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The Finite Continuous Jacobi Transform and Its Inverse

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Properties of the finite continuous Jacobi transform are given. Two inverse integral transforms are found whose kernels involve Jacobi functions. The range of the original transform is characterized to some extent and the inverse is shown to be both a left and a right inverse. © 1990 Academic Press, Inc.

1. INTRODUCTION

The Jacobi differential operator

$$(1-x^2)\, d^2/dx^2 + (\beta-\alpha-(\alpha+\beta+2)\,x)\, d/dx \tag{1.1}$$

has regular singularities at -1, 1, and ∞. Its spectral theory on the finite interval $(-1, 1)$ reduces to expanding a function as a (discrete) series of Jacobi polynomials. On the other hand, the spectrum on the infinite interval $(1, \infty)$ is continuous and leads to (continuous) expansions of Jacobi functions; cf. Koornwinder [7]. In the case of the finite interval, the integral defining the Fourier–Jacobi coefficient remains meaningful if one replaces the degree n in this integral by arbitrarily complex numbers. Thus one obtains a transform which maps quite general functions on $(-1, 1)$ into a class of entire functions of exponential type at most π. This transform is called the finite continuous Jacobi transform, which we will invert

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by means of an integral over the real axis instead of a series. We will also, to some extent, characterize the image of the transform.

The methods and results are quite different from the inversion problem for the Jacobi transform for the interval $(1, \infty)$, although in Section 3 some estimating techniques from the infinite case are used. The finite continuous Jacobi transform was already inverted for the special Gegenbauer cases $\alpha=\beta=$ integer $\geqslant 0$ by MacRobert in the 19th century; cf. Robin [8]. The interest in this transform was revived by the work of Butzer, Stens, and Wehrens [2], who dealt with the Legendre case $\alpha=\beta=0$ by methods different from those of MacRobert and who pointed out the relationship with sampling theory. Their results were extended to Jacobi transforms for values of the parameters α and β satisfying $\alpha+\beta=0$ by Deeba and Koh [3]. Subsequently Walter and Zayed [11] found similar results for values such that $\alpha+\beta$ is a non-negative integer.

In this work we remove the restriction on α and β, requiring only that $\alpha>-1$ and $\beta>-1$. We shall not use the standard Jacobi normalization since in that case, the transformed function would not always be entire. Rather we shall use the same normalization as the Legendre functions, viz., that the value of the Jacobi function at $x=1$ be 1. We shall derive a number of properties of the resulting Jacobi transform, and find two different expressions for an inverse transform. We then find sufficient conditions for a function to belong to the range of this transform.

Our principal concern will be to construct the inverse transform. A number of approaches are possible. The simplest perhaps involves contour integrals, in which a Jacobi series is written first as a sum of residues. This leads to an inverse integral transform in which the integration is over a contour in the complex plane. This is similar to the procedure followed in [6] for Laguerre transforms. However, we are interested in obtaining an inverse transform of the same form as in the references involving an integral over the real axis. Another approach used in [11] uses a series to obtain the kernel of the inverse transform. Our approach will differ from both of these; we will use the Poisson summation formula as our principal tool.

In Section 2 we present preliminary formulae and notation which because of our renormalization of the Jacobi function is not always standard. In addition we present a few facts from Fourier analysis which we shall require in the subsequent sections. In Section 3 we collect a number of estimates on the kernel and the transformed function of the Jacobi transform. We also introduce an appropriate space on which the transform operates, and derive some properties of the orthogonal system obtained by transforming the Jacobi polynomials.

In Section 4 we present our main results, namely the inversion formula for the continuous finite Jacobi transform. We obtain two integral transforms which recover the original function when applied to the transformed function. One involves the eigenfunction of (1.1) which is regular at 1 while the other involves the eigenfunction regular at -1 as part of the inverse kernel. The last section gives sufficient conditions for a function to be a Jacobi transform. This enables one to obtain a sampling theorem for such functions.

2. Preliminaries and Notation

For any complex numbers a, b, and c with $c \neq 0, -1, -2, \ldots$ the *hypergeometric function* is given by

$$F(a, b; c; z) := \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k \, k!} z^k, \qquad |z| < 1. \tag{2.1}$$

The *Jacobi function* $R_t^{(\alpha,\beta)}$ for a finite interval is defined by

$$R_t^{(\alpha,\beta)}(\cos\theta) := F(-t, t+\alpha+\beta+1; \alpha+1; \sin^2 \tfrac{1}{2}\theta), \qquad 0 \leqslant \theta < \pi, \tag{2.2}$$

where $\alpha, \beta > -1$ and $t \in \mathbb{C}$. A more usual normalization and notation (cf. [11]) is

$$P_t^{(\alpha,\beta)} := \frac{\Gamma(t+\alpha+1)}{\Gamma(\alpha+1)\,\Gamma(t+1)} R_t^{(\alpha,\beta)}. \tag{2.3}$$

Our choice avoids singularities in t and will yield the lowest possible exponential type (namely π) in t. We denote

$$\lambda := \tfrac{1}{2}(\alpha+\beta+1) \tag{2.4}$$

and observe the symmetry

$$R_{t-\lambda}^{(\alpha,\beta)} = R_{-t-\lambda}^{(\alpha,\beta)}. \tag{2.5}$$

The function $t \mapsto \mathbb{R}_{t-\lambda}^{(\alpha,\beta)}(\cos\theta)$ is an even entire analytic function. The hypergeometric differential equation (cf. [4, 2.1(1)]) translates into

$$D_\theta^{\alpha,\beta} R_{t-\lambda}^{(\alpha,\beta)}(\cos\theta) = (\lambda^2 - t^2)\, R_{t-\lambda}^{(\alpha,\beta)}(\cos\theta), \tag{2.6}$$

where

$$D_\theta^{\alpha,\beta} := (w_{\alpha,\beta}(\theta))^{-1} \frac{d}{d\theta}\left(w_{\alpha,\beta}(\theta) \frac{d}{d\theta} \right) \tag{2.7}$$

and

$$w_{\alpha,\beta}(\theta) := (\sin \tfrac{1}{2}\theta)^{2\alpha+1} (\cos \tfrac{1}{2}\theta)^{2\beta+1}. \tag{2.8}$$

For $n=0, 1, 2, \ldots$ the functions $P_n^{(\alpha,\beta)}$ given by (2.3) become the *Jacobi polynomials*. We will work with the renormalized Jacobi polynomials $R_n^{(\alpha,\beta)}$. They satisfy the orthogonality relations

$$\int_0^\pi R_m^{(\alpha,\beta)}(\cos\theta)\, R_n^{(\alpha,\beta)}(\cos\theta)\, w_{\alpha,\beta}(\theta)\, d\theta$$
$$=\delta_{m,n}(\omega_n^{(\alpha,\beta)})^{-1}, \qquad m,n=0, 1, 2, \ldots, \tag{2.9}$$

where

$$\omega_n^{(\alpha,\beta)}=\begin{cases}\dfrac{\Gamma(2\lambda+1)}{\Gamma(\alpha+1)\,\Gamma(\beta+1)}, & n=0,\\[2ex] \dfrac{(2n+2\lambda)\,\Gamma(n+\alpha+1)\,\Gamma(n+2\lambda)}{\Gamma^2(\alpha+1)\,\Gamma(n+1)\,\Gamma(n+\beta+1)}, & n=1, 2, \ldots;\end{cases} \tag{2.10}$$

cf. [4, 10.8(4)]. The functions $\theta\mapsto R_n^{(\alpha,\beta)}(\cos\theta)$, $n=0, 1, 2, \ldots$, form a complete orthogonal system in $L^2((0,\pi), w_{\alpha,\beta}(\theta)\, d\theta)$.

Let f be a function on $(0,\pi)$ such that

$$\hat{f}(t):=\int_0^\pi f(\theta)\, R_{t-\lambda}^{(\alpha,\beta)}(\cos\theta)\, w_{\alpha,\beta}(\theta)\, d\theta \tag{2.11}$$

is well-defined for all $t\in\mathbb{C}$. Then $\hat{f}$ is called the *finite continuous Jacobi transform* of f. In particular, we put

$$S_n^{(\lambda)}(t-\lambda):=\omega_n^{(\alpha,\beta)}(R_n^{(\alpha,\beta)}\circ\cos)^{\wedge}\,(t)$$

and obtain from [11, (2.9)] that

$$S_n^{(\lambda)}(t-\lambda)=\omega_n^{(\alpha,\beta)}\int_0^\pi R_n^{(\alpha,\beta)}(\cos\theta)\, R_{t-\lambda}^{(\alpha,\beta)}(\cos\theta)\, w_{\alpha,\beta}(\theta)\, d\theta$$
$$=\frac{(2n+2\lambda)\,\Gamma(n+2\lambda)}{\Gamma(n+1)}\cdot\frac{\sin\pi(t-\lambda-n)\,\Gamma(t-\lambda+1)}{\pi(t^2-(\lambda+n)^2)\,\Gamma(t+\lambda)}. \tag{2.12}$$

Clearly, $S_n^{(\lambda)}(t-\lambda)$ depends on α and β only through their sum, and by (2.9) we have

$$S_n^{(\lambda)}(m)=\delta_{nm}, \qquad m, n=0, 1, 2, \ldots. \tag{2.13}$$

We will make ample use of the formula

$$\Gamma(z)\,\Gamma(1-z)=\frac{\pi}{\sin(\pi z)}, \tag{2.14}$$

cf. [4, 1.2(6)], and the function

$$\psi(z) := \Gamma'(z)/\Gamma(z), \tag{2.15}$$

which satisfies the identity

$$\psi(z) - \psi(1-z) = -\pi \operatorname{cotg}(\pi z); \tag{2.16}$$

cf. [4, 1.7(11)]].

We will need a few but crucial facts from classical Fourier analysis. Define the Fourier transform $\mathscr{F}$ by

$$(\mathscr{F}\phi)(x) := \int_{-\infty}^{\infty} \phi(t)\, e^{-2\pi i t x}\, dt, \qquad \phi \in L^1(\mathbb{R}). \tag{2.17}$$

PROPOSITION 2.1. *Let ϕ be an entire analytic function such that, for some $p>1$ and γ, $C>0$,*

$$|\phi(t)| \leqslant C(1+|t|)^{-p}\, e^{2\pi\gamma|\operatorname{Im} t|}, \qquad t \in \mathbb{C}.$$

Then

$$(\mathscr{F}\phi)(x) = 0 \qquad \text{if} \quad |x| \geqslant \gamma.$$

PROPOSITION 2.2. *Let ϕ be a continuous function on $\mathbb{R}$ such that, for some $p>1$, there is a constant C_p such that*

$$|\phi(t)| \leqslant C_p(1+|t|)^{-p}, \qquad t \in \mathbb{R},$$
$$|(\mathscr{F}\phi)(x)| \leqslant C_p(1+|x|)^{-p}, \qquad x \in \mathbb{R}.$$

Then

$$\sum_{n=-\infty}^{\infty} \phi(\lambda+n) = \sum_{n=-\infty}^{\infty} (\mathscr{F}\phi)(n)\, e^{2\pi i n\lambda}, \qquad \lambda \in \mathbb{R}. \tag{2.18}$$

Formula (2.18) is the *Poisson summation formula*; cf. [9, Chap. 7, (2.7)].

3. ESTIMATES

In this section we collect all estimates which will be needed in the sequel. In particular, we will derive estimates for $R_{t-\lambda}^{(\alpha,\beta)}(\cos\theta)$ and $\hat{f}(t)$.

Let us start with the case of Jacobi series. From the equiconvergence theorem (cf. [10, Theorem 9.1.2]) and from well-known properties of Fourier-cosine series we obtain:

LEMMA 3.1. *Let $f \in L^2((0,\pi), w_{\alpha,\beta}(\theta)\, d\theta)$ and let f be once continuously differentiable on $(0,\pi)$. Then*

$$f(\theta) = \sum_{n=0}^{\infty} \hat{f}(\lambda + n)\, \omega_n^{(\alpha,\beta)} R_n^{(\alpha,\beta)}(\cos\theta), \qquad 0 < \theta < \pi,$$

uniformly on compact subsets of $(0,\pi)$.

Next we consider the integral representation

$$\begin{aligned} R_{t-\lambda}^{(\alpha,\beta)}(\cos\theta) = {} & \frac{2^{\alpha-1/2}\,\Gamma(\alpha+1)}{\pi^{1/2}\Gamma(\alpha+\frac{1}{2})} \left(\sin\frac{1}{2}\theta\right)^{-2\alpha} \left(\cos\frac{1}{2}\theta\right)^{-\beta-1/2} \\ & \times \int_0^{\theta} \cos(t\psi) \left(\cos\frac{1}{2}\psi - \cos\frac{1}{2}\theta\right)^{\alpha-1/2} \\ & \times F\left(\frac{1}{2}+\beta, \frac{1}{2}-\beta; \alpha+\frac{1}{2}; \frac{\cos\frac{1}{2}\theta - \cos\frac{1}{2}\psi}{2\cos\frac{1}{2}\theta}\right) d\psi, \\ & 0 < \theta < \pi,\ t \in \mathbb{C},\ \alpha > -\tfrac{1}{2},\ \beta > -1; \end{aligned} \tag{3.1}$$

cf. [7, (5.8)] (where the factor $(\cos\frac{1}{2}\psi - \cos\frac{1}{2}\theta)^{\alpha-1/2}$ is missing) or [7, (2.16), (2.19)] (by analytic continuation with respect to t) or [5, (6), (8)] (by quadratic transformation of the hypergeometric function). The hypergeometric function in the integrand of (3.1) can be rewritten as

$$\begin{aligned} & \left[\frac{\cos\frac{1}{2}\theta + \cos\frac{1}{2}\psi}{2\cos\frac{1}{2}\theta}\right]^{-1/2-\beta} \\ & \quad \times F\left(\frac{1}{2}+\beta, \alpha+\beta; \alpha+\frac{1}{2}; -\frac{\cos\frac{1}{2}\theta - \cos\frac{1}{2}\psi}{\cos\frac{1}{2}\theta + \cos\frac{1}{2}\psi}\right), \end{aligned}$$

cf. [4, 2.1(22)], which is non-negative for $\alpha > -\frac{1}{2}$, $\beta \geqslant -\frac{1}{2}$, $\alpha \geqslant -\beta$. Similarly it may be expressed as

$$\begin{aligned} & \left[\frac{\cos\frac{1}{2}\theta + \cos\frac{1}{2}\psi}{2\cos\frac{1}{2}\theta}\right]^{\alpha-1/2} \left[\frac{\cos\frac{1}{2}\psi}{\cos\frac{1}{2}\theta}\right]^{\beta-\alpha} \\ & \quad \times F\left[\frac{1}{2}(\alpha-\beta), \frac{1}{2}(\alpha-\beta+1); \alpha+\frac{1}{2}; 1 - \frac{\cos^2\frac{1}{2}\theta}{\cos^2\frac{1}{2}\psi}\right], \end{aligned}$$

cf. [4, 2.11(24)], which again is non-negative for $\alpha \geqslant \beta > -1$, $\alpha > -\frac{1}{2}$. Thus the function in (3.1) satisfies

$$\begin{aligned} & F\left(\frac{1}{2}+\beta, \frac{1}{2}-\beta; \alpha+\frac{1}{2}; \frac{\cos\frac{1}{2}\theta - \cos\frac{1}{2}\psi}{2\cos\frac{1}{2}\theta}\right) \\ & \quad \geqslant 0, \qquad 0 < \psi < \theta < \pi, \end{aligned}$$

if $\alpha > -\frac{1}{2}$, $\alpha \geqslant -|\beta|$, $\beta > -1$. For these values of α, β the integral in (3.1) is dominated by a simimar integral with $\cos(t\psi)$ being replaced by $\sup_{0<\psi<\theta} |\cos(t\psi)|$. Hence

$$|R_{t-\lambda}^{(\alpha,\beta)}(\cos\theta)| \leqslant e^{\theta|\mathrm{Im}\, t|}\, R_{-\lambda}^{(\alpha,\beta)}(\cos\theta) = e^{\theta|\mathrm{Im}\, t|}\, F(\lambda, \lambda; \alpha+1; \sin^2 \tfrac{1}{2}\theta). \tag{3.2}$$

The function $z \mapsto F(\lambda, \lambda; \alpha+1; z)$ is regular at 0 and is a solution of a hypergeometric differential equation for which the regular singularity at 1 has exponents 0 and $-\beta$. Thus $|F(\lambda, \lambda; \alpha+1; z)|$ is dominated on $[0, 1)$ by a constant multiple of $(1-z)^{-\beta}$ if $\beta > 0$, of $1 + |\log(1-z)|$ if $\beta = 0$, and of 1 if $\beta < 0$.

If $\alpha, \beta > -1$ and the inequalities $\alpha > -\frac{1}{2}$, $\alpha \geqslant -|\beta|$ are not both satisfied then they will be satisfied when α is replaced by $\alpha + 1$. We can express Jacobi functions of order (α, β) in terms of Jacobi functions of order $(\alpha+1, \beta)$ by

$$R_{t-\lambda}^{(\alpha,\beta)} = \frac{[(\lambda+t)(\lambda+t-\beta)\, R_{t-\lambda}^{(\alpha+1,\beta)} - (\lambda-t)(\lambda-t-\beta)\, R_{-t-\lambda}^{(\alpha+1,\beta)}]}{(2t(\alpha+1))}. \tag{3.3}$$

This in turn may be substituted in (3.1) to obtain

$$\begin{aligned} R_{t-\lambda}^{(\alpha,\beta)}&(\cos\theta) \\ &= \frac{2^{\alpha+1/2}\Gamma(\alpha+2)}{\Pi^{1/2}\Gamma(\alpha+3/2)} \left(\sin\frac{1}{2}\theta\right)^{-2\alpha-2} \left(\cos\frac{1}{2}\theta\right)^{-\beta-1/2} \\ &\times \int_0^\theta \frac{(\lambda+t)(\lambda+t-\beta)\cos((t+\frac{1}{2})\psi) - (\lambda-t)(\lambda-t-\beta)\cos((\frac{1}{2}-t)\psi)}{2(\alpha+1)t} \\ &\times \left(\cos\frac{1}{2}\psi - \cos\frac{1}{2}\theta\right)^{\alpha+1/2} F\left(\frac{1}{2}+\beta, \frac{1}{2}-\beta; \alpha+\frac{3}{2}; \frac{\cos\frac{1}{2}\theta - \cos\frac{1}{2}\psi}{2\cos\frac{1}{2}\theta}\right) d\psi. \end{aligned}$$

Since the hypergeometric function in the integrand is non-negative, and by (3.2) we have

$$|R_{t-\lambda}^{(\alpha+1,\beta)}(\cos\theta)| \leqslant e^{\theta|\mathrm{Im}\, t|}\, R_{-1/2-\lambda}^{(\alpha+1,\beta)}(\cos\theta),$$

we may conclude:

LEMMA 3.2. *Let* $\alpha, \beta > -1$; *put* $\varepsilon := 0$ *if* $\alpha > -\frac{1}{2}$, $\alpha \geqslant -|\beta|$ *and* $\varepsilon := 1$, *otherwise. Then there is a positive constant* $C_{\alpha,\beta}$ *such that, for* $0 \leqslant \theta < \pi$, $t \in \mathbb{C}$,

$$|R_{t-\lambda}^{(\alpha,\beta)}(\cos\theta)| \leqslant \begin{cases} C_{\alpha,\beta}(\cos\frac{1}{2}\theta)^{-2\beta}(1+|t|)^\varepsilon e^{\theta|\mathrm{Im}\, t|} & \text{if } \beta > 0, \\ C_{\alpha,\beta}(1+|\log(\cos\frac{1}{2}\theta)|)(1+|t|)^\varepsilon e^{\theta|\mathrm{Im}\, t|} & \text{if } \beta = 0, \\ C_{\alpha,\beta}(1+|t|)^\varepsilon e^{\theta|\mathrm{Im}\, t|} & \text{if } \beta < 0. \end{cases}$$

We now turn to estimates for $\hat{f}(t)$ (cf. (2.11)). If f is continuous on $[0, \pi]$ then, by Lemma 3.2, $\hat{f}$ is an even entire analytic function which for some constant $C'_{\alpha,\beta}$ satisfies the estimate

$$|\hat{f}(t)| \leqslant C'_{\alpha,\beta}(1+|t|)^{\varepsilon} e^{\pi|\mathrm{Im}\, t|}, \qquad t \in \mathbb{C}.$$

However, we will need functions f such that $\hat{f}$ decreases on the real axis as rapidly as some inverse power. For this purpose we restrict our functions somewhat more.

DEFINITION 3.3. Let $p = 0, 1, 2, \ldots$. The class C^{2p} consists of all even, $2p$ times continuously differentiable functions on $[-\pi, \pi]$ for which $f^{(k)}(\pi) = 0$, $k = 0, 1, \ldots, 2p-1$.

Let $f \in C^{2p}$, $p \geqslant 1$. Then, by (2.11) and (2.6),

$$\begin{aligned}(\lambda^2 - t^2)\hat{f}(t) &= \int_0^\pi f(\theta)(D_\theta^{\alpha,\beta} R_{t-\lambda}^{(\alpha,\beta)}(\cos\theta))\, w_{\alpha,\beta}(\theta)\, d\theta \\ &= \int_0^\pi (D_\theta^{\alpha,\beta} f(\theta))\, R_{t-\lambda}^{(\alpha,\beta)}(\cos\theta)\, w_{\alpha,\beta}(\theta)\, d\theta \\ &\quad + f(\theta)\, w_{\alpha,\beta}(\theta) \frac{d}{d\theta} R_{t-\lambda}^{(\alpha,\beta)}(\cos\theta)]_0^\pi \\ &\quad - R_{t-\lambda}^{(\alpha,\beta)}(\cos\theta)\, w_{\alpha,\beta}(\theta) \frac{d}{d\theta} f(\theta)]_0^\pi.\end{aligned}$$

The two integrated terms disappear. This follows from

$$\frac{d}{d\theta} R_{t-\lambda}^{(\alpha,\beta)}(\cos\theta) = \frac{\lambda^2 - t^2}{\alpha+1} \sin\frac{1}{2}\theta \cos\frac{1}{2}\theta\; R_{t-(1/2)(\alpha+\beta+3)}^{(\alpha+1,\beta+1)}(\cos\theta)$$

(cf. (2.2) and [4, 2.8(20)]), from Lemma 3.2, from the oddness of f', from $f(\pi) = 0$, and from $f'(\theta) = O(\cos\frac{1}{2}\theta)$ as $\theta \uparrow \pi$. Next we observe that $D^{\alpha,\beta} f \in C^{2p-2}$ if $f \in C^{2p}$, and conclude that

$$(\lambda^2 - t^2)^p \hat{f}(t) = ((D^{\alpha,\beta})^p f)^\wedge(t), \qquad f \in C^{2p}.$$

Hence:

LEMMA 3.4. *Let $\alpha, \beta > -1$ and $\varepsilon = 0$ or 1 as in Lemma 3.2, $p = 0, 1, \ldots$. Let $f \in C^{2p}$. Then there is a positive constant $C_{\alpha,\beta,p,f}$ such that*

$$|\hat{f}(t)| \leqslant C_{\alpha,\beta,p,f}(1+|t|)^{-2p+\varepsilon} e^{\pi|\mathrm{Im}\, t|}, \qquad t \in \mathbb{C}.$$

The following two lemmas will be useful.

LEMMA 3.5. *Let $a, b \in \mathbb{C}$. Then there is a positive constant $C_{a,b}$ such that*

$$|\Gamma(a+t)\,\Gamma(b-t)|^{-1} \leqslant C_{a,b}(1+|t|)^{1-a-b}\,e^{\pi|\mathrm{Im}\,t|}, \qquad t \in \mathbb{C}.$$

Proof. By (2.14) we have

$$\frac{1}{\Gamma(a+t)\,\Gamma(b-t)} = \frac{\Gamma(1-b+t)}{\Gamma(a+t)}\,\frac{\sin(\pi(b-t))}{\pi}$$
$$= \frac{\Gamma(1-a-t)}{\Gamma(b-t)}\,\frac{\sin(\pi(a+t))}{\pi}.$$

Now use the asymptotic formula [4, 1.18(4)] for the first factor. ∎

LEMMA 3.6. *$\pi^{-1}\sin(\pi t)\psi(-t)$ is an entire analytic function of t which equals $(-1)^t$ for $t=0, 1, 2, \ldots$ and 0 for $t=-1, -2, \ldots$; there is a positive constant C such that*

$$|\pi^{-1}\sin(\pi t)\,\psi(-t)| \leqslant C\log(1+|t|)\,e^{\pi|\mathrm{Im}\,t|}, \qquad t \in \mathbb{C}.$$

Proof. Use (2.16) and the asymptotic formula for $\psi(z)$ [4, 1.18(7)]. ∎

Finally, we estimate $S_n^{(\lambda)}(t-\lambda)$. By (2.12) and (2.14),

$$S_n^{(\lambda)}(t-\lambda) = \frac{(-1)^n\,(2n+2\lambda)\,\Gamma(n+2\lambda)}{\Gamma(n+1)(\lambda+n-t)(\lambda+n+t)\,\Gamma(\lambda-t)\,\Gamma(\lambda+t)}.$$

Hence, by Lemma 3.5:

LEMMA 3.7. *$S_n^{(\lambda)}(t-\lambda)$ is an even entire analytic function of t which satisfies the estimate*

$$|S_n^{(\lambda)}(t-\lambda)| \leqslant C_{n,\lambda}(1+|t|)^{-2\lambda-1}\,e^{\pi|\mathrm{Im}\,t|}, \qquad t \in \mathbb{C},$$

for some positive constant $C_{n,\lambda}$.

LEMMA 3.8. *Let $f \in C^{2p}$, where $2p > 1+\varepsilon+\max\{2\alpha+1, 2\lambda, \alpha+1/2\}$ and $\varepsilon=0$ or 1 as in Lemma 3.2. Then*

$$\hat{f}(t) = \sum_{n=0}^{\infty} \hat{f}(\lambda+n)\,S_n^{(\lambda)}(t-\lambda)$$

with absolute convergence, uniform on strips of finite width in $\mathbb{C}$ around $\mathbb{R}$.

Proof. By Lemma 3.1 we have

$$f(\theta)=\sum_{n=0}^{\infty}\hat{f}(\lambda+n)\,\omega_n^{(\alpha,\beta)}R_n^{(\alpha,\beta)}(\cos\theta),\qquad 0<\theta<\pi.$$

There exist constants such that

$$|\omega_n^{(\alpha,\beta)}|\leqslant \mathrm{const}(1+n)^{2\alpha+1}$$
$$\text{(by (2.10))},\tag{3.4}$$
$$|R_n^{(\alpha,\beta)}(\cos\theta)|\leqslant \mathrm{const}(1+n)^{\max\{0,\beta-\alpha,-1/2-\alpha\}}$$
$$\text{(by [10, Theorem 7.32.1])}.\tag{3.5}$$

Then, by these inequalities and the ones in Lemmas 3.2 and 3.4, we obtain

$$\omega_n^{(\alpha,\beta)}|\hat{f}(\lambda+n)|\int_0^\pi |R_n^{(\alpha,\beta)}(\cos\theta)|\,|R_{t-\lambda}^{(\alpha,\beta)}(\cos\theta)|\,w_{\alpha,\beta}(\theta)\,d\theta$$
$$\leqslant \mathrm{const}((1+|t|)^\varepsilon e^{\pi|\mathrm{Im}\,t|}(1+n)^{-2p+\varepsilon+\max\{2\alpha+1,2\lambda,\alpha+1/2\}}.$$

Now use (2.11) and (2.12). ∎

4. Analytic Continuation of the Inversion Formula

Let $f\in C^{2p}$ $(p\geqslant 1)$; by Lemma 3.1 we have

$$f(\theta)=\sum_{n=0}^{\infty}\hat{f}(\lambda+n)\,R_n^{(\alpha,\beta)}(\cos\theta)\,\omega_n^{(\alpha,\beta)},\qquad 0<\theta<\pi,\tag{4.1}$$

and, since

$$R_n^{(\alpha,\beta)}(\cos\theta)=(-1)^n\frac{(\beta+1)_n}{(\alpha+1)_n}R_n^{(\beta,\alpha)}(-\cos\theta),\qquad n=0,1,...,\tag{4.2}$$

cf. [4, 10.8(13)], (4.1) can also be written as

$$f(\theta)=\sum_{n=0}^{\infty}\hat{f}(\lambda+n)\,R_n^{(\beta,\alpha)}(-\cos\theta)$$
$$\times\frac{\tilde{\omega}_n^{(\lambda)}}{\Gamma(\alpha+1)\,\Gamma(\beta+1)},\qquad 0<\theta<\pi,\tag{4.3}$$

where

$$\tilde{\omega}_n^{(\lambda)}=\begin{cases}\Gamma(2\lambda+1), & n=0,\\ 2(-1)^n(n+\lambda)\,\Gamma(n+2\lambda)/n!, & n=1,2,\dots.\end{cases}\tag{4.4}$$

In their dependence on t, $\hat{f}(t)$, $R_{t-\lambda}^{(\alpha,\beta)}(\cos\theta)$, and $R_{t-\lambda}^{(\beta,\alpha)}(-\cos\theta)$ are even entire analytic functions with estimates given by Lemmas 3.2 and 3.4. It is our purpose to find $h_t^{(\alpha,\beta)}$ and $\tilde{h}_t^{(\lambda)}$, even entire analytic in t, such that (4.1) and (4.3), respectively, can be written as integrals

$$f(\theta)=\int_{-\infty}^{\infty}\hat{f}(t)\,R_{t-\lambda}^{(\alpha,\beta)}(\cos\theta)\,h_t^{(\alpha,\beta)}\,dt,\qquad 0<\theta<\pi,\quad(4.5)$$

$$f(\theta)=\int_{-\infty}^{\infty}\hat{f}(t)\,R_{t-\lambda}^{(\beta,\alpha)}(-\cos\theta)\,\frac{\tilde{h}_t^{(\lambda)}\,dt}{\Gamma(\alpha+1)\,\Gamma(\beta+1)},\qquad 0<\theta<\pi,\quad(4.6)$$

provided p is sufficiently large.

The idea of the derivation is as follows. Let us start, for instance, with (4.3), for which we write the right-hand side as

$$\sum_{n=0}^{\infty}F(\lambda+n)\,\tilde{\omega}_n^{(\lambda)},\qquad(4.7)$$

where F is an even entire analytic function satisfying the estimate

$$|F(t)|\leqslant\text{const}(1+|t|)^{-m}\,e^{2\pi|\text{Im}\,t|},\qquad t\in\mathbb{C},\qquad(4.8)$$

for a certain m. For $2\lambda\notin\mathbb{Z}$ we immediately find a natural analytic extension of $\tilde{\omega}_n^{(\lambda)}$ in the form of an entire analytic function

$$\tilde{\sigma}_t^{(\lambda)}:=\frac{2\pi t}{\sin(2\pi\lambda)\,\Gamma(1-\lambda+t)\,\Gamma(1-\lambda-t)},\qquad 2\lambda\notin\mathbb{Z},\qquad(4.9)$$

of t, satisfying the estimate

$$|\tilde{\sigma}_t^{(\lambda)}|\leqslant\text{const}(1+|t|)^{2\lambda}\,e^{\pi|\text{Im}\,t|},\qquad t\in\mathbb{C},\qquad(4.10)$$

by Lemma 3.5 and such that

$$\tilde{\sigma}_{\lambda+n}^{(\lambda)}=\begin{cases}\tilde{\omega}_n^{(\lambda)}, & n=0,1,2,...,\\ 0, & n=-1,-2,....\end{cases}\qquad(4.11)$$

Then (4.7) equals

$$\sum_{n=-\infty}^{\infty}F(\lambda+n)\,\tilde{\sigma}_{\lambda+n}^{(\lambda)}.\qquad(4.12)$$

Now let $m>2\lambda+1$. Then we can apply Proposition 2.2 in order to rewrite (4.12) as

$$\sum_{n=-\infty}^{\infty}e^{2\pi in\lambda}\int_{-\infty}^{\infty}F(t)\,\tilde{\sigma}_t^{(\lambda)}e^{-2\pi int}\,dt,\qquad(4.13)$$

which in turn, by applying Proposition 2.1 together with (4.8) and (4.10), can be written as

$$\int_{-\infty}^{\infty} F(t)\,\tilde{\sigma}_t^{(\lambda)}(1+2\cos(2\pi(t-\lambda)))\,dt. \tag{4.14}$$

Finally, we take the even part of $\tilde{\sigma}_t^{(\lambda)}(1+2\cos(2\pi(t-\lambda)))$ and denote it by $\tilde{h}_t^{(\lambda)}$:

$$\tilde{h}_t^{(\lambda)} = \frac{4\pi t\sin(2\pi t)}{\Gamma(1-\lambda+t)\,\Gamma(1-\lambda-t)}, \tag{4.15}$$

where $2\lambda \notin \mathbb{Z}$. Thus we have proved that

$$\sum_{n=0}^{\infty} F(\lambda+n)\,\tilde{\omega}_n^{(\lambda)} = \int_{-\infty}^{\infty} F(t)\,\tilde{h}_t^{(\lambda)}\,dt. \tag{4.16}$$

In particular, (4.6) will be valid with this choice of $\tilde{h}_t^{(\lambda)}$.

While the derivation was done for $2\lambda \notin \mathbb{Z}$, the singularities in (4.9) for $2\lambda = 0, 1, 2, \ldots$ drop out in (4.15). By continuity one can expect that (4.16) also holds for $2\lambda = 0, 1, 2, \ldots$ with the $\tilde{h}_t^{(\lambda)}$ given by (4.15). However, for such λ there is a simpler expression which may be derived from (4.15):

$$\tilde{h}_t^{(\lambda)} = \frac{t(2\cos(\pi(t-\lambda)) - 2\cos(3\pi t+\pi\lambda))(t-\lambda)_{2\lambda}}{t-\lambda},$$
$$2\lambda = 0, 1, 2, \ldots. \tag{4.17}$$

For this choice of $\tilde{h}_t^{(\lambda)}$, the terms with $e^{\pm 3\pi i t}$ in the right-hand side of (4.16) drop out because of Proposition 2.1. Hence, (4.16) is valid with

$$\tilde{h}_t^{(\lambda)} = \frac{2t\cos(\pi(t-\lambda))(t-\lambda)_{2\lambda}}{t-\lambda}, \qquad 2\lambda = 0, 1, 2, \ldots. \tag{4.18}$$

There is an analogous road from (4.1) to (4.5) by the identities

$$\begin{aligned}\sum_{n=0}^{\infty} F(\lambda+n)\,\omega_n^{(\alpha,\beta)} &= \sum_{n=-\infty}^{\infty} F(\lambda+n)\,\sigma_{\lambda+n}^{(\alpha,\beta)} \\ &= \int_{-\infty}^{\infty} F(t)\,\sigma_t^{(\alpha,\beta)}(1+2\cos(2\pi(t-\lambda)))\,dt \\ &= \int_{-\infty}^{\infty} F(t)\,h_t^{(\alpha,\beta)}\,dt,\end{aligned} \tag{4.19}$$

where F is an even entire analytic function satisfying the estimate (4.8) with $m > 2\alpha + 2$ and $t \mapsto \sigma_t^{(\alpha,\beta)}$ is an entire analytic function satisfying the estimate

$$|\sigma_t^{(\alpha,\beta)}| \leqslant \text{const}(1 + |t|)^{2\alpha + \delta + 1} e^{2\pi|\operatorname{Im} t|}, \qquad t \in \mathbb{C}, \tag{4.20}$$

for all $\delta > 0$. For $\alpha \neq 0, 1, 2, \ldots$ we find by reasoning as before that

$$\sigma_t^{(\alpha,\beta)} = \frac{\Gamma(-\alpha)\, \tilde{\sigma}_t^{(\lambda)}}{\Gamma(1+\alpha)\, \Gamma((\beta - \alpha + 1)/2 + t)\, \Gamma((\beta - \alpha + 1)/2 - t)}, \tag{4.21}$$

$$h_t^{(\alpha,\beta)} = \frac{\Gamma(-\alpha)\, \tilde{h}_t^{(\lambda)}}{\Gamma(1+\alpha)\, \Gamma((\beta - \alpha + 1)/2 + t)\, \Gamma((\beta - \alpha + 1)/2 - t)}, \qquad \alpha \notin \mathbb{Z} \tag{4.22}$$

($\tilde{h}_t^{(\lambda)}$ given by (4.15) or (4.18)). Hence we have

$$\sum_{n=0}^{\infty} F(\lambda + n)\, \omega_n^{(\alpha,\beta)} = \int_{-\infty}^{\infty} F(t)\, h_t^{(\alpha,\beta)}\, dt \tag{4.23}$$

with $h_t^{(\alpha,\beta)}$ given by (4.22).

This can be further simplified in a number of special cases. If $\alpha, \beta = -\frac{1}{2}, \frac{1}{2}, \ldots$ and $\alpha \geqslant \beta$ then (4.22) can be written as

$$h_t^{(\alpha,\beta)} = \frac{t(t-\lambda)_{2\lambda} \left(t - \dfrac{\alpha - \beta - 1}{2}\right)_{\alpha - \beta} (\sin(2\pi t - \pi\beta) + \sin(\pi(\alpha + 1)))}{\Gamma^2(\alpha + 1) \sin(\pi(\alpha + 1))(t - \lambda)}.$$

Hence, by Proposition 2.1 we also have (4.23) with

$$h_t^{(\alpha,\beta)} = \frac{t(t-\lambda)_{2\lambda} \left(t - \dfrac{\alpha - \beta - 1}{2}\right)_{\alpha - \beta}}{\Gamma^2(\alpha + 1)(t - \lambda)}, \qquad \alpha, \beta = -\tfrac{1}{2}, \tfrac{1}{2}, \ldots, \alpha \geqslant \beta. \tag{4.24}$$

If $\alpha = 0, 1, 2, \ldots$, however, the previous formulae are not valid and we take another extension of $\omega_n^{(\alpha,\beta)}$:

$$\sigma_t^{(\alpha,\beta)} = \frac{(-1)^\alpha\, t(t - \lambda + 1)_\alpha\, (-t - \lambda + 1)_\alpha \sin(2\pi(t - \lambda))\, \psi(\lambda - t)}{\pi \Gamma^2(\alpha + 1)},$$

$$\alpha = 0, 1, 2, \ldots. \tag{4.25}$$

Then

$$\sigma_t^{(\alpha,\beta)}(1+2\cos(2\pi(t-\lambda)))=\frac{(-1)^\alpha\, t(t-\lambda+1)_\alpha(-t-\lambda+1)_\alpha\,\psi(\lambda-t)}{\pi\Gamma^2(\alpha+1)}$$
$$\times(\sin(2\pi(t-\lambda))+\sin(4\pi(t-\lambda))).$$

Hence, by Proposition 2.1 and Lemma 3.6, (4.23) is valid with

$$h_t^{(\alpha,\beta)}=(-1)^{\alpha+1}\, t(t-\lambda+1)_\alpha(-t-\lambda+1)_\alpha$$
$$\times\frac{(\psi(\lambda-t)\sin(2\pi(\lambda-t))+\psi(\lambda+t)\sin(2\pi(\lambda+t)))}{2\pi\Gamma^2(\alpha+1)}. \qquad (4.26)$$

Formulas (4.23) and (4.16) cannot only be applied to (4.5) and (4.6) but can also be used to give orthogonality relations for the functions $S_n^{(\lambda)}$ as integrals. From (2.13) we obtain

$$\sum_{k=0}^{\infty} S_m^{(\lambda)}(k)\, S_n^{(\lambda)}(k)\, \omega_k^{(\alpha,\beta)}=\omega_n^{(\alpha,\beta)}\,\delta_{m,n}, \qquad (4.27)$$

$$\sum_{k=0}^{\infty} S_m^{(\lambda)}(k)\, S_n^{(\lambda)}(k)\, \tilde{\omega}_k^{(\lambda)}=\tilde{\omega}_n^{(\lambda)}\,\delta_{m,n}. \qquad (4.28)$$

Then, by Lemma 3.7, the function $F(t):=S_m^{\lambda}(t-\lambda)\, S_n^{(\lambda)}(t-\lambda)$ has the right behaviour in order that (4.23) and (4.16) hold.

Let us summarize our results in the following theorem.

THEOREM 4.1. (a) *Let* $\tilde{h}_t^{(\lambda)}$ *be given by* (4.15) (*general* λ) *or* (4.18). *Then* (4.16) *is valid for entire even analytic* F *satisfying* (4.8) *with* $m>2\lambda+1$. *In particular,*

$$f(\theta)=\int_{-\infty}^{\infty}\hat{f}(t)\, R_{t-\lambda}^{(\beta,\alpha)}(-\cos\theta)\,\frac{\tilde{h}_t^{(\lambda)}\,dt}{\Gamma(\alpha+1)\,\Gamma(\beta+1)}, \qquad 0<\theta<\pi, \quad (4.29)$$

if $f\in C^{2p}$ *with* $p>\lambda+\varepsilon+\frac{1}{2}$ *and* ε *as in Lemma* 3.2, *and*

$$\int_{-\infty}^{\infty} S_m^{(\lambda)}(t-\lambda)\, S_n^{(\lambda)}(t-\lambda)\,\tilde{h}_t^{(\lambda)}\,dt=\tilde{\omega}_n^{(\lambda)}\,\delta_{m,n}. \qquad (4.30)$$

(b) *Let* $h_t^{(\alpha,\beta)}$ *be given by* (4.22), (4.24), *or* (4.26). *Then* (4.23) *is valid for entire even analytic* F *satisfying* (4.8) *with* $m>2\alpha+2$. *In particular,*

$$f(\theta)=\int_{-\infty}^{\infty}\hat{f}(t)\, R_{t-\lambda}^{(\alpha,\beta)}(\cos\theta)\, h_t^{(\alpha,\beta)}\,dt, \qquad 0<\theta<\pi, \qquad (4.31)$$

if $f \in C^{2p}$ *with* $p > \alpha + \varepsilon + 1$ *and* ε *as in Lemma* 3.2, *and*

$$\int_{-\infty}^{\infty} S_m^{(\lambda)}(t-\lambda)\, S_n^{(\lambda)}(t-\lambda)\, h_t^{(\alpha,\beta)}\, dt = \omega_n^{(\alpha,\beta)}\, \delta_{m,n}. \tag{4.32}$$

Formula (4.29) with $\tilde{h}_t^{(\lambda)}$ given by (4.18) ($2\lambda = 1, 2, ...$) was obtained earlier in [11, (5.9), (5.11)]. Special cases of their result go back to [3, (3.4)] ($2\lambda = 1$) and MacRobert (cf. [8, Chap. IX, (151), (152)]) ($\alpha = \beta = 0, 1, 2, ...$); see also [2, Theorem 1] ($\alpha = \beta = 0$). The conditions on f vary and the methods differ from ours.

The cases $\alpha = \beta = \pm \frac{1}{2}$ of (4.31) with (4.24) also follow from the inversion formulas for the Fourier-cosine and Fourier-sine transform in view of the two formulae

$$R_t^{(-1/2,\,-1/2)}(\cos\theta) = \cos(t\theta), \tag{4.24}$$

$$R_{t-1}^{(1/2,\,1/2)}(\cos\theta) = \frac{\sin(t\theta)}{t \sin\theta}. \tag{4.25}$$

5. The Inverse Finite Continuous Jacobi Transform

Lemma 5.1. *Let* g *be an even entire analytic function satisfying*

$$|g(t)| \leqslant C(1+|t|)^{-m}\, e^{\pi|\operatorname{Im} t|}, \qquad t \in \mathbb{C},$$

with $m > 2\alpha + 2 + \varepsilon$, *where* $\varepsilon = 0$ *or* 1 *as in Lemma* 3.2 *for some constant* C. *Then*

$$\int_{-\infty}^{\infty} g(t)\, R_{t-\lambda}^{(\alpha,\beta)}(\cos\theta)\, h_t^{(\alpha,\beta)}\, dt$$

$$= \sum_{n=0}^{\infty} g(\lambda+n)\, R_n^{(\alpha,\beta)}(\cos\theta)\, \omega_n^{(\alpha,\beta)}, \qquad 0 \leqslant \theta < \pi.$$

Proof. Let $F(t) := g(t)\, R_{t-\lambda}^{(\alpha,\beta)}(\cos\theta)$. Then, in view of Lemma 3.2, $F(t)$ satisfies the estimate (4.8) with m replaced by $m - \varepsilon$ and $m - \varepsilon > 2\alpha + 2$. Hence (4.23) is valid. ∎

Lemma 5.2 (Cf. Boas [1, Theorem 9.6.11]). *Let* $\lambda \in \mathbb{R}$. *Suppose* g *is an even entire analytic function satisfying*

$$g(t) = o(|t|^{-2\lambda+1}\, e^{\pi|\operatorname{Im} t|}), \tag{5.1}$$

uniformly as $|t| \to \infty$, *and suppose* $g(\lambda+n) = 0$ *for* $n = 0, 1, 2, ...$. *If* $2\lambda = 0, -1, -2, ...$, *suppose moreover that the zeros at* $\lambda, \lambda+1, ..., -\lambda$ *are double. Then* g *is identically zero.*

The estimate for g in Lemma 5.2 cannot be further relaxed, as is shown by the example

$$g(t) := \frac{1}{\Gamma(\lambda+t)\,\Gamma(\lambda-t)}.$$

This satisfies all conditions on g in the lemma except that $g(t) = O(|t|^{-2\lambda+1} e^{\pi|\mathrm{Im}\, t|})$ rather than (5.1). However, g is not identically zero.

Fix $\alpha, \beta > -1$ and let $h_t^{(\alpha,\beta)}$ be given by (4.22), (4.24), or (4.26). Let g be an even function on $\mathbb{R}$ such that

$$\check{g}(\theta) := \int_{-\infty}^{\infty} g(t)\, R_{t-\lambda}^{(\alpha,\beta)}(\cos\theta)\, h_t^{(\alpha,\beta)}\, dt \tag{5.2}$$

is well-defined for $-\pi < \theta < \pi$. In view of (4.31) we call the transformation $g \mapsto \check{g}$ the inverse *finite continuous Jacobi transform.*

THEOREM 5.3. *Fix $\alpha, \beta > -1$ and let $\varepsilon = 0$ or 1 as in Lemma 3.2. Let $\mu > 6\lambda + 2\varepsilon + 3 + \max\{\alpha - \beta, 0\}$ and $p = \max\{1, [\lambda + (\varepsilon/2) + 1/2]\}$. Let g be an even entire analytic function satisfying, for some constant C,*

$$|g(t)| \leqslant C(1+|t|)^{-\mu} e^{\pi|\mathrm{Im}\, t|}, \qquad t \in \mathbb{C}, \tag{5.3}$$

and such that

$$\int_{-\infty}^{\infty} g(t)\, \tilde{h}_t^{(\lambda+k)}\, dt = 0, \qquad k = 0, 1, ..., 2p-1. \tag{5.4}$$

Then $\check{g} \in C^{2p}$ and $g = (\check{g})^{\wedge}$.

Proof. By Lemma 5.1 we have, since $\mu > 2(\alpha+1) + \varepsilon$,

$$\check{g}(\theta) = \sum_{n=0}^{\infty} g(\lambda+n)\, R_n^{(\alpha,\beta)}(\cos\theta)\, \omega_n^{(\alpha,\beta)}. \tag{5.5}$$

Now

$$\sum_{n=k}^{\infty} g(\lambda+n) \left(\frac{d}{d(\cos\theta)}\right)^k R_n^{(\alpha,\beta)}(\cos\theta)\, \omega_n^{(\alpha,\beta)}$$

is absolutely convergent, uniform on $\mathbb{R}$, for $k = 0, 1, ..., 2p$. This follows from the fact that the absolute values of the terms are dominated by a multiple of $(1+n)^{-1-\nu}$ by the formula

$$\omega_n^{(\alpha,\beta)} \frac{d}{d(\cos\theta)} R_n^{(\alpha,\beta)}(\cos\theta) = \tfrac{1}{2}(\alpha+1)\, \omega_{n-1}^{(\alpha+1,\beta+1)} R_{n-1}^{(\alpha+1,\beta+1)}(\cos\theta)$$

and by (5.3), (3.4), (3.5). Hence $\check{g}$ is $2p$ times continuously differentiable on $\mathbb{R}$. Also

$$\begin{aligned}
&\left(\frac{d}{d(\cos\theta)}\right)^k \check{g}(\pi) \\
&= 2^{-k}(\alpha+1)_k \sum_{n=k}^{\infty} g(\lambda+n)\, \omega_{n-k}^{(\alpha+k,\beta+k)} R_{n-k}^{(\alpha+k,\beta+k)}(-1) \\
&= \frac{2^{-k}}{\Gamma(\alpha+1)\,\Gamma(\beta+k+1)} \sum_{n=k}^{\infty} g(\lambda+n)\, \tilde{\omega}_{n-k}^{(\lambda+k)} \\
&= \frac{2^{-k}}{\Gamma(\alpha+1)\,\Gamma(\beta+k+1)} \int_{-\infty}^{\infty} g(t)\, \tilde{h}_t^{(\lambda+k)}\, dt = 0 \\
&\qquad \text{for} \quad k = 0, 1, ..., 2p-1,
\end{aligned}$$

where we used (2.10), (4.4), (5.4), and Theorem 4.1(a) since $\mu > 2\lambda + 1$. Hence, $\check{g} \in C^{2p}$. Now, by Lemma 3.4, $(\check{g})^\wedge$ satisfies (5.1) and by the uniqueness of the Jacobi series, $g - (\check{g})^\wedge$ satisfies the assumptions of Lemma 5.2. ∎

Remark. In [7, Theorem 5.1] necessary and sufficient conditions were given in order that $g = \hat{f}$ with f being an even C^∞-function with compact support inside $(-\pi, \pi)$.

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