Completeness of Combinations of Constructor Systems[†]

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ABSTRACT

A term rewriting system is called complete if it is both confluent and strongly normalizing. Barendregt and Klop showed that the disjoint union of complete term rewriting systems does not need to be complete. In other words, completeness is not a modular property of term rewriting systems. Toyama, Klop and Barendregt showed that completeness is a modular property of left-linear TRS's. In this paper we show that it is sufficient to impose the constructor discipline for obtaining the modularity of completeness. This result is a simple consequence of a quite powerful divide and conquer technique for establishing completeness of such constructor systems. Our approach is not limited to systems which are composed of disjoint parts. The importance of our method is that we may decompose a given constructor system into parts which possibly share function symbols and rewrite rules in order to infer completeness. We obtain a similar technique for semi-completeness, i.e. the combination of confluence and weak normalization.

Introduction

A property of term rewriting systems is *modular* if it is preserved under disjoint union. Starting with Toyama [19], several authors studied modular aspects of term rewriting systems. Toyama [19] showed that confluence is a modular property. In [20] Toyama refuted the modularity of strong normalization by means of the following term rewriting systems:

$$\mathcal{R}_1 = \{\ F(0, 1, x) \rightarrow F(x, x, x)\ \}$$

$$\mathcal{R}_2 = \begin{cases} g(x, y) & \rightarrow\ x \\ g(x, y) & \rightarrow\ y. \end{cases}$$

Both systems are terminating, but their union admits the following cyclic reduction:

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$$\begin{aligned} F(g(0,1),g(0,1),g(0,1)) &\to F(0,g(0,1),g(0,1)) \\ &\to F(0,1,g(0,1)) \\ &\to F(g(0,1),g(0,1),g(0,1)). \end{aligned}$$

His counterexample inspired Rusinowitch [18] to the formulation of sufficient conditions for the strong normalization of the disjoint union of strongly normalizing term rewriting systems $\mathcal{R}_1$ and $\mathcal{R}_2$ in terms of the distribution of collapsing and duplicating rules among $\mathcal{R}_1$ and $\mathcal{R}_2$. Rusinowitch's results were extended by Middeldorp [11]. Barendregt and Klop gave an example showing that completeness (i.e. the combination of confluence and strong normalization) is not a modular property, see Toyama [20]. Independently, Drosten [3] gave the following simpler counterexample:

$$\mathcal{R}_1 = \begin{cases} F(0,1,x) & \to & F(x,x,x) \\ F(x,y,z) & \to & 2 \\ 0 & \to & 2 \\ 1 & \to & 2 \end{cases}$$

$$\mathcal{R}_2 = \begin{cases} g(x,y,y) & \to & x \\ g(y,y,x) & \to & x. \end{cases}$$

Both systems are easily shown to be complete. However, because both $g(0,1,1) \twoheadrightarrow 0$ and $g(0,1,1) \twoheadrightarrow 1$, the term $F(g(0,1,1),g(0,1,1),g(0,1,1))$ has a cyclic reduction akin to the one in the previous counterexample. Toyama, Klop and Barendregt [22] showed that the restriction to left-linear term rewriting systems is sufficient for obtaining the modularity of completeness. Middeldorp [10] showed that the property of having unique normal forms is modular for general term rewriting systems. An interesting alternative approach to modularity is explored in Kurihara and Kaji [7]. Middeldorp [12, 13, 14] extended the above results to conditional term rewriting systems. Kurihara and Ohuchi [8] showed that strong normalization is a modular property of term rewriting systems whose strong normalization can be shown by a simplification ordering. They extended this result in [9] to term rewriting systems which share constructors. Constructors are function symbols which do not occur at the leftmost position in left-hand sides of rewrite rules. Dershowitz [1], Geser [4] and Toyama [21] give further results on combinations of term rewriting systems with common function symbols. A comprehensive survey of combinations of (conditional) term rewriting systems can be found in Middeldorp [15].

The starting point of the present paper is the refutation of the modularity of completeness. We show that instead of requiring left-linearity it is also possible to impose the so-called *constructor discipline* for obtaining the modularity of completeness. In a constructor system (a term rewriting system which obeys the constructor discipline) all function symbols occurring at non-leftmost positions in left-hand sides of rewrite rules are constructors. Many term rewriting systems that occur in practice follow this discipline, see e.g. O'Donnell [17]. Actually we prove a much stronger result. We show that a constructor system is complete if it can be *decomposed* into complete constructor systems. The important observation is that our notion of decomposition does not imply disjointness. Consider for example the constructor system

$$\mathcal{R} = \begin{cases} 0+x & \to x \\ S(x)+y & \to S(x+y) \\ 0 \times x & \to 0 \\ S(x) \times y & \to x \times y + y \\ f(0) & \to 0 \\ f(S(x)) & \to f(x)+S(x). \end{cases}$$

We can decompose $\mathcal{R}$ into

$$\mathcal{R}_1 = \begin{cases} 0+x & \to x \\ S(x)+y & \to S(x+y) \\ 0 \times x & \to 0 \\ S(x) \times y & \to x \times y + y \end{cases} \quad \text{and} \quad \mathcal{R}_2 = \begin{cases} 0+x & \to x \\ S(x)+y & \to S(x+y) \\ f(0) & \to 0 \\ f(S(x)) & \to f(x)+S(x). \end{cases}$$

Both systems are easily shown to be complete and our decomposition result yields the completeness of $\mathcal{R}$. Neither the result of Kurihara and Ohuchi [9] (because $\mathcal{R}_1$ and $\mathcal{R}_2$ share the non-constructor symbol +) nor the result of Dershowitz [1] (because $\mathcal{R}_1$ and $\mathcal{R}_2$ are not right-linear) applies.

In the next section we give a concise introduction to term rewriting. Extensive surveys are Dershowitz and Jouannaud [2] and Klop [6]. In Section 2 we introduce the concept of marked reduction which plays a crucial role in the proof of our main results. Section 3 contains our main results. We define a notion of decomposability and we show that completeness is a decomposable property of constructor systems. To appreciate the non-triviality of our result, it may be contrasted with the fact that neither confluence nor strong normalization is decomposable. We further show that semi-completeness (i.e the combination of confluence and weak normalization) is a decomposable property of constructor systems.

1. Preliminaries

Let $\mathcal{V}$ be a countably infinite set of *variables*. A *term rewriting system* (TRS for short) is a pair $(\mathcal{F}, \mathcal{R})$. The set $\mathcal{F}$ consists of *function symbols*; associated to every $F \in \mathcal{F}$ is a natural number denoting its arity. Function symbols of arity 0 are called *constants*. The set $\mathcal{T}(\mathcal{F}, \mathcal{V})$ of *terms* built from $\mathcal{F}$ and $\mathcal{V}$ is the smallest set such that $\mathcal{V} \subset \mathcal{T}(\mathcal{F}, \mathcal{V})$ and if $F \in \mathcal{F}$ has arity n and $t_1, \ldots, t_n \in \mathcal{T}(\mathcal{F}, \mathcal{V})$ then $F(t_1, \ldots, t_n) \in \mathcal{T}(\mathcal{F}, \mathcal{V})$. Identity of terms is denoted by $\equiv$. The *root symbol* of a term t is defined as follows: $root(t) = F$ if $t \equiv F(t_1, \ldots, t_n)$ and $root(t) = t$ if $t \in \mathcal{V}$. The set $\mathcal{R}$ consists of pairs (l, r) with $l, r \in \mathcal{T}(\mathcal{F}, \mathcal{V})$ subject to the following two constraints:

(1) the left-hand side l is not a variable,

(2) the variables which occur in the right-hand side r also occur in l.

Pairs (l, r) are called *rewrite rules* and will henceforth be written as $l \to r$. A rewrite rule $l \to r$ is *left-linear* if l does not contain multiple occurrences of the same variable. A *left-linear* TRS only contains left-linear rewrite rules.

A *substitution* σ is a mapping from $\mathcal{V}$ to $\mathcal{T}(\mathcal{F}, \mathcal{V})$ such that its *domain* $\{x \in \mathcal{V} \mid \sigma(x) \not\equiv x\}$ is finite. Substitutions are extended to morphisms from $\mathcal{T}(\mathcal{F}, \mathcal{V})$ to $\mathcal{T}(\mathcal{F}, \mathcal{V})$, i.e. $\sigma(F(t_1, \ldots, t_n)) \equiv F(\sigma(t_1), \ldots, \sigma(t_n))$ for every n-ary function symbol F and terms $t_1, \ldots, t_n$. We call $\sigma(t)$ an *instance* of t. We write t^σ instead of $\sigma(t)$. An instance of a left-hand side of a rewrite rule is a *redex* (reducible expression). Let $\square$ be a special constant symbol. A *context* $C[\,,\ldots,\,]$ is a term in $\mathcal{T}(\mathcal{F} \cup \{\square\}, \mathcal{V})$. If $C[\,,\ldots,\,]$ is a context with n occurrences of $\square$ and $t_1, \ldots, t_n$ are terms then $C[t_1, \ldots, t_n]$ is the result of replacing from left to right the occurrences of $\square$ by $t_1, \ldots, t_n$. A context containing precisely one occurrence of $\square$ is denoted by $C[\,]$. A term s is a *subterm* of a term t if there exists a context $C[\,]$ such that $t \equiv C[s]$. If $C[\,] \not\equiv \square$ then s is a *proper* subterm of t. We write $s \subseteq t$ to indicate that s is a subterm of t.

The *rewrite relation* $\rightarrow_{\mathcal{R}}$ is defined as follows: $s \rightarrow_{\mathcal{R}} t$ if there exists a rewrite rule $l \rightarrow r$ in $\mathcal{R}$, a substitution σ and a context $C[\,]$ such that $s \equiv C[l^{\sigma}]$ and $t \equiv C[r^{\sigma}]$. The transitive-reflexive closure of $\rightarrow_{\mathcal{R}}$ is denoted by $\twoheadrightarrow_{\mathcal{R}}$; if $s \twoheadrightarrow_{\mathcal{R}} t$ we say that s *reduces* to t. We write $s \leftarrow_{\mathcal{R}} t$ if $t \rightarrow_{\mathcal{R}} s$; likewise for $s \twoheadleftarrow_{\mathcal{R}} t$. The transitive closure of $\rightarrow_{\mathcal{R}}$ is denoted by $\rightarrow_{\mathcal{R}}^{+}$ and $\leftrightarrow_{\mathcal{R}}$ denotes the symmetric closure of $\rightarrow_{\mathcal{R}}$ (so $\leftrightarrow_{\mathcal{R}} = \rightarrow_{\mathcal{R}} \cup \leftarrow_{\mathcal{R}}$). The transitive-reflexive closure of $\leftrightarrow_{\mathcal{R}}$ is called *conversion* and denoted by $=_{\mathcal{R}}$. If $s =_{\mathcal{R}} t$ then s and t are *convertible.* Two terms t_1, t_2 are *joinable*, notation $t_1 \downarrow_{\mathcal{R}} t_2$, if there exists a term t_3 such that $t_1 \twoheadrightarrow_{\mathcal{R}} t_3 \twoheadleftarrow_{\mathcal{R}} t_2$. Such a term t_3 is called a *common reduct* of t_1 and t_2. We often omit the subscript $\mathcal{R}$.

A term s is a *normal form* if there is no term t with $s \rightarrow t$. A TRS is *weakly normalizing* if every term reduces to a normal form. A TRS is *strongly normalizing* if there are no infinite reduction sequences $t_1 \rightarrow t_2 \rightarrow t_3 \rightarrow \ldots$. In other words, every reduction sequence eventually ends in a normal form. A TRS is *confluent* or has the *Church-Rosser* property if for all terms s, t_1, t_2 with $t_1 \twoheadleftarrow s \twoheadrightarrow t_2$ we have $t_1 \downarrow t_2$. A well-known equivalent formulation of confluence is that every pair of convertible terms is joinable ($t_1 = t_2 \Rightarrow t_1 \downarrow t_2$). A TRS is *locally confluent* if for all terms s, t_1, t_2 with $t_1 \leftarrow s \rightarrow t_2$ we have $t_1 \downarrow t_2$. A *complete* TRS is confluent and strongly normalizing. A *semi-complete* TRS is confluent and weakly normalizing. These properties of TRS's specialize to terms in the obvious way. If a term t has a unique normal form then we denote this normal form by $t\!\downarrow$.

The following well-known result is due to Newman [16].

NEWMAN'S LEMMA. *Every strongly normalizing and locally confluent* TRS *is confluent.* $\square$

Let $l_1 \rightarrow r_1$ and $l_2 \rightarrow r_2$ be renamed versions of rewrite rules of a TRS $\mathcal{R}$ such that they have no variables in common. Suppose $l_1 \equiv C[t]$ with $t \notin \mathcal{V}$ such that t and l_2 are unifiable, i.e. $t^{\sigma} \equiv l_2^{\sigma}$ for a most general unifier σ. The term $l_1^{\sigma} \equiv C[l_2]^{\sigma}$ is subject to the reduction steps $l_1^{\sigma} \rightarrow r_1^{\sigma}$ and $l_1^{\sigma} \rightarrow C[r_2]^{\sigma}$. The pair of reducts $\langle C[r_2]^{\sigma}, r_1^{\sigma}\rangle$ is a *critical pair* of $\mathcal{R}$. If $l_1 \rightarrow r_1$ and $l_2 \rightarrow r_2$ are renamed versions of the same rewrite rule, we do not consider the case $C[\,] \equiv \square$. A critical pair $\langle s, t\rangle$ of a TRS $\mathcal{R}$ is *convergent* if $s \downarrow_{\mathcal{R}} t$. The following lemma of Huet [5] expresses the significance of critical pairs.

CRITICAL PAIR LEMMA. *A* TRS $\mathcal{R}$ *is locally confluent if and only if all its critical pairs are convergent.* $\square$

A *constructor system* (CS for short) is a TRS $(\mathcal{F}, \mathcal{R})$ with the property that $\mathcal{F}$ can be partitioned into disjoint sets $\mathcal{D}$ and $\mathcal{C}$ such that every left-hand side $F(t_1, \ldots, t_n)$ of a rewrite rule of $\mathcal{R}$ satisfies $F \in \mathcal{D}$ and $t_1, \ldots, t_n \in \mathcal{T}(\mathcal{C}, \mathcal{V})$. Function symbols in $\mathcal{D}$ are called *defined symbols* and those in $\mathcal{C}$ *constructors.* To emphasize the partition of $\mathcal{F}$ into $\mathcal{D}$ and $\mathcal{C}$ we write $(\mathcal{D}, \mathcal{C}, \mathcal{R})$ instead of $(\mathcal{F}, \mathcal{R})$ and $\mathcal{T}(\mathcal{F}, \mathcal{V})$ is denoted by $\mathcal{T}(\mathcal{D}, \mathcal{C}, \mathcal{V})$.

Since the behaviour of a Turing machine can be simulated by a CS (see Klop [6] for details), CS's have universal computing power. The restriction on the left-hand sides of rewrite rules of CS's enables a considerable simplification of many concepts and proofs. For instance, if $\langle s, t\rangle$ is a critical pair of a CS $(\mathcal{D}, \mathcal{C}, \mathcal{R})$ then there exist different rewrite rules $l_1 \to r_1, l_2 \to r_2 \in \mathcal{R}$ (with variables suitably renamed) and a most general unifier σ of l_1 and l_2 such that $s \equiv r_1^\sigma$ and $t \equiv r_2^\sigma$.

2. Marked Reduction

In this section we introduce a new rewrite relation which plays an essential role in the proofs of our decomposition results. Due to lack of space no proofs are presented in this section. They can be found in the full version of the paper. Throughout this section we will be dealing with an arbitrary CS $(\mathcal{D}, \mathcal{C}, \mathcal{R})$.

DEFINITION 2.1.

(1) The set $\mathcal{D}^* = \{F^* \mid F \in \mathcal{D}\}$ consists of *marked* defined symbols. Terms in $\mathcal{T}(\mathcal{D}^* \cup \mathcal{D}, \mathcal{C}, \mathcal{V})$ are called *marked terms*. An *unmarked term* belongs to $\mathcal{T}(\mathcal{D}, \mathcal{C}, \mathcal{V})$.

(2) If t is a marked term then $e(t) \in \mathcal{T}(\mathcal{D}, \mathcal{C}, \mathcal{V})$ denotes the term obtained from t by erasing all marks and t^* denotes the term obtained from t by marking every unmarked defined symbol in t.

(3) Two marked terms s and t are *similar*, notation $s \approx t$, if $e(s) \equiv e(t)$. If s and t are similar then their *intersection* is the unique term $s \wedge t$ such that $s \wedge t \approx s \approx t$ and a defined symbol occurrence in $s \wedge t$ is marked if and only if the corresponding symbols in s and t are marked.

(4) The set $\mathcal{R}^*$ of *marked rewrite rules* is defined as $\{l^* \to r^* \mid l \to r \in \mathcal{R}\}$.

EXAMPLE 2.2. Consider the CS $(\mathcal{D}_1, \mathcal{C}_1, \mathcal{R}_1)$ with $\mathcal{D}_1 = \{F, G\}$, $\mathcal{C}_1 = \{S, 0\}$,

$$\mathcal{R}_1 = \begin{cases} F(S(x), y) & \to \quad G(x) \\ G(x) & \to \quad S(0) \end{cases}$$

and the reduction sequence

$$t \equiv F(S(G(0)), G(0)) \to F(S(G(0)), S(0)) \to G(G(0)) \to S(0).$$

If we mark some defined symbols in t then we can easily mimic this sequence by a reduction sequence in $\mathcal{R}_1 \cup \mathcal{R}_1^*$, for instance

$$F^*(S(G^*(0)), G(0)) \to_{\mathcal{R}_1} F^*(S(G^*(0)), S(0)) \to_{\mathcal{R}_1^*} G^*(G^*(0)) \to_{\mathcal{R}_1^*} S(0).$$

This correspondence does not hold for non-left-linear CS's. Consider the CS $(\mathcal{D}_2, \mathcal{C}_2, \mathcal{R}_2)$ with $\mathcal{D}_2 = \{F\}$, $\mathcal{C}_2 = \{S\}$, $\mathcal{R}_2 = \{F(x, x) \to S(x)\}$ and the reduction step

$$F(F(0, S(0)), F(0, S(0))) \to S(F(0, S(0))).$$

The marked term $F^*(F^*(0, S(0)), F(0, S(0)))$ cannot be reduced in $\mathcal{R}_2 \cup \mathcal{R}_2^*$.

By modifying the rewrite relation associated to $\mathcal{R} \cup \mathcal{R}^*$ we are able to mimic every unmarked reduction sequence, irrespective of the marking attached to the starting term.

DEFINITION 2.3. We write $s \to_m t$ if there exists a context $C[\,]$, a rewrite rule

$$C_1[x_1, \ldots, x_n] \to C_2[y_1, \ldots, y_m]$$

in $\mathcal{R} \cup \mathcal{R}^*$ (with all variables displayed) and terms $s_1, \ldots, s_n, t_1, \ldots, t_m$ such that the

following three conditions are satisfied:
(1) $s \equiv C[C_1[s_1, \ldots, s_n]]$ and $t \equiv C[C_2[t_1, \ldots, t_m]]$,
(2) $s_i \approx s_j$ whenever $x_i \equiv x_j$ for $1 \le i < j \le n$,
(3) $t_i \equiv \wedge \{s_j \mid x_j \equiv y_i\}$ for $i = 1, \ldots, m$.
We call $C_1[s_1, \ldots, s_n]$ a *marked redex* and the relation $\to_m$ is called *marked reduction.*

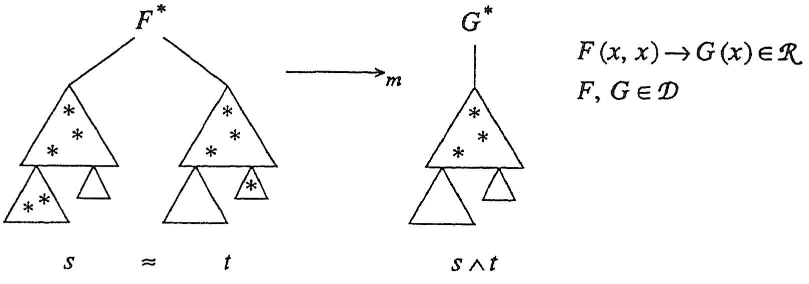

FIGURE 1.

Notice that $\to_m$ coincides with $\to_{\mathcal{R} \cup \mathcal{R}^*}$ whenever $\mathcal{R}$ is left-linear.

EXAMPLE 2.4. Consider the CS $(\mathcal{D}, C, \mathcal{R})$ with $\mathcal{D} = \{F, G\}$, $C = \{S, 0\}$,

$$\mathcal{R} = \begin{cases} F(x, x) & \to \quad S(x) \\ G(x) & \to \quad 0 \end{cases}$$

and the reduction sequence $F(G(F(0, S(0))), G(F(0, S(0)))) \to S(G(F(0, S(0)))) \to S(0)$. We have $F^*(G^*(F^*(0, S(0))), G^*(F(0, S(0)))) \to_m S(G^*(F(0, S(0)))) \to_m S(0)$.

The next proposition relates marked reduction to ordinary reduction. In part (2) it is essential that we restrict ourselves to CS's.

PROPOSITION 2.5.
(1) *If* $s \to_m t$ *then* $e(s) \to e(t)$.
(2) *If* $s \to t$ *and* $e(s') \equiv s$ *then there exists a term* t' *such that* $s' \to_m t'$ *and* $e(t') \equiv t$.
□

DEFINITION 2.6. If $t \equiv C[t_1, \ldots, t_n]$ such that all defined symbols in $C[\,, \ldots, \,]$ are marked and every t_i $(i = 1, \ldots, n)$ is unmarked then we call t a *capped* term. Furthermore, if $root(t_i) \in \mathcal{D}$ for $i = 1, \ldots, n$ then we write $t \equiv C{*}[t_1, \ldots, t_n]{*}$.

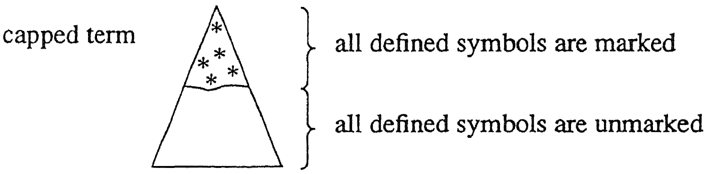
capped term
all defined symbols are marked
all defined symbols are unmarked

FIGURE 2.

DEFINITION 2.7. Let $s \equiv C*[s_1, \dots, s_n]*$ be a capped term.

(1) Suppose $s \to_m t$ by contraction of the marked redex Δ. We write $s \to_m^i t$ if Δ occurs in one of $s_1, \dots, s_n$ and we write $s \to_m^o t$ otherwise. The relation $\to_m^i$ is called *inner* marked reduction and $\to_m^o$ is called *outer* marked reduction.

(2) We call *s inside normalized* if it is a normal form with respect to $\to_m^i$.

DEFINITION 2.8. Let $t \equiv C*[t_1, \dots, t_n]*$ be a capped term. If $t_1, \dots, t_n$ are semi-complete then we define $\psi(t) \equiv C[t_1\downarrow, \dots, t_n\downarrow]$. Notice that $\psi(t)$ is inside normalized.

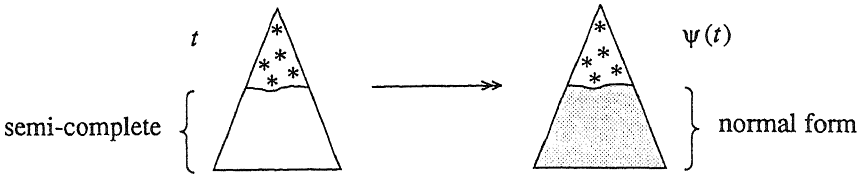

FIGURE 3.

LEMMA 2.9. *Let s be a capped term such that $\psi(s)$ is defined.*

(1) *If $s \to_m^o t$ then $\psi(t)$ is defined and $\psi(s) \to_m^{o+} \psi(t)$.*

(2) *If $s \to_m^i t$ then $\psi(t)$ is defined and $\psi(s) \equiv \psi(t)$.*

(3) *If s is a normal form then $\psi(s) \equiv s$.*

□

$$\begin{tikzcd}
s \arrow[r, "o", "m"'] \arrow[d, two heads] \arrow[dr, phantom, "(1)"] & t \arrow[d, dotted, two heads] & s \arrow[r, "i", "m"'] \arrow[d, two heads] \arrow[dr, phantom, "(2)"] & t \arrow[d, dotted, two heads] \\
\psi(s) \arrow[r, dashed, "o+", "m"'] & \psi(t) & \psi(s) \arrow[r, phantom, "\equiv"] & \psi(t)
\end{tikzcd}$$

FIGURE 4.

In the remainder of this section we give some further properties of marked reduction which are needed in the next section.

LEMMA 2.10. *Let s be a capped term and suppose $s \twoheadrightarrow_m t$. For every subterm u of t with $root(u) \in \mathcal{D}$ we can find terms $s' \subseteq s$ and $t' \subseteq t$ such that $root(s') \in \mathcal{D}$, $s' \twoheadrightarrow t'$ and $u \subseteq t'$.* □

DEFINITION 2.11. Let t be a marked term.

(1) The set $\{F \in \mathcal{D} \mid F^* \text{ occurs in } t\}$ is denoted by $\mathcal{D}^*(t)$.

(2) A subset $\mathcal{D}'$ of $\mathcal{D}$ is *unreachable* from t if $\mathcal{D}' \cap \mathcal{D}^*(t') = \varnothing$ whenever $t \twoheadrightarrow_m t'$.

DEFINITION 2.12. Let $\mathcal{D}'$ be a subset of $\mathcal{D}$.

(1) A set of pairs $\phi = \{\langle s_1, x_1\rangle, \dots, \langle s_n, x_n\rangle\}$ is a *$\mathcal{D}'$-replacement* if $x_1, \dots, x_n$ are mutually distinct variables and $s_1, \dots, s_n$ are mutually distinct unmarked terms such that $root(s_i) \in \mathcal{D}'$ for $i = 1, \dots, n$. Let $t \equiv C[t_1, \dots, t_m]$ such that all maximal subterms of t

with root symbol in $\mathcal{D}'$ are displayed. We say that ϕ is *applicable* to t if $x_1, \ldots, x_n$ do not occur in t and $\{t_1, \ldots, t_m\} \subseteq \{s_1, \ldots, s_n\}$. In this case we may write $t \equiv C[s_{i_1}, \ldots, s_{i_m}]$ with $1 \le i_1, \ldots, i_m \le n$ and we define $\phi(t) \equiv C[x_{i_1}, \ldots, x_{i_m}]$.

(2) Let $\phi = \{\langle s_1, x_1\rangle, \ldots, \langle s_n, x_n\rangle\}$ be a $\mathcal{D}'$-replacement. Suppose $t \equiv C[x_{i_1}, \ldots, x_{i_m}]$ such that all occurrences of the variables $x_1, \ldots, x_n$ in t are displayed. The term $C[s_{i_1}, \ldots, s_{i_m}]$ is denoted by $\phi^{-1}(t)$.

PROPOSITION 2.13. *Let $\mathcal{D}'$ be a subset of $\mathcal{D}$. For every term t there exists a $\mathcal{D}'$-replacement ϕ which is applicable to t.* □

PROPOSITION 2.14. *Let ϕ be a D'-replacement for some $\mathcal{D}' \subseteq \mathcal{D}$.*

(1) *If $s \twoheadrightarrow_m t$ then $\phi^{-1}(s) \twoheadrightarrow_m \phi^{-1}(t)$.*

(2) *If ϕ is applicable to t then $\phi^{-1}(\phi(t)) \equiv t$.*

(3) *If ϕ is applicable to a capped term t then $e(\phi(t)) \equiv \phi(e(t))$.*

□

LEMMA 2.15. *Let s be a capped term. Suppose $\mathcal{D}' \subseteq \mathcal{D}$ is unreachable from s and ϕ is a $\mathcal{D}'$-replacement applicable to s.*

(1) *If $s \rightarrow_m^o t$ then ϕ is applicable to t and $\phi(s) \rightarrow_m^o \phi(t)$.*

(2) *If s has an infinite $\rightarrow_m^o$-reduction then $\phi(s)$ has an infinite $\rightarrow_m^o$-reduction.*

□

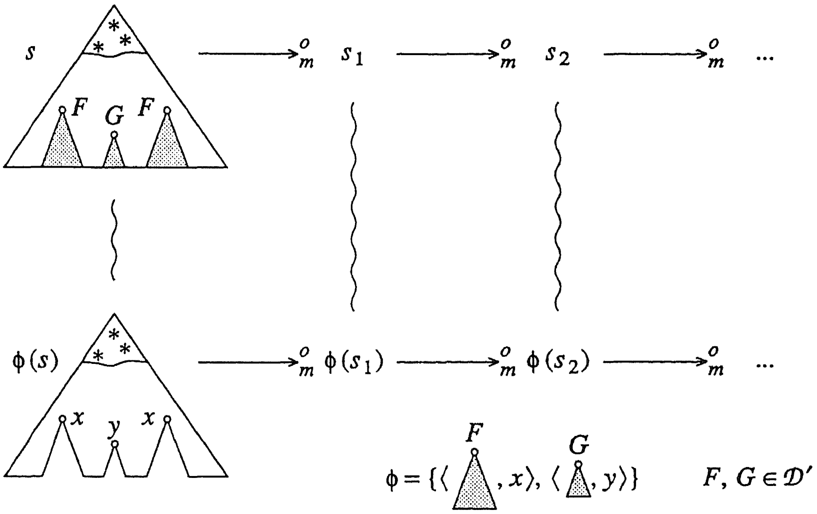

FIGURE 5.

3. Combinations of Constructor Systems

In this section we show that both completeness and semi-completeness exhibit the important compositional behaviour expressed in the next definition.

DEFINITION 3.1.

(1) Let $(\mathcal{D}, C, \mathcal{R})$ be a CS and suppose $\mathcal{D}' \subseteq \mathcal{D}$. The set $\{l \rightarrow r \in \mathcal{R} \mid root(l) \in \mathcal{D}'\}$ is

denoted by $\mathcal{R} \mid \mathcal{D}'$.

(2) Two CS's $(\mathcal{D}_1, C_1, \mathcal{R}_1)$ and $(\mathcal{D}_2, C_2, \mathcal{R}_2)$ are *composable* if $\mathcal{D}_1 \cap C_2 = \mathcal{D}_2 \cap C_1 = \varnothing$ and $\mathcal{R}_1 \mid \mathcal{D}_2 = \mathcal{R}_2 \mid \mathcal{D}_1$. The second requirement is equivalent to the condition that both CS's contain all rewrite rules which 'define' a defined symbol whenever that symbol is shared. The union of pairwise composable CS's $\mathcal{CS}_1, \ldots, \mathcal{CS}_n$ is denoted by $\mathcal{CS}_1 + \ldots + \mathcal{CS}_n$ and we say that $\mathcal{CS}_1, \ldots, \mathcal{CS}_n$ is a *decomposition* of $\mathcal{CS}_1 + \ldots + \mathcal{CS}_n$.

(3) A property $\mathcal{P}$ of CS's is *decomposable* if for all pairwise composable CS's $\mathcal{CS}_1, \ldots, \mathcal{CS}_n$ with the property $\mathcal{P}$ we have that $\mathcal{CS}_1 + \ldots + \mathcal{CS}_n$ has the property $\mathcal{P}$.

The counterexample of Toyama against the modularity of strong normalization shows that strong normalization is not a decomposable property of CS's. The following example of Huet [5] shows that also confluence is not decomposable.

EXAMPLE 3.2. Consider the CS $(\mathcal{D}, C, \mathcal{R})$ with $\mathcal{D} = \{F, C\}$, $C = \{S, A, B\}$ and

$$\mathcal{R} = \begin{cases} F(x, x) & \rightarrow A \\ F(x, S(x)) & \rightarrow B \\ C & \rightarrow S(C). \end{cases}$$

Let $\mathcal{D}_1 = \{F\}$, $C_1 = C$, $\mathcal{D}_2 = \{C\}$ and $C_2 = \{S\}$. The confluent CS's $(\mathcal{D}_1, C_1, \mathcal{R}_1)$, $(\mathcal{D}_2, C_2, \mathcal{R}_2)$ constitute a decomposition of $(\mathcal{D}, C, \mathcal{R})$, but $(\mathcal{D}, C, \mathcal{R})$ is not confluent since the term $F(C, C)$ can be reduced to the different normal forms A and B.

PROPOSITION 3.3. *let $\mathcal{P}$ be a property of CS's. The following statements are equivalent:*

(1) *$\mathcal{P}$ is decomposable;*

(2) *for all composable CS's $\mathcal{CS}_1$ and $\mathcal{CS}_2$ with the property $\mathcal{P}$ we have that $\mathcal{CS}_1 + \mathcal{CS}_2$ has the property $\mathcal{P}$.*

PROOF. Straightforward. □

LEMMA 3.4. *Local confluence is decomposable.*

PROOF. Let $(\mathcal{D}_1, C_1, \mathcal{R}_1)$ and $(\mathcal{D}_2, C_2, \mathcal{R}_2)$ be locally confluent and decomposable CS's. We have to show that their union $(\mathcal{D}, C, \mathcal{R})$ is locally confluent. According to the Critical Pair Lemma it sufficient to show that every critical pair of $(\mathcal{D}, C, \mathcal{R})$ is convergent. If $\langle s, t\rangle$ is a critical pair of $(\mathcal{D}, C, \mathcal{R})$ then there exist rewrite rules $l_1 \rightarrow r_1, l_2 \rightarrow r_2 \in \mathcal{R}$ and a substitution σ such that $l_1^\sigma \equiv l_2^\sigma$, $s \equiv r_1^\sigma$ and $t \equiv r_2^\sigma$. Choose $k \in \{1, 2\}$ such that $root(l_1) = root(l_2) \in \mathcal{D}_k$. We have $l_1 \rightarrow r_1, l_2 \rightarrow r_2 \in \mathcal{R}_k$ and because $(\mathcal{D}_k, C_k, \mathcal{R}_k)$ is locally confluent $\langle s, t\rangle$ is $\mathcal{R}_k$-convergent and hence also $\mathcal{R}$-convergent. □

THEOREM 3.5. *Completeness is decomposable.*

PROOF. Let $(\mathcal{D}_1, C_1, \mathcal{R}_1)$ and $(\mathcal{D}_2, C_2, \mathcal{R}_2)$ be complete and composable CS's. From Lemma 3.4 we obtain the local confluence of their union $(\mathcal{D}, C, \mathcal{R})$. According to Newman's Lemma it suffices to show the strong normalization of $(\mathcal{D}, C, \mathcal{R})$. This will be established by induction on the structure of terms $t \in \mathcal{T}(\mathcal{D}, C, \mathcal{V})$. If t is a variable or a constructor constant then t is a normal form. If t is a defined constant then t belongs to some $\mathcal{D}_k$ and because $(\mathcal{D}_k, C_k, \mathcal{R}_k)$ is strongly normalizing t cannot have an infinite reduction. For the induction step, let $t \equiv F(t_1, \ldots, t_n)$ such that $t_1, \ldots, t_n$ are strongly normalizing (and hence complete). If F is a constructor then t clearly is strongly normalizing. So assume that $F \in \mathcal{D}$. If t is not strongly normalizing then there exists an infinite reduction sequence

$$t \equiv s_1 \rightarrow s_2 \rightarrow s_3 \rightarrow \dots . \tag{1}$$

Let $t' \equiv F^*(t_1, \dots, t_n)$. According to Proposition 2.5 we can find terms s'_i with $e(s'_i) \equiv s_i$ such that

$$t' \equiv s'_1 \rightarrow_m s'_2 \rightarrow_m s'_3 \rightarrow_m \dots . \tag{2}$$

Using Lemma 2.10 and the assumption that $t_1, \dots, t_n$ are strongly normalizing, it is not difficult to show that sequence (2) contains infinitely many $\rightarrow_m^o$-steps. According to Lemma 2.9 we can transform sequence (2) into the marked reduction sequence

$$\psi(t') \equiv \psi(s'_1) \twoheadrightarrow_m^o \psi(s'_2) \twoheadrightarrow_m^o \psi(s'_3) \twoheadrightarrow_m^o \cdots \tag{3}$$

which contains infinitely many steps. Choose $k \in \{1, 2\}$ such that $F \in \mathcal{D}_k$ and let $\mathcal{D}' = \mathcal{D} - \mathcal{D}_k$. It is easy to show that $\mathcal{D}'$ is unreachable from $\psi(t')$. From Proposition 2.13 we obtain a $\mathcal{D}'$-replacement ϕ which is applicable to $\psi(t')$. By Lemma 2.15(2) the term $\phi(\psi(t'))$ has an infinite $\rightarrow_m^o$-reduction sequence

$$\phi(\psi(t')) \equiv \phi(\psi(s'_1)) \twoheadrightarrow_m^o \phi(\psi(s'_2)) \twoheadrightarrow_m^o \phi(\psi(s'_3)) \twoheadrightarrow_m^o \dots . \tag{4}$$

If we erase all markers in this sequence we obtain an infinite reduction sequence starting from the term $e(\phi(\psi(t')))$. This contradicts the strong normalization of the CS $(\mathcal{D}_k, C_k, \mathcal{R}_k)$. □

COROLLARY 3.6. *Completeness is a modular property of* CS'*s*. □

COROLLARY 3.7. *The union of complete* CS'*s which do not share defined symbols is complete.* □

We now consider a more challenging situation in which Theorem 3.5 can be applied.

EXAMPLE 3.8. Consider the CS $(\mathcal{D}, C, \mathcal{R})$ with $\mathcal{D} = \{+, \times, fib, <, \wedge\}$, $C = \{0, S, true, false\}$ and rewrite rules

r_1	$0+x$	$\rightarrow$	x	r_8	$x<0$	$\rightarrow$	$false$
r_2	$S(x)+y$	$\rightarrow$	$S(x+y)$	r_9	$0<S(x)$	$\rightarrow$	$true$
r_3	$0\times x$	$\rightarrow$	0	r_{10}	$S(x)<S(y)$	$\rightarrow$	$x<y$
r_4	$S(x)\times y$	$\rightarrow$	$x\times y+y$	r_{11}	$true \wedge false$	$\rightarrow$	$false$
r_5	$fib(0)$	$\rightarrow$	$S(0)$	r_{12}	$false \wedge true$	$\rightarrow$	$false$
r_6	$fib(S(0))$	$\rightarrow$	$S(0)$	r_{13}	$x \wedge x$	$\rightarrow$	x
r_7	$fib(S(S(x)))$	$\rightarrow$	$fib(S(x))+fib(x)$				

Consider the decomposition $(\mathcal{D}_i, C_i, \mathcal{R}_i)_{i=1}^4$ defined as follows:

i	$\mathcal{D}_i$	C_i	$\mathcal{R}_i$
1	$+\times$	$0\ S$	$r_1\ r_2\ r_3\ r_4$
2	$+fib$	$0\ S$	$r_1\ r_2\ r_5\ r_6\ r_7$
3	$<$	$0\ S\ true\ false$	$r_8\ r_9\ r_{10}$
4	$\wedge$	$true\ false$	$r_{11}\ r_{12}\ r_{13}$

Routine arguments show that every $(\mathcal{D}_i, C_i, \mathcal{R}_i)$ is complete. Theorem 3.5 yields the completeness of $(\mathcal{D}, C, \mathcal{R})$.

The proof of the decomposability of semi-completeness is comparable to the proof of Theorem 3.5. First we show the decomposability of weak normalization.

LEMMA 3.9. *Weak normalization is decomposable.*

PROOF. Suppose $(\mathcal{D}_1, C_1, \mathcal{R}_1)$ and $(\mathcal{D}_2, C_2, \mathcal{R}_2)$ are weakly normalizing and composable CS's and let $(\mathcal{D}, C, \mathcal{R}) = (\mathcal{D}_1, C_1, \mathcal{R}_1) + (\mathcal{D}_2, C_2, \mathcal{R}_2)$. We will show by induction on the structure of t that every term $t \in \mathcal{T}(\mathcal{D}, C, \mathcal{V})$ has a normal form. The case $t \in \mathcal{D} \cup C \cup \mathcal{V}$ is easy. Suppose $t \equiv F(t_1, \ldots, t_n)$ and $t_1, \ldots, t_n$ are weakly normalizing. Let s_i be a normal form of t_i for $i = 1, \ldots, n$ and define $t' \equiv F(s_1, \ldots, s_n)$. If $F \in C$ then t' is a normal form of t. If $F \in \mathcal{D}$ then there exists a $k \in \{1, 2\}$ such that $F \in \mathcal{D}_k$. Let $\mathcal{D}' = \mathcal{D} - \mathcal{D}_k$. From Proposition 2.13 we obtain a $\mathcal{D}'$-replacement ϕ which is applicable to t'. Since $(\mathcal{D}_k, C_k, \mathcal{R}_k)$ is weakly normalizing, the term $\phi(t')$ has a normal form, say t''. Using Proposition 2.14 we obtain $t \twoheadrightarrow t' \equiv \phi^{-1}(\phi(t')) \twoheadrightarrow \phi^{-1}(t'')$. It is easy to show that $\phi^{-1}(t'')$ is a normal form. □

THEOREM 3.10. *Semi-completeness is decomposable.*

PROOF. Let $(\mathcal{D}_1, C_1, \mathcal{R}_1)$ and $(\mathcal{D}_2, C_2, \mathcal{R}_2)$ be semi-complete and composable CS's. From Lemma 3.9 we obtain the weak normalization of their union $(\mathcal{D}, C, \mathcal{R})$. Hence it is sufficient to show that every term $t \in \mathcal{T}(\mathcal{D}, C, \mathcal{V})$ has at most one normal form. We use induction on the structure of t. The case $t \in \mathcal{D} \cup C \cup \mathcal{V}$ is easy. Suppose $t \equiv F(t_1, \ldots, t_n)$ such that every t_i is semi-complete. If $F \in C$ then $F(t_1{\downarrow}, \ldots, t_n{\downarrow})$ is the unique normal form of t. Suppose $F \in \mathcal{D}$ and let $t' \equiv F^*(t_1, \ldots, t_n)$. Define $\mathcal{D}_k$, $\mathcal{D}'$ and ϕ as in the proof of Theorem 3.5. First we show that if t has a normal form n then $\phi(e(\psi(t'))) \twoheadrightarrow \phi(n)$. With help of Proposition 2.5 and Lemma 2.9 we obtain a normal form n' such that $\psi(t') \twoheadrightarrow_m^o n'$ and $e(n') \equiv n$. Repeated application of Lemma 2.15(1) yields $\phi(\psi(t')) \twoheadrightarrow_m^o \phi(n')$. Erasing all markers in this sequences gives us $e(\phi(\psi(t'))) \twoheadrightarrow e(\phi(n'))$ and from Proposition 2.14(3) we obtain $e(\phi(\psi(t'))) \equiv \phi(e(\psi(t')))$ and $e(\phi(n')) \equiv \phi(e(n')) \equiv \phi(n)$. Now suppose that t has normal forms n_1 and n_2. From the above discussion we learn that $\phi(n_1) \twoheadleftarrow \phi(e(\psi(t'))) \twoheadrightarrow \phi(n_2)$. Notice that $\phi(n_1)$ and $\phi(n_2)$ are normal forms. We obtain $\phi(n_1) \equiv \phi(n_2)$ from the semi-completeness of $(\mathcal{D}_k, C_k, \mathcal{R}_k)$. Hence $n_1 \equiv \phi^{-1}(\phi(n_1)) \equiv \phi^{-1}(\phi(n_2)) \equiv n_2$ by Proposition 2.14(2). □

COROLLARY 3.11. *The union of semi-complete* CS'*s which do not share defined symbols is semi-complete.* □

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