forall i, j \in \{1, \ldots, n\},
\]
then we have automatically (4.2) and the surjectivity of \( \pi \). Also
\[
\kappa_K \circ \pi = \pi \circ \kappa_G,
\]
where \( \kappa_K \) and \( \kappa_G \) denote the coinverse (see [13], def. 1.1.) of \( K \) and \( G \), and \( \pi(\mathcal{A}) = \mathcal{B} \).

Because of proposition 4.1 it is sufficient to consider \( C_{j_1, j_2, \mu}^{l_1, l_2, \mu} \) with \( |l_1 - l_2| \leq l_1 + l_2; \ |j_1| \leq l_1; \ |j_2| \leq l_2; \ j = j_1 + j_2 \). Here \( |j| \leq l \) means \( j \in \{-l, \ldots, l\} \).

To narrow this area under consideration even more we will prove symmetry.
relations for the Clebsch-Gordan coefficients. We have the following relation in \( \mathcal{A}(\alpha, \beta, \gamma, \delta) \):

\[
(4.4) \quad \frac{\phi_{l_1,l_2,l_3}^{l_4,l_5,l_6} (\alpha, \beta, \gamma, \delta)}{\{ (\mu^{-2}; \mu^{-2})_{l_1} (\mu^{-2}; \mu^{-2})_{l_2} \}^{1/2}} = \sum_{l_1} \sum_{l_2} C_{l_1,l_2,l_3}^{l_4,l_5,l_6} (\alpha, \beta, \gamma, \delta, \mu) (\mu^{-2}; \mu^{-2})_{l_1} (\mu^{-2}; \mu^{-2})_{l_2} (\mu^{-2}; \mu^{-2})_{l_3}^{1/2}.
\]

Because \( \mu \in [-1, 1] \setminus \{0\} \) we see that the Clebsch-Gordan coefficients are real. Since \( \phi_{l_1,l_2,l_3}^{l_4,l_5,l_6} (\alpha, \beta, \gamma, \delta) = a^{l_1}_{l_4} a^{l_2}_{l_5} a^{l_3}_{l_6} (\alpha, \beta, \gamma, \delta) \) the symmetry relations for \( t_{n,m}^{l_1,l_2,l_3} (\alpha, \beta, \gamma, \delta) \) (see (2.8), (2.9) and (2.10)) yield symmetry relations for \( \phi_{l_1,l_2,l_3}^{l_4,l_5,l_6} (\alpha, \beta, \gamma, \delta) \).

\[
(4.5) \quad \frac{\phi_{l_1,l_2,l_3}^{l_4,l_5,l_6} (\alpha, \beta, \gamma, \delta)}{\{ (\mu^{-2}; \mu^{-2})_{l_1} (\mu^{-2}; \mu^{-2})_{l_2} \}^{1/2}} = \phi_{l_2,l_3}^{l_1,l_4,l_5,l_6} (\alpha, \beta, \gamma, \delta, \mu) \frac{1}{\{ (\mu^{-2}; \mu^{-2})_{l_1} (\mu^{-2}; \mu^{-2})_{l_2} \}^{1/2}}
\]

\[
(4.6) \quad \frac{\phi_{l_1,l_2,l_3}^{l_4,l_5,l_6} (\alpha, \beta, \gamma, \delta)}{\{ (\mu^{-2}; \mu^{-2})_{l_1} (\mu^{-2}; \mu^{-2})_{l_2} \}^{1/2}} = \phi_{l_2,l_3}^{l_1,l_4,l_5,l_6} (\alpha, \beta, \gamma, \delta, \mu) \frac{1}{\{ (\mu^{-2}; \mu^{-2})_{l_1} (\mu^{-2}; \mu^{-2})_{l_2} \}^{1/2}}
\]

Combination of (4.4) and (4.5), of (4.4) and (4.6) and of (4.4) and (4.7) gives the following symmetry relations for the Clebsch-Gordan coefficients. (Of course we use corollary 3.2 as well.)

**Proposition 4.3.** The Clebsch-Gordan coefficients \( C_{l_1,l_2,l_3}^{l_4,l_5,l_6} \) satisfy the following relations:

\[
C_{l_1,l_2,l_3}^{l_4,l_5,l_6} = C_{l_2,l_3,l_1}^{l_1,l_2,l_3} = C_{l_3,l_1,l_2}^{l_1,l_2,l_3} = C_{l_2,l_3,l_1}^{l_1,l_2,l_3} = C_{l_3,l_1,l_2}^{l_1,l_2,l_3} = C_{l_2,l_3,l_1}^{l_1,l_2,l_3}.
\]

Thus we can restrict ourselves to one of the following four subsets in the \((l_1, l_2, l_3, l_4, l_5, l_6)\)-space:

\[
(4.8) \quad \left\{ \begin{array}{ll}
(i) & l_1 - l_2 \leq j \leq l_3 - l_2 - l_1 - l_2 - l_1 + l_2; \quad -l_1 \leq l_1 \leq l_1 + l_2; \quad j = j_1 + j_2; \\
(ii) & l_2 - l_1 \leq j \leq l_3 - l_2 - l_1 - l_2 - l_1 + l_2; \quad -l_2 \leq l_2 \leq l_2 + l_1 + l_2; \quad j = j_1 + j_2; \\
(iii) & j \leq l_1 - l_2 \leq -j \leq l_1 - l_2 - l_1 - l_2; \quad -l_1 \leq j_1 \leq l_1; \quad -l_2 \leq j_2; \quad j = j_1 + j_2; \\
(iv) & -j \leq l_1 - l_2 \leq j \leq l_1 - l_2 - l_1 - l_2; \quad j_1 \leq l_1; \quad j_2 \leq l_2; \quad j = j_1 + j_2.
\end{array} \right.
\]
5. EXPRESSION OF THE CLEBSCH-GORDAN COEFFICIENTS IN TERMS OF q-HAHN POLYNOMIALS

We derive an explicit expression for the Clebsch-Gordan coefficients in terms of \( q \)-Hahn polynomials. This will allow us to prove the orthogonality relations for the \( q \)-Hahn polynomials.

First of all we need a generating function for the Clebsch-Gordan coefficients. We restrict ourselves to case (i) of (4.8). Then it follows from (4.4), (3.11), theorem 2.2 and (1.7) that

\[
\sum_{j_1+j_2 \leq J} \sum_{j_1=-l_1}^{l_1} C_{j_1, j_2, j}^{l_1, l_2, j, 1, j_1, j_1, l_1, j_2, j_2} = (-\mu) - l_1 - j_1 + l_1 - (l_1 + j_2)(l_2 - l_1 - j_2)(1 - \mu - 2(l_2 + j_2))^{1/2} \\
\times \left( \frac{(\mu^2; \mu^2)_{l_1} \cdot (\mu^2; \mu^2)_{l_2} \cdot (\mu^2; \mu^2)_{l_2 + j_2}}{(\mu^2; \mu^2)_{l_1 + l_2} \cdot (\mu^2; \mu^2)_{l_1 + l_2 + j_2}} \right)^{1/2} \\
\times \left\{ \frac{(\mu^2; \mu^2)_{l_1 - j_1} \cdot (\mu^2; \mu^2)_{l_2 - j_2}}{(\mu^2; \mu^2)_{l_1 - j_1} \cdot (\mu^2; \mu^2)_{l_1 + l_2 + j_2}} \right\}^{1/2} \\
\times \sum_{k=0}^{l_1 - j_1} \frac{(-\mu) - l_1 - j_1 + l_1 - (l_1 + j_2)(l_2 - l_1 - j_2)(1 - \mu - 2(l_2 + j_2))^{1/2}}{\mu^2(l_1 - j_1 + 1) \cdot \mu^2(l_1 + j_1) \cdot (-\mu \beta)^k}.
\]

In the right hand side we use

\[
\delta^{l_1 - j_1 + 2j_2} = (l_1 - j_1 + 2j_2) \rho(l_2 - l_1 - j_2)
\]

by lemma 3.3 and the commutation relations (2.7). Put \( j_1 = k + l_1 \) and change summation

\[
\sum_{k=0}^{l_1 - j_1} \sum_{j_1 = -l_1}^{l_1} \sum_{k=0}^{l_1 - j_1} \sum_{l_1 = -l_1}^{l_1} \sum_{k=0}^{l_1 - j_1} \sum_{j_1 = -l_1}^{l_1} .
\]

After some manipulation with \( q \)-shifted factorials we obtain

\[
\sum_{j_1+j_2 \leq J} C_{j_1, j_2, j}^{l_1, l_2, j, 1, j_1, j_1, l_1, j_2, j_2} = (-\mu) - l_1 - j_1 + l_1 - (l_1 + j_2)(l_2 - l_1 - j_2)(1 - \mu - 2(l_2 + j_2))^{1/2} \\
\times \left( \frac{(\mu^2; \mu^2)_{l_1} \cdot (\mu^2; \mu^2)_{l_2} \cdot (\mu^2; \mu^2)_{l_2 + j_2}}{(\mu^2; \mu^2)_{l_1 + l_2} \cdot (\mu^2; \mu^2)_{l_1 + l_2 + j_2}} \right)^{1/2} \\
\times \left\{ \frac{(\mu^2; \mu^2)_{l_1 - j_1} \cdot (\mu^2; \mu^2)_{l_2 - j_2}}{(\mu^2; \mu^2)_{l_1 - j_1} \cdot (\mu^2; \mu^2)_{l_1 + l_2 + j_2}} \right\}^{1/2} \\
\times \left\{ \frac{(\mu^2; \mu^2)_{l_1 - j_1} \cdot (\mu^2; \mu^2)_{l_2 - j_2}}{(\mu^2; \mu^2)_{l_1 - j_1} \cdot (\mu^2; \mu^2)_{l_1 + l_2 + j_2}} \right\}^{1/2} \\
\times \phi_2(\mu^2(l_1 - j_1 + 1), \mu^2(l_1 + j_1 + 2l_2 + j_2), \mu^2(l_1 - j_1 + 1), \mu^2(2l_2 - j_1)).
\]

454
Transform (5.2) using
\[(\mu^{-2} ; \mu^{-2})_k = (-1)^k \mu^{-k(k+1)}(\mu^2 ; \mu^2)_k.\]
Next we introduce
\[x = l_1 - j; \quad n = l_1 - l_2 + i; \quad N = 2l_1,\]
\[a = -l_1 + l_2 + j; \quad b = -l_1 + l_2 - j.\]
Then (4.8) (i) is equivalent to the condition that \(x, n, N, a\) and \(b\) are integers and
\[0 \leq x \leq N; \quad 0 \leq n \leq N; \quad a \geq 0; \quad b \geq 0.\]
If we also use a transformation rule for the \(3\psi_2\) (see (1.13) with \(a = \mu^{2(n+a+b+1)}\), \(b = \mu^{-2(N-x)}\), \(d = \mu^{-2N}\) and \(e = \mu^2(b+1)\)), then we can recognize a \(q\)-Hahn polynomial \(\varphi_n\). The result is
\[
\left\{ \begin{array}{l}
C_{1/2N,1/2N+x,a+b,n+1/2(a+b)} = (-1)^{N+n-x} \left( \frac{\mu^2 \mu^2}{\mu^2 \mu^2} \right)_N \\
\times \left\{ \left( \frac{\mu^2 \mu^2}{\mu^2 \mu^2} \right)_{N-x+a} \left( \frac{\mu^2 \mu^2}{\mu^2 \mu^2} \right)_n \left( \frac{1 - \mu^{1+2n+a+b}}{1 - \mu^{1+2n+a+b}} \right) \right\}^{1/2} \\
\times \left( \frac{\mu^2 \mu^2}{\mu^2 \mu^2} \right)_{N-x} \left( \frac{\mu^2 \mu^2}{\mu^2 \mu^2} \right)_{n+a+b+1} \left( \frac{\mu^2 \mu^2}{\mu^2 \mu^2} \right)_n \left( \frac{\mu^2 \mu^2}{\mu^2 \mu^2} \right)_{n+a} \\
\times \mu^{(N-x)(a+1)+n(a+n)} \left( \frac{\mu^{-2(n+a)}}{\mu^{-2(b+1)}} \right)_n \varphi_n(\mu^{-2x}; \mu^{-2a}; \mu^{-2b}; N | \mu^2).
\end{array} \right.
\]
Since the Clebsch-Gordan coefficients are matrix elements of a unitary matrix we have
\[
\sum_{x=0}^{N} C_{1/2N,1/2N+x,a+b,n+1/2(a+b)} = \delta_{mn}.
\]
Combination of (5.3) and (5.4) and \(q = \mu^2\) yields
\[
\sum_{x=0}^{N} \left( \frac{q; q}{q} \right)_{N-x+a} \left( \frac{q; q}{q} \right)_{N-x} \varphi_n(q^{-x}) \varphi_m(q^{-x}) = c_n \delta_{mn},
\]
where \(\varphi_n(q^{-x}) = \varphi_n(q^{-x}; q^a, q^b; N | q)\) and
\[
\left\{ \begin{array}{l}
c_n = \left( \frac{q; q}{q} \right)_{N-a} \left( \frac{q; q}{q} \right)_{N+a+b+1} \left( \frac{q; q}{q} \right)_{n+a} \\
\times \left( \frac{q; q}{q} \right)^2_2 \left( \frac{q; q}{q} \right)_{N-a} \left( \frac{q; q}{q} \right)^2_2 \\
\times \left( \frac{q; q}{q} \right)_{N-a} \left( \frac{q; q}{q} \right)_{N-a} \left( \frac{q; q}{q} \right)_{N-a}.
\end{array} \right.
\]
When we divide both sides of (5.5) by \((q; q)_a(q; q)_b\) we obtain the orthogonality relations (1.9) for the \(q\)-Hahn polynomials with \(a, b\) replaced by \(q^a, q^b\). Since \(q = \mu^2 \in (0, 1)\) we can use analytic continuation to obtain (1.9) for arbitrary \(a, b\).
Of course we also have orthogonality relations dual to (5.4):

\[
\sum_{n=0}^{N} C_{1/2N}^{1/2N+1/2(a+b), \mu 1/2N} C_{1/2N}^{1/2N+1/2(a+b), \mu 1/2N} = \delta_{xy}.
\]

Substitution of (5.3) and (1.12) in (5.7) yields the orthogonality relations (1.11) for the dual \(q\)-Hahn polynomials with \(a, b\) replaced by \(qa, qb\) and \(n\) and \(x\) interchanged.

REFERENCES