ASYMPTOTIC BEHAVIOUR FOR WALL POLYNOMIALS AND THE ADDITION FORMULA FOR LITTLE $q$-LEGENDRE POLYNOMIALS*

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Abstract. Wall polynomials $W_n(x; b, q)$ are considered and their asymptotic behaviour is described when $q = e^{1/n}$ and $n$ tends to infinity. The results are then used to derive the addition and product formulas for the Legendre polynomials from the recently obtained addition and product formulas for little $q$-Legendre polynomials.

Key words. Wall polynomials, addition formula, product formula, basic hypergeometric polynomials, Legendre polynomials

AMS(MOS) subject classifications. 33A65, 42C05

1. Introduction. The Wall polynomials $W_n(x; b, q)$ are defined by the recurrence formula

$$W_{n+1}(x; b, q) = \left( x - [b + q - (1 + q) bq^n]q^n \right) W_n(x; b, q) - b(1-q^n)(1-bq^{n-1})q^n W_{n-1}(x; b, q), \quad n = 0, 1, 2, \ldots$$

with initial values $W_{-1} = 0$ and $W_0 = 1$. Clearly $W_n(x; b, q)$ is a monic polynomial of degree $n$ in the variable $x$. Some properties of Wall polynomials are given in Chihara’s book [4, p. 198]. These polynomials are closely related to the continued fraction

$$\frac{1 + x}{1 + \frac{(1-b)qx}{1 + \frac{(1-q)bx}{1 + \frac{(1-bq)x}{1 + \frac{(1-bq^2)x}{1 + \ldots}}}},$$

which was studied by H. S. Wall [16]. The Wall polynomials were also studied by Chihara [5] because they have a Brenke-type generating function, i.e.,

$$\sum_{n=0}^{\infty} W_n(x; b, q) \frac{z^n}{(b; q)_n(q; q)_n} = A(z) B(zx),$$

where

$$A(z) = \sum_{n=0}^{\infty} (-1)^n q^{n(n+1)/2} \frac{z^n}{(q; q)_n} = (zq; q)_\infty,$$

$$B(z) = \sum_{n=0}^{\infty} \frac{z^n}{(b, q)_n(q; q)_n}.$$

We have used the notation

$$(b; q)_n = (1-b)(1-bq) \cdots (1-bq^{n-1}),$$

$$(b; q)_\infty = \lim_{n \to \infty} (b; q)_n.$$
the latter limit exists whenever $|q| < 1$. From this generating function we easily find

$$W_n(x; b, q) = (-1)^n(b; q)q^{n(n+1)/2} \sum_{k=0}^{n} \frac{(q; q)_n}{(q; q)_{n-k}} q^{k(k-1)/2} \frac{(-q^{-1})^k}{(b; q)_k}$$

where the $q$-hypergeometric (or basic hypergeometric [6]) function is defined by

$$r+1 \phi_r(a_1, \ldots , a_{r+1}; b_1, \ldots , b_r; q, z) = \frac{\prod_{k=0}^{\infty} (a_1; q)_k \cdots (a_{r+1}; q)_k}{\prod_{k=0}^{\infty} (b_1; q)_k \cdots (b_r; q)_k (q; q)_k} z^k.$$

If $0 < q < 1$ and $0 < b < 1$ then the Wall polynomials are orthogonal with respect to a positive measure supported on the geometric sequence $\{q^n: n = 1, 2, 3, \ldots\}$ and we have

$$\sum_{k=0}^{\infty} W_n(q^{k+1}; b, q) W_m(q^{k+1}; b, q) \frac{b^k}{(q; q)_k} = 0, \quad n \neq m.$$

The orthonormal polynomials are given by

$$w_n(x; b, q) = \frac{q^{-n(n+1)/2}}{\sqrt{b^n(q; q)_n(b; q)_n}} W_n(x; b, q),$$

and they satisfy

$$(b; q)_\infty \sum_{k=0}^{\infty} w_n(q^{k+1}; b, q) w_m(q^{k+1}; b, q) \frac{b^k}{(q; q)_k} = \delta_{n,m}, \quad n, m \geq 0$$

and the three-term recurrence relation (1.1) becomes

$$x w_n(x; b, q) = a_n w_{n+1}(x; b, q) + b_n w_n(x; b, q) + c_n w_{n-1}(x; b, q)$$

with $w_{-1} = 0$, $w_0 = 1$, and

$$a_n = a_n(b, q) = q^n \sqrt{b(1-q^n)(1-bq^{n-1})}, \quad n = 1, 2, 3, \ldots,$$

$$b_n = b_n(b, q) = q^n [b + q - (1 + q)bq^n], \quad n = 0, 1, 2, \ldots.$$

Sometimes it is convenient to use the notation

$$(b; q)_\infty \sum_{k=0}^{\infty} f(q^{k+1}) \frac{b^k}{(q; q)_k} = \int_0^1 f(z) d\mu(z; b, q), \quad f \in C[0, 1]$$

so that $\mu(\cdot; b, q)$ is the orthogonality measure for the Wall polynomials $W_n(x; b, q)$.

Recently Koornwinder [8] obtained the addition formula for little $q$-Legendre polynomials by using the fact that the matrix elements of the irreducible unitary representations of the quantum group $S_q U(2)$ (see, e.g., Woronowicz [17], [18]) can be expressed in terms of little $q$-Jacobi polynomials (Masuda et al. [9], Vaksman and Soibelman [13], Koornwinder [7]). The little $q$-Jacobi polynomials are defined in terms of $q$-hypergeometric functions by

$$p_n(x; a, b| q) = \phi_1(q^{-n}, abq^{n+1}; q, qx).$$

If $a = q^a$ and $b = q^b$ then these little $q$-Jacobi polynomials approach the Jacobi polynomials $P_n^{(a,b)}(1-2x)/P_n^{(a,b)}(1)$ as $q$ tends to 1 [1], [3]. If $a = b = 1$ then we have
the little $q$-Legendre polynomials. Notice that for $b = 0$ we essentially have the Wall polynomials:

\begin{equation}
\begin{aligned}
p_n \left( \frac{x - b}{q - q^b} \right) &= (-1)^n q^{n(\alpha + 1)/2} W_n(x; b, q) \\
&= (-1)^n \left( \frac{b^n(q; q)_n}{(b; q)_n} \right)^{1/2} W_n(x; b, q).
\end{aligned}
\end{equation}

The addition formula for little $q$-Legendre polynomials is

\begin{equation}
\begin{aligned}
p_m(q^x; 1, 1 | q)p_r(q^y; 1, 1 | q)
p_s(q^z; q^*, 0 | q) \\
&= \sum_{k=1}^{m} (q; q)_k(q^{x+y+k}(q; q)_{m+k}q^{k(y-m+k)} - p_{m-k}(q^{x+y}; q^*, q^k | q)k^{x+y-m+k}(q; q)_kq^2
\end{aligned}
\end{equation}

\begin{equation}
\begin{aligned}
&\cdot p_{m-k}(q^y; q^k | q)p_{r+k}(q^z; q^*, 0 | q) \\
&+ \sum_{k=1}^{m} (q; q)_k(q^{x-y-k}(q; q)_{m-k}(q; q)_{m-k}q^{k(x+y-m+1)} - p_{m-k}(q^{x+y-k}; q^*, q^k | q)k^{x+y-m-k}(q; q)_kq^2
\end{aligned}
\end{equation}

\begin{equation}
\begin{aligned}
&\cdot p_{m-k}(q^{-y-k}; q^k | q)p_{r-k}(q^z; q^*, 0 | q)
\end{aligned}
\end{equation}

with $x, y, z = 0, 1, 2, \ldots$. Rahman [11] has given an analytic proof of this addition formula while Rahman and Verma [12] have given similar formulas for the continuous $q$-ultraspherical polynomials. The right-hand side of the above formula can be considered as an expansion of the left-hand side in terms of Wall polynomials. For $q \uparrow 1$ we should get the familiar addition formula for Legendre polynomials (see, e.g., [2, pp. 29–38]), but this limit involves some interesting asymptotic formulas for the Wall polynomials $W_n(x; b, c1_{\text{c}})$ with $0 < c < 1$ and $n$ tending to infinity. This was the main reason for investigating such asymptotic formulas for Wall polynomials. In § 2 we establish some weak asymptotics for Wall polynomials. In § 3 we show how the addition formula for Legendre polynomials can be obtained from the addition formula for little $q$-Legendre polynomials by letting $q \rightarrow 1$, and in § 4 we obtain the familiar product formulas for Legendre polynomials from the product formulas for little $q$-Legendre polynomials.

2. Weak asymptotics for Wall polynomials. For little $q$-Jacobi polynomials $p_n(x; a, b | q)$ we can put $a = q^a$ and $b = q^b$ and let $q \uparrow 1$ to find Jacobi polynomials on $[0, 1]$. However, if either $a$ or $b$ is zero, which is exactly what happens for Wall polynomials, then the limit as $q \uparrow 1$ is $(1 + (x/(a - 1)))^n$. Therefore another approach is needed to handle the behaviour of Wall polynomials as $q \uparrow 1$. It turns out that we can find some relevant results if we consider the polynomials $W_n(x; b, c1_{\text{c}})$ for $n \to \infty$. We will prove a more general result for orthonormal polynomials \{ $p_n(x; n)$: $k = 0, 1, 2, \ldots$ \} with $k$ the degree of the polynomial and $n$ an extra (discrete) parameter. The recurrence formula for these polynomials is given by

\begin{equation}
\begin{aligned}
xp_k(x; n) = a_{k+1,n}p_{k+1}(x; n) + b_k,n p_k(x; n) + a_{k,n} p_{k-1}(x; n),
\end{aligned}
\end{equation}

where $a_{k,n} > 0$, $b_k,n \in \mathbb{R}$, $p_0(x; n) = 1$, and $p_{-1}(x; n) = 0$. Orthogonal polynomials with regularly varying recurrence coefficients [15] are of this type.
THEOREM 1. Assume that \([r, s]\) is a finite interval that, for all \(n\), contains the support of the orthogonality measure for \(\{p_n(x; n)\}\). Assume moreover that
\[
\lim_{n \to \infty} a_{n,n} = A > 0, \quad \lim_{n \to \infty} b_{n,n} = B \in \mathbb{R}
\]
and that
\[
\lim_{n \to \infty} (a_{n,n}^2 - a_{k-1,n}^2) = 0, \quad \lim_{n \to \infty} (b_{n,n} - b_{k-1,n}) = 0,
\]
uniformly in \(k\), then
\[
\lim_{n \to \infty} \frac{p_{n+1}(x; n)}{p_n(x; n)} = \rho \left( \frac{x - B}{2A} \right),
\]
uniformly on compact sets of \(\mathbb{C}\setminus[r, s]\), where \(\rho(x) = x + \sqrt{x^2 - 1}\) (the square root here is defined to be the one for which \(|\rho(x)| > 1\) for \(x \in \mathbb{C}\setminus[-1, 1]\)).

Proof. Let \(K\) be a compact set in \(\mathbb{C}\setminus[r, s]\); then the distance between \(K\) and \([r, s]\) is strictly positive. Denote this distance by \(\delta > 0\). A decomposition into partial fractions gives
\[
p_{k-1}(x; n) = \frac{p_k(x; n)}{p_k(x; n)} \sum_{j=0}^{k} d_{k,j} n \quad \text{where} \quad \{x_{k,j} : 1 \leq j \leq k\} \quad \text{are the zeros of} \quad p_k(x; n) \quad \text{and} \quad \{d_{k,j} : 1 \leq j \leq k\} \quad \text{are positive numbers adding up to} \quad 1.
\]
Since all the zeros of \(p_k(x; n)\) are in \([r, s]\) we have \(|x - x_{k,j}| > \delta\) for \(x \in K\) and therefore
\[
\left| \frac{p_{k-1}(x; n)}{p_k(x; n)} \right| < \frac{a_{k,n}}{\delta}
\]
holds uniformly for \(x \in K\). Consider the Turán determinant
\[
D_k(x; n) = p_k^2(x; n) - \frac{a_{k+1,n}}{a_{k,n}} p_{k+1}(x; n) p_{k-1}(x; n).
\]
By using the recurrence relation (2.1) we find
\[
D_k(x; n) = D_{k-1}(x; n) + \frac{b_{k,n} - b_{k-1,n}}{a_{k,n}} p_k(x; n) p_{k-1}(x; n)
\]
(see [14, Thm. 4.10, p. 117]). If we define
\[
R_{k,n}(x) = \frac{D_k(x; n)}{p_{k+1}(x; n) p_k(x; n)},
\]
then by (2.6)
\[
|R_{k,n}(x)| \leq \left| R_{k-1,n}(x) \right| \left| \frac{p_{k-1}(x; n)}{p_k(x; n)} \right| + \left| \frac{b_{k,n} - b_{k-1,n}}{a_{k,n}} \right| \left| \frac{p_{k-1}(x; n)}{p_k(x; n)} \right|
\]
so that by (2.5) we have for \(x \in K\)
\[
|R_{k,n}(x)| \leq \frac{a_{k,n}^2 + b_{k,n}^2}{\delta^2} |R_{k-1,n}(x)| + b_{k,n} - b_{k-1,n} \frac{a_{k,n}^2 + a_{k-1,n}^2}{\delta^2}.
\]
By the conditions imposed there exists a constant $C$ such that $a_{k,n} < C$ for every $n$ and $k$ (cf. [4, Chap. IV, Example 2.12]). Therefore, by (2.3),

$$|R_{k,n}(x)| \leq \left( \frac{C}{\delta} \right)^2 |R_{k-1,n}(x)| + A_n, \quad x \in K,$$

where $A_n \to 0$ as $n \to \infty$. Iteration gives

$$|R_{n,n}(x)| \leq A_n \left( \frac{C}{\delta} \right)^{2n} + |R_{n,n}(x)| \left( \frac{C}{\delta} \right)^{2n}, \quad x \in K.$$

If $\delta > C$ then obviously $R_{n,n}(x) \to 0$ as $n \to \infty$ (use $|R_{n,n}| = |p_n(x; n)/p_n(x; n)| < a_{n,n} / \delta$), which by (2.2), (2.3), and (2.5) leads to

$$\lim_{n \to \infty} \left| \frac{p_n(x; n)}{p_{n+1}(x; n)} - \frac{p_{n-1}(x; n)}{p_n(x; n)} \right| = 0,$$

uniformly for $x \in K$ (provided $\delta > C$). By (2.5) the sequence of analytic functions $p_n(x; n)/p_{n+1}(x; n)$ is uniformly bounded on compact sets of $\mathbb{C}\setminus [r, s]$ and thus there exists a subsequence converging to some function $L(x)$, uniformly on $K$. Use the recurrence formula (2.1) and the properties (2.2), (2.3), and (2.7) to find that this limit satisfies

$$x = \frac{A}{L(x)} + B + AL(x),$$

and since $|p_n(x; n)/p_{n+1}(x; n)| < C/\delta < 1$ for $x \in K$ by (2.5) we have

$$\frac{1}{L(x)} = \rho \left( \frac{x - B}{2A} \right).$$

This gives the result for $\delta > C$. This can be extended to hold for $\delta > 0$ by using the Stieltjes-Vitali theorem (cf. [4, p. 121]) and the uniform bound (2.5).

Remark. The asymptotic behaviour actually holds uniformly on compact sets of $\mathbb{C}\setminus \Omega$, where $\Omega$ is the closure of the set of zeros of $p_n(x; n)$ as $n$ runs through the integers. Clearly, $\Omega$ is a subset of $[r, s]$ since the zeros of $p_n(x; n)$ are all inside the interval $[r, s]$. The condition that the joint supports of the orthogonality measures should be contained in the finite interval $[r, s]$ can also be relaxed. Only the zeros of $p_k(x; n)(k \geq n+1, n = 0, 1, 2, \ldots)$ must lie in $[r, s]$.

**Corollary 1.** Suppose $0 < b < 1$ and $0 < c < 1$. Then

$$\lim_{n \to \infty} \frac{W_{n+k}(x; b, c^1/n)}{W_n(x; b, c^1/n)} = \frac{b(1-c)(1-bc) c^{k/2}}{2c \sqrt{b(1-c)(1-bc)}},$$

uniformly on compact sets of $\mathbb{C}\setminus [0, 1]$.

**Proof.** The proof follows immediately from

$$\lim_{n \to \infty} \frac{W_{n+k}(x; b, c^1/n)}{W_{n+k-1}(x; b, c^1/n)} = \frac{b(1-c)(1-bc) c^{1/2}}{2c \sqrt{b(1-c)(1-bc)}},$$

which in turn can be proved by using Theorem 1 with recurrence coefficients $a_{k,n} = a_k(b, c^1/n)$ and $b_{k,n} = b_k(b, c^1/n)$ given by (1.6).

**Corollary 2.** Suppose $0 < b < 1$ and $0 < c < 1$. Then

$$\lim_{n \to \infty} \frac{p_{n+k}(z; b, 0|c^1/n)}{p_n(z; b, 0|c^1/n)} = (-1)^k \left( \frac{b(1-c)}{1-bc} \right)^{k/2} \rho^k \left( \frac{z - [b+1-2bc]c}{2c \sqrt{b(1-c)(1-bc)}} \right).$$
uniformly for $z$ on compact subsets of $\mathbb{C}[0, 1]$, where $p_n(x; a, b | q)$ are the little $q$-Jacobi polynomials.

Proof. This follows immediately from (1.8) and Corollary 1. □

It is important in the asymptotic formula (2.4) that the variable $x$ stays away from the zeros of $p_n(x; n)$. On the set $\Omega$, the closure of the zeros of $p_n(x; n)$, the orthogonal polynomials will oscillate. The following theorem gives a result about the weak convergence of measures involving the polynomials $p_n(x; n)$ on $[r, s]$ in terms of their orthogonality measures.

**Theorem 2.** Assume that $[r, s]$ is a finite interval that, for all $n$, contains the support of the orthogonality measure $\mu_n$ for the orthonormal polynomials $\{p_k(x; n): k = 0, 1, 2, \cdots\}$. Assume, moreover, that for all $k \in \mathbb{Z}$

$$
\lim_{n \to \infty} a_{n+k,n} = A, \quad \lim_{n \to \infty} b_{n+k,n} = B;
$$

then for every continuous function $f$ on $[r, s]$,

$$
\lim_{n \to \infty} \int_r^s f(z)p_n(z; n)p_{n+k}(z; n)\,d\mu_n(z) = \frac{1}{\pi} \int_{B-2A}^{B+2A} \frac{f(z)T_k((z-B)/(2A))}{\sqrt{4A^2 - (z-B)^2}} \,dz,
$$

where $T_n(x)$ are the Chebyshev polynomials of the first kind.

Proof. We follow the ideas of Nevai and Dehesa [10, Lemma 3]. Let $m$ be a positive integer and apply the recurrence formula (2.1) repeatedly to get

$$
z^m p_n(z; n) = \sum_{k=0}^{n} \alpha_{m+k,n} \alpha_{n+k,n+1} p_{n+k}(z; n),
$$

where

$$
\alpha_{j,k} = \begin{cases} 
\alpha_{n+k,n} & \text{if } k = j - 1, \\
\alpha_{n+k,n+1} & \text{if } k = j, \\
\alpha_{j+1,n} & \text{if } k = j + 1.
\end{cases}
$$

Hence

$$
\int_r^s z^m p_n(z; n)p_{n+k}(z; n)\,d\mu_n(z) = \sum_{k=0}^{m} \alpha_{m+k,n} \alpha_{n+k,n+1} \cdots \alpha_{n+m+1,n+k}.
$$

Because of this equation and by (2.10) it follows that the limit as $n \to \infty$ of $\int_r^s z^m p_n(z; n)p_{n+k}(z; n)\,d\mu_n(z)$ is the same as the limit of

$$
\frac{1}{2A^2} \pi \int_{B-2A}^{B+2A} z^m U_n \left( \frac{z-B}{2A} \right) U_{n+k} \left( \frac{z+B}{2A} \right) \sqrt{4A^2 - (z-B)^2} \,dz
$$

since the Chebyshev polynomials of the second kind $U_n((z-B)/2A)$ are the orthogonal polynomials with constant recurrence coefficients $a_n = A$ and $b_n = B$. Use the identity

$$
U_n(x) U_{n+k}(x) = \frac{1}{2} T_k(x) T_{2n+k+2}(x)
$$

to find

$$
\frac{1}{2A^2} \pi \int_{B-2A}^{B+2A} z^m T_n \left( \frac{z-B}{2A} \right) U_{n+k} \left( \frac{z+B}{2A} \right) \sqrt{4A^2 - (z-B)^2} \,dz
$$

$$
= \frac{1}{\pi} \int_{B-2A}^{B+2A} \frac{z^m T_n((z+B)/2A)}{\sqrt{4A^2 - (z-B)^2}} \,dz - \frac{1}{\pi} \int_{B-2A}^{B+2A} \frac{z^m T_{n+k+2}((z+B)/2A)}{\sqrt{4A^2 - (z-B)^2}} \,dz.
$$
If $2n + k + 2 > m$ then the second term on the right-hand side vanishes because of orthogonality, and thus we have the result when $f(x) = x^m$. The general result follows from the Hahn–Banach theorem: let the operators $L_{k,n}(k, n = 0, 1, 2, \cdots)$, defined on the Banach space $C[r, s]$ of continuous functions equipped with the supremum norm, be given by

$$L_{k,n}f = \int_r^sf(z)p_n(z; n)p_{n+k}(z; n)\, d\mu_n(z).$$

These are uniformly bounded operators because, by Schwarz’s inequality and the orthonormality,

$$\left| \int_r^sf(z)p_n(z; n)p_{n+k}(z; n)\, d\mu_n(z) \right|^2 \leq \int_r^s|f(z)|^2p_n^2(z; n)\, d\mu_n(z) \int_r^s|f(z)|^2p_{n+k}^2(z; n)\, d\mu_n(z)$$

$$= \|f\|_2^2.$$

Now use Weierstrass’s result that the polynomials form a dense subspace of $C[r, s]$.

**Corollary 3.** Suppose $0 < b < 1$ and $0 < c < 1$. Then for every continuous function $f$ on $[0, 1]$,

$$\lim_{n \to \infty} \int_0^1 f(z)w_n(z; b, c^{1/n})w_{n+k}(z; b, c^{1/n})\, d\mu(z; b, c^{1/n}) = \frac{1}{\pi} \int_{B-2A}^{B+2A} f(z)T_k((z-B)/(2A))dz,$$

where $A = \sqrt{b(1-c)(1-bc)}$, $B = (b+1-2bc)c$, and $T_k(x)$ are the Chebyshev polynomials of the first kind.

**Proof.** The proof follows because the Wall polynomials $w_n(x; b, c^{1/n})$ satisfy the conditions of Theorem 2, with recurrence coefficients $a_{k,n} = a_k(b, c^{1/n})$ and $b_k(b, c^{1/n})$ given by (1.6). □

**3. The addition formula.** The little $q$-Legendre polynomials $p_n(z; 1, 1|q)$ and the Wall polynomials $p_n(z; a, 0|q)$ are analytic functions of $z$ and the addition formula (1.9) holds for every $z \in \{ q^n: n = 0, 1, 2, \cdots \}$ (which is a set with an accumulation point). Therefore it follows that

$$p_n(z; 1, 1|q)p_n(z; q^{x+y}, 0|q)$$

$$= p_m(q^{x+y}; 1, 1|q)p_m(q^{x+y}; 1, 1|q)p_m(z; q^n, 0|q)$$

$$+ \sum_{k=1}^m \left( \begin{array}{c} q; q; q_{x+y-k}(q; q)_{m-k}q^{k(1-y-m+k)} \\ q; q; q_{x+y+k}(q; q)_{m+k}q^{k(1-y-m+k)} \\ q; q; q \end{array} \right) p_{m-k}(q^{x+y}; q^k, q^k|q)$$

$$\cdot p_{m-k}(q^{x+y}; q^k, q^k|q)p_{n+k}(z; q^n, 0|q)$$

$$+ \sum_{k=1}^m \left( \begin{array}{c} q; q; q_{x+y-k}(q; q)_{m-k}q^{k(1-y-m+k)} \\ q; q; q_{x+y+k}(q; q)_{m+k}q^{k(1-y-m+k)} \\ q; q; q \end{array} \right) p_{m-k}(q^{x+y-k}; q^k, q^k|q)$$

$$\cdot p_{m-k}(q^{x+y-k}; q^k, q^k|q)p_{n-k}(z; q^n, 0|q)$$

holds for every $z \in \mathbb{C}$ and $x, y = 0, 1, 2, \cdots$. It is well known that

$$\lim_{q \to 1} p_n(z; q^n, q^\beta|q) = R_n^{(a, \beta)}(1-2z),$$

where $R_n^{(a, \beta)}(1-2z)$ is the Macdonald polynomial.
where \( R_n^{(\alpha, \beta)}(x) \) are Jacobi polynomials with the normalization \( R_n^{(\alpha, \beta)}(1) = 1 \), i.e., \( R_n^{(\alpha, \beta)}(x) = P_n^{(\alpha, \beta)}(x)/P_n^{(\alpha, \beta)}(1) \). Fix \( b, c \in (0, 1) \) such that \( \log b / \log c = \beta / \gamma \) with \( \beta, \gamma \) positive integers, substitute in (3.1) \( q = b^{1/(\beta \gamma)} = c^{1/(\beta \gamma)} \), \( x = n \beta, \ y = n \gamma \), and let \( n \to \infty \) through the integers. Then by (2.9), (3.1), and (3.2)

\[
R_m^{(0, 0)}(1 - 2z) = R_m^{(0, 0)}(1 - 2bc)R_m^{(0, 0)}(1 - 2b)
+ \sum_{k=1}^{m} \frac{(m + k)!}{(m - k)! (k!)^2} (1 - bc)^k \frac{k!}{(1 - bc)^{k/2}} R_m^{(k, k)}(1 - 2b)R_m^{(k, k)}(1 - 2c) \left[ \frac{z - [b + 1 - 2bc]c}{2c\sqrt{b(1 - c)(1 - bc)}} \right]^k.
\]

Now use the formula \( T_t(x) = [\rho^x(x) + \rho^{-x}(x)]/2 \); then

\[
R_m^{(0, 0)}(1 - 2z) = R_m^{(0, 0)}(1 - 2bc)R_m^{(0, 0)}(1 - 2b)
+ 2 \sum_{k=1}^{m} (-1)^k \frac{(m + k)!}{(m - k)! (k!)^2} c^k [b(1 - c)(1 - bc)]^{k/2}
\cdot R_m^{(k, k)}(1 - 2bc)R_m^{(k, k)}(1 - 2c) T_t \left( \frac{z - [b + 1 - 2bc]c}{2c\sqrt{b(1 - c)(1 - bc)}} \right).
\]

Finally, choose

\[
1 - 2z = xy - \sqrt{1 - x^2} \sqrt{1 - y^2} t,
1 - 2bc = x,
1 - 2c = y;
\]

then

\[
R_m^{(0, 0)}(xy - \sqrt{1 - x^2} \sqrt{1 - y^2} t) = R_m^{(0, 0)}(x)R_m^{(0, 0)}(y)
+ 2 \sum_{k=1}^{m} (-1)^k \frac{(m + k)!}{(m - k)! (k!)^2} 2^{-2k} \left[ \sqrt{1 - x^2} \sqrt{1 - y^2} \right]^k
\cdot R_m^{(k, k)}(x)R_m^{(k, k)}(y) T_t(t),
\]

which is the familiar addition formula for Legendre polynomials. By our method of proof this formula only holds for \( t \in \mathbb{C} \setminus \mathbb{R} \) (because we use Corollary 2), but since all the functions considered are analytic in \( t \), the result definitely holds for every \( t \in \mathbb{C} \).

4. Product formulas. If we multiply both sides of the addition formula (1.9) by \( p_{s+k}(q^x, q^y, 0, 0, q)^{r^x+r^y}/(q, q)_z \) and sum from \( z = 0 \) to \( \infty \), then by the orthogonality (1.4) and by (1.8)

\[
\sum_{z=0}^{\infty} p_m(q^x, 1, 1 | q) p_x(q^x, q^y, 0 | q) p_{s+k}(q^x, q^y, 0 | q) q^{(x+y)z} / (q, q)_z
= \frac{(q; q)_{x+y+k} (q; q)_{m+s} (q; q)_{z+k} (q; q)_{x+y+k} (q; q)_{m+k} (q; q)_{z+k}}{(q; q)_{x+y+k} (q; q)_{m+k} (q; q)_{z+k}} p_{m-k}(q^{x+y}, q^x, q^y | q) p_{s+k}(q^x, q^y, q^z | q)
\cdot \sum_{z=0}^{\infty} p^x_{s+k} (q^x, q^y, 0 | q) q^{(x+y+z)z} / (q, q)_z,
\]
which holds whenever $k \{0, 1, \cdots, m\}$. In terms of orthonormal Wall polynomials we have by (1.8)

$$
\begin{align*}
\frac{(-1)^k}{(q; q)_{m+k}} \sum_{z=0}^{\infty} p_m(q; \frac{z}{q}; 1, 1 | q) w_y(q^{z+1}; q^{x+1}, q) w_{y+k}(q^{z+1}; q^{x+1}, q) \\
\cdot \frac{q^{(x+1)z}}{(q; q)_z}
\end{align*}
$$

which can be considered as a product formula for the little $q$-Legendre polynomials and which (for $k = 0$) is equivalent with the product formula given by Koornwinder [8]. If we use the notation (1.7) then

$$
\begin{align*}
\frac{(-1)^k}{(q; q)_{m+k}} \sum_{z=0}^{\infty} p_m(q; \frac{z}{q}; 1, 1 | q) w_y(q^{z+1}; q^{x+1}, q) w_{y+k}(q^{z+1}; q^{x+1}, q) \\
\cdot \frac{q^{(x+1)z}}{(q; q)_z}
\end{align*}
$$

Fix $b, c$ in $(0, 1)$ such that $\log b / \log c = \beta / \gamma$ with $\beta$ and $\gamma$ positive integers and let $q = b^{1/n} = c^{1/(n\gamma)}$, $1 + x = n\beta$, $y = n\gamma$. Then as $n \to \infty$ we have by Corollary 3 and by the uniform convergence in (3.2) (keep in mind that $p_m((z/q); 1, 1 | q)$ is a polynomial of degree $m$)

$$
\begin{align*}
\frac{(-1)^k}{(m-k)! (k)!^2} \cdot \frac{c^{-k}(b(1-c)(1-bc))^{-k/2}}{(m+k)!} \\
\frac{1}{\pi} \int_{\frac{B+2A}{B-2A}}^{B+2A} R_m^{(0,0)}(1-2z) T_k\left(\frac{(z-B)/2A}{\sqrt{4A^2-(z-B)^2}} \right) dz,
\end{align*}
$$

where $A = c\sqrt{b(1-c)(1-bc)}$ and $B = (b+1-2bc)c$. Setting $bc = x$, $c = y$ gives the familiar product formulas for Legendre polynomials:

$$
\begin{align*}
R_m^{(k,k)}(1-2x) R_m^{(k,k)}(1-2y) = (-1)^k \frac{(m-k)! (k)!^2}{(m+k)!} \cdot \frac{\{xy(1-y)(1-x)\}^{-k/2}}{\pi} \\
\frac{1}{\pi} \int_{\frac{B+2A}{B-2A}}^{B+2A} R_m^{(0,0)}(1-2z) T_k\left(\frac{(z-B)/2A}{\sqrt{4A^2-(z-B)^2}} \right) dz,
\end{align*}
$$

with $A = \sqrt{xy(1-x)(1-y)}$ and $B = x+y-2xy$.

REFERENCES


