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ON THE ZEROS OF THE ERROR TERM FOR THE MEAN SQUARE OF $|\zeta(\frac{1}{2}+it)|$

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ABSTRACT. Let $E(T)$ denote the error term in the asymptotic formula for

$$\int_0^T \left|\zeta\left(\frac{1}{2}+it\right)\right|^2 dt.$$

The function $E(T)$ has mean value π. By t_n we denote the nth zero of $E(T)-\pi$. Several results concerning t_n are obtained, including $t_{n+1}-t_n \ll t_n^{1/2}$. An algorithm is presented to compute the zeros of $E(T)-\pi$ below a given bound. For $T \le 500000$, 42010 zeros of $E(T)-\pi$ were found. Various tables and figures are given, which present a selection of the computational results.

1. INTRODUCTION

Let, as usual, for $T \ge 0$

$$E(T)=\int_0^T \left|\zeta\left(\frac{1}{2}+it\right)\right|^2 dt - T\log\left(\frac{T}{2\pi}\right)-(2\gamma-1)T$$

denote the error term in the asymptotic formula for the mean square of the Riemann zeta function on the critical line (γ is Euler's constant). In view of F. V. Atkinson's explicit formula for $E(T)$ (see [2] and [11, Chapter 15]) and its important consequences, this function plays a central role in the theory of $\zeta(s)$.

It is also of interest to consider $E(T)$ in mean square, and one has

$$(1)\quad \int_2^T E^2(t)\,dt = CT^{3/2}+\mathcal{O}(T\log^5 T) \qquad \left(C=\frac{2\zeta^4(3/2)}{3\sqrt{2\pi}\zeta(3)}\sim 10.3047\right).$$

This formula is due independently to T. Meurman [16] and Y. Motohashi [17], who improved the previous error term $\mathcal{O}(T^{5/4}\log^2 T)$ of D. R. Heath-Brown [10]. One consequence of (1) is the omega result $E(T)=\Omega(T^{1/4})$ [6], which was sharpened by Hafner and Ivić [7, 8] to

$$(2)\quad E(T)=\Omega_+\{T(\log T)^{1/4}(\log\log T)^{(3+\log 4)/4} \times \exp(-B\sqrt{\log\log\log T})\} \qquad (B>0)$$

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and

$$E(T) = \Omega_-\left\{T^{1/4}\exp\left(\frac{D(\log\log T)^{1/4}}{(\log\log\log T)^{3/4}}\right)\right\} \qquad (D > 0), \tag{3}$$

where $f(x) = \Omega_+(g(x))$ (resp. Ω_-) means that $f(x) > Cg(x)$ (resp. $f(x) < -Cg(x)$) for some $C > 0$ and some arbitrarily large values of x; furthermore, $f(x) = \Omega(g(x))$ means that $|f(x)| = \Omega_+(g(x))$. These omega results are analogous to the sharpest omega results for $\Delta(x)$, the error term in the classical Dirichlet divisor problem. This suggests the analogy between $E(T)$ and $2\pi\Delta(\frac{T}{2\pi})$ (see [11, Chapter 15]), which was one of the principal motivations for Atkinson's pioneering work [2]. However, there is an important difference between $E(T)$ and $2\pi\Delta(\frac{T}{2\pi})$. While $E(T)$ is a continuous function of T (with derivatives of any order), $\Delta(\frac{T}{2\pi})$ is certainly not, since $\sum_{n\le x} d(n)$ ($d(n)$ is the number of divisors of n) has jumps for integral x which may be as large as $\exp(\frac{C\log x}{\log\log x})$. (Here and later, $C, C_1, \ldots$ denote positive, absolute constants). From (2), (3) and continuity it is immediate that $E(T)$ has an infinity of zeros, and the purpose of this paper is to study these zeros and related topics, both from the theoretical and numerical viewpoint.

It seems expedient, especially from the numerical viewpoint, to study the zeros of $E(T) - \pi$ rather than those of $E(T)$. This is because $E(T)$ has the mean value π. More precisely, Hafner and Ivić [7] prove that, for $T \ge 2$,

$$\begin{aligned}\int_2^T E(t)\,dt = \pi T &+ 2^{-3/2}\sum_{n\le N}(-1)^n\frac{d(n)}{\sqrt{n}}\left(\operatorname{arcsinh}\sqrt{\frac{\pi n}{2T}}\right)^{-2} \\ &\times\left(\frac{T}{2\pi n}+\frac{1}{4}\right)^{-1/4}\sin(f(T, n)) \\ &-2\sum_{n\le N'}\frac{d(n)}{\sqrt{n}}\left(\log\frac{T}{2\pi n}\right)^{-2}\sin\left(T\log\frac{T}{2\pi n} - T + \frac{\pi}{4}\right) \\ &+\mathcal{O}(T^{1/4}),\end{aligned} \tag{4}$$

where

$$f(T, n) = 2T\operatorname{arcsinh}\sqrt{\frac{\pi n}{2T}} + (2\pi nT + \pi^2n^2)^{1/2} - \pi/4,$$

$$\operatorname{arcsinh} x = \log\left(x + \sqrt{x^2+1}\right),$$

$$N' = N'(T, N) = \frac{T}{2\pi} + \frac{N}{2} - \left(\frac{N^2}{4} + \frac{NT}{2\pi}\right)^{1/2},$$

$AT < N < A'T$ for any two fixed constants $0 < A < A'$. Note that formal differentiation of the sine terms in (4) leads to Atkinson's formula [2] for $E(T)$ itself (without the error term $\mathcal{O}(\log^2 T)$). Also, on simplifying (4) by Taylor's formula, one may deduce

$$(5)\quad \int_2^T E(t)\,dt = \pi T + 2^{-1/4}\pi^{-3/4}T^{3/4}\sum_{n=1}^{\infty}(-1)^n d(n) n^{-5/4}\sin\left(\sqrt{8\pi n T}-\frac{\pi}{4}\right) + \mathcal{O}(T^{2/3}\log T),$$

which we shall need in §3. A nice feature of (5) is that the series is absolutely convergent, so that

$$\int_2^T (E(t)-\pi)\,dt = \mathcal{O}(T^{3/4}),$$

and, as shown in [7], the above integral is also $\Omega_\pm(T^{3/4})$.

The plan of the paper is as follows. In §2 we shall study the general problem of gaps between consecutive zeros of $E(t)-f(t)$, where $f(t)$ ($\ll t^{1/4-\eta}$ for any fixed $0<\eta<\frac{1}{4}$) is continuous. In §3 we turn to zeros of $E(t)-\pi$ and show, by using (5), that $E(t)-\pi$ always has a zero in $[T, T+c\sqrt{T}]$ ($c>0$, $T\ge T_0$). Some other results involving the zeros of $E(t)-\pi$, $E'(t)$ and related topics, are discussed in §4. In §5 we describe the algorithms we used for the computation of $E(t)-\pi$ and its zeros, including an estimate of the errors involved. For $t\le 500000$, 42010 zeros of $E(t)-\pi$ were found. In §6 we present tables and figures with a selection of the results of the computations. These results elucidate the behavior of $E(t)-\pi$, but clearly much more extensive computations will be needed to examine the most important conjectures concerning the order of $E(t)-\pi$ and the distribution of its zeros.

2. Gaps between general zeros

In this section we consider the zeros of the general function

$$E_f(t) := E(t) - f(t),$$

where we shall assume that $f(t)$ is continuous for $t\ge t_0(f)$ and satisfies

$$f(t) = \mathcal{O}(t^{1/4-\eta})$$

for any fixed η such that $0<\eta<1/4$ (note that the sign of $f(t)$ is unimportant). From (2)–(3) one has trivially $E(T)=\Omega_\pm(T^{1/4})$, hence also $E_f(T) = \Omega_\pm(T^{1/4})$, so that by continuity each $E_f(t)$ has infinitely many distinct zeros in $(t_0(f),\infty)$, which we shall denote by $t_1(f) < t_2(f) < \cdots$. Our aim is to estimate the quantity

$$(6)\quad \kappa(f) = \inf\{c\ge 0 : t_{n+1}(f) - t_n(f) \ll t_n^c(f)\},$$

or, in other words, to estimate the gaps between consecutive zeros of $E_f(t)$, since (6) implies

$$t_{n+1}(f)-t_n(f) \ll (t_n(f))^{\kappa(f)+\varepsilon} \qquad (n\ge n_0(\varepsilon, f))$$

for any $\varepsilon>0$. Here and subsequently, $a(n)\ll b(n)$ means that $a(n) = \mathcal{O}(b(n))$, $n\to\infty$. Determining the exact value of $\kappa(f)$ for any f seems a difficult problem. Our main result on $\kappa(f)$ is contained in

Theorem 1. *Let*

$$\alpha = \inf\{c > 0 : E(t) \ll t^c\}.$$

Then

$$\alpha \le \kappa(f) \le \frac{1}{2}. \tag{7}$$

Proof. Note first that from known results on α (see [11, Chapter 15] and [7]) one has

$$\frac{1}{4} \le \alpha \le \frac{139}{429} = 0.324009324\ldots,$$

so that unconditionally $\kappa(f) \ge \frac{1}{4}$. Since, in analogy with the classical conjecture $\Delta(x) \ll x^{1/4+\varepsilon}$ for the divisor problem, one conjectures that $\alpha = \frac{1}{4}$, perhaps even $\kappa(f) = \frac{1}{4}$ for all f. If true, the last conjecture is very strong, since it implies [11, Chapter 15] that $\zeta(\frac{1}{2} + it) \ll t^{1/8+\varepsilon}$, which is not proved yet.

Now we turn to the proof of the lower bound in (7). Suppose that $\alpha > \kappa(f)$. Then for $\varepsilon > 0$ sufficiently small, $E_f(t)$ must vanish in $[T, T + T^{\alpha-\varepsilon}]$ for $T \ge T_0(\varepsilon)$, which we shall presently show to be impossible. By the definition of α, there exist arbitrarily large T such that for any given $\varepsilon > 0$ we have either $E(T) > T^{\alpha-\varepsilon/2}$ or $E(T) < -T^{\alpha-\varepsilon/2}$. In both cases the analysis is similar, so we shall consider only the former case. From

$$E(T+H) - E(T) = \int_T^{T+H} \left|\zeta\left(\frac{1}{2} + it\right)\right|^2 dt - t\left(\log\frac{t}{2\pi} + 2\gamma - 1\right)\Bigg|_T^{T+H}$$

it follows that for some absolute $C > 0$,

$$E(T+H) - E(T) \ge -CH\log T \quad (T \ge 2,\ 0 \le H \le \tfrac{1}{2}T). \tag{8}$$

Let $0 \le H \le T^{\alpha-\varepsilon}$. Then

$$\begin{aligned} E_f(T+H) - E_f(T) &= E(T+H) - E(T) + \mathcal{O}(T^{1/4-\eta}) \\ &\ge -CT^{\alpha-\varepsilon}\log T - C_1 T^{1/4-\eta}, \end{aligned}$$

which implies

$$E_f(T+H) \ge T^{\alpha-\varepsilon/2} - CT^{\alpha-\varepsilon}\log T - 2C_1 T^{1/4-\eta} > C_2 T^{\alpha-\varepsilon/2} > 0$$

for some $C_1, C_2 > 0$, $0 < \varepsilon \le 2(\alpha - \frac{1}{4} + \eta)$ and $T \ge T_0(\varepsilon)$, since $\alpha \ge \frac{1}{4}$. Therefore, $E_f(t)$ does not vanish in $[T, T + T^{\alpha-\varepsilon}]$, which is a contradiction if $\alpha > \kappa(f)$ and $\varepsilon > 0$ is sufficiently small.

To prove the upper bound in (7), suppose that

$$T^{1/2}\log^6 T \le H \le T^{1/2+\eta}. \tag{9}$$

Assuming that $E_f(t)$ does not change sign in $[T, T+H]$, we shall obtain a contradiction with suitable H, which will yield $\kappa(f) \le \frac{1}{2}$. If $E_f(t)$ does not change sign in $[T, T+H]$, then

$$\int_T^{T+H} |E_f(t)|\, dt = \left|\int_T^{T+H} E_f(t)\, dt\right|. \tag{10}$$

From (1), (5) and (9) we infer that

$$\int_T^{T+H} E_f^2(t)\,dt = \left(\frac{3}{2}C + o(1)\right) T^{1/2}H \qquad (T\to\infty) \tag{11}$$

and

$$\int_T^{T+H} E_f(t)\,dt \ll T^{3/4}, \tag{12}$$

since by hypothesis $f(t) \ll t^{1/4-\eta}$ and, for $T \le t \le T+H$,

$$E_f^2(t) = E^2(t) + \mathcal{O}(T^{1/4-\eta}|E(t)|) + \mathcal{O}(T^{1/2-2\eta}).$$

We shall use Hölder's inequality for integrals, letting $0 < \delta < \frac{1}{2}$ be a fixed number which may be chosen arbitrarily small. We have

$$\begin{aligned} HT^{1/2} &\ll \int_T^{T+H} E_f^2(t)\,dt = \int_T^{T+H} E_f^\delta(t)E_f^{2-\delta}(t)\,dt \\ &\le \left(\int_T^{T+H} |E_f(t)|\,dt\right)^\delta \left(\int_T^{T+H} |E_f(t)|^{(2-\delta)/(1-\delta)}\,dt\right)^{1-\delta} \\ &\ll T^{3\delta/4} I^{1-\delta}, \end{aligned} \tag{13}$$

where we used (10) and (12), and where

$$I = \int_T^{T+H} |E_f(t)|^{(2-\delta)/(1-\delta)}\,dt = \int_T^{T+H} |E_f(t)|^{2+\delta/(1-\delta)}\,dt.$$

We need an upper bound estimate for I, and to this, we shall use the large values technique discussed in Chapters 13 and 15 of [11]. Namely, let $T^{1/4} \ll V \ll T^{1/3}$ and let $R_0 = R_0(V, T, T_0)$ be the number of points t_i in $[T, T+T_0]$ such that $|E_f(t_i)| \ge V$ and $|t_i - t_j| \ge CV$ for $i \ne j$ and any fixed $C > 0$. Then

$$|E(t_i)| \ge |E_f(t_i)| - |f(t_i)| \ge V - C_1T^{1/4-\eta} \ge V/2$$

for $i = 1, \dots, R_0$. Analogously as in (13.66) of [11], we obtain

$$R_0 \ll T^\varepsilon(TV^{-3} + R_0T_0^{2/9}T^{7/18}V^{-2}) \tag{14}$$

for any given $\varepsilon > 0$. This gives

$$R_0 \ll T^{1+\varepsilon}V^{-3} \tag{15}$$

for

$$V \ge C_1T_0^{4/36}T^{7/36+\varepsilon/2} \qquad (C_1 > 0). \tag{16}$$

In our case, $T_0 = H \ge T^{1/2}\log^6 T$, thus $V \gg T^{1/4}$ holds trivially if (16) is satisfied, and $V > T^{1/3}$ is impossible since $\alpha < \frac{1}{3}$. In I we divide the interval of integration $[T, T+H]$ into subintervals of length V (except perhaps the last such subinterval, which may be shorter, but whose contribution to I is clearly negligible), and write $I = I_1 + I_2$. In I_1 the maximum of $|E(t)|$ in each of the subintervals is at most $C_1 H^{4/36} T^{7/36+\varepsilon/2}$, while in I_2 it is larger. We estimate I_1 trivially, using (11), and obtain

$$I_1 \ll HT^{1/2}(H^{4/36}T^{7/36+\varepsilon/2})^{\delta/(1-\delta)}.$$

To bound I_2, we use the large values estimate (15) (since (16) holds), considering separately subintervals with even and odd indices, so that $|t_i - t_j| \ge V$ is fulfilled. Hence, by the definition of α, we obtain (considering $\mathcal{O}(\log T)$ possible values of V of the form $V = 2^m$)

$$\begin{aligned} I_2 &\ll \log T \max_{T^{1/4} \le V \le T^{\alpha+\varepsilon}} R_0 V^{3+\delta/(1-\delta)} \\ &\ll T^{1+\varepsilon} \log T \max_{V \le T^{\alpha+\varepsilon}} V^{\delta/(1-\delta)} \ll T^{1+3\varepsilon+\alpha\delta/(1-\delta)} \end{aligned}$$

if $0 < \delta < \frac{1}{2}$. Therefore, (13) gives, for $0 < \varepsilon < \frac{1}{2}$,

$$HT^{1/2} \ll T^{3\delta/4} T^{1-\delta+\varepsilon+\alpha\delta} + T^{3\delta/4} H^{1-\delta} T^{(1-\delta)/2} H^{4\delta/36} T^{7\delta/36+\delta\varepsilon/2}.$$

Simplifying, it follows that

$$H \ll T^{1/2+\delta(\alpha-1/4)+\varepsilon} + T^{1/2+\varepsilon} \ll T^{1/2+\delta(\alpha-1/4)+\varepsilon}, \tag{17}$$

since $\alpha \ge \frac{1}{4}$. Thus, if we take

$$H = T^{1/2+\delta(\alpha-1/4)+2\varepsilon},$$

then for δ and ε sufficiently small, (9) holds but (17) is impossible. This contradiction shows that $\kappa(f) \le \frac{1}{2}$, as asserted. □

3. Zeros of $E(t) - \pi$

The upper bound in Theorem 1 applies to the case when $f(t) = \pi$, that is, to the zeros of $E(t) - \pi$. Henceforth, we let $t_n = t_n(\pi)$, so that $0 < t_1 < t_2 < \cdots$ denote the distinct zeros of $E(t) - \pi$ (these bear no relation to the points t_i in §2). Theorem 1 gives

$$t_{n+1} - t_n \ll t_n^{1/2+\varepsilon} \qquad (n \ge n_0(\varepsilon)), \tag{18}$$

but we shall use a special method to improve (18) by removing "ε". The result is

Theorem 2. *There exists a constant $c > 0$ (effectively computable) such that $E(t) - \pi$ has a zero of odd order in $[T, T + c\sqrt{T}]$ for $T \ge T_0$.*

Proof. We shall use the fact that, if $h(t) \in \mathbb{C}[2, T]$, and $N_h(T)$ is the number of zeros of $h(t)$ in $[2, T]$, then $N_h(T) \ge N_H(T) - 1$, where $N_H(T)$ is the number of zeros in $[2, T]$ of the function

$$H(t, a) = \int_2^t u^a h(u)\, du$$

for any fixed real a. For, if T' and T'' are two zeros of $H(t, a)$ in $[2, T]$, then

$$0 = H(T'', a) - H(T', a) = \int_{T'}^{T''} u^a h(u)\, du .$$

Hence $u^a h(u)$, and consequently $h(u)$, must change sign in $[T', T'']$ and have a zero of odd order in $[T', T'']$. In proving Theorem 2, we shall make use of the asymptotic formula (5), and we shall consider the function

$$g(t) := \sum_{n=1}^{\infty} d(n) n^{-5/4} \sin\left(\sqrt{8\pi n t} - \frac{\pi}{4}\right) .$$

Clearly, $g(t) \in \mathbb{C}[0, \infty)$, and $g(t)$ has an infinity of zeros, since $g(t) = \Omega_{\pm}(1)$ (follows by the method of Hafner and Ivić [7]). Further, for $0 \le H \le T$, we have

$$\int_T^{T+H} g^2(t)\, dt = \frac{1}{2} \sum_{n=1}^{\infty} d^2(n) n^{-5/2} \int_T^{T+H} \left(1 - \cos\left(2\sqrt{8\pi n t} - \frac{\pi}{2}\right)\right) dt$$
$$+ \mathcal{O}\left(\sum_{m,\, n=1;\, m \ne n}^{\infty} (mn)^{\varepsilon - 5/4} \left| \int_T^{T+H} e^{i\sqrt{8\pi t}(\sqrt{m} \pm \sqrt{n})}\, dt \right| \right) .$$

To estimate the integrals on the right-hand side, we use the simplest result on exponential integrals (see Lemma 2.1 of [11]): Let $F(x)$ be a real differentiable function such that $F'(x)$ is monotonic with either $F'(x) \ge m$, or $F'(x) \le -m < 0$ for $a \le x \le b$. Then

$$\left| \int_a^b e^{iF(x)}\, dx \right| \le 4m^{-1} . \tag{19}$$

Using (19), we obtain

$$\int_T^{T+H} g^2(t)\, dt = \left(\frac{1}{2} \sum_{n=1}^{\infty} d^2(n) n^{-5/2} \right) H + \mathcal{O}(\sqrt{T})$$
$$+ \mathcal{O}\left\{ T^{1/2} \left(\sum_{n=1}^{\infty} n^{2\varepsilon - 2} \sum_{n < m \le 2n} \frac{1}{m - n} \right.\right.$$
$$\left.\left. + \sum_{n=1}^{\infty} n^{\varepsilon - 5/4} \sum_{m > 2n} \frac{m^{\varepsilon - 3/4}}{m - n} \right) \right\}$$
$$= CH + \mathcal{O}(\sqrt{T}),$$

where

$$C = \frac{1}{2} \sum_{n=1}^{\infty} d^2(n) n^{-5/2} = \frac{\zeta^4(5/2)}{2\zeta(5)} \approx 1.561592 .$$

Hence, for $T \ge T_0$, suitable $C_1, C_2, C_3 > 0$ and $C_3\sqrt{T} \le H \le T$, we have

$$C_1 H \le \int_T^{T+H} g^2(t)\, dt \le C_2 H . \tag{20}$$

However,

$$E_1(T) := \int_2^T (E(t) - \pi)\, dt = \frac{1}{2}\left(\frac{2}{\pi}\right)^{3/4} T^{3/4}(g(T) + \gamma(T)),$$

where $\gamma(t)$ is continuous, and by (5),

$$\gamma(t) = \mathscr{O}(t^{-1/12}\log t). \tag{21}$$

Suppose now that $E_1(t)$ does not change sign in $[T, T+H]$, where $H = D\sqrt{T}$ for some sufficiently large $D > 0$. Then $g(t) + \gamma(t)$ also does not change sign in $[T, T+H]$, and we have

$$\begin{aligned}\int_T^{T+H} (g(t) + \gamma(t))^2\, dt &\le \max_{t\in[T, T+H]} |g(t) + \gamma(t)| \cdot \left|\int_T^{T+H} (g(t) + \gamma(t))\, dt\right| \\ &\le C_4 \left|\int_T^{T+H} (g(t) + \gamma(t))\, dt\right|.\end{aligned} \tag{22}$$

Estimating the last integral by (19) and using (21), we obtain

$$\int_T^{T+H} (g(t) + \gamma(t))^2\, dt \le C_5\sqrt{T}, \tag{23}$$

where $C_5 > 0$ is an absolute constant. On the other hand, using (20) and (21), we obtain

$$\begin{aligned}\int_T^{T+H} (g(t) + \gamma(t))^2\, dt &= \int_T^{T+H} (g^2(t) + 2g(t)\gamma(t) + \gamma^2(t))\, dt \\ &= \int_T^{T+H} g^2(t)\, dt \\ &\quad + \mathscr{O}\left\{T^{-1/12}H^{1/2}\log T\left(\int_T^{T+H} g^2(t)\, dt\right)^{1/2}\right\} + o(H) \\ &\ge C_1 H + o(H) \qquad (T \to \infty).\end{aligned} \tag{24}$$

Comparing (23) and (24), it follows that

$$C_1 H + o(H) \le C_5\sqrt{T},$$

which is impossible for $D > C_5/C_1$. Hence $g(t) + \gamma(t)$, and consequently $E_1(t)$, must change sign in $[T, T + D\sqrt{T}]$. By the discussion at the beginning of the proof it follows that $E(t) - \pi$ must change sign in $[T, T + 2D\sqrt{T}]$, and Theorem 2 follows with $c = 2D$. A more careful estimation of the preceding integrals (working out explicitly all the $\mathscr{O}$-constants) would yield an explicit value for c. □

The method of proof of Theorem 2 is fairly general and can be used to yield results on sign changes in short intervals for certain types of arithmetic error terms. The key ingredient is the existence of a sharp formula for the integral of the error term in question (the analogue of (5)). In particular, it follows by our method that $\Delta(x)$ changes sign in $[x, x+C_1\sqrt{x}]$ for $x \ge x_0$. This also follows from general results of J. Steinig [22], whose method is different from ours and cannot be used to yield Theorem 2.

Another important problem is the estimation of t_n, the nth zero of $E(t)-\pi$, as a function of n. Alternatively, one may consider the estimation of the counting function

$$K(T) := \sum_{t_n \le T} 1.$$

Since $[T, T+c\sqrt{T}]$ contains a t_n for $T \ge T_0$, it follows that $K(T) \gg \sqrt{T}$. Setting $T = t_n$, we have $n = K(t_n) \gg \sqrt{t_n}$, or

$$t_n \ll n^2. \tag{25}$$

This upper bound seems very crude to us, and we proceed to deduce a lower bound for t_n, which appears to be somewhat closer to the truth. Note that $K(T) \ll M(T)$, where $M(T)$ denotes the number of zeros in $[0, T]$ of the function

$$E'(t) = \left|\zeta\left(\frac{1}{2}+it\right)\right|^2 - \log\left(\frac{t}{2\pi}\right) - 2\gamma = Z^2(t) - \log\left(\frac{t}{2\pi}\right) - 2\gamma.$$

Here, as usual, we denote by $Z(t)$ the real-valued function

$$Z(t) = \chi^{-1/2}\left(\frac{1}{2}+it\right)\zeta\left(\frac{1}{2}+it\right)$$
$$\left(\chi(s) = \frac{\zeta(s)}{\zeta(1-s)} - 2^s\pi^{s-1}\sin\left(\frac{\pi s}{2}\right)\Gamma(1-s)\right),$$

so that $|Z(t)| = |\zeta(\frac{1}{2}+it)|$ and the real zeros of $Z(t)$ are precisely the ordinates of the zeros of $\zeta(s)$ on the critical line $\operatorname{Re} s = \frac{1}{2}$. But $M(T) = M_1(T)+M_2(T)$, where $M_1(T)$ and $M_2(T)$ denote the number of zeros of

$$Z(t) - \left(\log\frac{t}{2\pi}+2\gamma\right)^{1/2}, \qquad Z(T) + \left(\log\frac{t}{2\pi}+2\gamma\right)^{1/2}$$

in $[0, T]$, respectively. Note that $M_j(T) \ll L_j(T)$, where $L_j(T)$ is the number of zeros of

$$Z'(t) + \frac{(-1)^j}{2t\sqrt{\log(t/2\pi)}}$$

in $[0, T]$. It was shown by R. J. Anderson [1] that the number of zeros of $Z'(t)$ in $[0, T]$ is asymptotic to $\frac{T}{2\pi}\log T$, and by the same method it follows that $L_j(T) = \mathscr{O}(T\log T)$. Hence $K(T) \ll T\log T$, and taking $T = t_n$, we obtain

$$t_n \gg n/\log n. \tag{26}$$

Our numerical results (cf. Table 2 in §6) indicate that both (25) and (26) are far from the truth. In the range we have investigated numerically, t_n behaves roughly like $n\log n$, but we have no idea how to prove this in general.

4. Some further results

As before, let t_n be the nth distinct zero of $E(t)-\pi$. In this section we present some further results on the t_n's and related subjects.

First observe that

$$\begin{aligned}(t_{n+1}-t_n)\max_{t_n\le t\le t_{n+1}}\left|\zeta\left(\frac{1}{2}+it\right)\right|^2 &\ge \int_{t_n}^{t_{n+1}}\left|\zeta\left(\frac{1}{2}+it\right)\right|^2 dt\\ &= t\left(\log\frac{t}{2\pi}+2\gamma-1\right)\Bigg|_{t_n}^{t_{n+1}} + E(t_{n+1})-E(t_n)\\ &= t\left(\log\frac{t}{2\pi}+2\gamma-1\right)\Bigg|_{t_n}^{t_{n+1}} \ge (t_{n+1}-t_n)\left(\log\frac{t_n}{2\pi}+2\gamma\right).\end{aligned}$$

Therefore, it follows that

$$\max_{t_n\le t\le t_{n+1}}\left|\zeta\left(\frac{1}{2}+it\right)\right| \ge \left(\log\frac{t_n}{2\pi}+2\gamma\right)^{1/2}. \tag{27}$$

This inequality shows that the maximum of $|\zeta(\frac{1}{2}+it)|$ between consecutive zeros of $E(T)-\pi$ cannot be too small, even if the gap between such zeros is small. On the other hand, the maxima of $|\zeta(\frac{1}{2}+it)|$ can be larger[1] over long intervals. Namely, Balasubramanian [3] proved that

$$\max_{t\in[T,T+H]}\left|\zeta\left(\frac{1}{2}+it\right)\right| \ge \exp\left(\frac{3}{4}\sqrt{\frac{\log H}{\log\log H}}\right) \tag{28}$$

for $(\log T)^\varepsilon \le H \le T$. We recall that the best upper bound for $\zeta(\frac{1}{2}+it)$ is, under the truth of the Riemann hypothesis (see E. C. Titchmarsh [24, p. 354, Theorem 14.14(A)]),

$$\zeta\left(\frac{1}{2}+it\right) \ll \exp\left(\frac{C\log t}{\log\log t}\right), \tag{29}$$

so that the gap between (28) (for $H=T$) and (29) is not so large.

Another useful inequality is

$$\max_{t\in[t_n,t_{n+1}]}|E(t)-\pi| \ll (t_{n+1}-t_n)\log t_n, \tag{30}$$

the proof of which is analogous to the proof of $\kappa(f)\ge\alpha$ in Theorem 1, and which is actually more precise than the inequality $\kappa(\pi)\ge\alpha$ (for $f(t)=\pi$ in Theorem 1). Namely, let

$$|E(\bar t)-\pi| = \max_{t\in[t_n,t_{n+1}]}|E(t)-\pi|.$$

[1] Take, for example, $H=T=t_n=10^{20}$; then (27) yields $\max\ge 6.73$ and (28) yields $\max\ge 13.47$. Practically speaking, the bounds in (27) and (28) are weak: in [14] we have $\max Z(t) = 116.88$; the corresponding right-hand sides (27) and (28) yield 4.39 and 6.77, respectively! For recent results concerning large values of $Z(t)$, cf. [18] and [19].

Suppose $E(\bar{t})-\pi>0$ (if $E(\bar{t})-\pi<0$, then we use

(31) $$E(T-H)-E(T)\le CH\log T \qquad (C>0,\ 0\le H\le \tfrac{1}{2}T),$$

which is proved analogously as (8)), and use (8). Then

$$E(\bar{t}+H)-\pi\ge E(\bar{t})-\pi-CH\log t_n\ge 0$$

for $0\le H\le (E(\bar{t})-\pi)/(2C\log t_n)$. Thus $E(t)-\pi$ has no zeros in $[\bar{t},\bar{t}+H]$ with $H=(E(\bar{t})-\pi)/(2C\log t_n)$. Consequently,

$$(E(\bar{t})-\pi)/(2C\log t_n)=H\le t_{n+1}-t_n,$$

and (30) follows.

Using (30), we may investigate sums of powers of consecutive gaps $t_{n+1}-t_n$. Namely from (1) we have, as $T\to\infty$,

(32) $$C_1T^{3/2}\sim\int_T^{2T}E^2(t)\,dt\sim\sum_{T<t_n\le 2T,\ T^{1/4}\log^{-2}T\le t_{n+1}-t_n}\int_{t_n}^{t_{n+1}}E^2(t)\,dt.$$

The contribution of gaps less than $T^{1/4}\log^{-2}T$ is negligible by (30) and trivial estimation. From (32) we infer by (30) that

(33) $$\begin{aligned}T^{3/2}&\ll\sum_{T<t_n\le 2T,\ t_{n+1}-t_n\ge T^{1/4}\log^{-2}T}(t_{n+1}-t_n)\left(\max_{t\in[t_n,t_{n+1}]}|E(t)-\pi|^2+1\right)\\&\ll\log^2 T\sum_{T<t_n\le 2T,\ t_{n+1}-t_n\ge T^{1/4}\log^{-2}T}(t_{n+1}-t_n)^3+T,\end{aligned}$$

which gives (replacing T by $T2^{-j}$ and summing over $j\ge 1$)

(34) $$T^{3/2}\log^{-2}T\ll\sum_{t_n\le T}(t_{n+1}-t_n)^3.$$

In general, for any fixed $\alpha\ge 1$ and any given $\varepsilon>0$,

(35) $$T^{(3+\alpha-\varepsilon)/4}\ll_{\varepsilon,\alpha}\sum_{t_n\le T}(t_{n+1}-t_n)^{\alpha}.$$

(Here, $a(n)\ll_{\varepsilon,\alpha}b(n)$ means that the constant implied by the relation $a(n)=O(b(n))$ depends on ε and α.) This follows along the same lines as (34), using

$$T^{1+a/4-\varepsilon}\ll_{a,\varepsilon}\int_2^T|E(t)|^a\,dt \qquad (a\ge 0,\ \varepsilon>0)$$

with $a=\alpha-1$. The last bound for $a>2$ (without "ε") follows easily from (1) and Hölder's inequality, and for $0<a<2$ it follows from

$$T^{3/2}\ll\int_2^T E^{a/2}(t)E^{2-a/2}(t)\,dt\le\left(\int_2^T|E(t)|^a\,dt\right)^{1/2}\left(\int_2^T|E(t)|^{4-a}\,dt\right)^{1/2}$$

on bounding the last integral by Theorem 15.7 of [11]. It may be conjectured that the lower bound in (35) is close to the truth, that is, we expect that

$$\sum_{t_n \le T} (t_{n+1} - t_n)^{\alpha} = T^{(3+\alpha+o(1))/4} \qquad (\alpha \ge 1,\, T \to \infty), \tag{36}$$

but unfortunately we cannot at present prove this for any specific $\alpha > 1$ (for $\alpha = 1$ it is trivial).

The lower bound estimate (34) may be compared to corresponding results for $\gamma_{n+1} - \gamma_n$, where $0 < \gamma_1 \le \gamma_2 \le \cdots$ are the positive zeros of $\zeta(\frac{1}{2} + it)$ (or $Z(t)$). Large values of $\gamma_{n+1} - \gamma_n$ were (unconditionally) investigated by Ivić and Jutila [13] and Ivić [12], where it was shown that

$$\sum_{\gamma_n \le T} (\gamma_{n+1} - \gamma_n)^3 \ll T \log^6 T, \tag{37}$$

while from

$$T \ll \sum_{\gamma_n \le T} (\gamma_{n+1} - \gamma_n) \le \left(\sum_{\gamma_n \le T} (\gamma_{n+1} - \gamma_n)^3 \right)^{1/3} \left(\sum_{\gamma_n \le T} 1 \right)^{2/3}$$
$$\ll \left(\sum_{\gamma_n \le T} (\gamma_{n+1} - \gamma_n)^3 \right)^{1/3} (T \log T)^{2/3}$$

it follows that

$$\sum_{\gamma_n \le T} (\gamma_{n+1} - \gamma_n)^3 \gg T \log^{-2} T.$$

Thus, it follows that the gaps between the t_n's are, on the average, much larger than the gaps between the γ_n's.

If one assumes RH, then (37) may be sharpened. From the work of A. Fujii (see p. 246 of [24]) it follows that, for any fixed integer $k \ge 1$,

$$T \log^{1-k} T \ll \sum_{\gamma_n \le T} (\gamma_{n+1} - \gamma_n)^k \ll T \log^{1-k} T, \tag{38}$$

and presumably the bounds in (38) may be replaced by an asymptotic equality. Of course, the lower bound in (38) is unconditional and follows as in the case $k = 3$ which we already discussed.

It is known (see H. M. Edwards [4]) that $Z(t)$ cannot attain positive local minima or negative local maxima if RH is true. In other words, the zeros of $Z(t)$ and $Z'(t)$ are interlacing, and on RH, R. J. Anderson [1] proved that the zeros of $Z'(t)$ and $Z''(t)$ are also interlacing. However, the situation with the zeros of $E(t)$ (or $E(t) - \pi$) in this respect is (unconditionally) quite different. We have

$$E'(t) = Z^2(t) - \left(\log \frac{t}{2\pi} + 2\gamma \right), \qquad E''(t) = 2Z(t)Z'(t) - \frac{1}{t},$$
$$E^{(r)}(t) = 2 \sum_{j=0}^{r-2} \binom{r-2}{j} Z^{(j)}(t) Z^{(r-1-j)}(t) + (-1)^{r-1} (r-2)! t^{1-r}$$

for $r \geq 3$. If $E'(t) = 0$, then

$$\left|\zeta\left(\frac{1}{2}+it\right)\right| = \sqrt{\log\frac{t}{2\pi} + 2\gamma}, \tag{39}$$

and heuristically, (39) should hold quite often. Since we have by the remark following Theorem 1 that $\kappa(f) \geq \frac{1}{4}$, this would mean that $E(t) - \pi$ must have many positive local minima and negative local maxima between large gaps between its zeros, regardless of the truth of RH. This will be confirmed by the numerical experiments described in §6. If $0 < \tau_1 \leq \tau_2 \leq \cdots$ are the zeros of $E'(t)$, let

$$\xi = \inf\{c \geq 0 \colon \tau_{n+1} - \tau_n \ll \tau_n^c\}$$

and

$$\theta = \inf\{c \geq 0 \colon \gamma_{n+1} - \gamma_n \ll \gamma_n^c\}$$

denote the exponents for the gaps between the corresponding consecutive zeros of $E'(t)$ and $Z(t)$. Determining the true values of θ and ξ seems almost hopeless. Under RH, $\theta = \xi = 0$, and unconditionally, A. Ivić in [11] proved that $\theta \leq 0.15594583\ldots$ and in [12] indicated how $\theta \leq 0.15594578\ldots$ may be attained. These are the sharpest hitherto published results.

5. Computation of the zeros of $E(T) - \pi$

In this section we shall describe how we have computed the zeros of $E(T) - \pi$. We write

$$E(T) - \pi = E_\pi(T) = I(T) - T\left(\log\frac{T}{2\pi} + 2\gamma - 1\right) - \pi, \tag{40}$$

where

$$I(T) = \int_0^T \left|\zeta\left(\frac{1}{2}+it\right)\right|^2 dt. \tag{41}$$

Each time a value of $E_\pi(T)$ is computed, the corresponding $I(T)$-value is saved, since this can be used in the computation of neighboring $E_\pi(T)$-values, in view of the relation

$$I(T+h) = I(T) + \int_T^{T+h} \left|\zeta\left(\frac{1}{2}+it\right)\right|^2 dt. \tag{42}$$

In §5.1 the formulas used to compute the values of $|\zeta(\frac{1}{2}+it)|$ are given. The integral in (42) is computed by means of the Simpson quadrature formula with extrapolation; this is described in §5.2.

We have developed a numerical algorithm to find as many zeros of $E_\pi(T)$ as possible, starting at $T = 0$, and proceeding with small steps on T. We cannot be *absolutely* sure that this algorithm does find *all* the zeros, but special provisions have been made in case of doubt. The algorithm is described in §5.3. The error control of the computations is explained in §5.4.

5.1. Computation of $|\zeta(\frac{1}{2}+it)|$. There are two formulas suitable for the computation of $|\zeta(\frac{1}{2}+it)|$: the *Euler-Maclaurin* and the *Riemann-Siegel* formula.

The Euler-Maclaurin formula enables us to compute $\zeta(s)$ to any desired accuracy, by taking m and n large enough in

$$\zeta(s) = \sum_{j=1}^{n-1} j^{-s} + \frac{1}{2}n^{-s} + \frac{n^{1-s}}{s-1} + \sum_{k=1}^{m} T_{k,n}(s) + U_{m,n}(s), \tag{43}$$

where

$$T_{k,n}(s) = \frac{B_{2k}}{(2k)!} n^{1-s-2k} \prod_{j=0}^{2k-2} (s+j)$$

and

$$|U_{m,n}(s)| < \left| T_{m+1,n}(s) \frac{s+2m+1}{\mathrm{Re}(s)+2m+1} \right|$$

for all $m \geq 0$, $n \geq 1$, and $\mathrm{Re}(s) > -(2m+1)$. Here, $B_2 = 1/6$, $B_4 = -1/30, \ldots$ are the Bernoulli numbers. In order to obtain $\zeta(\frac{1}{2}+it)$ to within a specified absolute tolerance, we may take $n \approx t/(2\pi)$. Thus, the computational work required is roughly proportional to t. The precise choice we made for m and n is as described in [20] and [21] (see also §5.4). As an example of the use of (43), take $s = \frac{3}{2}$ (cf. (1)); for $m = 2$, $n = 5$ we find $\zeta(\frac{3}{2}) = 2.612375056$ with an error which is less than 3.1×10^{-7}, and for $m = 2$, $n = 6$ we find the value 2.612375259 with an error less than 9.4×10^{-8}. For the computation of C in (1) we took $\zeta(\frac{3}{2}) = 2.612375$ (and $\zeta(3) = 1.202057$).

The Riemann-Siegel formula is a substantial improvement over the Euler-Maclaurin formula for not too small t, since its computing time is proportional to $t^{1/2}$ rather than t. Write the function $Z(t) = \chi^{-1/2}(\frac{1}{2}+it)\zeta(\frac{1}{2}+it)$ as

$$Z(t) = \exp(i\theta(t))\zeta\left(\frac{1}{2}+it\right),$$

where

$$\theta(t) = \mathrm{Im}\left(\log\Gamma\left(\frac{1}{4}+\frac{1}{2}it\right)\right) - \frac{1}{2}t\log\pi,$$

and let $\tau = t/(2\pi)$, $m = \lfloor \tau^{1/2} \rfloor$ and $z = 2(\tau^{1/2} - m) - 1$. Then the Riemann-Siegel formula for $Z(t)$ with $n+1$ terms in its asymptotic expansion is given by

$$\begin{aligned} Z(t) &= 2\sum_{k=1}^{m} k^{-1/2}\cos(\theta(t) - t\log k) \\ &\quad + (-1)^{m-1}\tau^{-1/4}\sum_{j=0}^{n} \Phi_j(z)(-1)^j\tau^{-j/2} + R_n(\tau), \end{aligned} \tag{44}$$

when $R_n(\tau) = \mathcal{O}(\tau^{-(2n+3)/4})$ for $n \geq 0$ and $\tau > 0$. Here, the $\Phi_j(z)$ are certain entire functions which can be expressed in terms of the derivatives of

$$\Phi_0(z) \equiv \Phi(z) = \frac{\cos(\pi(4z^2+3)/8)}{\cos(\pi z)}.$$

Φ_1 and Φ_2 are given by

$$\Phi_1(z) = \frac{\Phi^{(3)}(z)}{12\pi^2}$$

and

$$\Phi_2(z) = \frac{\Phi^{(2)}(z)}{16\pi^2} + \frac{\Phi^{(6)}(z)}{288\pi^4}.$$

The coefficients of rapidly converging power series expansions of $\Phi_j(z)$ are given in [9]. Gabcke [5] has obtained error bounds for $R_n(\tau)$, for $t \geq 200$ and $0 \leq n \leq 10$. For $n \leq 4$ these bounds are optimal, and they are given by $|R_n(\tau)| < c_n t^{-(2n+3)/4}$, where $c_0 = 0.127$, $c_1 = 0.053$, $c_2 = 0.011$, $c_3 = 0.031$, $c_4 = 0.017$.

5.2. Computation of $I(T+h)$ from $I(T)$. In order to compute $I(T+h)$ from $I(T)$ for some step h, we use Simpson's rule with extrapolation as follows. Let

$$I(T, h) := \int_T^{T+h} f(t)\,dt, \tag{45}$$

where $f(t) = |\zeta(\frac{1}{2}+it)|^2$. We first compute two approximations I_1 and I_2 to $I(T, h)$ based on applying Simpson's quadrature rule to the interval $[T, T+h]$, and to the two intervals $[T, T+h/2]$ and $[T+h/2, T+h]$, respectively:

$$I_1 = \frac{h}{6}\{f(T) + 4f(T+h/2) + f(T+h)\}$$

and

$$I_2 = \frac{h}{12}\{f(T) + 4f(T+h/4) + 2f(T+h/2) + 4f(T+3h/4) + f(T+h)\}.$$

Using the technique of extrapolation (cf., e.g., [23, §3.3]), these two values can be combined to yield the better approximation (provided that h is sufficiently small):

$$I_{\text{extr}} = I_2 + (I_2 - I_1)/15, \tag{46}$$

where $(I_2 - I_1)/15$ is a good approximation of the error in I_2. This error is used in our computations as a (very pessimistic) estimate of the error in I_{extr}.

A possible alternative to (46) might be a Gauss-Legendre quadrature rule. For example, some experiments revealed that a 3-point Gauss-Legendre rule would yield roughly the same accuracy as the above 5-point Simpson rule (which, effectively, is a 4-point rule since the end point value $f(T+h)$ on $[T, T+h]$ is used as starting point value on $[T+h, T+2h]$). However, in order to get an *estimate* of the error in the 3-point Gauss-Legendre rule, we know of no better way (cf. [23, p. 127]) than to apply a 4-point Gauss-Legendre rule, and compare the results; this would require *four* extra function evaluations, since the f-values needed in the 3-point rule cannot be used in the 4-point rule. This is our motivation for choosing (46). Professor W. Gautschi has kindly pointed out the alternative of using the 7-point Gauss-Kronrod formula for estimating the error in the 3-point Gauss-Legendre formula. This also requires four additional points, but is more accurate than the 4-point Gauss-Legendre formula, since it has maximum degree of exactness. See, e.g., W. Gautschi, *Gauss-Kronrod quadrature—A survey*, in "Numerical Methods and Approximation Theory III" (G. V. Milovanović, ed.), Faculty of Electronic Engineering, Univ. of Niš, Niš, 1988, pp. 39–66.

5.3. The zero-searching algorithm. Our algorithm proceeds with a step h to find zeros of the function $E_\pi(t)$, i.e., after the search has been completed for $t \le T$, the interval $[T, T+h]$ is searched (in certain cases combined with a second search on $[T-h, T]$). Now and then, small parts of the computations are repeated with a smaller, and also with a larger step. This is in order to check whether the step has to be decreased or may be increased, respectively, in view of the required accuracy.

Let $T_j := jh$, $I_j := I(T_j)$ and $E_j := E_\pi(T_j)$, $j = 0, 1, \dots$. Suppose that the interval $[0, T_i]$ has already been treated. This implies that I_j and E_j are known, for $j = 0, 1, \dots, i$. We now compute $I(T_{i+1})$ from $I(T_i)$ (by means of (42) and Simpson's rule as described in §5.2) and then E_{i+1} (with the help of (40)).

If $E_iE_{i+1} < 0$, then by continuity there is at least one zero between T_i and T_{i+1}. This zero is found by a rootfinder described at the end of this section.

If $E_iE_{i+1} \ge 0$ and $E_{i-1}E_i < 0$, then we are finished with the interval $[T_i, T_{i+1}]$.

If $E_iE_{i+1} \ge 0$ and $E_{i-1}E_i \ge 0$, then E_{i-1}, E_i and E_{i+1} have the same sign. We check whether $|E_i| \le |E_{i-1}|$ and $|E_i| \le |E_{i+1}|$. If so, this means we have a local extremum; if not, we are finished on $[T_i, T_{i+1}]$.

In case of a local extremum, we check whether

$$|E_i| \le h\left(\log\frac{T_{i+1}}{2\pi} + 2\gamma\right). \tag{47}$$

If not, we know that there can be no zero on $[T_{i-1}, T_{i+1}]$ because of

$$\frac{d}{dt}E_\pi(t) = \left|\zeta\left(\frac{1}{2}+it\right)\right|^2 - \left(\log\frac{t}{2\pi} + 2\gamma\right) \ge -\left(\log\frac{T_{i+1}}{2\pi} + 2\gamma\right) \tag{48}$$

and the mean value theorem, and we are finished on $[T_i, T_{i+1}]$.

If we do have a local extremum such that (47) holds, we fit a quadratic polynomial through the three points (T_{i-1}, E_{i-1}), (T_i, E_i), (T_{i+1}, E_{i+1}) and compute the point $\tilde{T}_e$ where this polynomial has its extremum. If h is small enough, there should be a zero, $T = T_e$, of $E'_\pi(t)$ very close to $T = \tilde{T}_e$. This zero is found with the Newton process. Next, $I(T_e)$ and $E_e := E_\pi(T_e)$ are computed, and if $E_eE_i < 0$, then there are zeros on $[T_{i-1}, T_e]$ and $[T_e, T_{i+1}]$, which are found by the rootfinder described below. This completes the description of our algorithm, apart from the rootfinder.

The *rootfinder* is designed to find a zero of $E_\pi(t)$ on the interval $[a, b]$, where $E_\pi(a)$ and $E_\pi(b)$ have different sign. First, the intersection point

$(c, E_\pi(c))$ of the line through $(a, E_\pi(a))$ and $(b, E_\pi(b))$ and the horizontal axis is found. Next, a quadratic polynomial is fitted through the three points $(a, E_\pi(a))$, $(c, E_\pi(c))$, $(b, E_\pi(b))$, and its zero on $[a, b]$ is taken as the starting point to find a zero of $E_\pi(t)$ on $[a, b]$ with the Newton process.

5.4. Error control. The numerical computations were carried out on the CDC Cyber 995 computer of SARA (the Academic Computer Centre Amsterdam), which has a floating-point mantissa of 48 bits, i.e., a machine accuracy of about 14 decimal digits.

Our aim was to compute as many zeros as possible of the function $E_\pi(t)$ on the interval $[0, 5\times 10^5]$, each with an absolute error of about 10^{-4}. This means an accuracy of at least 5 decimal digits for the smallest zero, and 10 decimal digits for the largest zero below 5×10^5.

The error in the computation of $|\zeta(\frac{1}{2}+it)|$ was controlled as follows.

For $t\in[0, 5\times 10^3]$ we applied the *Euler-Maclaurin* formula (43) in single precision. If we assume $|U_{m,n}(\frac{1}{2}+it)| < 10^{-A}$, then it follows (cf. [20, pp. 151–152]) that

$$n \approx (2\pi)^{-1} 10^{A/(2m+2)}(t+m+1).$$

This still leaves freedom to choose one of either n or m, given t and A. We took $A = 15$ and (for most t) $m = 100$, so that $n\approx 0.2244(t+101)$. The actual error is dominated by the machine errors in the computation of the terms $j^{-(1/2+it)}$ in (43). A pessimistic upper bound for this error is $10^{-14}tn$, and for the value of n given above, and for $t\le 5000$, this is less than 5.8×10^{-8}.

For $t\in[5\times 10^3, 5\times 10^5]$ we applied the *Riemann-Siegel* formula (44) with $n = 3$, in double precision (i.e., with an accuracy of about 28 decimal digits), and the result was truncated to single precision. We denote this numerical approximation of $Z(t)$ by $Z_d(t)$. An extensive error analysis for $t\in[3.5\times 10^7, 3.72\times 10^8]$ is given in [15]. A similar analysis shows that for $t\in[5\times 10^3, 5\times 10^5]$ the error is dominated by the inherent error in (44), i.e.,

$$|Z_d(t) - Z(t)| < 0.031t^{-2.25} < 1.5\times 10^{-10} \quad \text{for } t\in[5\times 10^3, 5\times 10^5].$$

In order to get an idea of the *actual* error, we computed $|\zeta(\frac{1}{2}+it)|$ by (43) and compared it with $|Z_d(t)|$, for $t = 4900(0.1)5100$. The maximum difference we found was 3.9×10^{-10} at $t = 5067.2$.

Since the function $E_\pi(t)$ measures how well the integral $I(t)$ is approximated by the function $t(\log\frac{t}{2\pi}+2\gamma-1)+\pi$, we can expect a loss of significant digits when we subtract the two terms for the computation of $E_\pi(t)$. Therefore, we computed the integral $I(t, h)$ in (45) so that its contribution to the total error in $I(t+h)$ $(= I(t) + I(t, h))$ was as small as the machine accuracy allows. Thus, the number h was chosen such that

$$\frac{(I_2 - I_1)/15}{I(t+h)} \approx 10^{-12}. \tag{49}$$

(Recall that $(I_2 - I_1)/15$ is a very pessimistic estimate of the quadrature error in $I(t, h)$.) Actually, we took $h = 0.01$ for $0 \le t \le 500$, $h = 0.02$ for $500 \le t \le 2000$ and $h = 0.05$ for $2000 \le t \le 500000$. Several spot checks were carried out locally for smaller values of h. To summarize, we estimate that the number of correct digits in our computation of $E_\pi(t)$ varies between at least 13 decimal digits near $t = 0$ and about 7 near $t = 5 \times 10^5$. The *absolute* error is about $10^{-12} t(\log \frac{t}{2\pi} + 2\gamma - 1) \approx 5.7 \times 10^{-6}$ for $t = 5 \times 10^5$.

In the rootfinder used in the zero-searching algorithm described in §5.3, the Newton process was iterated to machine precision. Usually, no more than two Newton iterations were needed for this purpose. The influence of the error in $E_\pi(t)$ on its zeros may be quantified as follows. Suppose that in the neighborhood of a zero $t = t_0$ of $E_\pi(t)$ we compute with $\widetilde{E}_\pi(t)$ rather than with $E_\pi(t)$, where $\widetilde{E}_\pi(t) = E_\pi(t) + \varepsilon$, ε being a fixed small number. Then the Newton process for the computation of $t = t_0$ is given by

$$t^{i+1} = t^i - \frac{\widetilde{E}_\pi(t^i)}{E'_\pi(t^i)} = t^i - \frac{E_\pi(t^i)}{E'_\pi(t^i)} - \frac{\varepsilon}{E'_\pi(t^i)},$$

so there is a systematic error $\varepsilon / E'_\pi(t^i)$ in the computation of the zero $t = t_0$. In particular, when $E'_\pi(t)$ is small for t close to t_0, then the error in this zero may be large. We found

$$\max_{t \le 500000,\, E_\pi(t)=0} \frac{1}{|E'_\pi(t)|} \approx 3.015,$$

where the maximum is assumed for $t = 137538.499969$. For $\varepsilon = 10^{-5}$, this means a maximum absolute error in the zeros of $E_\pi(t)$ of about 3×10^{-5}.

6. Results and conjectures

In this section we present a selection of our computational results. We have found 42010 zeros of the function $E_\pi(t)$ on the interval $[0, 500000]$. The first 100 of them are listed in Table 1.

For selected values of n, Table 2 compares $\log t_n$ with $\log n$, and t_n with $n \log n$. The quotient $\log t_n / \log n$ is slowly changing, with a global tendency to decrease. We believe it converges to 1, although very large t_n will certainly have to be computed in order to corroborate this. The quotient $t_n / n \log n$ first decreases to 0.8904, and then increases slowly to 1.1180; no possible conclusion about a limit is apparent from these data. Perhaps $n \log n$ is just a rough approximation to t_n, much as $n \log n$ is a rough approximation to p_n, the nth prime.

Data on gaps between consecutive zeros of $E_\pi(t)$ are shown in Tables 3, 4, 5 and 6. It appears that the gaps $d_n := t_n - t_{n-1}$, $n = 2, 3, \ldots$ behave in a very irregular way. Although we cannot exclude the possibility that $\kappa = \kappa(\pi) = \frac{1}{2}$, this seems unlikely to us. In fact, we believe $\kappa = \frac{1}{4}$ to hold. Maxima and minima of the quotient $d_n / t_{n-1}^{1/4}$ are presented in Tables 4 and 5, respectively.

TABLE 1
The first 100 *zeros of* $E_\pi(t)$

n	t_n	n	t_n	n	t_n	n	t_n
1	1.199593	26	99.048912	51	190.809257	76	318.788055
2	4.757482	27	99.900646	52	192.450016	77	319.913514
3	9.117570	28	101.331134	53	199.646158	78	321.209365
4	13.545429	29	109.007151	54	211.864426	79	326.203904
5	17.685444	30	116.158343	55	217.647450	80	330.978187
6	22.098708	31	117.477368	56	224.290283	81	335.589281
7	27.706900	32	119.182848	57	226.323460	82	339.871410
8	31.884578	33	119.584571	58	229.548079	83	343.370082
9	35.337567	34	121.514013	59	235.172515	84	349.890794
10	40.500321	35	126.086783	60	239.635323	85	354.639224
11	45.610584	36	130.461139	61	245.494672	86	358.371624
12	50.514621	37	136.453527	62	256.571746	87	371.554495
13	51.658642	38	141.371299	63	262.343301	88	384.873869
14	52.295421	39	144.418515	64	267.822499	89	390.001409
15	54.750880	40	149.688528	65	280.805140	90	396.118200
16	56.819660	41	154.448617	66	289.701637	91	399.102390
17	63.010778	42	159.295786	67	290.222188	92	402.212210
18	69.178386	43	160.333263	68	294.912620	93	406.737516
19	73.799939	44	160.636660	69	297.288651	94	408.735190
20	76.909522	45	171.712482	70	297.883251	95	417.047725
21	81.138399	46	179.509721	71	298.880777	96	430.962383
22	85.065503	47	181.205224	72	299.919407	97	434.927645
23	90.665198	48	182.410680	73	308.652004	98	439.425963
24	95.958639	49	182.899197	74	314.683833	99	445.648250
25	97.460878	50	185.733682	75	316.505614	100	448.037348

TABLE 2
Some data concerning the order of t_n

n	t_n	$\log t_n / \log n$	$t_n / n \log n$
2	4.757482	2.2502	3.4318
5	17.685444	1.7849	2.1977
10	40.500321	1.6075	1.7589
20	76.909522	1.4496	1.2837
50	185.733682	1.3355	0.9496
100	448.037348	1.3257	0.9729
200	978.559572	1.2997	0.9235
500	2766.863752	1.2753	0.8904
1000	6174.307534	1.2635	0.8938
2000	13807.257919	1.2542	0.9083
5000	39310.200279	1.2421	0.9231
10000	89563.343441	1.2380	0.9724
20000	204737.805598	1.2349	1.0337
42010	499993.656034	1.2326	1.1180

For $4 \le n \le 42010$ we observed that $d_n < t_{n-1}^{1/4} \log t_{n-1}$, which in general is close to best possible, in view of (2), (3), and (30). Combined with the data in Table 5, this supports the conjecture that $\kappa(\pi) = \frac{1}{4}$ (where κ is defined in (6)). The data on g_n and $g_n / \log t_n$ in Table 3 support (36) for $\alpha = 2$. Table 6 gives a

TABLE 3
Various data related to the gaps between consecutive zeros

n	$d_n := t_n - t_{n-1}$	$d_n/t_{n-1}^{1/2}$	$d_n/t_{n-1}^{1/4}$	$\log d_n/\log t_n$	$\log d_n/\log n$	$g_n := t_n^{-5/4}\sum_{i=2}^n d_i^2$	$g_n/\log t_n$
2	3.557889	3.2484	3.3996	0.8137	1.8310	1.8016	1.1551
5	4.140015	1.1249	2.1580	0.4945	0.8827	1.8864	0.6566
10	5.162754	0.8685	2.1175	0.4435	0.7129	1.7165	0.4638
20	3.109583	0.3620	1.0609	0.2612	0.3787	1.5146	0.3488
50	2.834485	0.2096	0.7708	0.1994	0.2663	1.3773	0.2636
100	2.389098	0.1132	0.5200	0.1427	0.1891	1.4164	0.2320
200	0.075980	0.0024	0.0136	-0.3743	-0.4864	1.3835	0.2009
500	3.624824	0.0690	0.5000	0.1625	0.2072	1.2801	0.1615
1000	0.753268	0.0096	0.0850	-0.0325	-0.0410	1.2895	0.1477
2000	0.596044	0.0051	0.0550	-0.0543	-0.0681	1.3067	0.1371
5000	7.983033	0.0403	0.5670	0.1964	0.2439	1.3173	0.1245
10000	22.172542	0.0741	1.2818	0.2718	0.3365	1.3931	0.1222
20000	1.240345	0.0027	0.0583	0.0176	0.0217	1.4619	0.1195
42010	1.636594	0.0023	0.0615	0.0375	0.0463	1.5505	0.1182

TABLE 4
Maxima of $d_n/t_{n-1}^{1/4}$

n	t_n	d_n	$d_n/t_{n-1}^{1/4}$
2	4.757482	3.557889	3.3996
370	1992.136994	24.861362	3.7330
510	2850.462567	31.291596	4.2943
1176	7420.277407	42.085752	4.5410
1321	8475.806973	43.841653	4.5751
1322	8520.092619	44.285645	4.6155
1472	9708.104280	54.053035	5.4531
2074	14365.716667	61.751030	5.6465
4224	32120.209803	76.460074	5.7148
4692	36685.948268	82.898386	5.9933
4848	38070.374558	88.990702	6.3746
5006	39518.339822	96.093410	6.8196
6058	49552.122137	104.276659	6.9928
8230	71699.441192	123.858798	7.5724
17138	170654.832030	165.382076	8.1389
18198	183304.147130	169.425143	8.1900
21804	227502.378144	186.717169	8.5512
23764	252647.958173	213.951458	9.5451
39084	457431.381229	261.454651	10.0549

frequency distribution of the computed values of $d_n/t_{n-1}^{1/4}$, in classes of length 0.1. For example, we found 10641 values in the interval $[0, 0.1)$, 818 in the interval $[0.9, 1.0)$ and 1 value (the largest) in the interval $[10.0, 10.1)$ (cf. the last entry in Table 4). To summarize: 82% of all values are in $[0.0, 1.0)$, 11% in $[1.0, 2.0)$, 4% in $[2.0, 3.0)$, 2% in $[3.0, 4.0)$ and 1% in $[4.0, 10.1)$.

Table 7 presents maximal values of $|E_\pi(t)|$ in intervals of length 25000, and the location of the adjacent zeros. The computed values of $E_\pi(t)/t^{1/4}$ confirm the order results of $E(t)$ as discussed at the beginning of §1.

TABLE 5
Minima of $d_n/t_{n-1}^{1/4}$

n	t_n	d_n	$d_n/t_{n-1}^{1/4}$
2	4.757482	3.557889	3.3996
3	9.117570	4.360087	2.9522
4	13.545429	4.427859	2.5481
5	17.685444	4.140015	2.1580
6	22.098708	4.413263	2.1521
8	31.884578	4.177677	1.8209
9	35.337567	3.452989	1.4531
13	51.658642	1.144021	0.4291
14	52.295421	0.636779	0.2375
33	119.584571	0.401722	0.1216
44	160.636660	0.303397	0.0853
159	753.427349	0.280739	0.0536
200	978.559572	0.075980	0.0136
301	1604.012827	0.063653	0.0101
628	3569.014754	0.062385	0.0081
1030	6389.011638	0.038008	0.0043
2674	18818.622459	0.037263	0.0032
3616	27076.314671	0.031137	0.0024
6841	57197.581870	0.022931	0.0015
8088	70009.242085	0.021013	0.0013
11857	110163.040870	0.006778	0.0004
11987	111649.073447	0.004789	0.0003
27021	294421.287720	0.005105	0.0002

TABLE 6
Frequency distribution of the $d_n/t_{n-1}^{1/4}$*-values, in classes of length* 0.1

10641	7208	5243	3192	1829	1812	1760	1082	912	818
752	591	561	503	415	389	390	373	320	242
210	201	196	190	195	170	143	143	130	121
114	99	83	81	73	75	57	45	39	54
68	46	30	40	23	30	35	36	19	21
20	15	13	19	13	10	12	13	9	6
2	2	7	4	8	6	5	4	7	5
2	3	2	1	3	5	2	3	1	0
0	2	1	1	0	1	0	0	0	0
0	0	0	0	0	1	0	0	0	0
1									

Graphs of $E_\pi(t)$ and its derivatives are presented in Figures 1–5, which cover the intervals $[0, 50]$, $[123400, 123600]$, $[456999.4, 457431.4]$, $[277514.8, 277661.5]$, and $[495151.35, 495321.95]$, respectively. Figure 2 shows how $E_\pi(t)$ behaves on an arbitrarily chosen interval. Figure 3 shows this function near the largest observed d_n-value (cf. the last entry in Table 4). Figures 4 and 5 show the behavior of $E_\pi(t)$ near its smallest and largest observed values, respectively (cf. Table 7). The function $E_\pi(t)$ may increase sharply, but in view of (8) and (31) we see that it decreases relatively slowly, which is also reflected in the graphs. The function $E'_\pi(t)$ has sharp peaks which roughly correspond to large values of $|\zeta(\frac{1}{2}+it)|$. Note that Figure 5 displays many local extrema of $E_\pi(t)$ in the large intervals between its consecutive zeros.

TABLE 7
Maxima of $|E_\pi(t)|$ *in intervals* $[i \times 25000, (i+1) \times 25000]$, *and adjacent zeros* t_{n-1}, t_n

i	t	$E_\pi(t)$	$E_\pi(t)/t^{1/4}$	n	t_{n-1}	d_n
4	105730.30	-294.972917	-16.36	11511	105652.215612	78.840885
5	130061.30	342.688448	18.05	13728	130060.547162	130.903167
6	152359.00	-364.453147	-18.45	15615	152263.935681	95.831001
7	183134.00	-355.682389	-17.19	18197	182999.306300	135.415687
8	221488.30	367.810105	16.95	21310	221487.533725	104.953578
9	225005.15	367.801167	16.89	21628	225004.360683	104.423051
10	263358.05	404.632562	17.86	24614	263357.300497	135.522875
11	277660.65	-436.894699	-19.03	25701	277514.752120	146.699293
12	304718.75	-379.461854	-16.15	27807	304616.030686	103.513435
13	328768.95	489.881453	20.46	29574	328768.180260	134.489310
14	367120.55	-387.994451	-15.76	32491	366950.894252	170.362297
15	379395.50	394.115535	15.88	33424	379394.773446	134.268815
16	415716.60	-381.854476	-15.04	36021	415610.911107	106.341938
17	428843.30	474.148290	18.53	36986	428842.537715	195.257502
18	457170.60	430.409601	16.55	39084	457169.926578	261.454651
19	495152.05	506.242025	19.08	41688	495151.305121	170.597059

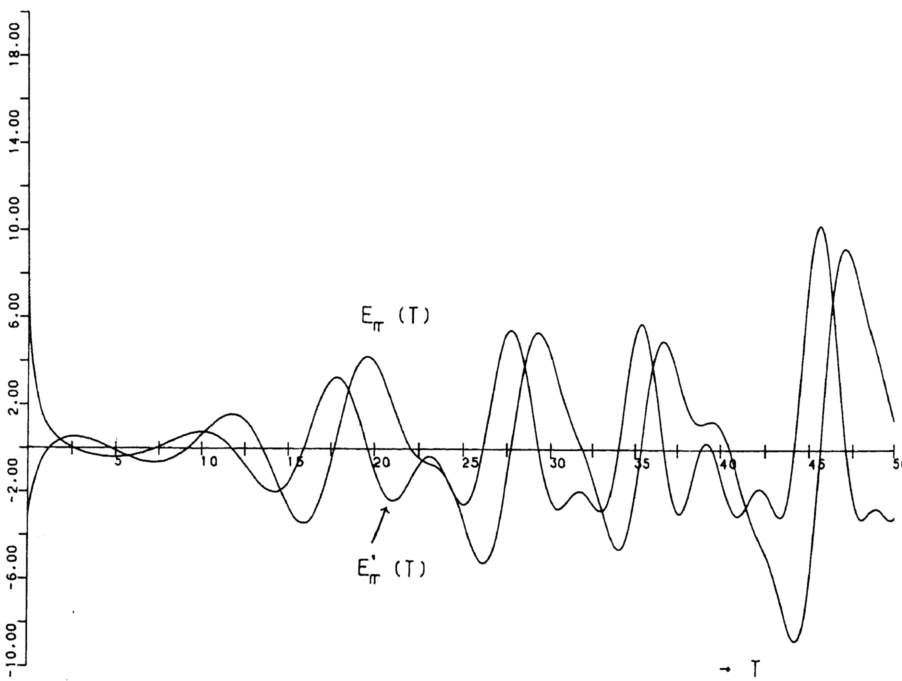

FIGURE 1
$E_\pi(t)$ *and* $E'_\pi(t)$ *on the interval* $[0, 50]$

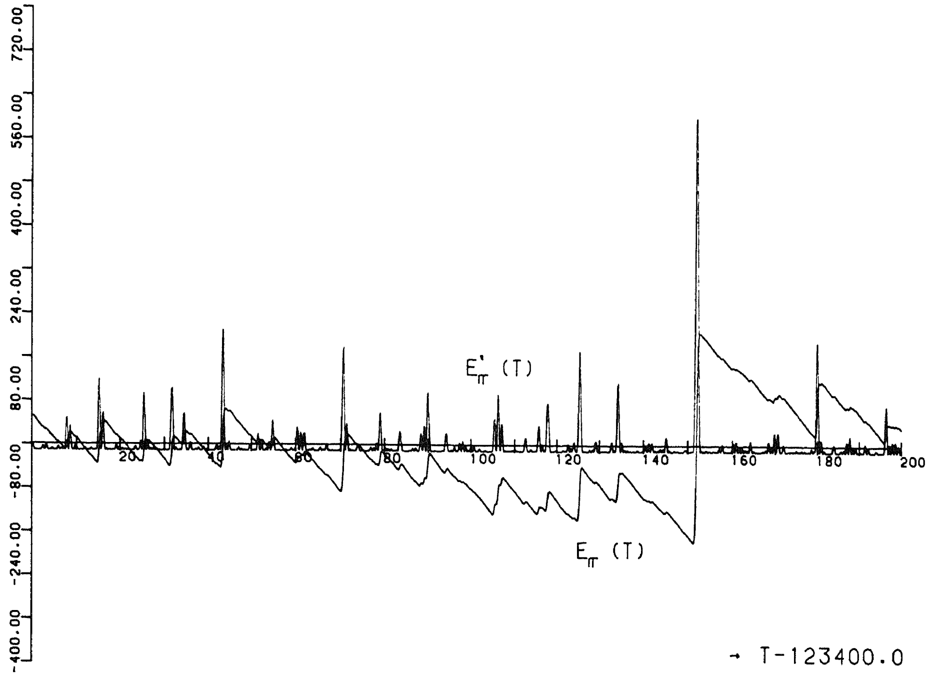

FIGURE 2
$E_\pi(t)$ and $E'_\pi(t)$ on the interval [123400, 123600]

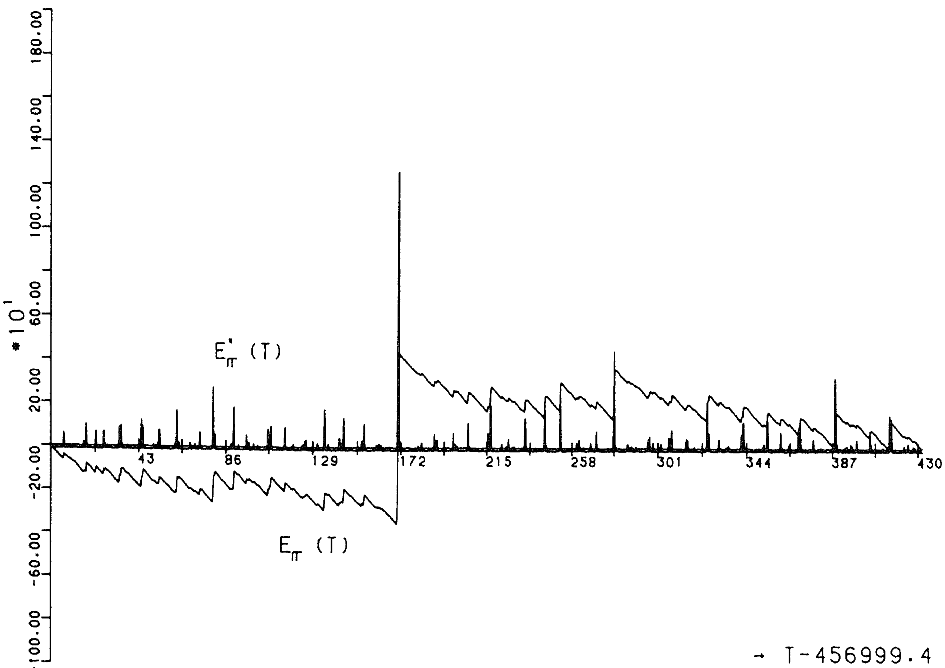

FIGURE 3
$E_\pi(t)$ and $E'_\pi(t)$ on the interval [456999.4, 457431.4]

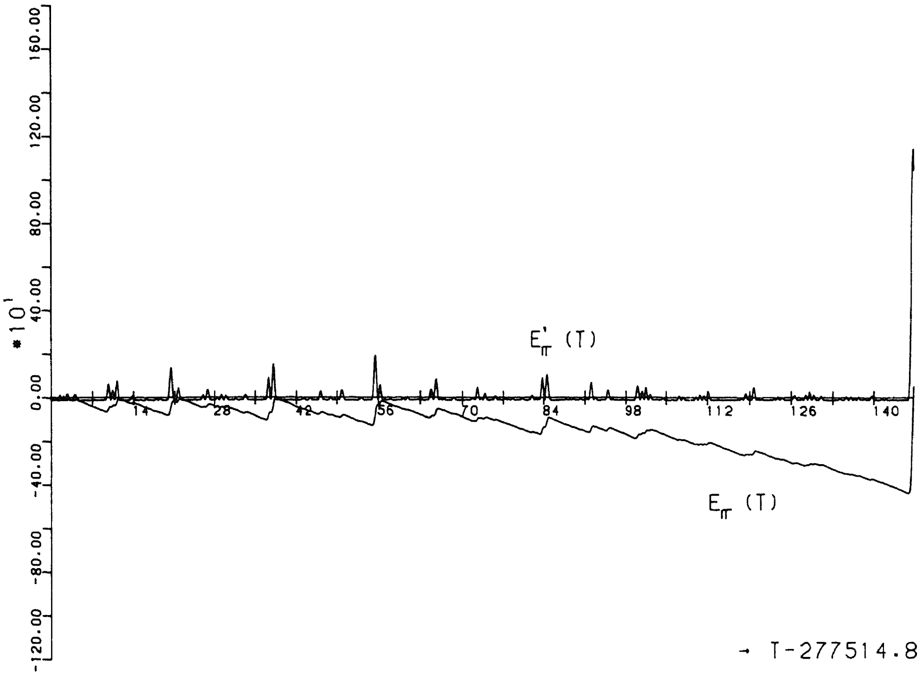

FIGURE 4
$E_\pi(t)$ *and* $E'_\pi(t)$ *on the interval* [277514.8, 277661.5]

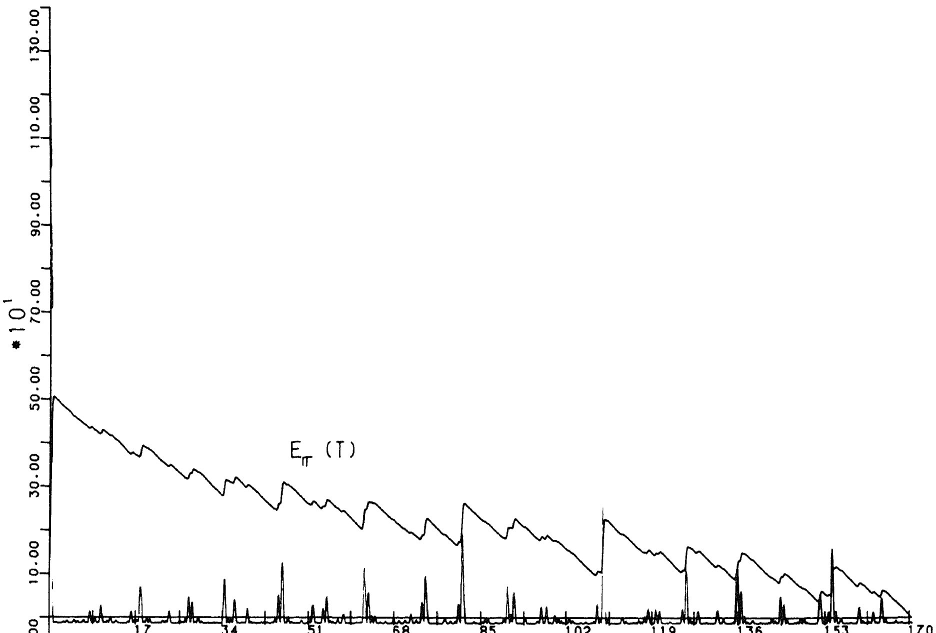

FIGURE 5
$E_\pi(t)$ *and* $E'_\pi(t)$ *on the interval* [495151.35, 495321.95]

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