THE LOCAL MAXIMAL SUBGROUPS OF EXCEPTIONAL GROUPS OF LIE TYPE, FINITE AND ALGEBRAIC

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Introduction

Following the classification of finite simple groups, one of the major problems in finite group theory is the determination of the maximal subgroups of the *almost simple* groups—that is, of the groups X such that $L \leqslant X \leqslant \mathrm{Aut}\, L$ for some non-abelian finite simple group L. In the investigation of the maximal subgroups M of X, the analysis is often divided into two parts: the *local* case, in which $M = N_X(E)$ for some elementary abelian subgroup E of X; and the *non-local* case, in which the socle of M is a direct product of non-abelian simple groups. In this paper we determine the local maximal subgroups of the finite exceptional groups of Lie type in the families G_2, F_4, E_6, E_7, E_8, ${}^2G_2$, ${}^2F_4$ and ${}^2E_6$. (The maximal subgroups of the other families of exceptional groups, ${}^2B_2$ and ${}^3D_4$, can be found in [**29**, **18**].) It is a consequence of our main result, Theorem 1 (stated in § 1), together with the results of [**23**] and work on the other simple groups discussed in [**21**, I–III], that the local maximal subgroups of the almost simple groups are all explicitly known, apart from the 2-locals of the sporadic groups BM and M. Theorem 1 is used in the proof of [**22**, Theorem 2], where the study of maximal subgroups of finite exceptional groups of Lie type is reduced to the case of almost simple subgroups. We remark that the 'maximal local subgroups' of X form a larger class than the 'local maximal subgroups', and we make no attempt to determine the former class.

For most of our proof of Theorem 1 we work in the simple algebraic group G corresponding to the finite exceptional group L. Our methods also yield the determination of the local maximal subgroups of simple algebraic groups of exceptional type over algebraically closed fields. The results are stated in Theorem 2 (the positive-dimensional subgroups) and Theorem 3 (the zero-dimensional subgroups). Theorem 2 is used in [**22**, Theorem 1], which determines all positive-dimensional maximal subgroups of G.

One of the local subgroups occurring in the conclusion of Theorem 1 is a subgroup $5^3.\mathrm{SL}_3(5)$ of $E_8(p^a)$ (see Table 1). This turns out to be non-maximal when $p = 2$, because it lies in a subgroup $L_4(5)$ of $E_8(4)$ in that case. This embedding $L_4(5) < E_8(4)$, which may be of independent interest, is exhibited in § 5.

Finally, we remark that our proofs are all independent of the classification of finite simple groups.

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1. *Statement of results and notation*

Let L be a finite exceptional simple group of Lie type over $\mathbf{F}_q$, where $q = p^a$ and p is prime. As described in [**27**], there is a simple adjoint algebraic group G over the algebraic closure of $\mathbf{F}_q$, and a surjective endomorphism σ of G such that $L = O^{p'}(G_\sigma)$. Also $G_\sigma = \mathrm{Inndiag}(L)$, the group generated by all inner and diagonal automorphisms of L.

Let X be a group such that $L \leqslant X \leqslant \mathrm{Aut}\, L$, and let M be a local maximal subgroup of X. Then $M = N_X(E)$ for some elementary abelian r-subgroup E of X with r prime. Choosing E minimal, we may assume that

$$M \text{ normalizes no proper non-trivial subgroup of } E. \qquad (*)$$

Assume that $E \leqslant G_\sigma$ (the case where $E \not\leqslant G_\sigma$ is discussed in the note after Theorem 1 below). If $r = p$ then by [**5**, 3.12] (see also [**8**]), M lies in a parabolic subgroup of X. Otherwise, we may take it that one of the following holds:

(I) E lies in a σ-stable maximal torus of G; or

(II) M normalizes no non-trivial subgroup of a torus of G_σ.

In Case (I), let $D = C_G(E)^0$. Then D is a σ-stable closed connected reductive subgroup of G containing a maximal torus, and $M = N_X(D_\sigma \cap L)$. In the situation of the previous sentence, we say that M is a *subgroup of maximal rank* in X (and also that D is a subgroup of maximal rank in G). The subgroup D has a root system relative to the maximal torus which is a subsystem of the root system of G. The possibilities for such subsystems are given in Tables A and B of [**23**, § 2], and the results of [**23**] include a complete determination of the maximal subgroups of maximal rank in exceptional groups of Lie type.

Theorem 1 determines all the subgroups M in Case (II).

THEOREM 1. *Let L, X, G be as above, and let $M = N_X(E)$ be a local maximal subgroup of X, with $E \leqslant G_\sigma$, E an elementary abelian r-group. Then either*

(I) *M is a parabolic subgroup or a subgroup of maximal rank (determined in [**23**]), or*

(II) *the pair (L, E) is as in Table 1; in each case $r \neq p$ and there is just one G_σ-conjugacy class of such subgroups E.*

In Table 1 we use the notation $E_6^\varepsilon(q)$ for $E_6(q)$ if $\varepsilon = +1$, and ${}^2E_6(q)$ if $\varepsilon = -1$. Also for a prime r, we write just r^e for an elementary abelian group of that order.

TABLE 1

L	E	$C_{G_\sigma}(E)$	$N_{G_\sigma}(E)/C_{G_\sigma}(E)$	Conditions
$G_2(p)$	2^3	E	$\mathrm{SL}_3(2)$	
${}^2G_2(3)'$	2^3	E	Z_7	
$F_4(p)$	3^3	E	$\mathrm{SL}_3(3)$	$p \geqslant 5$
$E_6^\varepsilon(p)$	3^3	special, of order 3^6	$\mathrm{SL}_3(3)$	$\varepsilon = \pm 1$, $3 \mid p - \varepsilon$, $p \geqslant 5$
$E_7(q)$	2^2	$E \times (P\Omega_8^+(q).2^2)$	S_3	$P\Omega_8^+(q).2^2 = \mathrm{Inndiag}(D_4(q))$
$E_8(p)$	2^5	special, of order 2^{15}	$\mathrm{SL}_5(2)$	
$E_8(p^a)$	5^3	E	$\mathrm{SL}_3(5)$	$p \neq 2, 5$, $a = \begin{cases} 1, & \text{if } 5 \mid p^2 - 1 \\ 2, & \text{if } 5 \mid p^2 + 1 \end{cases}$
${}^2E_6(2)$	3^2	$E \times G_2(2)$	$N_L(E) = U_3(2) \times G_2(2)$	
$E_7(3)$	2^2	$E \times F_4(3)$	$N_L(E) = L_2(3) \times F_4(3)$	

REMARK. In the maximal rank case in (I), our proof in fact shows that M normalizes a non-trivial subgroup of a torus of G_σ (see Lemma 2.2).

Constructions of the subgroups in Table 1 can be found in the proof of Theorem 1 in § 2. It is of interest to note that the condition that $p \neq 2$, when E is 3^3 or 5^3 and L is $F_4(p)$ or $E_8(p^a)$, occurs because of the embeddings $L_4(3) < F_4(2)$ and $L_4(5) < E_8(4)$; the subgroups $L_4(3)$ and $L_4(5)$ contain $N_L(E)$ in each case. The embedding $L_4(3) < F_4(2)$ is known [**11, 25**], but the fact that $L_4(5) < E_8(4)$ is new, and we give a proof in § 5.

Note. In Theorem 1 we assume that $E \leqslant G_\sigma$. When $E \nleqslant G_\sigma$, we have $E \cap G_\sigma = 1$ by $(*)$, and so $M \cap L = C_L(\alpha)$ for some automorphism $\alpha \in (\mathrm{Aut}\, L) \backslash G_\sigma$ of prime order r. Then α is a field, graph-field or graph automorphism (see [**16**, § 7]). The conjugacy classes of such automorphisms are known, by [**16**, § 7; **3**, § 19] and Proposition 2.7 of this paper. The centralizers arising are subgroups of the same type as L over a maximal subfield of $\mathbf{F}_q$, subgroups ${}^2G_2(q)$, ${}^2F_4(q)$, ${}^2E_6(q^{\frac{1}{2}})$ in $G_2(q)$, $F_4(q)$, $E_6(q)$, and subgroups $C_4(q)$ (with q odd) and $F_4(q)$ in $E_6^\varepsilon(q)$.

Next we turn to algebraic groups. We prove Theorems 2 and 3, which determine all local subgroups (that is, normalizers of finite abelian subgroups) of simple algebraic groups of exceptional type over algebraically closed fields, subject to certain maximality conditions. Theorem 2 handles the subgroups of positive dimension and Theorem 3 those of dimension zero.

THEOREM 2. *Let G be a simple adjoint algebraic group of exceptional type (G_2, F_4, E_6, E_7 or E_8) over an algebraically closed field of characteristic l. Let S be a subgroup of* Aut G *such that $D = (S \cap G)^0$ is a non-trivial closed connected subgroup of G, and assume that*

(1) *$N_G(D)/D$ is finite,*

(2) *for some prime r, $O_r(C_G(D)) \neq 1$, and*

(3) *D is maximal among closed connected S-invariant subgroups of G.*

Then either D is parabolic or a subgroup of maximal rank, or $G = E_7$, $l \neq 2$, $D = D_4$ and $N_G(D) = (2^2 \times D_4).S_3$ (with $C_G(D) = 2^2$ as in Table 1).

THEOREM 3. *Let G be as in Theorem* 2, *and suppose A is a subgroup of G satisfying the following conditions:*

(a) *A is an elementary abelian r-group, where r is prime and $r \neq l$,*

(b) *$N_G(A)$ is finite,*

(c) *$N_G(A)$ normalizes no proper non-trivial subgroup of A,*

(d) *$N_G(A)$ is maximal with respect to* (a), (b) *and* (c),

(e) *there is no proper non-trivial connected $N_G(A)$-invariant subgroup of G.*

Then A is given in Table 2, *and is uniquely determined up to G-conjugacy.*

Theorem 3 generalizes to arbitrary characteristic a result stated for characteristic zero in [**1**]. We give the proof, which runs along the same lines as that of Theorem 1, in § 4. The subgroups A in Table 2 are called *Jordan* subgroups of G in [**1**], and a result similar to Theorem 3 concerning Jordan subgroups is proved in [**7**].

TABLE 2

G	A	$N_G(A)$
G_2	2^3	$2^3.SL_3(2)$
F_4	3^3	$3^3.SL_3(3)$
E_6	3^3	$3^{3+3}.SL_3(3)$
E_8	2^5	$2^{5+10}.SL_5(2)$
	5^3	$5^3.SL_3(5)$

COROLLARY. *Let G be as in Theorem* 2.

(i) *The local maximal subgroups of G of positive dimension are those given in Theorem* 2.

(ii) *Suppose H is a finite local subgroup of G which is contained in no proper closed connected subgroup of G. Then H is one of the groups $N_G(A)$ in Table* 2.

The corollary is immediate from Theorems 2 and 3.

The actions of the local subgroups $N_G(A)$ in Table 2 on the Lie algebra $L(G)$ of G yield interesting orthogonal decompositions of $L(G)$, which are studied in [**6, 19**].

The layout of the rest of the paper is as follows. Sections 2, 3 and 4 contain the proofs of Theorems 1, 2 and 3, respectively. Finally, in § 5 we exhibit the embedding $L_4(5) < E_8(4)$.

2. *Proof of Theorem* 1

Assume the hypotheses of Theorem 1. Thus G is a simple adjoint group of exceptional type, $L = O^{p'}(G_\sigma)$ is a simple group of Lie type over $\mathbf{F}_q$, $q = p^a$, and X is a group with $L \leq X \leq \operatorname{Aut} L$. Also $M = N_X(E)$ is a non-parabolic local maximal subgroup of X normalizing no non-trivial subgroup of a torus of G_σ, and also normalizing no non-trivial proper subgroup of E, where E is an elementary abelian r-subgroup of G_σ consisting of semisimple elements. Notice that we are not excluding the possibility that M is of maximal rank here; in fact, Lemma 2.2 rules this out (see the Remark after Theorem 1).

We introduce some further notation. Let $\hat{G}$ be the simply connected cover of G, let $\hat{E}$ be the full preimage of E in $\hat{G}$, and let $\hat{g}$ be a preimage of an element $g \in G$. Write also $E^\# = E \backslash \{1\}$, and $W = W(G)$, the Weyl group of G, and denote by T_i a torus of rank i in G.

If K is a connected reductive subgroup of G, define a *homogeneous factor* of K to be the product of all the simple connected normal subgroups of K in a single ($\operatorname{Aut} G$)-conjugacy class; if $Z(K)^0 \neq 1$, define $Z(K)^0$ also to be a homogeneous factor. Thus K is the commuting product of its homogeneous factors.

Note that $\operatorname{Aut} L$ is generated by inner, diagonal, field and graph automorphisms (see [**9, 28**]), all of which extend to automorphisms of the abstract group G which commute with σ. Thus there is a subgroup $\tilde{X}$ of $C_{\operatorname{Aut} G}(\sigma)$ such that $X = \tilde{X}/\langle\sigma\rangle$, and so X acts on the set of σ-stable subsets of G. For a σ-stable subset Y, we write $N_X(Y)$ for the stabilizer in X of Y.

We now embark upon the proof of Theorem 1. This is carried out in a series of lemmas (2.1 to 2.17). Lemmas 2.1 to 2.9 are concerned with restricting the structure of $C_G(e)$ for $e \in E^\#$. The remaining lemmas deal separately with the various possibilities for G and r.

LEMMA 2.1. *Suppose that $M\langle\sigma\rangle$ normalizes a non-trivial, proper, connected, semisimple subgroup K of G. Then one of the following holds:*

(i) $[K, E]=1$;

(ii) $q=3$ *and K has components of type A_1 such that if C is the product of these components then* $E \lhd O^{p'}(C_\sigma)\cong L_2(3)^x$, *a direct product of x copies of* $L_2(3)$*; in particular, the rank of E is even;*

(iii) $q=2$ *and K has components of type A_2 such that if C is the product of these components then* $E\lhd O^{p'}(C_\sigma)\cong U_3(2)^y$ *for some* $y\geq 1$.

Proof. As $K_\sigma\neq 1$, by the maximality of M we have $M=N_X(K_\sigma)=N_X(E)$. Thus $K_\sigma\leq M$ and $E\leq N(K_\sigma)$. It follows that

$$[K_\sigma, E]\lhd K_\sigma,\quad [K_\sigma, E]\leq K_\sigma\cap E.$$

Assume first that every factor of K'_σ is quasisimple, where K'_σ means $(K_\sigma)'$. We show that conclusion (i) holds. Since $[K'_\sigma, E]\lhd K'_\sigma$, we deduce that $[K'_\sigma, E]\leq Z(K'_\sigma)$, so that $[K'_\sigma, E, K'_\sigma]=[E, K'_\sigma, K'_\sigma]=1$. By the three-subgroup lemma, $[K'_\sigma, E]=1$. Now let $e\in E^{\#}$. Then $K'_\sigma\leq C_K(e)$.

We claim that $C_K(e)$ is reductive. For if not, let $Q=R_u(C_K(e))$, the unipotent radical of $C_K(e)$. Then $K'_\sigma\leq N_K(Q)\leq P$, where P is the canonical parabolic subgroup of K determined by Q as in **[5]**. Moreover P is σ-stable and P_σ is a parabolic subgroup of K_σ. But K'_σ normalizes $R_u(P)_\sigma$ and $R_u(P)_\sigma\leq O^{p'}(K_\sigma)=K'_\sigma$, so $R_u(P)_\sigma\lhd K'_\sigma$, a contradiction. Thus $C_K(e)$ is reductive, as claimed. Now $K'_\sigma=(C_K(e)_\sigma)'$, so, in particular,

$$|K'_\sigma|_p=|C_K(e)'_\sigma|_p.$$

Since $|K'_\sigma|_p$ is q^n, where n is the number of positive roots in the root system of K, we deduce that $C_K(e)=K$. Hence $[K, E]=1$, giving conclusion (i).

Now suppose that (i) fails. Then some factor of K_σ is not quasisimple, so is of type $A_1(2)$, $A_1(3)$, ${}^2A_2(2)$ or ${}^2B_2(2)$. If there is a factor $A_1(2)$ or ${}^2B_2(2)$ then the product A of all such factors is M-invariant, so $M\leq N(O_r(A))$ where r is 3 or 5; but $O_r(A)$ intersects each factor in a cyclic group, and so lies in a maximal torus of G, contrary to our hypothesis on M. If there is a factor $A_1(3)$ or ${}^2A_2(2)$ of the form $SL_2(3)$ or $SU_3(2)$ then M normalizes the product of the centres of all such factors, which again lies in a torus. Thus K must have components of type A_1 or A_2, such that if C is the product of these components then $O^{p'}(C_\sigma)\cong L_2(3)^x$ or $U_3(2)^y$ for some positive integers x, y. Moreover, if $[C_\sigma, E]=1$ then $|C_\sigma|_p=|C_C(e)_\sigma|_p$, and hence as above, $[C, E]=1$; consequently $[K, E]=1$, which is false. So $[C_\sigma, E]\neq 1$. Since M normalizes C_σ and E, we have $1\neq[C_\sigma, E]\leq C_\sigma\cap E$, and hence $E\leq C_\sigma$ as M normalizes no proper subgroup of E. Thus (ii) or (iii) holds.

LEMMA 2.2. *Suppose that $M\langle\sigma\rangle$ normalizes a connected subgroup D of G, such that D is normalized by a maximal torus of G. Then D is 1 or G.*

Proof. Suppose that D is not 1 or G, and take D maximal with respect to the hypotheses of the lemma. Let T be a maximal torus in $N_G(D)$. If $Q=R_u(D)\neq 1$ then $M\langle\sigma\rangle\leq N_G(Q)\leq P$, where P is the canonical parabolic subgroup of G determined by Q as in **[5]**. But then M normalizes the parabolic P_σ, so M must be parabolic, a contradiction.

Thus D is reductive, so $D = D'Z$ with D' semisimple and $Z = Z(D)^0$. The maximality of D implies that $D = (DC_G(D))^0$. We claim that $DT \leq DC_G(D)$: for DT is reductive and so corresponds to a subsystem of roots relative to T, from which the claim is clear. Thus $T \leq D$, and hence $Z = C_T(D)^0$. Since M normalizes Z, our hypothesis on M forces $Z_\sigma = 1$. Moreover, we may take T to be σ-stable, by [**26**, I, 2.9].

Suppose that $D' \neq 1$. Since $T \leq D$, components A_1 and A_2 of the root system of D' (relative to T) are generated by root subgroups of G, so cannot have σ-fixed point groups of the form $L_2(3)$ or $U_3(2)$. Thus by Lemma 2.1 we have $[D', E] = 1$. Note also that $Z \neq 1$, since otherwise $T \leq D = D'$, giving $E \leq C_G(D')_\sigma \leq C_G(T)_\sigma = T_\sigma$, contrary to hypothesis. Also $Z(D) \leq C(T) = T$. We have now

$$D = D'Z, \quad D' \neq 1, \quad Z \neq 1.$$

We next claim that $C_G(D') = Z(D')Z$. To see this, let $D' = H_1 \ldots H_m$ be the expression for D' as a commuting product of its homogeneous factors H_i (see the beginning of this section for the definition). By the maximality of D, for each i, $C_G(H_i)^0$ is the product of those H_j with $j \neq i$, and also $C_G(Z)^0 = D$. Moreover, the rank of D' is less than the rank of G, as $Z \neq 1$. Let Δ be the root system of D'. The possibilities for Δ satisfying the above conditions are not hard to determine, using the lists of all closed subsystems of the root system of G given in [**10**, Tables 7–11]. Indeed, the possibilities are as follows:

G	G_2	F_4	E_6	E_7	E_8
Δ	none	B_3 (p odd)	$D_5, D_4, 4A_1, 2A_1 + A_3$	$E_6, A_6, 3A_2$	$D_7, A_7, 2A_3$

Assume that $G \neq F_4$. Then $C_W(\Delta) = 1$ (see [**10**, Tables 7–11]). Since $N_G(D) \leq DN_G(T)$, this forces $C_G(D') = C_T(D') = Z(D')Z$, as claimed. And if $G = F_4$, $\Delta = B_3$ and p is odd, then M normalizes $Z(D) \cong Z_2$, contrary to hypothesis. This establishes the claim. It follows that $E \leq C_G(D')_\sigma = (Z(D')Z)_\sigma \leq T_\sigma$, a contradiction.

We have now shown that $D' = 1$, and so $D = T$. Consequently M normalizes T_σ, so by hypothesis we must have $T_\sigma = 1$. By [**26**, II, 1.7], $|T_\sigma| = f(q)$, where f is the characteristic polynomial of some $w \in W$ acting on the associated Euclidean space. It follows that $w = 1$ and $q = 2$, so that $M \cap G_\sigma \cong W$. Now for $G \neq G_2$, $E \lhd W$ forces $r = q = 2$, a contradiction. And if $G = G_2$ then $L = G_2(2)$ and M is clearly non-maximal. This completes the proof.

LEMMA 2.3. *Suppose that K is a connected subgroup of G normalized by $M\langle \sigma \rangle$. Then K is semisimple and $Z(K) = 1$. In particular, $C_G(E)^0$ is semisimple with trivial centre.*

Proof. First, K is reductive: for if not, we can use [**5**], as at the beginning of the proof of Lemma 2.2, to show that M is parabolic, a contradiction. Thus $K = K'Z(K)$ with K' semisimple and $Z(K)$ contained in a torus. We must have $Z(K) = 1$ by Lemma 2.2. The result follows.

LEMMA 2.4. (i) *If $e \in E^\#$ then $E = \langle e^G \cap E \rangle$.*

(ii) *r is 2, 3 or 5. Moreover, r is 5 only if $G = E_8$.*

(iii) *The rank $m(E)$ of E is at least 2. Further, if (G, r) is not $(E_6, 3)$ or $(E_7, 2)$ then $m(E) \geq 3$.*

Proof. Part (i) is immediate from the hypothesis that M normalizes no proper non-trivial subgroup of E. By assumption, E does not lie in a torus of G. Hence (ii) follows from [**26**, II, 5.8 and 5.11], as the torsion primes for G (see [**26**, I, 4.4]) are 2, 3 and 5, with 5 occurring only for E_8. Finally, for (iii), suppose that $E = \langle e, f\rangle$ and (G, r) is not $(E_6, 3)$ or $(E_7, 2)$. Then $C_G(e)$ is connected, by [**26**, II, 4.6]. Hence there is a torus of $C_G(e)$ containing f and e, again a contradiction. Thus $m(E) \geqslant 3$, as required.

LEMMA 2.5. (i) *$C_G(E)$ contains no normal torus.*

(ii) *Let $e \in E^{\#}$. Then $C_G(e)$ does not contain a central torus. Moreover, if $C_G(e)$ contains a normal torus then $E \nleqslant C_G(e)^0$.*

Proof. Part (i) is immediate from Lemma 2.3. Part (ii) follows from (i), since if T is a torus in $Z(C(e))$ then $T \leqslant Z(C(E))$.

The possibilities for the centralizers in G of semisimple elements can be calculated using [**16**, 14.1] and its proof. Provided (G, r) is not $(E_6, 3)$ or $(E_7, 2)$, $C_G(e)$ is connected, and hence has no normal torus by Lemma 2.5(ii). From these observations we deduce:

LEMMA 2.6. *For $e \in E^{\#}$, the group $C_G(e)$ has one of the structures given in* Table 3 *below. In the table, w_2 and w_3 denote elements of $W = W(G)$ of orders* 2 *and* 3, *respectively.*

TABLE 3

G	$r = 2$	$r = 3$	$r = 5$
G_2	$A_1 \circ A_1$	A_2	
F_4	$A_1 \circ C_3$ or B_4	$A_2 \circ A_2$	
E_6	$A_1 \circ A_5$	$(A_2 \circ A_2 \circ A_2)\langle w_3\rangle$ or $T_2 D_4\langle w_3\rangle$	
E_7	$A_1 \circ D_6$, $A_7\langle w_2\rangle$ or $T_1 E_6\langle w_2\rangle$	$A_2 \circ A_5$	
E_8	$A_1 \circ E_7$ or D_8	$A_2 \circ E_6$ or A_8	$A_4 \circ A_4$

In order to restrict further the possibilities for $C_G(e)$ $(e \in E^{\#})$, it is convenient to handle first the local subgroups $U_3(2) \times G_2(2)$ in ${}^2E_6(2)$ and $L_2(3) \times F_4(3)$ in $E_7(3)$ given in Table 1 of Theorem 1. For this we require a proposition concerning graph automorphisms of groups of type A_n and E_6.

PROPOSITION 2.7. *Assume that p is odd and that Y is a simple adjoint algebraic group of type A_{2n-1} or E_6 over the algebraic closure of $\mathbf{F}_p$. Let τ be the standard involutory graph automorphism of Y given in* [**9**, Chapter 12] (*with centralizer C_n or F_4 respectively*). *Then there are precisely two classes of involutions in $Y\tau$, with representatives τ and τh, where h is an involution in Y. The connected centralizers of τ, τh have types C_n, D_n if $Y = A_{2n-1}$, and types F_4, C_4 if $Y = E_6$.*

Proof. Let $\hat{Y}$ be the simply connected cover of Y. Pick $\gamma \in \hat{Y}\tau$. By [**27**, 7.5], γ normalizes a Borel subgroup B of $\hat{Y}$ and a maximal torus T of B. Moreover $N_{\hat{Y}\langle\tau\rangle}(T) \cap N_{\hat{Y}\langle\tau\rangle}(B) = T\langle\delta\rangle$ with δ a conjugate of τ, so we may take $\delta = \tau$ and $\gamma = \tau t$ for some $t \in T$. Further, T is the direct product of 1-dimensional tori T_α, one for each fundamental root α of $\hat{Y}$.

Suppose that $Y = E_6$. Here τ interchanges two pairs of groups T_α and fixes the other two. Write $T = T_n T_f$, where T_n (respectively, T_f) is the product of those T_α not fixed (respectively, fixed) by τ, and corresponding to this decomposition, write $t = t_n t_f$. Since τt has order 2, τt is conjugate to τt_f by an element of T, so we replace τt by τt_f. Now $t_f^2 = 1$ and t_f lies in the product of the two fixed tori T_α, so t_f is contained in a subgroup A_2 centralized by τ. All involutions in this A_2 are conjugate. Thus γ is conjugate to either τ or τt_α, where t_α is the involution in a fundamental subgroup SL_2 centralized by τ. Conjugating by an element of the Weyl group of $C_Y(\tau)$, we see that α can be replaced by the highest root α_0. Finally, we identify $C_Y(\tau)$ and $C_Y(\tau t_{\alpha_0})$. We know that $C_Y(\tau) \cong F_4$. We now claim that $C_Y(\tau t_{\alpha_0}) \cong C_4$. First note that $C_Y(\tau)$ contains $D = (A_1)^4$, and this can be chosen with $t_{\alpha_0} \in Z(D)$. So it is clear that $C_Y(\tau t_{\alpha_0})$ contains D. Let α be a root of E_6 and U_α the corresponding T-root subgroup. If $\alpha\tau = \alpha$ then one checks from the known action of the graph automorphism τ that $U_\alpha \leqslant C_Y(\tau t_{\alpha_0})$ if and only if t_{α_0} centralizes U_α. A direct check of roots shows that such root subgroups U_α span D. Now suppose that $\alpha\tau \neq \alpha$. One checks from the Chevalley commutator relations that in each case $\langle U_\alpha, U_{\alpha\tau} \rangle = U_\alpha \times U_{\alpha\tau}$, and hence $C_Y(\tau t_{\alpha_0}) \cap \langle U_\alpha, U_{\alpha\tau} \rangle$ is 1-dimensional. There are 24 pairs of roots interchanged by τ, and hence

$$\dim L(C_Y(\tau t_{\alpha_0})) \geqslant 24 + \dim L(D) = 36.$$

On the other hand, the same arguments at the level of the Lie algebra show that $\dim(L(Y) \cap C(\tau t_{\alpha_0})) = 36$. Thus $C_Y(\tau t_{\alpha_0})$ is a reductive group of dimension 36, and D is a maximal commuting set of fundamental subgroups SL_2 (this is already true in Y). It follows that $C_Y(\tau t_{\alpha_0}) \cong C_4$, as claimed.

Now let $Y = A_{2n-1}$, so $\hat{Y} = SL_{2n}$. Here it will be convenient to replace τ by $\tau' = h\delta$, where δ is the inverse-transpose map and h is the $n \times n$ matrix $(\delta_{i,n+1-i})$ (that is, the matrix with entries 1 on the opposite diagonal and entries 0 elsewhere). Then $\tau' \in \hat{Y}\tau$. Now τ' normalizes the lower triangular group and also the diagonal group T. An easy computation shows that any element of $\tau' T$ is T-conjugate to an element of the form $\tau' t$, where $t = \mathrm{diag}(1, \ldots, 1, c_{n+1}, \ldots, c_{2n})$ for some $c_i \in \bar{\mathbf{F}}_p$. If we also assume that $\tau' t$ corresponds to an involution in $Y\langle \tau \rangle$, then $(\tau' t)^2 \in Z(\hat{Y})$. It then follows that $t = \mathrm{diag}(1, \ldots, 1, c, \ldots, c)$ with $c = \pm 1$. Then $C_{\hat{Y}}(\tau' t)$ is D_n or C_n according as c is 1 or -1.

LEMMA 2.8. *There are local subgroups* $U_3(2) \times G_2(2)$ *in* ${}^2E_6(2)$, *and* $L_2(3) \times F_4(3)$ *in* $E_7(3)$. *These subgroups are unique up to* G_σ*-conjugacy.*

Proof. First consider $L = E_7(3)$. The adjoint algebraic group G contains three conjugacy classes of involutions, with centralizers $A_1 D_6$, $A_7\langle w_2 \rangle$ and $T_1 E_6 \langle w_2 \rangle$ (see [**17**]). Let a be an involution in G_σ with $C_G(a) = T_1 E_6 \langle w_2 \rangle$. Here w_2 induces a graph automorphism on the E_6. By Proposition 2.7, we may pick an involution $b \in C_{G_\sigma}(a)$ such that $C_G(a, b)^0 \cong F_4$. Write $F = C_G(a, b)^0$. Then $T_1 \leqslant C_G(F)$. Since $C_G(b)$ contains F, b must be conjugate to a, so $C_G(b) = T_1' E_6' \langle w_2' \rangle$. Then $F \leqslant E_6'$ and so $C_G(F)$ contains $\langle T_1, T_1' \rangle$. We claim that $C_G(F)$ is reductive. For otherwise, $FC_G(F)$ lies in a parabolic subgroup P of G. Since P contains $F \cong F_4$, P must be an E_6-parabolic with F fixing a 1-space of the unipotent radical. But F also fixes a 1-space of the unipotent radical of the opposite parabolic, contradicting the fact that $C_G(F)^0 \leqslant P$. This proves the claim. Also $C_G(F)^0 \cap C_G(a) = T_1$, so

$C_G(F)^0$ has rank at most 1, and so $C_G(F)^0 \cong A_1$. Thus we have constructed a subgroup $A_1 \times F_4$ in $G = E_7$. The subgroup A_1 here is of adjoint type, as $A_1 \times F_4$ does not centralize any involution. Taking fixed points under σ then, we have a subgroup $L_2(3) \times F_4(3)$ of $E_7(3)$. This subgroup is $N_L(\langle a, b\rangle)$. Since the above proof shows that $\langle a, b\rangle$ is unique up to G_σ-conjugacy, the result is proved for this case.

The argument for $L = {}^2E_6(2)$ is similar. Here we take an element $a \in L$ of order 3 with $C_G(a) = T_2D_4\langle w_3\rangle$. Choosing $b = w_3$, we find that $C_G(a, b)^0 = G_2$, and a, b are conjugate. Further, if T_2' is the centre of $C_G(b)^0$, then $C_G(G_2)$ contains $\langle T_2, T_2'\rangle$, and we calculate as above that $C_G(G_2)^0 = A_2$. Thus we obtain a subgroup $A_2 \times G_2$ in E_6, and, taking fixed points, a subgroup $U_3(2) \times G_2(2)$ in ${}^2E_6(2)$. Uniqueness follows as before.

For the remainder of the proof we assume that $N_L(E)$ is not one of the subgroups given in Lemma 2.8.

LEMMA 2.9. (i) *Suppose that $Z(\hat{G})$ has order r. Then $\hat{E}$ (the preimage of E in $\hat{G}$) is either elementary abelian or extraspecial.*
(ii) *For $e \in E^\#$, $C_G(e)^0$ is semisimple.*

Proof. Now $\hat{E}$ is the commuting product of an abelian group and an extraspecial group, and M normalizes $Z(\hat{E})$. Thus our hypothesis on M, together with Lemma 2.4(i), implies Part (i).

For (ii), assume that $e \in E^\#$ is such that $C_G(e)^0$ is not semisimple. Then $C_G(e)^0 = TD$, a commuting product of a non-trivial torus T and a semisimple group D. By Lemma 2.6, there are precisely two possibilities: $TD = T_2D_4$ with $(G, r) = (E_6, 3)$, and $TD = T_1E_6$ with $(G, r) = (E_7, 2)$. By Lemma 2.5(ii), we have $E \nleq C_G(e)^0$, and so Part (i) implies that $\hat{E}$ is extraspecial. Thus for each $f \in E^\#$ there is an element $\hat{z} \in Z(\hat{G})^\#$ such that $\hat{f}$ is $\hat{E}$-conjugate to $\hat{f}\hat{z}$. Since the order of the multiplier of D is not divisible by r, it follows that $E \cap D = 1$, and hence $|E| = r^2$. Write $E = \langle e, f\rangle$. Define $Y = C_G(E)^0$ and $K = C_G(Y)^0$. Then Y and K are both semisimple by Lemma 2.3, and we have $Y \leqslant D$, $T \leqslant K$. Moreover $Y \neq 1$, since f induces an element in the coset of a graph automorphism of D. Thus $K < G$.

Suppose first that $[K, E] = 1$. Then $K \leqslant C_G(E)^0 = Y$ and so $K \leqslant Z(Y)$, which is absurd as K and Y are semisimple.

Thus $[K, E] \neq 1$, so by Lemma 2.1, L is $E_7(3)$ or $E_6^\varepsilon(2)$ and K_σ has factors $L_2(3)$ or $U_3(2)$, respectively. Since $\hat{E}$ is extraspecial, the multiplier of L has order divisible by r, so when $q = 2$ we have $L = {}^2E_6(2)$. Consider $L = E_7(3)$. Now $C_G(e) = T_1E_6\langle w_2\rangle$ and f induces an element in the coset of a graph automorphism of $D = E_6$. Hence by Proposition 2.7, $Y = C_G(E)^0$ is C_4 or F_4. Moreover, $K \cap C_G(e) = T_1$, so as K_σ has a factor $L_2(3)$, we must have $K = A_1$ and $K_\sigma' = L_2(3)$. Thus $E = (Z_2)^2 \lhd K_\sigma$. If $Y = C_4$ then E is not fused in G (see the proof of Lemma 2.15 below, third paragraph), which is absurd. Thus $Y = F_4$ and $N_L(E) = L_2(3) \times F_4(3)$. This is conjugate to the subgroup of Lemma 2.8, which we have excluded by assumption. An entirely similar argument for $L = {}^2E_6(2)$ (using here [**16**, 9.1] for the classes of graph automorphisms of D of order 3), shows that $N_L(E)$ is the subgroup of Lemma 2.8 in this case also.

In the remaining lemmas we deal separately with the various possibilities for G and r.

LEMMA 2.10. (i) *If* $G = G_2$ *then* $r = 2$ *and* $N_G(E) = 2^3.\mathrm{SL}_3(2)$.

(ii) *Let* G *be* F_4 *or* E_8 *with* $r = 3$ *or* 5, *respectively. Then* $N_G(E) = r^3.\mathrm{SL}_3(r)$.

In both cases (i) *and* (ii), L *and* $N_{G_\sigma}(E)$ *are as in Table* 1, *and* E *is determined uniquely up to* G_σ*-conjugacy.*

Proof. Fix $e \in E^{\#}$. First assume that $G = G_2$ and $r = 3$. Then $E \leqslant C_G(e) \cong \mathrm{SL}_3$ by Lemma 2.6, so E lies in a maximal torus of G, contrary to hypothesis. Thus r is 2, 3 or 5 according as G is G_2, F_4 or E_8 (by the hypothesis of the lemma). From Lemma 2.6 we see that $C_G(e) = X_1X_2$, a commuting product with $X_1 \cong X_2$ of type A_1, A_2 or A_4, respectively. Moreover $Z(X_1X_2) = \langle e \rangle$.

Let $x \in X_i \backslash \langle e \rangle$ for some i, with x of order r (this is not possible if $G = G_2$ as then $X_i \cong \mathrm{SL}_2$). If $G = F_4$ then x is contained in a fundamental subgroup SL_2, and if $G = E_8$ then x lies in a product of two commuting fundamental subgroups SL_2. Consequently $C_G(x)$ contains C_3, D_6 in the F_4, E_8 cases, respectively, and we deduce that x is not G-conjugate to e. Hence in any case, we may by Lemma 2.4(i) choose $e' \in E \backslash \langle e \rangle$ with $e' = a_1a_2$ and $a_i \in X_i \backslash \langle e \rangle$ for $i = 1, 2$. Note that a_1 has distinct eigenvalues on the natural module for X_1: for otherwise $C(e, e')$ has a central torus, whence $C(E)$ has a normal torus, contrary to Lemma 2.5.

By Lemma 2.4(iii) there exists $e'' \in E \backslash \langle e, e' \rangle$, and by Lemma 2.4(i) we may take $e'' = b_1b_2$ with $b_i \in X_i \backslash \langle e \rangle$. We have $[a_1, b_1] \neq 1$. A straightforward calculation in X_1X_2 shows that $\langle e, e', e'' \rangle$ is self-centralizing in $C_G(e)$, so

$$E = C_G(E) = \langle e, e', e'' \rangle.$$

Now the group $F = \langle a_1, a_2, b_1, b_2 \rangle$ is extraspecial of order r^5 and normalizes E, inducing a group of automorphisms of E of order r^2 and centralizing $E/\langle e \rangle$ (this is a group of transvections in $\mathrm{SL}_3(r)$). Considering the above configuration, we see that beginning with the group $C_G(e') \cong C_G(e)$ yields a group normalizing E and inducing a group of order r^2 centralizing $E/\langle e' \rangle$. We conclude that

$$N_G(E)/E \geqslant \mathrm{SL}_3(r).$$

Since $C_G(e) \cap C_G(e') = T\langle e'' \rangle$ with T a maximal torus of X_1X_2, it follows that no element of $N_G(E)$ can centralize a hyperplane of E without centralizing E. Therefore $N_G(E)/E = \mathrm{SL}_3(r)$. This proves (i) and (ii). To complete the proof of the lemma, we must analyse the situation in the finite group $L = G_\sigma$.

In the case where $L = {}^2G_2(q)'$ we have $N_L(E)/E \cong Z_7$ (see [**31**]), and $q = 3$ by the maximality of M, as in Table 1. And if $L = {}^2F_4(q)'$ then the 3-rank of L is only 2 (see [**16**, 10.2]), so $E \nleqslant L$, a contradiction.

Now let L be $G_2(q)$ or $F_4(q)$, or $E_8(q)$ with $5 \mid p^2 - 1$. The subgroup E lies in the σ-stable group $Y = X_1X_2$, and $O^{p'}(Y_\sigma) = \mathrm{SL}_r^\varepsilon(q) \circ \mathrm{SL}_r^\varepsilon(q)$. Write $Z = \langle e \rangle$, so that $Y = C_G(Z)$. If $E \leqslant O^{p'}(Y_\sigma)$ then the above argument shows that E is conjugate to a subgroup E_1 in $\mathrm{SL}_r^\varepsilon(p) \circ \mathrm{SL}_r^\varepsilon(p)$, where $r \mid p - \varepsilon$, and $N_G(E_1)$ lies in $G_2(p)$, $F_4(p)$ or $E_8(p)$ in the respective cases. Hence $q = p$ by the maximality of M, and E is determined up to L-conjugacy here. Also when $L = F_4(2)$ we have $N(E) < L_4(3) < L$ (see [**11**, p. 170] or [**25**]), so this case is excluded from Table 1. Now suppose that $E \nleqslant O^{p'}(Y_\sigma)$. We may take it that $a_1a_2 \in O^{p'}(Y_\sigma)$, $b_1b_2 \notin O^{p'}(Y_\sigma)$. Now M induces an irreducible subgroup of $\mathrm{SL}_3(r)$ on E containing a transvection (one exists in $N_{Y_\sigma}(E)$). Hence M induces $SL_3(r)$ on E (see [**24**]). Consequently $N_M(Z)$ is transitive on $(E/Z)^\#$. But this is impossible as $a_1a_2 \in O^{p'}(C(Z)_\sigma)$ and $b_1b_2 \notin O^{p'}(C(Z)_\sigma)$.

Finally, let $G = E_8$ with $5 \nmid p^2 - 1$. If $L = E_8(p)$ then $C_L(e)$ must be $SU_5(p^2)$ (see [**13**, p. 215]); but then E lies in a maximal torus of $C_L(e)$, which is not so. Hence $q > p$. As above we see that $E \leqslant O^{p'}(Y_\sigma)$. Then E is conjugate to a subgroup E_1 in $SU_5(p^2) \circ SU_5(p^2)$, and $N_G(E_1) \leqslant E_8(p^2)$. Consequently $q = p^2$ here. All parts of the lemma are now proved, apart from the non-maximality of $N_L(E)$ when $p = 2$ and $G = E_8$. As stated in § 1, this is due to the embedding $N_L(E) < L_4(5) < E_8(4)$, which is demonstrated in § 5.

LEMMA 2.11. *Suppose that $G = E_6$ with $r = 3$. Then $E \cong (Z_3)^3$, $C_G(E)$ is special of order 3^6 with derived group E, and $N_G(E)/C_G(E) \cong SL_3(3)$. Moreover, L and $N_{G_\sigma}(E)$ are as in Table* 1, *and E is determined uniquely up to G_σ-conjugacy.*

Proof. Fix $e \in E^\#$. By Lemmas 2.6 and 2.9(ii), we have

$$C := C_G(e) = (X_1X_2X_3)\langle w_3 \rangle,$$

where each $X_i \cong SL_3$, $Z(X_i) = \langle e \rangle$ and $\langle w_3 \rangle$ is transitive on $\{X_1, X_2, X_3\}$. Moreover, $E^\# = e^G \cap E$. Set $D = X_1X_2X_3$. For $i = 1, 2, 3$ let $Z(\hat{X}_i') = \langle \hat{e}_i \rangle$. Notation may be chosen so that $\hat{e}_1\hat{e}_2\hat{e}_3 = 1$, $Z(\hat{G}) = \langle \hat{e}_1\hat{e}_2^{-1} \rangle$ and $e = \hat{e}_i Z(\hat{G})$ for $i = 1, 2, 3$. Hence $e_1 = e_2 = e_3 = e$.

We next claim that if f is an element of order 3 in $X_iX_j \backslash \langle e \rangle$ for some $i \neq j$, then f is not conjugate to e. For let J be a fundamental subgroup SL_2 within the third SL_3. Viewing f as an element of $C_G(J) = A_5$, we compute that $C_{A_5}(f)$ contains SL_4 or $(SL_2)^3$, and hence that $C_G(f)$ contains SL_4 or $(SL_2)^4$, which proves the claim. Therefore, if $f \in (E \cap D) \backslash \langle e \rangle$ then f has the form $a_1a_2a_3$ with $a_i \in X_i \backslash \langle e \rangle$ for $i = 1, 2, 3$.

We now show that $E \leqslant D$. For suppose that there exists $f \in E \backslash D$. It is easy to check that $C \backslash \langle e \rangle$ contains precisely one conjugacy class of subgroups of order 3 lying outside $D \backslash \langle e \rangle$. Consequently C contains a unique class of elementary abelian subgroups of order 9 which contain e and are not contained in D, and $\langle e, f \rangle$ is a representative of this class. For $i = 1, 2, 3$ let a_i be an element of order 3 in $X_i \backslash \langle e \rangle$ and let $a = a_1a_2a_3$. Then $(C_G(e) \cap C_G(a))^0 = T_6$, a maximal torus. The possibilities for $C_G(a)^0$ can be read off from [**16**, 14.1]. Viewing e as an element of $C_G(a)$ we see that the fact that $(C_G(e) \cap C_G(a))^0 = T_6$ forces $C_G(a)^0 = (A_2)^3$, and hence a is conjugate to e. For $i = 1, 2$ choose b_i of order 3 in $X_i \backslash \langle e \rangle$ such that $[a_i, b_i] = e$, and set $b = b_1b_2^{-1}$. Then $\langle a, b \rangle$ is elementary abelian of order 9, but $\langle \hat{a}, \hat{b} \rangle$ is not abelian. It follows that $b \notin C_G(a)^0$, and hence by the above, $\langle a, b \rangle$ is conjugate to $\langle e, f \rangle$. But this is impossible since $\langle e, f \rangle^\#$ is fused, while $\langle a, b \rangle^\#$ is not. This contradiction shows that $E \leqslant D$.

Let $f = a_1a_2a_3 \in E \backslash \langle e \rangle$. As $C_G(e, f)^0 = T_6$, Lemma 2.3 implies that $E > \langle e, f \rangle$, so we may pick $g = b_1b_2b_3 \in E \backslash \langle e, f \rangle$. By the second paragraph, $b_i \notin \langle a_i, e \rangle$ for $i = 1, 2, 3$. Replacing g by g^2 if necessary, we may assume that $[a_i, b_i] = e$ for $i = 1, 2, 3$ (since $[a_i, b_i] = 1$ implies that $b_i \in \langle a_i, e \rangle$). It follows that $E = \langle e, f, g \rangle$.

Let $K = \langle a_1, a_2, a_3, b_1, b_2, b_3 \rangle$, an extraspecial group of order 3^7 normalizing E. We have

$$C_K(E) = \langle E, a_1a_2^2, b_1b_2^2 \rangle, \quad C_K(E)' = \langle e \rangle = [K, E],$$

and $C_G(E) = C_K(E)\langle t\rangle$ where $\langle t\rangle$ is transitive on $\{X_1, X_2, X_3\}$. Replacing e by other elements of $E^\#$ in the above arguments, we see that $N_G(E)/C_G(E) \cong SL_3(3)$.

We now consider the finite group L. Here $L = E_6^\varepsilon(q)$ with $\varepsilon = \pm 1$. Now E lies in the σ-stable subgroup $D = X_1X_2X_3$. First suppose that $3 \mid p - \varepsilon$. We see, as in the proof of Lemma 2.10, that $E \leqslant O^{p'}(D_\sigma)$, so E is conjugate to a subgroup E_1 in $SL_3^\varepsilon(p) \circ SL_3^\varepsilon(p) \circ SL_3^\varepsilon(p)$, which lies in $E_6^\varepsilon(p)$. Moreover the above argument gives $N_G(E) \leqslant E_6^\varepsilon(p)$ and hence $q = p$ and E is determined up to G_σ-conjugacy here. Finally, let $3 \mid p + \varepsilon$. Then E lies in a subgroup $N(SL_3(p^2) \circ SL_3^{-\varepsilon}(p))$ of $E_6^\varepsilon(p)$ (see [**14**, p. 50]), and we find that E is self-centralizing in $E_6^\varepsilon(p)$. Then by Lemma 2.10 the normalizer of E in $E_6^\varepsilon(p)$ lies in a subgroup $F_4(p)$. Again the G_σ-class of E is determined, so $N_X(E)$ is non-maximal here, a contradiction. To conclude, note that $L \neq {}^2E_6(2)$ here also, since in this case $N_L(E)$ lies in a subgroup $\Omega_7(3)$ of L (see [**11**, p. 191]). Thus L and $N_{G_\sigma}(E)$ are as in Table 1, and the proof is complete.

LEMMA 2.12. *If $r = 3$ then G is not E_7 or E_8.*

Proof. First suppose that $G = E_7$. Let $e \in E^\#$. By Lemma 2.6, $C_G(e) = A_2A_5$. Write $C_G(e)/\langle e\rangle = \bar{A}_2\bar{A}_5$ correspondingly, and let E_0 be the projection of E in $\bar{A}_5$. If the preimage of E_0 in A_5 is abelian then E is contained in a maximal torus, which is not so. Otherwise, by 2.5, the preimage is extraspecial and $C_G(E)^0$ is a subgroup A_1 of A_5. The centralizer of this A_1 in G contains A_2A_2 (within $C_G(e)$), and is normalized by $M = N_X(E)$. As M is maximal, it therefore contains $C_G(A_1)_\sigma \cap L$. But this group does not normalize E.

Thus we assume that $G = E_8$. Then $E \leqslant K_1R$, a commuting product with $K_1 \cong SL_3$ and $R \cong \hat{E}_6$. Let $\langle e_1\rangle = Z(K_1R)$. Now $R \geqslant K_2K_3K_4 \cong (A_2)^3$, and we write $\langle e_i\rangle = Z(K_i)$ for $i = 1, 2, 3, 4$. Let $K = K_1K_2K_3K_4$. One checks that $N_G(K)$ induces S_4 on $\{K_1, \ldots, K_4\}$ and $GL_2(3)$ on $Z(K) = \langle e_1, e_2\rangle$. We may choose notation so that the relations on $\langle e_1, e_2, e_3, e_4\rangle$ are spanned by $e_1e_2e_3^{-1} = e_2e_3e_4 = 1$.

Suppose that $g \in E^\#$ with $B = C_G(g)$ of type A_8. Let $f \in E\backslash\langle g\rangle$. Now $C_B(f)^0$ contains a normal torus, so Lemma 2.5 implies that there exists $h \in E \cap (C_B(f)\backslash C_B(f)^0)$. It follows that $C_B(f)^0 = (A_2)^3$. Hence $O_3(Z(C(E)))$ contains conjugates of e_1, and by Lemma 2.4(i) we may (and do) replace E by $\langle e_1^G \cap O_3(Z(C(E)))\rangle$. Hence we may assume that $e_1 \in E$.

Thus $E \leqslant C_G(E) \leqslant C_G(e_1) = K_1R$, and without loss of generality, we may assume that $E \leqslant K\langle t\rangle$, where $\langle t\rangle$ centralizes K_1 and permutes $\{K_2, K_3, K_4\}$ transitively. By Lemma 2.3, E does not centralize K_1. Hence there is an element $f = a_1x \in e_1^G \cap E$ with $a_1 \in K_1\backslash\langle e_1\rangle$, $x \in K_2K_3K_4\langle t\rangle$. By Lemma 2.3 also, there exists $g = b_1y \in E$ with $[a_1, b_1] = e_1$, $b_1 \in K_1$ and $y \in K_2K_3K_4\langle t\rangle$.

Let $b \in K_j$ with $|b| = 9$ and $b^3 \in Z(K_j)$. Then a direct check shows that b is not K_j-conjugate to $be_j^{\pm 1}$. In particular, the relation $[a_1, b_1] = e_1$ implies that the group $\langle a_1, b_1\rangle$ has exponent 3.

Now R is the simply connected group $\hat{E}_6$, which has three classes of centralizers of non-central elements of order 3, of types $(A_2)^3$, T_2D_4 and T_1A_5 (since the action of $W \cong SO_5(3)$ on $\Omega_3(T)$, where T is a maximally split torus, has three orbits on 1-spaces). Thus $C_R(x)$ is of one of these types. If $C_R(x) = (A_2)^3$ then $f = a_1x$ is conjugate by an element of R to $a_1e_2^\varepsilon$, where $\varepsilon = \pm 1$. Now $N_G(K)$ is

3-transitive on $\{K_1, \ldots, K_4\}$, so $K_1K_2K_3$ is contained in a subgroup A_8, and working within this A_8 we see clearly that $C_G(a_1e_2^\varepsilon)$ contains a subgroup $(A_3)^2$; consequently, by Lemma 2.6, $C_G(f)$ must be A_8, contradicting the fact that f is conjugate to e_1. Thus $C_R(x)$ is not $(A_2)^3$. If $C_R(x)$ is T_1A_5 then $x\langle e_1\rangle/\langle e_1\rangle$ is contained in a fundamental SL_2 of $R/\langle e_1\rangle$, so its centralizer in $R/\langle e_1\rangle$ is connected and $C_G(e_1, f)$ contains a central torus, contrary to Lemma 2.5. Hence $C_R(x)$ must be T_2D_4.

Now at least one of the elements f, g, fg and fg^{-1} lies in K. Let h be such an element, with $h = k_1k_2k_3k_4$ and $k_i \in K_i$. If $E \cap K = \langle e_1, h\rangle$ then $E = \langle e_1, h, e'\rangle$, where either $e' = 1$ or $e' \in E\backslash K$. In either case $C(E)$ contains a normal torus, contrary to Lemma 2.5. Therefore $\langle e_1, h\rangle < E \cap K$ and so we may rechoose h if necessary so that $|k_i| = 3$ for some $i > 1$. Previous remarks (in the fifth paragraph of this proof) imply that $|k_1| = 3$, and so from the relations on the elements e_j we conclude that $|k_j| \leqslant 3$ for each j. Set $r = k_2k_3k_4$. We may write $h = k_1k_ik_j$, as otherwise $C_R(r) = (A_2)^3$, contradicting the previous paragraph. As $E \leqslant C(h)$ we conclude that $E \leqslant K$. We may thus take $x = a_2a_3$ with $a_i \in K_i$ and a_i of order 3. If $K_4 \leqslant C(E)$ then $(C_G(E)^0)' = K_4$, contradicting Lemma 2.3. Hence there exists $d \in e_1^G \cap E$ such that $d = d'd_4$, where $d' \in K_1K_2K_3$ and $d_4 \in K_4$. Then $d = d_id_jd_4$ where $i \neq j$, $i, j \in \{1, 2, 3\}$ and $d_i \in K_i$, $d_j \in K_j$. If $[a_i, d_i] \neq 1$ or $[a_j, d_j] \neq 1$ then $[f, d] \neq 1$ since $\langle e_i, e_j\rangle \cong (Z_3)^2$. Thus $d_i \in \langle a_i, e_i\rangle\backslash\langle e_i\rangle$ and $d_j \in \langle a_j, e_j\rangle\backslash\langle e_j\rangle$.

Since $[x, y] = e_1^{-1}$, we have $y = b_2b_3b_4$ with $b_i \in K_i$ and $[b_2, a_2] \in \langle e_2\rangle^\#$, $[b_3, a_3] \in \langle e_3\rangle^\#$. But then $[g, d] \neq 1$ unless $[b_4, d_4] \in \langle e_4\rangle^\#$. Now earlier remarks imply that $|b_i| = 3$ for all i. We have $g = b_1b_2b_3b_4$ with each b_i of order 3 in $K_i\backslash\langle e_i\rangle$. But then $C_R(b_2b_3b_4) = (A_2)^3$, whereas we have seen that this centralizer should be T_2D_4.

This completes the proof of the lemma.

LEMMA 2.13. *If $r = 2$ then G is not F_4 or E_6.*

Proof. Suppose that the lemma is false. First let $G = E_6$. Let z and e be representatives of the two classes of involutions in G, with $C_G(z) = T_1D_5$ and $C_G(e) = A_1A_5$. By Lemma 2.6 we have $E^\# = e^G \cap E$, and, in particular, we may assume that $e \in E$. Write $C_G(e) = XD$ with $X = A_1$, $D = A_5$. If d is an involution in $(D \cap E)\backslash\langle e\rangle$, then $C_D(d) = A_1 \times A_3$, and setting $c = d$ or de, we have $c \in Z(A_3)$. Now consideration of $C_G(c)$ yields $c \in z^G$, a contradiction.

Thus $E \cap D = \langle e\rangle$, and it follows that E has rank at most 3. Consequently, by Lemma 2.4, E has rank 3. Write $E = \langle e, f, g\rangle$. Then $C_G(e, f) \cap X = T_1$, a 1-dimensional torus inverted by g. As E projects to a quaternion subgroup of D acting homogeneously on the usual 6-dimensional module, $(C_G(E)^0)' = K \cong A_2$. Lemma 2.3 implies that $C_G(K)^0$ is semisimple, and as E does not centralize T_1, it cannot centralize $C_G(K)^0$. Hence by Lemma 2.1, $q = 3$ and $(C_G(K)^0)_\sigma$ has factors $L_2(3)$. But E has rank 3, so this contradicts Lemma 2.1(ii).

Now suppose $G = F_4$. Again G has two classes of involutions, with representatives e and z such that $C_G(e) = A_1C_3$ and $C_G(z) = B_4$. If $E^\# = e^G \cap E$ then we obtain a contradiction as above—the only change is that here K is A_1 rather than A_2.

So assume that $z \in E = \langle z^G \cap E\rangle$. Let $y \in (z^G \cap E)\backslash\langle z\rangle$. Then $C_G(y, z)$ is D_4 or $A_1A_1B_2$. In the latter case $Z(C_G(y, z)) = Z(A_1 \times A_1)$ contains only one conjugate of z, a contradiction. Thus $C_G(y, z) = D_4$ and y has eight eigenvalues

-1 on the usual 9-dimensional module for B_4. The product of two commuting involutions of this type cannot again be of this type, so it follows that E has rank at most 2. This contradicts Lemma 2.4.

It remains to deal with the cases where $r=2$ and G is E_7 or E_8. These require considerably more work than the previous cases.

LEMMA 2.14. *Suppose that p is odd, $G=E_8$, and that $J_1 \dots J_8$ is a maximal commuting product of fundamental subgroups* SL_2 *in G. Let $Z(J_i)=\langle e_i\rangle$ for $1 \leqslant i \leqslant 8$. Then with suitable ordering,*

(i) $e_1 \dots e_8 = 1$,

(ii) $\langle e_1, \dots, e_8\rangle \cong (\langle e_1\rangle \times \dots \times \langle e_8\rangle)/R$, *where*

$$R=\langle e_1e_2e_3e_8,\ e_1e_4e_5e_8,\ e_2e_4e_6e_8,\ e_3e_4e_7e_8\rangle,$$

(iii) $\langle e_1, \dots, e_8\rangle = \langle e_1\rangle \times \langle e_2\rangle \times \langle e_7\rangle \times \langle e_8\rangle$.

Proof. Choose a subsystem $(A_1)^8$ in the E_8 root system. The elements e_i are the elements $h_\alpha(-1)$ as described in [**9**, Chapter 6] for the roots α in the subsystem. A direct check using these elements gives the assertions.

LEMMA 2.15. *Let $r=2$ and $G=E_7$. Then $E=(Z_2)^2$, $C_G(E)=E \times D_4$ (D_4 of adjoint type), and $N_G(E)/C_G(E) \cong S_3$. Moreover, L and $N_{G_\sigma}(E)$ are as in Table* 1, *and E is determined up to G_σ-conjugacy.*

Proof. First note that G has precisely three classes of involutions, with centralizers A_1D_6, $A_7\langle w_2\rangle$ and $T_1E_6\langle w_2\rangle$. Of these, the first lifts to a class of involutions in $\hat{E}_7$, while the other two lift to elements of order 4.

Let $\langle z\rangle = Z(\hat{G})$. By Lemma 2.9(i), $\hat{E}$ is either extraspecial or elementary abelian. Suppose that $\hat{E}$ is extraspecial. Then $\hat{e}$ is conjugate to $\hat{e}z$ for every $\hat{e} \in \hat{E}$, and hence $C_G(e)^0 < C_G(e)$ for each $e \in E^\#$. Hence Lemmas 2.6 and 2.9(ii) imply that each element of $E^\#$ has centralizer $A_7\langle w_2\rangle$. Consequently, every element of $\hat{E}\backslash\langle z\rangle$ has order 4, and so $\hat{E} \cong Q_8$ and $\hat{E}=\langle \hat{e}, \hat{f}\rangle$ with $[\hat{e}, \hat{f}]=z$.

Let y be an involution of G with centralizer $T_1E_6\langle w_2\rangle$ and let d be an involution in $C_G(y)\backslash C_G(y)^0$. (To see that d exists, let T be a maximal torus of T_1E_6 and let $T \leqslant J_1 \dots J_7$, a maximal commuting product of fundamental subgroups SL_2. Let s_i be a fundamental reflection in J_i and set $d=s_1 \dots s_7$. Then $\alpha^d=-\alpha$ for each root α and so d induces a graph automorphism on the factor E_6. Finally, d is an involution.) Write $C=(C_G(y)^0)' \cong E_6$. The proof of Proposition 2.7 shows that there is an involution h in a fundamental subgroup SL_2 of $K=C \cap C_G(d)$ such that d, dh are representatives of the two classes of involutions in Cd. Suppose that $d, dh \in y^G$. Then

$$C_G(d, dh)^0=(C_G(d)^0 \cap C_G(h))^0=T_1A_1A_5.$$

Hence $d^g=dh$ implies that g normalizes T_1. Then g normalizes $C_G(T_1)=C_G(y)^0$. But d, dh are not conjugate in $N_G(C_G(y)^0)=C_G(y)$, a contradiction. Thus d, dh are not both conjugates of y. Now $d \in C(y)\backslash C(y)^0$, so $[\hat{d}, \hat{y}] \neq 1$ and therefore $C_G(d)^0 < C_G(d)$. Similarly $C_G(dh)^0 < C_G(dh)$. It follows that at least one of d, dh conjugate to an element of $E^\#$. Consequently, we may choose $e \in E^\#$ such that y commute but $\hat{e}$, $\hat{y}$ do not commute.

We now have the two non-conjugate involutions $f, y \in C(e)\backslash C(e)^0$. Moreover, since $e \in C(y)\backslash C(y)^0$, Proposition 2.7 implies that $(C(e) \cap C(y)^0)'$ is of type F_4 or C_4; as this subgroup lies in $C(e)^0 = A_7$, it must be of type C_4. Hence, also by Proposition 2.7, $C(e)^0 \cap C(f)^0 = D_4$, that is, $C_G(E)^0 = D_4$ (of adjoint type). Further, $E^\# = e^G \cap E$ and f, fe are conjugate in $C_G(e) = A_7\langle w_2\rangle$ (regarding the action of f as the 'inverse-transpose' action, f and fe are conjugate by a suitable diagonal element of A_7). Repeating this for each element of $E^\#$, we see that $N_G(E)/C_G(E) \cong S_3$. We claim that the group S_3 here acts as graph automorphisms on the factor $D = C_G(E)^0 \cong D_4$. For suppose to the contrary that there is a 3-element x centralizing D. Now D has two composition factors on the minimal 56-dimensional module for G, each of dimension 28 (each is the skew-square of a natural 8-dimensional module). It follows that x has precisely two distinct eigenvalues on the module, and $C_G(x)$ acts on the corresponding eigenspaces, each of which has dimension 28. But D is irreducible on each eigenspace, so by [**16**, 14.1] the only possibility for $C_G(x)^0$ is A_7. This is absurd as A_7 centralizes only 2-elements. Hence our claim is proved.

To conclude this case (assuming $\hat{E}$ extraspecial), we consider the situation in the finite group G_σ. We have proved that the G-class of E is uniquely determined, and $N_G(E) = (E \times D_4).S_3$. Applying [**26**, I, 2.7 and 2.8], we see that the G_σ-classes of subgroups $(E_1)_\sigma$ for σ-stable $E_1 \in E^G$ are in bijective correspondence with the classes of elements in the coset $S_4\sigma = (E.S_3)\sigma$. The only such classes giving Klein 4-groups $(E_1)_\sigma$ are represented by σ and $e\sigma$, where $e \in E^\#$. In the latter case, however, the corresponding Klein 4-group does not have an element of order 3 acting on it. Hence the G_σ-class of E is uniquely determined, and $N_{G_\sigma}(E) = (E \times \mathrm{Inndiag}(D_4(q)).S_3$, as in Table 1.

We may now assume that $\hat{E}$ is elementary abelian. Thus each involution in $E^\#$ has centralizer of type A_1D_6. We view $\hat{G}$ as a subgroup of E_8, centralizing a fundamental subgroup SL_2, say J_8 (in the notation of Lemma 2.14). We may then use the relations in Lemma 2.14, setting $\langle z\rangle = \langle e_8\rangle = Z(\hat{G})$, and $\hat{e} = e_1$. Let $C_{\hat{G}}(\hat{e}) = J_1Y$, a commuting product of $J_1 \cong A_1$ and $Y \cong D_6$. Note that e_1 and e_1z are not conjugate in $\hat{G}$, since otherwise $C_G(e)$ would not be connected, a contradiction.

Suppose that there exists $\hat{f} \in (\hat{E} \cap Y)\backslash\langle \hat{e}, z\rangle$. Choose a maximal torus T of J_1Y containing $\hat{e}$ and $\hat{f}$. Then J_1 is contained in $J_1T \leqslant C_{\hat{G}}(\hat{f}) = J_1^gY^g$ for some $g \in G$. Since J_1 is T-invariant, it follows that $J_1 \leqslant Y^g$ and so $[J_1, J_1^g] = 1$. Hence $\hat{f} \in J_1^g \leqslant C(J_1) = Y$. Thus $\hat{f}$ lies in a fundamental SL_2 of Y, so

$$\hat{E} \leqslant C_G(\hat{e}, \hat{f}) = C_{\hat{G}}(\hat{e}, \hat{f})^0 = J_1J_2J_3R,$$

where J_1, J_2, J_3 are conjugate fundamental subgroups SL_2, and $R \cong D_4$. Letting $\langle e_i\rangle = Z(J_i)$, we have $Z(J_1J_2J_3R) = \langle e_1\rangle \times \langle e_2\rangle \times \langle e_3\rangle$ and $Z(J_1J_2J_3) \cap Z(R) = \langle e_1e_2, e_2e_3\rangle$. By Lemma 2.3, $C_{J_i}(\hat{E})^0 = 1$ for $i = 1, 2, 3$, so there exist elements $a_1a_2r_1$, $b_1b_2r_2$, $a_2'a_3r_3$, $b_2'b_3r_4 \in \hat{E}$ such that a_i and b_i (also a_2' and b_2') are elements of order 4 in J_i with $[a_i, b_i] = e_i$ (also $[a_2', b_2'] = e_2$), and $r_1, r_2, r_3, r_4 \in R$. But then either $[a_2, a_2'] \neq 1$ or $[a_2, b_2'] \neq 1$, say the latter. This forces $[a_1a_2r_1, b_2'b_3r_4] \neq 1$, a contradiction. Thus there is no such element $\hat{f}$.

By Lemmas 2.3 and 2.5, there exist $x_1 = a_1y_1$, $x_2 = b_1y_2 \in \hat{E}$ with $a_1, b_1 \in J_1$, $y_1, y_2 \in Y$ and $a_1^2 = b_1^2 = y_1^2 = y_2^2 = [a_1, b_1] = [y_1, y_2] = e_1$. The previous paragraph implies that

$$\hat{E} = \langle e_1, z, x_1, x_2\rangle.$$

Now $Y/\langle z\rangle$ is a half-spin group: to see this, regard $Y/\langle z\rangle$ as a Levi factor of G; the corresponding unipotent radical is a 32-dimensional spin module for a half-spin group of type D_6. Since z is conjugate to e_1 in $N_{E_8}(Y)$, $Y/\langle e_1\rangle$ is also a half-spin group, and hence $Y/\langle ze_1\rangle \cong \mathrm{SO}_{12}$. Moreover, $y_1\langle ze_1\rangle$ and $y_2\langle ze_1\rangle$ are elements of order 4 squaring to the central involution. Consequently

$$C_{\hat{G}}(\hat{e}, z, x_1)^0 = T_2 A_5.$$

Further, $x_2 = b_1 y_2$ inverts T_2 and acts as a graph automorphism on A_5. Thus by Proposition 2.7, $K = C_G(E)^0$ is of type C_3 or D_3. Also $T_2 \leqslant C(K)^0$. Since x_2 inverts T_2, E does not centralize $C(K)^0$, so by Lemma 2.1, $q = 3$ and $(C(K)^0)_\sigma$ has factors $L_2(3)$. But the rank of E is 3, which contradicts Lemma 2.1(ii).

For the final lemma, handling the case where $G = E_8$ and $r = 2$, we require the following elementary proposition.

PROPOSITION 2.16. *Let D be an elementary abelian group of order $2^\alpha \geqslant 4$, and let V be a module for D over a field of odd characteristic. Then*

$$V = C_V(D) \oplus \sum C_V^0(D_0),$$

where the sum is over the subgroups D_0 of index 2 in D, and $C_V^0(D_0)$ is the unique D-invariant complement to $C_V(D)$ in $C_V(D_0)$. In particular, if $f = \dim C_V(D)$ then

$$\dim V + (2^\alpha - 2)f = \sum \dim C_V(D_0).$$

Proof. This follows from [**15**, 3.3.3].

LEMMA 2.17. *Suppose that $G = E_8$ and $r = 2$. Then $E = (Z_2)^5$, $C_G(E)/E \cong (Z_2)^{10}$, $C_G(E)' = E$ and $N_G(E)/C_G(E) \cong \mathrm{SL}_5(2)$. Moreover, $L = E_8(p)$ and $N_L(E) = N_G(E)$ as in Table 1, and E is uniquely determined up to L-conjugacy.*

Proof. Let e, z be involutions in G with $C_G(z) = D \cong D_8$ and $C_G(e) = J_1 Y$, where J_1 is a fundamental subgroup SL_2 and Y is simply connected of type E_7. Note that D is a half-spin group (see [**17**]).

Let $F = J_1 \dots J_8$ be as in Lemma 2.14, with $\langle e_i\rangle = Z(J_i)$ for $1 \leqslant i \leqslant 8$. Take $F \leqslant D$, so that $z \in Z(F)$. The group $N_G(F)/F$ induces $2^3.\mathrm{SL}_3(2)$ on F (see [**2**]), and $Z(F) = \langle e_1, e_2, e_7, e_8\rangle$ has two $N_G(F)$-classes of involutions: $e^G \cap Z(F) = \{e_i \mid 1 \leqslant i \leqslant 8\}$, and $z^G \cap Z(F) = \{e_i e_j \mid i \neq j\}$. Take $e = e_1$ and $z = e_1 e_8$.

Throughout, a_i and b_i denote elements of order 4 in J_i satisfying $a_i^2 = b_i^2 = [a_i, b_i] = e_i$. We divide the proof into steps.

Step 1. *Involutions in D.* We make some observations concerning involutions in D. If x is an involution in D such that $C_D(x) = S_1 S_2$ with S_1 and S_2 of type D_4, then $Z(C_G(z, x)) = \langle z, x\rangle$ and z, x, xz are all conjugate. We may take $J_1 J_2 J_3 J_8 \leqslant S_1$ and $J_4 J_5 J_6 J_7 \leqslant S_2$. Then $Z(S_1) = Z(S_2) = \langle e_1 e_2, e_2 e_3\rangle = \langle e_4 e_7, e_4 e_5\rangle$.

Next, D contains two classes of subgroups A_7. In one class the groups A_7 are isomorphic to SL_8, with central involution z. The groups A_7 in the other class have centre of order 4 and do not contain z; the central involutions in these latter groups A_7 are conjugates of e. Note that a subgroup A_7 of $\hat{E}_7$ has centre of order 4.

Step 2. *Involutions in F.* Let $\Delta = \{\{i, j, k, l\} \mid e_i e_j e_k e_l = 1\}$. The 4-sets in Δ are determined by Lemma 2.14(ii). We have $|\Delta| = 14$ and $N_G(F)/F = \mathrm{AGL}_3(2)$ is transitive on Δ. Each 4-set in Δ corresponds to a product of two groups SO_4 in an $N(F)$-conjugate of D. For an involution $t \in D$ corresponding to an element of SO_{16} with a eigenvalues -1 and b eigenvalues $+1$ (with $a + b = 16$), we say that t is of type $(-1)^a(1)^b$. Thus

$$e = e_1 \text{ is of type } (-1)^4(1)^{12},$$

$$e_1 e_8 = z \text{ is of type } (-1)^8(1)^8.$$

The involutions in $F \backslash Z(F)$ are $a_1 \dots a_8$ and $a_i a_j a_k a_l e_r^s$, where $\{i, j, k, l\} \in \Delta$ and $r \notin \{i, j, k, l\}$, $s = 0$ or 1. Calculation in D gives:

$a_1 \dots a_8$ is of type $(-1)^8(1)^8$, conjugate to $e_1 e_8$,

$a_i a_j a_k a_l$ is of type $(-1)^4(1)^{12}$, conjugate to e_1,

$a_i a_j a_k a_l e_r$ $(r \notin \{i, j, k, l\})$ is of type $(-1)^8(1)^8$, conjugate to $e_1 e_8$.

Note also that $C_D(a_1 \dots a_8)^0 \cong C_D(a_i a_j a_k a_l e_r)^0 \cong D_4 D_4$.

The remainder of the proof falls into two sections: in Part A we show that $E^\# = z^G \cap E$, that is, E is a *z-group*; and in Part B we show that the z-group E is the group $(Z_2)^5$ in the conclusion of the lemma.

PART A. *E is a z-group.*

Suppose that this is false, and take $e \in E$. Write $E_0 = E \cap Y$ (recall that $C_G(e) = J_1 Y$ with $Y = \hat{E}_7$).

Step 3. *There exists* $f \in E_0 \backslash \langle e \rangle$. By Lemmas 2.3 and 2.5, there exist $a_1 y$, $b_1 y' \in E$ such that $a_1, b_1 \in J_1$, $y, y' \in Y$ and

$$a_1^2 = b_1^2 = y^2 = (y')^2 = [a_1, b_1] = [y, y'] = e.$$

Suppose that $E_0 = \langle e \rangle$. Then $E = \langle e, a_1 y, b_1 y' \rangle$. Now $C_G(a_1 y) \cap J_1 = T_1$ and b_1 inverts T_1. Moreover, $C(e, a_1 y)^0$ is $T_1 A_7$ or $T_1 T_1' E_6$ (see Lemma 2.15 for the involution classes in Y), and y' acts as a graph automorphism on $(C(e, a_1 y)^0)'$. Thus by Proposition 2.7, $(C(E)^0)' = K$ is of type F_4, D_4 or C_4. Clearly K centralizes T_1, so $T_1 \leq C(K)^0$. Moreover $E \cap C(K)^0$ contains e, so $E \leq C(K)^0$. Since $b_1 y'$ inverts T_1, E does not centralize $C(K)^0$. Hence Lemma 2.1(ii) applies to give a contradiction.

Step 4. *The subgroup* $C_G(e, f)$. In this step we calculate $C_G(e, f)$ (where f is as given by Step 3). As Y is simply connected, $C_Y(f)$ is $A_1 D_6$, A_7 or $T_1 E_6$. The last possibility does not hold, by Lemma 2.5(i); and in the second case $C_Y(f) \cong SL_8/Z_2$, forcing $f = e$, which is false. Hence $C_G(e, f) = J_1 C_Y(f) \cong A_1 A_1 D_6$. We may take

$$C_G(e, f) = J_1 J_8 R$$

where $R \cong D_6$. Then $\langle e, f \rangle = Z(C_G(e, f)) = \langle e_1, e_8 \rangle$, and $z \in E$. Also $J_1 J_8 \cap R = \langle e_1, e_8 \rangle$.

Step 5. *We have* $E \cap R = \langle e_1, e_8 \rangle$. Suppose that the claim is false, and pick $g \in (E \cap R) \backslash \langle e_1, e_8 \rangle$. Then $C_R(g) \cong A_1 A_1 D_4$. We may take $g = e_2$, $C_R(g) = J_2 J_3 S_2$, so that $C_G(e_1, e_8, g) = J_1 J_2 J_3 J_8 S_2$ (recall that $S_2 \cong D_4$).

First assume that there also exists $h \in E \cap (S_2 \backslash Z(S_2))$. We may take $C_{S_2}(h) = J_4 J_5 J_6 J_7$, so that $E \leqslant F$ and $Z = Z(F) \leqslant E$. For $1 \leqslant i \leqslant 8$ choose a_i, $b_i \in J_i$ of order 4 with $[a_i, b_i] = e_i$. For a 4-set $A = \{\alpha, \beta, \gamma, \delta\} \in \Delta$, write

$$a_A = a_\alpha a_\beta a_\gamma a_\delta, \quad b_A = b_\alpha b_\beta b_\gamma b_\delta.$$

Define the subgroups U, Y_A $(A \in \Delta)$ of F as follows:

$$U = \langle Z, a_B, b_1 \dots b_8 \mid \text{all } B \in \Delta \rangle,$$
$$Y_A = \langle Z, a_A, b_A, a_{\bar{A}}, b_{\bar{A}} \rangle,$$

where $\bar{A}$ denotes the complement of A in $\{1, \dots, 8\}$. Thus $U \cong (Z_2)^9$ and $Y_A \cong (Z_2)^8$. Moreover, if $U_0 = \langle Z, a_B \mid B \in \Delta \rangle$, the coset $U_0 b_1 \dots b_8$ in U consists entirely of conjugates of z.

We claim now that for some $A \in \Delta$ and choice of a_i, b_i, the group E is contained in either U or Y_A. To see this, note that if E contains a_B and a_C with B, $C \in \Delta$ and $|B \cap C| = 2$, then E can contain no element b_A (as E is abelian), and hence $E \leqslant U$. Otherwise, the set $\{a_B \mid a_B \in E\}$ is contained in $\{a_A, a_{\bar{A}}\}$ for some $A \in \Delta$, and then clearly $E \leqslant Y_A$, proving the claim.

If $E \leqslant U$ then $\langle e^G \cap E \rangle \leqslant U_0$, and hence $E \leqslant U_0$ by Lemma 2.4; but clearly U_0 lies in a maximal torus of F, so this contradicts our hypothesis on E. Hence $E \leqslant Y_A$ for some $A \in \Delta$ and choice of a_i, b_i. Now E centralizes no J_i by Lemma 2.3, and hence we may assume that $a_A a_{\bar{A}} = a_1 \dots a_8 \in E$. Further, $C_F(a_1 \dots a_8)^0 = T_8$. By Lemma 2.5, E centralizes no torus in T_8, and hence E also contains $b_A b_{\bar{A}} = b_1 \dots b_8$. Thus

$$\langle Z, a_1 \dots a_8, b_1 \dots b_8 \rangle \leqslant E \leqslant Y_A.$$

If $|E| = 2^6$ then $\langle e^G \cap E \rangle \leqslant Z < E$, a contradiction. If $|E| = 2^7$ then we may take $E = \langle Z, a_A, a_{\bar{A}}, b_1 \dots b_8 \rangle$ and then $b_1 \dots b_8 \notin \langle e^G \cap E \rangle$, again a contradiction. Thus $|E| = 2^8$ and $E = Y_A$. Write $A = \{i, j, k, l\}$. One checks now that $\langle e_i e_j, e_i e_k \rangle$ is the unique largest z-pure subgroup of Y_A whose involutions t satisfy $tx \in z^G \cup \{1\}$ for all $x \in E \cap z^G$. Consequently $N_G(E)$ normalizes $\langle e_i e_j, e_i e_k \rangle$, which lies in a torus, contrary to hypothesis.

Thus there is no such element $h \in E \cap (S_2 \backslash Z(S_2))$.

We have $E \leqslant J_1 J_2 J_3 J_8 S_2$ and $\langle e_1, e_2, e_8 \rangle \leqslant E$. Since $E \cap (S_2 \backslash Z(S_2)) = \varnothing$, elements z' of $(E \cap z^G) \backslash \langle e_1, e_2, e_8 \rangle$ have the form $a_1 a_2 a_3 a_8 s$ or $a_i a_j s$ with $s \in S_2$: for if $xs \in E$ with $x \in Z(J_1 J_2 J_3 J_8)$ then $s \in E$, a contradiction. If $z' = a_1 a_2 a_3 a_8 s$, then $z' \langle e_1, e_2, e_8 \rangle$ is fused (that is, consists entirely of z-conjugates), and if $z' = a_i a_j s$ then $z' \langle e_i, e_j \rangle$ is fused.

Suppose that E contains $a_i a_j s_1$ with $s_1 \in S_2$, and let $\{i, j, k, l\} = \{1, 2, 3, 8\}$. Then there exist s_2, s_3, $s_4 \in S_2$ such that

$$E \leqslant \langle e_1, e_2, e_8, a_i a_j s_1, b_i b_j s_2, a_k a_l s_3, b_k b_l s_4 \rangle.$$

It follows that the intersection of all maximal z-pure subgroups of E must contain $e_i e_j$, and hence must be a proper non-trivial subgroup of E normalized by M, which is a contradiction. Consequently E contains no such element $a_i a_j s_1$, and so there exist $s_1, s_2 \in S_2$ such that

$$E \leqslant \langle e_1, e_2, e_8, a_1 a_2 a_3 a_8 s_1, b_1 b_2 b_3 b_8 s_2 \rangle.$$

Lemmas 2.3 and 2.5 force equality here, and Lemma 2.4(i) allows us to take $a_1a_2a_3a_8s_1$, $b_1b_2b_3b_8s_2 \in z^G$. But then $\langle E \cap e^G \rangle < E$, a contradiction.

Thus the element g does not exist, and so $E \cap R = \langle e_1, e_8 \rangle$, completing Step 5.

From Step 5 it follows that

$$E \leqslant \langle e_1, e_8, a_8r_1, b_8r_2, a_1r_3, b_1r_4 \rangle,$$

where $r_i \in R \cong D_6$ for $1 \leqslant i \leqslant 4$.

Step 6. *We have* $a_8r_1 \notin E$. Suppose that $a_8r_1 \in E$. Recall that $a_8r_1 \in Y \cong E_7$, where $C_G(e_1) = J_1Y$. Since Y is simply connected, we have $C_Y(a_8r_1) = A_1D_6$ (see Lemma 2.15 for the involution classes of Y), and $Z(D_6) = \langle a_8r_1, e_1 \rangle$. From Step 2 we see that a_8r_1 and $a_8r_1e_1$ are not conjugate in G, and hence we may assume that $a_8r_1 \in z^G$. Similarly we may take b_8r_2, a_1r_3 and b_1r_4 to lie in z^G. We now compute connected centralizers as follows. First, we have

$$C_G(e_8, a_8r_1)^0 = T_1A_7,$$

since $a_8r_1 \in z^G$ and r_1 is an element of order 4 in the factor E_7 of $C_G(e_8)$. Next,

$$C_G(e_8, a_8r_1, e_1)^0 = T_2J_1A_5,$$

since $C(e_1, e_8)^0 = J_1J_8R$ and r_1 is an element of order 4 in R. Also,

$$C_G(e_8, a_8r_1, e_1, b_8r_2)^0 = J_1C_3 \text{ or } J_1D_3,$$

since r_2 inverts r_1, hence interchanges the two distinct 6-dimensional eigenspaces of r_1 on the natural 12-dimensional R-module, and hence induces a graph automorphism on the factor A_5 in $C_G(e_8, a_8r_1, e_1)^0$: now use Proposition 2.7. Next,

$$C_G(e_8, a_8r_1, e_1, b_8r_2, a_1r_3)^0 = T_2A_2,$$

since r_3 lies in C_3 or D_3. Finally,

$$C_G(e_8, a_8r_1, e_1, b_8r_2, a_1r_3, b_1r_4)^0 = A_1,$$

since r_4 inverts r_3.

Now recall that $\langle e_1, e_8, a_8r_1 \rangle \leqslant E$. Let $T_1 = Z(C_G(e_8, a_8r_1)^0)$. Since $C_G(E)$ has no normal torus by Lemma 2.5, E contains an element inverting T_1. From the above calculations we see that $K = C_G(E)^0 \neq 1$ and $T_1 \leqslant C_G(K)^0$. Moreover $e_8 \in E \cap C_G(K)^0$, so $E \leqslant C_G(K)^0$. Now E does not centralize $C_G(K)^0$, so Lemma 2.1(ii) applies. Thus the rank $m(E)$ of E is even, so it is 4 or 6. If $m(E) = 6$ then $K = C_G(E)^0 \cong A_1$ and M normalizes $K_\sigma \cong \mathrm{PGL}_2(3)$. But then M normalizes $O_2(K_\sigma) \cong (Z_2)^2$, which lies in a torus of G_σ.

Therefore $m(E) = 4$. Now $C(e_1, e_8, a_8r_1)^0 = T_2J_1A_5$, so if $C = C(E)^0 \cap A_5$ then C' is a group of rank at least 2, normalized by M. Then $C_G(C')$ contains T_1, so does not centralize E, and hence Lemma 2.1(ii) applies to $C_G(C')$. This is impossible as $C_G(C')$ contains J_1J_8.

Thus $a_8r_1 \notin E$, and Step 6 is complete.

Similarly b_8r_2, a_1r_3, $b_1r_4 \notin E$. Consequently, by Lemma 2.5 we may assume that

$$E = \langle e_1, e_8, a_1a_8r, b_1b_8r' \rangle$$

where $r, r' \in R \cong D_6$. The cosets $a_1a_8r\langle e_1, e_8\rangle$, $b_1b_8r'\langle e_1, e_8\rangle$ and $a_1a_8rb_1b_8r'\langle e_1, e_8\rangle$ are each fused in J_1J_8. Hence it is not possible that $E = \langle e^G \cap E\rangle = \langle z^G \cap E\rangle$, contrary to Lemma 2.4.

This completes Part A. Thus from now on we assume that E is a z-group, and that $z = e_1e_8 \in E$.

PART B. *$E \cong (Z_2)^5$ is as in the conclusion of the lemma.*

We begin by producing the z-pure subgroups $(Z_2)^5$ of the conclusion.

Step 7. Let $F = J_1 \dots J_8$ with each J_i a fundamental subgroup SL_2, *and $Z(J_i) = \langle e_i\rangle$, as above. Let $Z_0 = \langle e_ie_j \mid$ all $i, j\rangle$. Define*

$$A = \langle Z_0, a_1 \dots a_8, b_1 \dots b_8\rangle.$$

Then:

(i) *A is a z-group and $A \cong (Z_2)^5$; any z-pure subgroup $(Z_2)^5$ of F containing Z_0 is F-conjugate to A;*

(ii) *$C_G(A)$ is a special group of order 2^{15}, with $C_G(A)' = A$ and $C_G(A)/A \cong (Z_2)^{10}$;*

(iii) *$X = N_G(A)/C_G(A) \cong \mathrm{SL}_5(2)$, and the action of X on $C_G(A)/A$ is that of $\mathrm{SL}_5(2)$ on the skew-square of a 5-dimensional module $V_5(2)$;*

(iv) *A lies in no larger z-pure subgroup of G.*

Proof. We have $Z_0 \cong (Z_2)^3$, so $A \cong (Z_2)^5$. Moreover A is z-pure by Step 2. It is clear from Step 2 that any z-pure $(Z_2)^5$ in F containing Z_0 is of the form $\langle Z_0, a_1' \dots a_8', b_1' \dots b_8'\rangle$, and hence is conjugate in F to A. Thus (i) is proved.

We next calculate $C_G(A)$. Since $C_G(Z_0)^0 = F$, we have $C_G(A) \leq N_G(F)$. Moreover $N_G(F)/FC_G(F) \cong (Z_2)^3.\mathrm{SL}_3(2)$, and $C_G(Z_0)/F$ lies in the normal subgroup $(Z_2)^3$ acting regularly on $\{J_1, \dots, J_8\}$. It follows from (i) using a Frattini argument that

$$N_G(F) = F(N_G(A) \cap N_G(F)). \qquad (*)$$

Moreover, $F \cap N_G(A)$ induces S_3 on the Klein group A/Z_0. Thus there is a subgroup $\langle r, s, t\rangle \leq N_G(A) \cap N_G(F)$ inducing the regular normal subgroup $(Z_2)^3$ on $\{J_1, \dots, J_8\}$. Now there exist i, j, k, l such that

$$[r, a_1 \dots a_8] = (e_ie_j)^{s_1}, \quad [r, b_1 \dots b_8] = (e_ke_l)^{s_2},$$

where $s_1, s_2 \in \{0, 1\}$. Replacing r by $r(b_ib_j)^{s_1}(a_ka_l)^{s_2}$, we have $r \in C_G(A)$. Similarly we may take $s, t \in C_G(A)$, and so we have the subgroup

$$\langle r, s, t\rangle \leq C_G(A).$$

Recall the set Δ of 4-sets defined in Step 2, and that for $R = \{i, j, k, l\} \in \Delta$, we set $a_R = a_ia_ja_ka_l$. Direct calculation gives

$$C_F(A) = \langle A, e_1, a_R, b_R \mid R \in \Delta\rangle,$$

and hence

$$C_G(A) = \langle A, e_1, a_R, b_R, r, s, t \mid R \in \Delta\rangle.$$

Thus $|C_G(A)/A| = 2^{10}$ and $C_G(A)/A$ contains the elementary abelian subgroup $C_F(A)/A$ of order 2^7.

We now show that $X = N_G(A)/C_G(A) \cong L_5(2)$. First note that for any $a \in A \backslash Z_0$, there is a maximal torus T_a of F, and hence of G, such that $\langle Z_0, a\rangle \leqslant T_a$. Now $N_G(T_a)/T_a \cong W(E_8) \cong 2.O_8^+(2)$. The non-trivial central element here is w_0, the longest element of $W(E_8)$, and the corresponding element n_{w_0} of $N_G(T_a)$ inverts every element of T_a. Moreover, A induces $\langle n_{w_0}\rangle$ on T_a. Write $Q = \Omega_1(O_2(T_a)) \cong (Z_2)^8$. Obviously $\langle Z_0, a\rangle \leqslant Q$, and $N_G(T_a)$ induces the group $O_8^+(2)$ acting naturally on Q. Since $\langle Z_0, a\rangle$ is z-pure, it is a totally singular 4-space in Q. Let Y be any 3-space in $\langle Z_0, a\rangle$. Then there exists $g \in N_G(T_a)$ such that $Z_0^g = Y$. Further, $C_G(Y)^0 = F^g$. Also g normalizes $\langle T_a, n_{w_0}\rangle$ $(= C_G(Q))$, a group containing A, and hence $A \leqslant F^g$. Also $A^g \leqslant F^g$, and so by the uniqueness of the conjugacy class of A^g in F^g given by (i), there exists $f \in F^g$ such that $A^f = A^g$. Then

$$gf^{-1} \in N_G(A), \quad Z_0^{gf^{-1}} = Y.$$

As a was an arbitrary element of $A\backslash Z_0$, we deduce that $X = N_G(A)/C_G(A)$ is transitive on the set of 3-spaces in A. Moreover, the element a_1a_2 lies in $N_G(A)$ and induces a transvection in X. Consequently $X \cong L_5(2)$.

By $(*)$ there is an element $u \in N_G(A)$ of order 3 which acts on $\{J_1, \ldots, J_8\}$ as a product of two 3-cycles. Write $V_1 = C_F(A)/A$, $V_2 = C_G(A)/C_F(A)$. Elementary calculation shows that $\dim C_{V_1}(u) = 3$, $\dim C_{V_2}(U) = 1$, that $[V_1, u]$ is a direct sum of two 2-spaces, and that $\dim[V_2, u] = 2$. Now the non-trivial irreducible modules in characteristic 2 for $X = L_5(2)$ of dimension 10 or less are the natural module W of dimension 5, its dual W^*, and the skew-squares Λ^2W, Λ^2W^*. Since $\dim[W, u] = \dim[W^*, u] = 2$, it follows from the above information that $C_G(A)/A$ must be the irreducible X-module Λ^2W or Λ^2W^* (and in particular that $C_G(A)/A \cong (Z_2)^{10}$). It is now immediate that $C_G(A)$ is a special group of order 2^{15}, and so Parts (ii) and (iii) are proved.

It remains to prove (iv). Now $X = L_5(2)$ has precisely two orbits on the non-zero vectors of Λ^2W or Λ^2W^* (see [**20**, 2.5]). In $C_G(A)/A$ these orbits are represented by e_1A and $a_{R_1}b_{R_2}A$, where $R_1, R_2 \in \Delta$ with $|R_1 \cap R_2| = 2$. Since $e_1 \in e^G$ and $a_{R_1}b_{R_2}$ is an element of order 4 in $C_G(A)$, (iv) follows. This completes Step 7.

Recall now that $z = e_1e_8 \in E$. Pick $z_1 \in E\backslash\langle z\rangle$.

Step 8. *We have* $(C_G(z, z_1))^0 = S_1S_2 \cong D_4D_4$, *and* $Z(S_1) = Z(S_2) = \langle z, z_1\rangle$. To see this, note that any z-pure Klein 4-subgroup V of G can be embedded in a maximal torus T of G, and $N_G(T)$ induces $O_8^+(2)$ on the 8-space $\Omega_1(O_2(T))$. Since it is z-pure, V is a totally singular 2-space here, and $O_8^+(2)$ is transitive on such 2-spaces. Consequently G has just one class of z-pure Klein 4-groups. Since $\langle z, z_1\rangle$ is one such, Step 8 follows.

Step 9. *E contains no element interchanging* S_1 *and* S_2. Suppose this is false, and pick $z_2 \in E$ interchanging S_1 and S_2. Then $C_G(z, z_1, z_2) = \langle z, z_1, z_2\rangle N$ with $N \cong D_4$. Let $V = L(G)$, the 248-dimensional Lie algebra of G. Then $C_V(z, z_1, z_2)$ contains $L(N)$, so if $f = \dim C_V(z, z_1, z_2)$ then $f \geqslant 28$. On the other hand, by Step

8, for each hyperplane D_0 of $\langle z, z_1, z_2\rangle$, we have, using [4, Corollary 9.2],

$$\dim C_V(D_0) = \dim (L(D_4) \oplus L(D_4)) = 56.$$

Thus by Proposition 2.16, $6f + 248 = 7 \times 56$. This yields $f = 24$, a contradiction. This proves Step 9.

Step 10. *There exists $z_2 \in E$ such that $C_G(z, z_1, z_2)^0$ is conjugate to F.* By Step 9, we have $E \leq S_1S_2 = C_G(z, z_1)^0$. Thus for $z_2 \in E \backslash \langle z, z_1\rangle$ the group $\langle z, z_1, z_2\rangle$ is contained in a maximal torus T of G. Since E is z-pure, this is a totally singular 3-subspace of $Q = \Omega_1(O_2(T))$ regarded as an orthogonal space for $W^Q = O_8^+(2)$. Since W is transitive on the set of totally singular 3-spaces in Q, we deduce that $\langle z, z_1, z_2\rangle$ is conjugate to $\langle e_ie_j \mid \text{all } i, j\rangle = \langle Z(F) \cap z^G\rangle = Z_0$. Hence $C_G(z, z_1, z_2)^0$ is conjugate to $C_G(Z_0)^0 = F$, as required for Step 10.

By Step 10 we may take

$$C_G(z, z_1, z_2)^0 = F = J_1 \dots J_8,$$

and also $\langle z, z_1, z_2\rangle = Z_0$ and $E \leq N_G(F) \cap C_G(Z(F))$. Further, we may assume that

$$J_1J_2J_3J_8 \leq S_1, \quad J_4J_5J_6J_7 \leq S_2.$$

Step 11. *We have $E \leq F$.* Suppose this is false, and pick $x \in E \backslash F$. As $x \in z^G$ and also $x \in S_1S_2$, $\langle x\rangle$ must act semiregularly on both $\{J_1, J_2, J_3, J_8\}$ and $\{J_4, J_5, J_6, J_7\}$. Clearly then there exists $R \in \Delta$ such that x interchanges $\prod_{i\in R} J_i$ and $\prod_{i\notin R} J_i$. Consequently we can find $V \leq Z_0$ with $V \cong (Z_2)^2$ such that $C_G(V) = R_1R_2 \cong D_4D_4$, R_1 contains $\prod_{i\in R} J_i$, R_2 contains $\prod_{i\notin R} J_i$, and such that x interchanges R_1 and R_2. This cannot happen, by Step 9.

Step 12. *Completion of the proof for the group G.* We have $Z_0 = \langle z, z_1, z_2\rangle = \langle e_ie_j \mid \text{all } i, j\rangle \leq E \leq F$. By Step 2, any conjugate of z in $F\backslash Z(F)$ is of the form $a_1 \dots a_8$ or a_Re_r with $R \in \Delta$, $r \notin R$. Since $\langle Z_0, a_Re_r\rangle$ contains elements of e^G, any element of $E\backslash Z_0$ must be of the form $a_1 \dots a_8$. Further, by Lemmas 2.3 and 2.5, E must contain elements $a_1 \dots a_8$ and $b_1 \dots b_8$. Thus by Step 7(iv), we have

$$E = A = \langle Z_0, a_1 \dots a_8, b_1 \dots b_8\rangle.$$

All conclusions of the lemma for the algebraic group $G = E_8$ are now proved.

Step 13. *Completion of the proof for the finite group L.* We finally verify the statements of the lemma for the finite group $L = G_\sigma$. The group E is contained in the σ-stable group $F = J_1 \dots J_8$. Since σ centralizes E, it centralizes $Z_0 = \langle e_ie_j \mid \text{all } i, j\rangle$. It follows that σ either fixes each J_i, or has four orbits of length 2 on the J_i. Correspondingly, $O^{p'}(F_\sigma)$ is a central product of either eight copies of $\mathrm{SL}_2(q)$ or four copies of $\mathrm{SL}_2(q^2)$.

Suppose first that $O^{p'}(F_\sigma) = \mathrm{SL}_2(q) \circ \dots \circ \mathrm{SL}_2(q)$ (eight copies). If $E \leq O^{p'}(F_\sigma)$ then E is conjugate to a subgroup E_1 of $\mathrm{SL}_2(p) \circ \dots \circ \mathrm{SL}_2(p)$. Moreover we can argue as before that $N_G(E_1) \leq E_8(p)$ and that E_1 is unique up to $E_8(p)$-conjugacy. Thus $L = E_8(p)$ by the maximality of M, and the conclusion of the lemma holds in this case. Now assume that $E \not\leq O^{p'}(F_\sigma)$. We may then take

$a = a_1 \dots a_8 \in O^{p'}(F_\sigma)$ and $b = b_1 \dots b_8 \notin O^{p'}(F_\sigma)$. Now M acts irreducibly on E and $M/C_M(E) \leq SL_5(2)$. Moreover $M/C_M(E)$ contains 2-elements such as $a_1 a_2$, so $M/C_M(E) = SL_5(2)$ (see [**30**]). Thus $N_M(Z_0)$ is transitive on $(E/Z_0)^\#$. But $N_M(Z_0)$ cannot send a to b, since $a \in O^{p'}(C(Z_0))$ and $b \notin O^{p'}(C(Z_0))$. Hence this case does not occur.

Finally, suppose that $O^{p'}(F_\sigma) = SL_2(q^2) \circ \dots \circ SL_2(q^2)$ (four copies). As in the previous paragraph, the irreducibility of M on E and the existence of 2-elements in $M/C_M(E)$ imply that $M/C_M(E) = SL_5(2)$. Consequently $N_L(Z_0)$ induces $SL_3(2)$ on Z_0. This is impossible since $O^{p'}(C_L(Z_0)) = O^{p'}(F_\sigma)$, a central product of only four quasisimple groups, which cannot admit $SL_3(2)$.

Lemma 2.17 completes the proof of Theorem 1, apart from the demonstration of the embedding $L_4(5) < E_8(4)$, which, as pointed out in the Introduction and in the proof of Lemma 2.10, is needed to show the non-maximality of the subgroup $5^3.SL_3(5)$ in Table 1 when $p = 2$. This embedding is exhibited in § 5.

3. *Proof of Theorem* 2

In this section we give a proof of Theorem 2. Thus let G be a simple exceptional adjoint algebraic group in characteristic l, and let S be a subgroup of $\operatorname{Aut} G$ such that $D = (S \cap G)^0$ is non-trivial, closed, and satisfies Conditions (1), (2) and (3) of Theorem 2. Suppose that D is not parabolic or of maximal rank, that is, D does not contain a maximal torus of G.

By Condition (2), the group $E = \Omega_1(O_r(C_G(D)))$ is non-trivial. Clearly S normalizes E, so by Condition (3) we have $D = C_G(E)^0$ and $N_G(D) = N_G(E)$. Let $Q = R_u(D)$. If $Q \neq 1$ or if $r = l$, then by [**5**, 3.12], S normalizes a parabolic subgroup P containing $N(Q)$ or $N(E)$, forcing $D = P$; but then D contains a maximal torus, a contradiction. Thus $Q = 1$, and so D is reductive. Moreover, if S normalizes a connected subgroup $K \neq 1, G$ then by (3), $D = N(K)^0$, and hence $N(K)$ contains no maximal torus. In particular $Z(D)^0 = 1$, so D is semisimple. As S normalizes $DC(D)$, (3) gives $C(D)^0 \leq D$, whence $C(D)$ is finite. Since $r \neq l$, E consists of semisimple elements.

We claim next that if $1 \neq E_1 \leq E$, then $C_G(E_1)$ contains no non-trivial normal torus. For otherwise, if $T \lhd C_G(E_1)$ with T a non-trivial torus, then $D = C_G(E)^0$ normalizes T, and hence centralizes T, contrary to $C(D)$ being finite.

As E lies in no torus, we see as in Lemma 2.4 that $r = 2$, 3 or 5 and $m(E) \geq 2$. Suppose that for some $e \in E^\#$, $C_G(e)$ is connected and has a factor A_n. Then since for semisimple $x \in A_n \backslash Z(A_n)$, $C_{A_n}(x)$ has a normal torus, $C(E)$ must contain the factor A_n. But then S normalizes the product of these factors A_n, and the normalizer of this product contains a maximal torus, a contradiction. This establishes the fact that for $e \in E^\#$, $C(e)$ has no normal torus, and if $C(e)$ is connected, it has no factor A_n. We conclude (cf. Lemma 2.6) that $C(e)$ is as follows:

G	r	$C(e)$
F_4	2	B_4
E_6	3	$(A_2 \circ A_2 \circ A_2)\langle w_3 \rangle$
E_7	2	$A_7\langle w_2 \rangle$
E_8	2	D_8

When $G = F_4$, we argue as in the last paragraph of the proof of Lemma 2.13 that $m(E) \leqslant 2$, whence E lies in a maximal torus, a contradiction. For $G = E_6$, let $f \in E \backslash \langle e \rangle$. If $f \in C(e)^0$ then $C_G(e, f)$ has a normal torus, so $f \in C(e) \backslash C(e)^0$ and $E = \langle e, f \rangle$. We now obtain a contradiction as in the third paragraph of the proof of Lemma 2.11. When $G = E_7$, the same argument gives $E = \langle e, f \rangle$ with $f \in C(e) \backslash C(e)^0$. Then as in Lemma 2.15, $D = C(E)^0 = D_4$, $N_G(D) = (2^2 \times D_4).S_3$. This is in the conclusion of Theorem 2. Finally, if $G = E_8$, we argue as in Steps 8–11 of Lemma 2.17 that $E \leqslant F = (A_1)^8$ and E contains $Z_0 = \langle Z(F) \cap z^G \rangle$. As E does not lie in a torus, $E \neq Z_0$, so there exists $f \in E \backslash Z_0$. Then $C(Z_0, f)$ has a normal torus, which is a contradiction. This completes the proof of Theorem 2.

4. *Proof of Theorem* 3

Here we give the proof of Theorem 3. Thus let G be an exceptional simple adjoint algebraic group in characteristic l, and let A be an elementary abelian r-subgroup satisfying Conditions (a)–(e) of Theorem 3. The proof runs parallel to that of Theorem 1 in § 2. Since $N_G(A)$ is finite, it is clear that A does not lie in a torus of G, and $C_G(A)^0 = 1$.

First, the proof of Lemma 2.4 gives

LEMMA 4.1. (i) $A = \langle a^G \cap A \rangle$ *for* $a \in A^{\#}$.

(ii) r *is* 2, 3 *or* 5, *and* $r = 5$ *only if* $G = E_8$.

(iii) $m(A) \geqslant 2$; *and* $m(A) \geqslant 3$ *if* (G, r) *is not* $(E_6, 3)$ *or* $(E_7, 2)$.

The fact that $C_G(A)^0 = 1$ implies the following analogue of Lemma 2.5.

LEMMA 4.2. *For* $a \in A^{\#}$, $C_G(a)$ *does not contain a central torus. Moreover, if* $C_G(a)$ *has a normal torus then* $A \nleqslant C_G(a)^0$.

Lemmas 4.1 and 4.2 give

LEMMA 4.3. *For* $a \in A^{\#}$, *the possibilities for* $C_G(a)$ *are those given in Table* 3 *of Lemma* 2.6.

LEMMA 4.4. (i) *Suppose* $|Z(\hat{G})| = r$. *Then* $\hat{E}$ *is elementary abelian or extraspecial.*

(ii) *For* $a \in A^{\#}$, $C_G(a)^0$ *is semisimple.*

Part (i) of Lemma 4.4 is proved as in Lemma 2.9(i). If (ii) fails, we argue as in Lemma 2.9 that $|A| = r^2$; but then $C_G(A)^0 \neq 1$.

Now the proofs of Lemmas 2.10–2.17 give Theorem 3. Note that these proofs are sometimes greatly simplified by the hypothesis of Theorem 3, since $C_G(A)^0 = 1$ and the conclusions of Lemmas 2.1–2.3 are subsumed by the much stronger hypothesis (e) of Theorem 3.

5. *The embedding* $L_4(5) < E_8(4)$

In this final section we show that $E_8(4)$ contains a subgroup $L_4(5)$, and also that for $p \neq 2$, 5, the algebraic group E_8 over the algebraic closure of $\mathbf{F}_p$ does not contain $L_4(5)$. As a consequence, the local subgroup $N(E) \cong 5^3.\mathrm{SL}_3(5)$ of $E_8(4)$ constructed in Lemma 2.10 is non-maximal (see Table 1 of Theorem 1).

THEOREM 5.1. *Let G be the simple algebraic group of type E_8 over the algebraic closure of $\mathbf{F}_p$, where p is prime and $p \neq 5$. Then G contains a subgroup isomorphic to $L_4(5)$ if and only if $p = 2$. Moreover, this embeds $L_4(5)$ in $E_8(4)$.*

The subgroup $L_4(5)$ will be constructed as the group generated by two conjugates of the local subgroup $N_G(E) \cong 5^3.\mathrm{SL}_3(5)$ constructed in Lemma 2.10. Write

$$P = N_G(E).$$

We require a preliminary lemma concerning the group P.

LEMMA 5.2. *P is a split extension of E by* $\mathrm{SL}_3(5)$.

Proof. Let $e \in E^{\#}$, and consider $C_P(e)$. This has the form $F.\mathrm{SL}_2(5)$, where, in the notation of the proof of Lemma 2.10, $F = \langle a_1, b_1, a_2, b_2 \rangle \cong 5^{1+4}$, an extraspecial group. By the construction of E (Lemma 2.10), $C_G(e) = X_1X_2$ with each $X_i \cong \mathrm{SL}_5$. Since $C_P(e)$ normalizes each X_i, we conclude that $C_P(e)/\langle e \rangle$ acts completely reducibly on $F/\langle e \rangle$. Thus $F/\langle e \rangle$ has precisely six proper non-trivial $C_P(e)/\langle e \rangle$-invariant subgroups. Those lifting to extraspecial groups 5^{1+2} come in pairs (orthogonal complements under the action of $\mathrm{Sp}_4(5)$ on F). Since E is abelian, $C_P(e)$ therefore normalizes a complement $\bar{E}/\langle e \rangle$ to $E/\langle e \rangle$ with $\bar{E}$ abelian. As $\mathrm{SL}_2(5)$ has trivial first cohomology on the natural 2-dimensional module, we have $\bar{E} = \langle e \rangle \times S$ with S invariant under $\mathrm{SL}_2(5)$. It now follows that a Sylow 5-subgroup of P splits over E, and hence P splits, as required.

Proof of Theorem 5.1. In view of Lemma 5.2 we may write $P = ED$, with $D \cong \mathrm{SL}_3(5)$. Let $T \cong Z_4 \times Z_4$ be a Cartan subgroup of D, and let U be a Sylow 5-subgroup of P such that $T \leqslant N_P(U)$ and $U = E(U \cap D)$. Consider the six T-composition factors of U. Each has centralizer Z_4 in T, and a direct check shows that these six subgroups of T are distinct.

It follows that there are precisely six T-invariant subgroups of U of order 5. We call these positive root subgroups; thus a positive root subgroup is determined by its centralizer in T. The group D contains three other T-invariant subgroups of order 5. These are the usual root groups in $\mathrm{SL}_3(5)$ corresponding to negative roots. Each has centralizer in T equal to that of some positive root group, and we call such a subgroup the opposite root group.

Label root subgroups as follows: $E = U_\alpha U_{\alpha+\beta} U_{\alpha+\beta+\gamma}$, with $\langle e \rangle = U_{\alpha+\beta+\gamma}$. Write $D = \langle U_{\pm\beta}, U_{\pm\gamma} \rangle$, where notation is chosen so that $S = U_\gamma U_{\beta+\gamma}$ (here S is the group constructed in the proof of Lemma 5.2). The usual commutator relations hold among the root groups in P, this group being isomorphic to a maximal parabolic subgroup of $L_4(5)$.

Now $\bar{E} = U_\gamma U_{\beta+\gamma} U_{\alpha+\beta+\gamma}$ (where $\bar{E}$ is as in Lemma 5.2). Consider $\bar{E}\langle U_{\pm\beta}\rangle \leq C_G(e) = X_1X_2 \cong \mathrm{SL}_5 \circ \mathrm{SL}_5$. The group $\langle U_{\pm\beta}\rangle$ is diagonally embedded here (that is, it intersects each X_i trivially), and it normalizes the intersection of $\bar{E}$ with each X_i. But F intersects each X_i in an extraspecial 5^{1+2}, and the action of $\langle U_{\pm\beta}\rangle$ on $\bar{E}/\langle e\rangle$ is irreducible. Consequently $\bar{E} \cap X_i = \langle e\rangle$ for $i = 1, 2$. From our construction of E in Lemma 2.10, we thus see that $\bar{E}$ is conjugate to E.

Hence $\bar{P} = N_G(\bar{E}) \cong \bar{E}.\mathrm{SL}_3(5)$, and there is a unique complement $\bar{D}$ to $\bar{E}$ which contains T. Then $\bar{D}$ contains precisely six T-invariant subgroups of order 5, and we have $\bar{D} = \langle U_{\pm\alpha}, U_{\pm\beta}\rangle$. That is, we have a new root subgroup $U_{-\alpha}$, opposite to U_α.

Set $M = \langle U_{\pm\alpha}, U_{\pm\beta}, U_{\pm\gamma}\rangle$. We shall show that if $p = 2$ then $M \cong L_4(5)$. To do this, it suffices to show that $[\langle U_{\pm\gamma}\rangle, U_{-\alpha}] = 1$; for then $[\langle U_{\pm\alpha}\rangle, \langle U_{\pm\gamma}\rangle] = 1$ and an application of the Curtis–Tits relations [**12**, Theorem 1.4] implies that M is an image of $L_4(5)$.

To this end, set $V = U_{\alpha+\beta}U_{\alpha+\beta+\gamma}$, $\bar{V} = U_{\beta+\gamma}U_{\alpha+\beta+\gamma}$ and $R = V\bar{V}U_\beta = U_{\alpha+\beta}U_{\alpha+\beta+\gamma}U_\beta U_{\beta+\gamma}$. We check that $R \cong 5^4$, and also $N_P(V)^{(\infty)} = R\langle U_{\pm\gamma}\rangle$ and $N_{\bar{P}}(\bar{V})^{(\infty)} = R\langle U_{\pm\alpha}\rangle$.

We now consider $N_G(R)$. Working in $C_G(e)$ we see that $C_G(V)^0$ is a maximal torus, say H, so $N_P(V) \leq N_G(H)$. As $N_P(V)/V$ is irreducible on R/V and 5^3 does not divide $|W(E_8)|$, we have $R \leq H$. Hence $N_G(R)$ normalizes $C_G(R)^0 = H$. Consider the actions of $\langle U_{\pm\alpha}\rangle$ and $\langle U_{\pm\gamma}\rangle$ on R. We claim that these groups commute modulo H. To see this, note that $N_G(R)'$ induces on R an irreducible subgroup of $\mathrm{SL}_4(5)$ of order dividing $|W(E_8)|$, and containing a 5-element centralized by $\mathrm{SL}_2(5)$ (corresponding to the image of $U_\alpha \times \langle U_{\pm\gamma}\rangle \leq P$). It follows that $N_G(R)'/H \cong \mathrm{SL}_2(5) \circ \mathrm{SL}_2(5)$, corresponding to a subgroup $\Omega_4^-(2) \times \Omega_4^-(2)$ or $\Omega_4^+(4)$ in $\Omega_8^+(2)$. The former subgroup comes from $W(A_4) \times W(A_4)$, which does not act faithfully and irreducibly on R. Hence $N_G(R)'/H$ corresponds to $\Omega_4^+(4)$.

Let t be the involution in T centralizing $\langle U_{\pm\alpha}\rangle$ and $\langle U_{\pm\gamma}\rangle$. Then $\langle tH\rangle = Z(N(H)/H)$. Hence $C_{N(R)'}(t) = \Omega_1(O_2(H)).(\mathrm{SL}_2(5) \circ \mathrm{SL}_2(5))$, showing that $\langle U_{\pm\alpha}\rangle$ and $\langle U_{\pm\gamma}\rangle$ commute modulo $\Omega_1(O_2(H))$.

If $p = 2$ then $O_2(H) = 1$, so $\langle U_{\pm\alpha}\rangle$ and $\langle U_{\pm\gamma}\rangle$ commute, which implies that $M \cong L_4(5)$. Moreover, all the groups occurring in the proof exist in $E_8(4)$, so this embeds $L_4(5)$ in $E_8(4)$, as required.

It remains to show that if $p \neq 2, 5$ then G has no subgroup $L_4(5)$. To do this, we shall demonstrate that the extension $C_{N(R)'}(t) = 2^8.(\mathrm{SL}_2(5) \circ \mathrm{SL}_2(5))$ does not split. This implies that $\langle U_{\pm\alpha}\rangle$ and $\langle U_{\pm\gamma}\rangle$ do not commute. Hence if G had a subgroup $L_4(5)$, we could have performed all the above calculations within this subgroup to obtain a contradiction.

Thus suppose for a contradiction that the extension splits. We have $C_{N(R)'}(t) = 2^9.(\mathrm{Alt}_5 \times \mathrm{Alt}_5)$ with central involution t. This is a subgroup of $C_G(t) = D_8$. Let $\bar{N} \cong 2^8$ be the image of the normal 2^9 in $C_G(t)/\langle t\rangle = P\Omega_{16}(K)$, where $K = \bar{\mathbb{F}}_p$. Let N be the preimage of $\bar{N}$ in $\Omega_{16}(K)$, and let C be the preimage of $C_{N(R)'}(t)/\langle t\rangle$ in $\Omega_{16}(K)$.

We claim that N is extraspecial. Otherwise, it is abelian (since the top factor $\mathrm{Alt}_5 \times \mathrm{Alt}_5$ acts irreducibly on $\bar{N}$), and an easy argument using the action of elements of order 5 shows that $N = N_1 \times \langle z\rangle$, where z is the central involution in $\Omega_{16}(K)$ and N_1 is normal in C. The action of C on N_1 is $\Omega_4^+(4)$, which has orbits of size 75 and 180 on the linear characters of N_1, and hence by Clifford's theorem, C cannot lie in $\Omega_{16}(K)$, a contradiction. Thus N is extraspecial, as claimed.

We now use an argument of R. L. Griess to complete the proof. We are assuming that the extension $C = N.\Omega_4^+(4)$ splits. Take a subgroup A of type $\Omega_3(4)$. There are an involution $a \in A$ and a singular vector $s \in N/\langle z \rangle$ for which $s^a = s + n$, where n is a non-singular vector fixed by A and n lifts to an element of order 4 in N. In the group N the preimage of $\langle s, n \rangle$ is a dihedral group with a inverting the element of order 4, which corresponds to n. But A normalizes this Z_4 and $A = A'$. This is a contradiction, completing the proof of Theorem 5.1.

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