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Generating Functions and Lie Groups

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0. Introduction

In their listings [McK], [Bre] of decompositions of characters of semisimple Lie subgroups obtained by restriction from overgroups or by tensor products of irreducible representations, McKay et al. often use generating functions that turn out to be rational. In this paper, we prove that they are always rational and provide an example of how to derive an explicit expression for this rational function in the case $G_2 \downarrow A_2$.

Let G be a semisimple complex connected Lie group of Lie rank n with maximal torus T. The group of all rational characters of T, called *weights*, is denoted by $\Lambda(T)$. As groups, we have $\Lambda(T) \cong \mathbf{Z}^n$. The set of all *roots* (that is, all nonzero weights occurring in the restriction to T of the adjoint representation of G), is denoted by Φ_G. A set of fundamental roots $\alpha_1, \ldots, \alpha_n$ is chosen with respect to a fixed Borel subgroup B of G containing T. We write $W_G = N_G(T)/T$ and $(\cdot, \cdot)$ for the canonical W_G-invariant inner product on $\Lambda(T)$. We also fix the fundamental weights $\omega_1, \ldots, \omega_n$ as the basis dual to the fundamental roots in the following sense: $2(\omega_i, \alpha_j)/(\alpha_j, \alpha_j) = \delta_{i,j}$, $1 \leq i, j \leq n$. The set $\Lambda^+(G,T)$ of dominant weights is the $\mathbf{N}$-span of these fundamental weights. As semigroups we have $\Lambda^+(G,T) \cong \mathbf{N}^n$. The set of positive roots is $\Phi_G^+ = \{\alpha \in \Phi_G | (\alpha, \omega_i) \geq 0 \text{ for } i = 1, \ldots, n\}$ and $\rho_G = \sum_{i=1}^n \omega_i$ is the half sum of all roots in Φ_G^+. A partial ordering $\leq$ on $\Lambda(T)$ is given by $\lambda \leq \mu$ if and only if $\mu - \lambda$ is a non-negative integral linear combination of positive roots. $\Lambda^+(G,T)$ is used to indicate irreducible representations of G and V_λ is the irreducible G-module with highest weight λ. This module can be obtained as $\{f \in \mathbf{C}[G] \mid f(gb) = \lambda(b)f(g)\}$; here λ is viewed as a character of B by $\lambda(b) = \lambda(t)$ for $b = tu$, with $t \in T$ and $u \in U$, where U is the maximal unipotent subgroup of B. There is a straightforward extension from semisimple to reductive groups. If G is a reductive group we also use elements of $\Lambda^+(G,T)$ to indicate the set of all weights that are dominant with respect to the semisimple part of the torus. Thus for example $\Lambda^+(T,T) = \Lambda(T)$.

1. Rational generating functions

Assume that G is a semisimple Lie group. Let $\mu_1, \ldots, \mu_p \in \Lambda^+ = \Lambda^+(G,T)$ and set $M = \mathbf{N}^p$ with standard basis $e_1, \ldots, e_p$. For the G-module $V \cong V_{\mu_1} \oplus \cdots \oplus V_{\mu_p}$, the algebra $\mathbf{C}[V^*]$ of polynomial functions on the dual V^* of V can be M-graded in such a way that $\mathbf{C}[V^*]_{e_i} = V_{\mu_i}$ for each $i \in \{1, \ldots, p\}$. Given $m = (m_1, \ldots, m_p) \in M$, the homogeneous part $\mathbf{C}[V^*]_m$ is a homomorphic image of $V_{\mu_1}^{m_1} \otimes \cdots \otimes V_{\mu_p}^{m_p}$ in which $V_{m_1\mu_1 + \cdots + m_p\mu_p}$ occurs with multiplicity 1 and has a unique G-stable complement J_m. Clearly $\mathbf{C}[V^*]_m \cdot J_{m'} \subset J_{m+m'}$, so $J = \oplus_{m \in M} J_m$ is an M-graded G-stable ideal in $\mathbf{C}[V^*]$. Since the algebra $\mathbf{C}[V^*]$ is Noetherian, J must be finitely generated. In fact

.1 Theorem ([Bri, 4.1]) *The ideal J is generated by the $J_{e_i+e_j}$ for all $1 \leq$ $\leq j \leq n$.*

The quotient algebra $A = \mathbf{C}[V^*]/J$ can be provided with the induced M-grading and is preserved by the induced G-action. By construction, $A_m \cong V_{m_1\mu_1 + \cdots + m_p\mu_p}$. In particular, putting $p = n$, $\mu_i = \omega_i$ for all $i \in \{1, \ldots, n\}$ and $M = \Lambda^+$, the direct sum $A = \bigoplus_{\lambda \in \Lambda^+} V_\lambda$ over all irreducible representations of G is a Λ^+-graded G-algebra, with $A_\lambda \cong V_\lambda$. The Poincaré series of A is the expression

$$P_{dim}(x) = \sum_{\lambda \in \Lambda^+} (\dim V_\lambda) x^\lambda,$$

where x stands for $(x_1, \ldots, x_n)$ and x^λ stands for $x_1^{\lambda_1} \cdots x_n^{\lambda_n}$ if $\lambda = \sum_{i=1}^n \lambda_i \omega_i$. Thus, $x_i = x^{\omega_i}$. As A is finitely generated, $P_{dim}(x)$ is a rational function in $x_1, \ldots, x_n$ (this will later be abbreviated to: rational in x). The rational function can be explicitly given:

1.2 Theorem (Weyl's Dimension Formula, cf. [Hum]) *The dimensions of the highest weight modules of G are given by the formula*

$$P_{dim}(x) = \sum_{\lambda \in \Lambda^+} \prod_{\alpha \in \Phi^+} \frac{(\lambda + \rho_G, \alpha)}{(\rho_G, \alpha)} x^\lambda$$

$$= \prod_{\alpha \in \Phi^+} \frac{\sum_{i=1}^n (\omega_i, \alpha) \frac{\partial}{\partial x_i} \cdot x_i}{(\rho_G, \alpha)} \left(\prod_{i=1}^n \frac{1}{1 - x_i} \right).$$

The first identity gives an explicit formula for the dimension of V_λ and is more convenient for actual computation. The second identity expresses $D_{dim}(x)$ as a rational function in x; it can easily be derived from the first by use of $\frac{\partial}{\partial x_i} \cdot x_i(x^m) = (m_i + 1)x^m$ and $(\lambda + \rho_G, \alpha) = \sum_{i=1}^n (\lambda_i + 1)(\omega_i, \alpha)$.

1.3 Example Let G be a Lie group of type A_1. Then $\rho_G = \omega_1$ and $\Phi^+ = \{\alpha_1\} = \{2\,\omega_1\}$, so

$$P_{dim}(x) = \sum_{m \geq 0} \frac{((m+1)\omega_1, 2\omega_1)}{(\omega_1, 2\omega_1)} x^m = \sum_{m \geq 0} (m+1)x^m = \frac{\partial}{\partial x} \cdot x \left(\frac{1}{(1-x)} \right)$$
$$= \frac{1}{(1-x)^2}.$$

We shall extend these observations to Weyl's Character Formula. Let H be a reductive closed Lie subgroup of the semisimple Lie group G. The fact that H is reductive ensures that any finite-dimensional rational representation of H decomposes into a direct sum of irreducibles. *Branching* is the decomposition of a representation of H that is obtained by restriction from a highest weight module of G. Let S be a maximal torus of H and m the Lie rank of H. Then, there is a maximal torus of G containing S, which we may take (up to conjugacy) to be T. Thus for dominant weights $\lambda = (\lambda_1, \ldots, \lambda_n)$ of G and $\mu = (\mu_1, \ldots, \mu_m)$ of H, we are after the multiplicity $(V_\mu, V_\lambda \downarrow_H)$ of the highest weight representation V_μ of H in the highest weight representation V_λ of G restricted to H. In terms of formal power series in the indeterminates $x_1, \ldots, x_n, y_1, \ldots, y_m$, we want to find an explicit description of the *branching series*

$$P_{G\downarrow H}(x; y) = \sum_{\lambda, \mu} (V_\mu, V_\lambda \downarrow_H) x^\mu y^\lambda$$

of H in G.

The coefficients of the branching series have a second interpretation. Given a highest weight module V_μ of the subgroup H there is a unique (possibly infinite dimensional) induced G-module $V_\mu \uparrow^G$. The multiplicity of the highest weight module V_λ in the module $V_\mu \uparrow^G$ is denoted by $(V_\mu \uparrow^G, V_\lambda)$. Frobenius Reciprocity gives that $(V_\mu \uparrow^G, V_\lambda) = (V_\mu, V_\lambda \downarrow_H)$, which is a second interpretation of the power series. However, considered as a power series in x, the coefficients of the series are power series in y that need not be polynomials.

1.4 Lemma *$P_{G\downarrow H}(x;y)$ is a rational function in x and y.*

Proof Denote by $B = \bigoplus_{\mu\in\Lambda^+(H,S)} V_\mu^*$ the $\Lambda^+(H,S)$-graded H-algebra of all dual irreducible H-representations, and by $A = \bigoplus_{\lambda\in\Lambda^+(G,T)} V_\lambda$ the $\Lambda^+(G,T)$-graded G-algebra of all irreducible G-representations. Then the tensor product $A\otimes B$ is a $\Lambda^+(G,T)\oplus\Lambda^+(H,S)$-graded algebra with a $G\times H$-action preserving the grading. Considering H as a diagonally embedded subgroup of $G\times H$, we get for $\lambda\in\Lambda^+(G,T)$ and $\mu\in\Lambda^+(H,S)$

$$(A\otimes B)^H_{(\lambda,\mu)} \cong (V_\lambda\downarrow_H \otimes V_\mu^*)^H \cong \mathrm{Hom}_H(V_\lambda\downarrow_H, V_\mu).$$

Since the dimension of the latter complex vector space is the multiplicity $(V_\mu, V_\lambda\downarrow_H)$, the Poincaré series of $(A\otimes B)^H$ is precisely $P_{G\downarrow H}(x;y)$. On the other hand, it is a rational function too, as $(A\otimes B)^H$ is finitely generated, for H is reductive and acts grade preserving on the finitely generated graded ring $A\otimes B$. (cf. [Spri, Proposition 2.4.14]). □

Weyl's Dimension Formula handles the special case $H=1$. In the case of the reductive subgroup $H=T$, an explicit rational form is known. For any $\lambda\in\Lambda^+(G,T)$ set $\Theta_\lambda(x)=\sum_{w\in W_G}\det w\; x^{w\lambda}$.

1.5 Theorem (Weyl's Character Formula, cf. [Hum]) *The branching series of the maximal torus T in the semisimple Lie group G is*

$$\begin{aligned} P_{G\downarrow T}(x;y) &= \sum_{\lambda\in\Lambda^+(G,T)} \frac{\Theta_{\lambda+\rho_G}(x)}{\Theta_{\rho_G}(x)} y^\lambda \\ &= \frac{1}{x^{\rho_G}\prod_{\alpha\in\Phi^+}(1-x^{-\alpha})} \sum_{w\in W_G} sgn(w) x^{w\rho_G} \Big(\sum_{\lambda\in\Lambda^+(G,T)} x^{w\lambda}y^\lambda \Big) . \end{aligned}$$

This is indeed a rational function since $\sum_{\lambda\in\Lambda^+(T)} x^{w\lambda}y^\lambda$ is rational for each $w\in W_G$. If G is of type E_8, the expression consists of $|W_G| = 696729600$ summands, which is unrealistically high for computations.

1.6 Example Let G be a simply connected Lie group of type A_1. Then

$$\begin{aligned} P_{G\downarrow T}(x;y) &= \frac{1}{x(1-x^{-2})}\left(\frac{x}{1-xy} - \frac{x^{-1}}{1-x^{-1}y}\right) \\ &= \frac{1}{(1-xy)(1-x^{-1}y)} \\ &= \sum_m (x^m + x^{m-2} + \cdots + x^{2-m} + x^{-m}) y^m , \end{aligned}$$

which is a well-known fact.

2. Tensor decomposition and plethysms

Computing the decomposition of the tensor product $V_\lambda\otimes V_\mu$ of two irreducible G-modules can be viewed as branching the irreducible $G\times G$-module

$V_\lambda \otimes V_\mu$ to the diagonal subgroup isomorphic to G. Denote by $(V_\lambda, V_\mu \otimes V_\nu)$ the multiplicity of V_λ in $V_\mu \otimes V_\nu$ and identify $\Lambda^+(G \times G, T \times T)$ with $\Lambda^+(G,T) \times \Lambda^+(G,T)$. Then, as we have seen in the previous section, the power series in $x_1, \ldots, x_n, y_1, \ldots, y_n, z_1, \ldots, z_n$

$$P_\otimes(x; y; z) = \sum_{\lambda,\mu,\nu \in \Lambda^+(G,T)} (V_\lambda, V_\mu \otimes V_\nu) x^\lambda y^\mu z^\nu$$

is rational in x, y, z. Again, let $A = \bigoplus_{\lambda \in \Lambda^+(G,T)} V_\lambda$ and $A^* = \bigoplus_{\lambda \in \Lambda^+(G,T)} V_\lambda^*$. By the proof of the lemma in the previous section, the power series can be considered as the Poincaré series of $(A \otimes A \otimes A^*)^G$, where G must be considered as the diagonal subgroup of $G \times G \times G$. If we use an automorphism of G to identify A^* with A, the entries of the multidegrees in $\Lambda^+(G,T)$ must be permuted in an appropriate way. Therefore the series can be considered as the Poincaré series of $(A \otimes A \otimes A)^G$ and is invariant under permutation of the names x, y, z.

2.1 Example Take G a Lie group of type A_1. Then $A = \bigoplus_{n \geq 0} V_n$ is a polynomial algebra in two variables. $(A \otimes A)_{(1,1)} \cong V_2 \oplus V_0$, and if we take $p \in (A \otimes A)_{(1,1)}$ a generator for V_0, then it follows by Theorem 2.1 that $A \otimes A/(p) \cong \oplus_{n,m \geq 0} V_{n+m}$. On the other hand $(A \otimes A)^G \cong \mathbf{C}[p]$, because $(V_n \otimes V_m)^G$ has dimension 1 if $n = m$ and dimension 0 otherwise. Therefore, $A \otimes A$ is a free $(A \otimes A)^G$–module, or equivalently $A \otimes A \cong (A \otimes A/(p)) \otimes (A \otimes A)^G$. This yields the generating function for the tensor product of G:

$$\frac{1}{(1-xy)(1-xz)(1-yz)}.$$

This formula can also be used to compute the power series of the l-fold tensor products. If $P_l(x_1; \ldots; x_l; y) = \sum c_{k,m_1,\ldots,m_l} y^k x_1^{m_1} \cdots x_l^{m_l}$, where the sum is taken over all $k, m_1, \ldots, m_l \geq 0$ and $c_{k,m_1,\ldots,m_l}$ denotes the multiplicity of V_k in $V_{m_1} \otimes \cdots \otimes V_{m_l}$, then $P_0 = 1$ and, for $l > 0$,

$$P_{l+1} = \frac{x_{l+1} P_l(x_1; \ldots; x_l; x_{l+1}) - y P_l(x_1; \ldots; x_l; y)}{(1 - y x_{l+1})(x_{l+1} - y)}.$$

The factor $x_{l+1} - y$ in the denominator always divides the numerator.

As we will see, also symmetric and skew-symmetric powers, and more general plethysms, lead to rational functions. Let $d \in \mathbf{N}$. Identify $\Lambda^+(G^d, T^d)$ with $(\Lambda^+(G,T))^d$, and set $\mu_i = \omega_i^d = (\omega_i, \ldots, \omega_i)$, d times ω_i, for $i = 1, \ldots, n$. Then $B = \bigoplus_{m \in M} V_{m_1\mu_1 + \cdots + m_n\mu_n}$, where $M = \mathbf{N}^n$, is a M-graded algebra, which is preserved by the G^d action. Restricted to the subgroup G, embedded diagonally in G^d, we get $B = \bigoplus_{m \in M} V_{m_1\omega_1 + \cdots + m_n\omega_n}^{\otimes d}$. The action of Sym_d on B, given by permuting the factors of the d-fold tensor product in each degree, preserves the degree and commutes with the G action. Suppose τ is any irreducible character of Sym_d. Denote by V_μ^τ the τ-homogeneous part of the Sym_d-module $V_\mu^{\otimes d}$, and by (V_λ, V_μ^τ) the multiplicity of V_λ in V_μ^τ. The *Plethysm of V_μ with respect to τ* is the decomposition of V_μ^τ as a G-module. The symmetric and skew-symmetric d-tensors are special cases corresponding to the trivial character $\tau = d+$ and the sign character $\tau = d-$, respectively.

2.2 Theorem *Let τ be a character of Sym_d. The power series*

$$\sum_{\lambda,\mu\in\Lambda^+(G,T)} (V_\lambda, V_\mu^\tau) x^\lambda y^\mu$$

is a rational function in x and y.

Proof Note that for given τ the power series is the Poincaré series of $(B^\tau \otimes A^*)^G$. The algebra $C = (B^{Sym_d} \otimes A^*)^G$ is finitely generated and has rational Poincaré series. This proves the case where τ is the trivial character. $(B \otimes A^*)^G$ is finitely generated and integral over C, thus is a finitely generated C-module. We have the C-module decomposition $(B \otimes A^*)^G = \bigoplus_\tau (B^\tau \otimes A^*)^G$, where the sum is taken over all irreducible representations τ of Sym_d. Thus $(B^\tau \otimes A^*)^G$ is a finitely generated C-module for each τ, and therefore its Poincaré series is rational. □

2.3 Example Take $G = A_1$, $d = 2$ and set $S = Sym_2 \cong \{\pm 1\}$. We have $B = \bigoplus_{k\geq 0} V_k \otimes V_k$ and $A^* = \bigoplus_{k\geq 0} V_k^*$. Let $C = \mathbf{C}[(V_1 \otimes V_1^*]$ an $\mathbf{N}$-graded polynomial algebra provided with the natural $S \times G$ action. There is the natural surjective homomorphism $C \to B$, which preserves the degree and commutes with the $S \times G$ action. The kernel, I say, is graded and $S \times G$ stable. Let p be a generator of the skew-symmetric part V_0 of $C_1 \cong V_2 \oplus V_0$. We have $C = C^S \oplus C^S p$. By Brion's theorem I is generated by elements of degree 2 and from that it follows $I \cap (C^S p) = (I \cap C^S)p = I^S p$. But then $B \cong C/I \cong C^S/I^S \oplus C^S/I^S p \cong B^S \oplus B^S p$. Thus if P_{2+} is the Poincaré series of $(B^S \otimes A^*)^G$ corresponding to the series for the symmetric 2-tensors, then $P_{2-} = yP_{2+}$ is the series corresponding to the skew-symmetric 2-tensors. The Poincaré series $P_{2\otimes} = P_{2+} + P_{2-}$ of $(B \otimes A^*)^G$ can easily be derived from the tensor product series of G above:

$$P_{2\otimes} = \frac{1}{(1-x^2y)(1-y)}.$$

The series of the symmetric 2-tensors becomes

$$P_{2+} = \frac{1}{(1-x^2y)(1-y^2)}$$

and for the skew-symmetric 2-tensors

$$P_{2-} = \frac{y}{(1-x^2y)(1-y^2)}.$$

Write $P_{3\otimes} = \sum_{k,l\geq 0} c_{k,l} x^k y^l$, where $c_{k,l}$ is the multiplicity of V_k in $V_l^{3\otimes}$, then straightforward computations using the above formulas give

$$P_{3\otimes} = \frac{x^2y^2 + xy + 1}{(1-x^3y)(1-xy)(1-y^2)}.$$

3. Branching

We now return to the general situation. G is a reductive group with maximal torus T and H a reductive subgroup with maximal torus S, such that $S \subset T$. The most straightforward way to compute a coefficient $(V_\mu, V_\lambda \downarrow_H)$ of the branching series $P_{G\downarrow H}(x;y)$ is by determining the set of all weights of the G-module V_λ, next computing their restrictions to S and then decomposing this set with the inverse of Freudenthal's formula as an H-module. In this section we give an explicit formula for the coefficients of the branching series. Let $r : \Lambda(G) \to \Lambda(H)$ denote the linear map restricting the weights of T to weights on S. Also, by choosing appropriate Borel subgroups, we may assume that for $\alpha \in \Phi_G^+$ we have $r(\alpha) \notin \Phi_H^-$. Let $\Phi = \{\alpha \in \Phi_G \mid r(\alpha) = 0\}$, $\Phi^+ = \Phi \cap \Phi_G^+$ and W_Φ the subgroup of W_G generated by Φ. Each coset in W_G relative to W_Φ has a unique representative in W_G of minimal length, the set of these representatives is denoted by W. Put $A = r(\Phi_G^+)\backslash\{0\}$ and provide each element $\alpha \in A$ with a finite multiplicity $m_\alpha = |\{\beta \in \Phi_G^+ \mid r(\beta) = \alpha\}|$ if $\alpha \notin \Phi_H^+$ and $m_\alpha = |\{\beta \in \Phi_G^+ \mid r(\beta) = \alpha\}| - 1$ if $\alpha \in \Phi_H^+$. Let L be the lattice of non-negative integral linear combinations of elements in A. Kostant's partition function p_A on L is defined by

$$\frac{1}{\prod_{\alpha\in A}(1-z^\alpha)^{m_\alpha}} = \sum_{\beta\in L} p_A(\beta) z^\beta$$

and is extended to the real span of L by putting $p_A(\beta) = 0$ if $\beta \notin L$. Finally put

$$D(\lambda) = \prod_{\alpha\in\Phi^+} \frac{(\lambda,\alpha)}{(\rho_\Phi,\alpha)}.$$

3.1 Theorem ([Hec])

$$(V_\mu, V_\lambda \downarrow_H) = \sum_{w\in W} det(w) D(w(\lambda+\rho_G)) p_A(r(w(\lambda+\rho_G)) - (\mu + r(\rho_G))).$$

The theorem can be proved using Weyl's dimension and character formulas above. Conversely Weyl's formulas are special cases of the theorem. The theorem suggests how the branching series can be written as a sum over W of power series, which represent rational functions. Below we indicate by means of a rank 2 example how the actual rational functions can be obtained. Again, a higher rank case such as E_8 seems intractible. Here is a simpler one.

3.2 Example Let G be a Lie group of type G_2, with root system Φ_G and fundamental roots β_1, β_2, where β_1 is long and β_2 is short. There is a subgroup H of type A_2, whose root system Φ_H is the root subsystem of long roots of Φ_G, and with fundamental roots β_1 and $\beta_1 + 3\beta_2$. We want to give the power series $P_{G\downarrow H} = \sum_{\lambda,\mu} (V_\mu, V_\lambda \downarrow_H) x^{\lambda_1} y^{\lambda_2} z^{\mu_1} u^{\mu_2}$, where the sum is taken over all $\lambda \in \Lambda^+(G,T)$ and $\mu \in \Lambda^+(H,T)$. The restriction map with respect to the bases of fundamental weights is given by $r((1,0)) = (1,1)$ and $r((0,1)) = (0,1)$. Thus Φ is empty, so $D \equiv 1$, $W_\Phi = \{1\}$ and $W = W_G$. The multiplicities of the elements in $A = r(\Phi_G^+)$ are one for the images of the short roots and zero for the long roots. The short positive roots are $\gamma_1 = \beta_1 + \beta_2$, $\gamma_2 = \beta_2$ and $\gamma_1 + \gamma_2$. Kostant's partition function p_A at the lattice points $m\gamma_1 + n\gamma_2$, $m, n \geq 0$, is given by

$$\frac{1}{(1-a)(1-b)(1-ab)} = \sum_{m,n\geq 0} p_A(m\gamma_1 + n\gamma_2)a^m b^n$$

and is zero outside these points. We need the following more general formal power series expansion defining the function q_A on the same lattice, whose values are polynomials in z and u.

$$\frac{1}{(1-a)(1-b)(1-ab)(1-az)(1-abu)} = \sum_{m,n\geq 0} q_A(m\gamma_1 + n\gamma_2)a^m b^n \quad (*)$$

The values of q_A are taken to be zero outside the lattice. Thus,

$$q_A(v) = \sum_{\mu} p_A(v - (\mu_1\gamma_1 + \mu_2(\gamma_1 + \gamma_2)))z^{\mu_1}u^{\mu_2}.$$

Since the fundamental weights of the A_2 subsystem of long roots are γ_1 and $\gamma_1 + \gamma_2$, we have $\mu = \mu_1\gamma_1 + \mu_2(\gamma_1 + \gamma_2)$. Consequently, substitution of the formula of Theorem 3.1 in the formal power expansion $P_{G\downarrow H}$, yields

$$P_{G\downarrow H} = \sum_{w\in W}\sum_{\lambda} q_A(r(w(\lambda + \rho_G)) - r(\rho_G))x^{\lambda_1}y^{\lambda_2}.$$

Now $P_{G\downarrow H}$ is computed by finding rational functions for the power series corresponding to each $w \in W$ separately. Let s_1 and s_2 denote the reflections in W_G corresponding to the fundamental roots β_1 and β_2 of G, respectively. In light of the support of q_A, a non-zero series occurs only when w is one of the four elements 1, s_1, s_2, s_1s_2.

We indicate how to compute the rational function corresponding to $w = 1$. As $r(\lambda) = \lambda_1(2\gamma_1 + \gamma_2) + \lambda_2(\gamma_1 + \gamma_2)$, we have to compute the rational function expression of

$$\sum_{\lambda} q_A(\lambda_1(2\gamma_1 + \gamma_2) + \lambda_2(\gamma_1 + \gamma_2))x^{\lambda_1}y^{\lambda_2}. \quad (**)$$

But, writing $x = a^2b$ and $y = a\,b$, we obtain the subseries of (*) in which precisely those monomials $a^m b^n$ occur that can be written in the form $(a^2b)^i(ab)^j$ for certain $i, j \geq 0$. The following general identity is useful in finding the required rational function

$$\sum_{\substack{n_1,n_2,\ldots,n_k\geq 0\\ m_1,m_2,\ldots,m_l\geq 0\\ n_1+n_2+\cdots+n_k\geq m_1+m_2+\cdots+m_l}} a_1^{n_1}a_2^{n_2}\cdots a_k^{n_k}b_1^{m_1}b_2^{m_2}\cdots b_l^{m_l} = \quad (***)$$

$$\sum_{i=1}^{k}\sum_{j=1}^{l}\frac{a_i^{k-1}b_j^{l-1}}{\prod_{p\neq i}(a_i - a_p)\prod_{q\neq j}(b_j - b_q)}\frac{1}{(1-a_ib_j)(1-a_i)}.$$

We first compute a rational expression for the subseries of (*) in which only monomials $a^m b^n$ occur that are at the same time monomials in the variables ab and a. To this end we need only consider the fraction

$$\frac{1}{(1-a)(1-b)(1-az)}$$

of (*). Letting $k = 2$ and $l = 1$ and substituting $a_1 = a$, $a_2 = az$ and $b_1 = b$ in (***) we obtain the rational expression for the relevant subseries of the above fraction of (*). Thus, the rational expression for the subseries of (*) itself becomes

$$\left(\frac{a}{(1-ab)(1-a)} - \frac{az}{(1-abz)(1-az)}\right)\frac{1}{a-az}\,\frac{1}{(1-ab)(1-abu)}$$
$$= \frac{(1-a^2bz)}{(1-a)(1-az)(1-ab)(1-abz)(1-ab)(1-abu)}.$$

A look at the denominator of this function shows that a similar step, with $k = 4$ and $l = 2$, and substitution $a_1 = a_2 = ab$, $a_3 = abz$, $a_4 = abu$, $b_1 = a$, and $b_2 = az$ in (***) yields the required expression for (**) upon substitution of x for a^2b and y for ab. The resulting rational function for the $w = 1$ summand of $P_{G\downarrow H}$ is

$$\frac{\begin{aligned}&-(-1+z^2ux^2-xz+x^2z^2+yx+x^2z-z^2ux^3+z^3y^2x^3-z^3ux^3-z^3x^2y-\\&x^2yu+y^2x^2z^2-ux^3z-y^2xu-y^2xz-z^2y^2x+3yxz+2x^2zu+2z^2yx+\\&2xyu+3yxzu+z^3u^2x^3y+u^2yx^3z+z^2x^3yu^2+2yx^3zu-3x^2z^2y+z^4u^2x^5y^2+\\&4z^2x^2y^2u+2z^2x^2y^2u^2-2z^3ux^2y+z^3x^3y^2u^2-2x^2zy-6x^2zyu+z^2xyu-\\&2x^2zyu^2-2u^2x^4z^3y^2-u^2x^2z^2y+z^4x^4y^3u^2-x^4z^3y^2u-y^3x^2z^2u^2+z^3x^4y^3u^2-\\&z^4x^4y^2u-2z^4u^2x^4y^2+z^3y^2x^3u+2x^2y^2uz+x^2y^2u^2z-6ux^2z^2y+z^2y^3ux+\\&z^4x^3yu+3x^3z^3yu+3x^3z^2yu-u^2y^2x^4z^2-2y^2xuz^2+y^3xzu-3y^2xzu+\\&y^2x^2z^3u-z^3y^3x^3u-y^3x^2z^2u-z^3y^3x^3u^2)\end{aligned}}{(1-x)^2(1-zux)(1-xz)(1-y)^2(1-z^2x)(1-yz)(1-xu)(1-yu)}$$

For $w \in \{s_1, s_2, s_1s_2\}$ one can follow the same procedure. In these cases, an additional summand $r(w(\rho_G)-\rho_G)$ occurs in the argument of q_A. However this requires only a shift in the grading or removing some terms of the series. The corresponding rational functions are, respectively,

$$\frac{y^2x^3zu+y^3x^2zu-2x^2y^2uz-y^2x^2u-y^2x^2z+xyu+yxz+2yx-x-y}{(1-xu)(1-xz)(1-x)^2(1-yu)(1-yz)(1-y)^2}$$

$$\frac{-x^2}{(1-y)(1-x)^2(1-xz)(1-xu)}$$

$$\frac{\begin{aligned}-x(z^4x^3yu-z^3ux^2y-z^3ux^3-z^3x^2y+x^3z^3yu+2z^2x-z^2yx+yz^2-z^2ux^2\\-z^2+z^2xu-z^2xyu+xz+yz-z+zux-1)\end{aligned}}{(1-zux)(1-xz)(1-z^2x)(1-yz)(1-xu)(1-x)^2(1-y)}.$$

Adding these series gives the rational form of the branching series of G_2 to the subgroup A_2:

$$P_{G_2\downarrow A_2}(x,y;z,u) = \frac{1-xyzu}{(1-yu)(1-xu)(1-yz)(1-y)(1-xz)(1-zux)}.$$

An immediate consequence of the obtained rational function expression is the following recursive expression for the coefficient $q(\lambda_1,\lambda_2)$ of $x^{\lambda_1}y^{\lambda_2}$ in $P_{G_2\downarrow A_2}$.

$$q(\lambda_1,\lambda_2) = \begin{cases} \sum_{\ell=0}^{\lambda_2}\sum_{m=0}^{\lambda_2-\ell} z^\ell u^m & \text{if } \lambda_1 = 0 \\ \sum_{\ell=0}^{\lambda_1}\sum_{m=0}^{\lambda_1-\ell} z^{\lambda_1-m}u^{\ell+m} & \text{if } \lambda_1 > 0,\ \lambda_2 = 0 \\ q(\lambda_1,0)q(0,\lambda_2) - zuq(\lambda_1-1,0)q(0,\lambda_2-1) & \text{if } \lambda_1 > 0, \lambda_2 > 0 \end{cases}$$

We recall that $q(\lambda_1,\lambda_2)$ is a polynomial in z and u expressing the decomposition into irreducibles of the restriction to A_2 of the G_2 representation with highest weight (λ_1,λ_2). The computation of $q(\lambda_1,\lambda_2)$ via this method is much faster than the general method as implemented in, e.g., the software package LiE (cf. [Co]).

4. References

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