JOURNAL OF COMBINATORIAL THEORY, Series A 58, 301-305 (1991)

Note

A Simpler Proof and a Generalization of the Zero-Trees Theorem

A. SCHRIJVER*

Mathematical Centre, Kruislaan 413, 1098 SJ Amsterdam, The Netherlands and University of Amsterdam, The Netherlands

AND

P. D. SEYMOUR

Bellcore, 445 South Street, Morristown, New Jersey 07960

Communicated by the Managing Editors

Received December 30, 1989

Z. Füredi and D. J. Kleitman proved that if an integer weight is assigned to each edge of a complete graph on $p+1$ vertices, then some spanning tree has total weight divisible by p. We obtain a simpler proof by generalizing the result to hypergraphs. © 1991 Academic Press, Inc.

1. INTRODUCTION

The following theorem is due to Z. Füredi and D. J. Kleitman [2]. (It was conjectured by A. Bialostocki and P. Dierker [1], who proved the case when p is prime.)

THEOREM (1.1). *Let Γ be a finite abelian group of order p, and let w: $E(K_{p+1}) \to \Gamma$ be some function. Then there is a spanning tree T of K_{p+1} with $w(T)=0$.*

(K_n denotes the complete graph with n vertices; $E(G)$ denotes the set of edges of a graph G; $w(T)$ means $\Sigma(w(e): e \in E(T))$, where the summation is in Γ.)

* This research was performed under a consulting agreement with Bellcore.

0097-3165/91 $3.00

We shall give a simpler proof of (1.1). For inductive purposes, it is advantageous to prove a version of (1.1) for complete uniform hypergraphs, because it is then easy to reduce the general problem to the case when p is prime.

Thus, let V be a finite set. A *hypergraph* in V is a collection of subsets of V; and it is *r-uniform* if each of these subsets has cardinality r. (In this paper, all our hypergraphs will be r-uniform for some r.) If H is a hypergraph, we denote $\bigcup\ (e\colon e \in H)$ by $V(H)$. A hypergraph T is *connected* if $T \neq \varnothing$ and for every partition (A, B) of $V(T)$ such that A and B are both nonempty there is a member $e \in T$ with $e \cap A$, $e \cap B$ both nonempty. It is easy to see that if T is connected and r-uniform then $|V(T)| \leqslant (r-1)\,|T| + 1$; and if equality holds we say that T is a *tree*. (If $r = 2$, this coincides with the usual definition of a tree for graphs, except for trees with $\leqslant 1$ vertex.) If H is r-uniform, and $T \subseteq H$ is a tree, we call it a *tree* of H; and if $V(T) = V(H)$ we call it a *spanning tree* of H. If V is a finite set with $|V| \geqslant r$, we denote by $\binom{V}{r}$ the collection of all r-element subsets of V. We shall prove the following generalization of (1.1).

THEOREM (1.2). *Let Γ be a finite abelian group of order p, let $r \geqslant 2$ be an integer, let V be a set of cardinality $p(r-1)+1$, and let $w\colon \binom{V}{r} \to \Gamma$ be some function. Then there is a spanning tree T of $\binom{V}{r}$ with $w(T) = 0$.*

($w(T)$ means $\Sigma(w(e)\colon e \in T)$.)

2. THE PROOF OF (1.2)

We require several lemmas. First, we shall need the following, which is a special case of the Cauchy–Davenport theorem (see [3]). (It can also be proved directly in a couple of lines, as the reader may verify.)

LEMMA (2.1). *Let p be prime, let $A \subseteq \mathbf{Z}_p$, and let $b, c \in \mathbf{Z}_p$ be distinct. If $1 \leqslant |A| \leqslant p-1$ then*

$$|\{a+b : a \in A\} \cup \{a+c : a \in A\}| > |A|.$$

If T is an r-uniform tree, we say that $f \in T$ is a *leaf* of T if there exists $u \in f$ such that $e \cap f \subseteq \{v\}$ for every $e \in T - \{f\}$. We call such an element u a *root* of the leaf e. If T, T' are trees in $\binom{V}{r}$ with leaves e, e', respectively, and $T - \{e\} = T' - \{e'\}$, we say that T' is obtained from T by *shifting a leaf*. If $T, T' \subseteq \binom{V}{r}$ are trees, we say that T is *shiftable* to T' if there is a sequence

$$T = T_1, T_2, ..., T_k = T'$$

of trees in $\binom{V}{r}$ such that T_{i+1} is obtained from T_i by shifting a leaf for $1 \leqslant i \leqslant k-1$. This is evidently an equivalence relation, and in fact all trees in $\binom{V}{r}$ of the same cardinality are shiftable to one another, but we only need a weaker result, the following.

LEMMA (2.2). *Let $r \geqslant 2$, $k \geqslant 1$ be integers, let $|V| \geqslant k(r-1)+2$, and let $v_0 \in V$. Let T_0 be a tree in $\binom{V}{r}$ with $|T_0| = k$. Then T_0 is shiftable to a tree T with $v_0 \notin V(T)$.*

Proof. We may assume that $k \geqslant 2$, for the result is clear if $k = 1$. If T is a tree in $\binom{V}{r}$ with $v_0 \in V(T)$ and f is a leaf of T, we define $d(T, f)$ to be the unique $d \geqslant 1$ such that there is a sequence

$$v_0 = v_1, e_1, v_2, e_2, ..., v_d, e_d = f$$

satisfying

(i) $v_1, v_2, ..., v_d \in V(T)$ are all distinct, and so are $e_1, e_2, ..., e_d \in T$

(ii) $v_i \in e_{i-1}$ for $2 \leqslant i \leqslant d$, and $v_i \in e_i$ for $1 \leqslant i \leqslant d$.

Let us choose a tree T in $\binom{V}{r}$ such that T_0 is shiftable to T and $v_0 \in V(T)$, and a leaf f of T, in such a way that $d(T, f)$ is maximum. Let u be a root of f. Since $|T| \geqslant 2$ it follows that T has at least two leaves; let f' be another leaf, with root u'. Since $d(T, f') \leqslant d(T, f)$ it follows that $v_0 \notin f - \{u\}$. Choose $v \in f - \{u\}$, and let $e = (f' - \{u'\}) \cup \{v\}$. Now $T' = (T - \{f'\}) \cup \{e\}$ is shiftable from T and hence from T_0, and e is a leaf of it, and if $v_0 \notin f' - \{u'\}$ then $d(T', e) > d(T, f)$, a contradiction. Thus $v_0 \in f' - \{u'\}$ and, since $V(T) \neq V$, the result follows. ∎

Again, let $r \geqslant 2$, $k \geqslant 1$ and let $|V| \geqslant k(r-1)+1$. We say that $S \subseteq \binom{V}{r}$ is a (V, k)-*blocker* if $|S \cap T| \neq \varnothing$ for every tree T in $\binom{V}{r}$ with $|T| = k$. Our third lemma is the following.

LEMMA (2.3). *Let $r \geqslant 2$, $k \geqslant 1$ be integers, and let $|V| = k(r-1)+1$. If $S \subseteq \binom{V}{r}$ is a (V, k)-blocker then S includes a spanning tree of $\binom{V}{r}$.*

Proof. The result holds if $k = 1$, and so we may assume that $k \geqslant 2$ and proceed by induction on k. Since there is a spanning tree and we may assume that it is not included in S, it follows that $\varnothing \neq S \neq \binom{V}{r}$. Thus, we may choose $e, f \in \binom{V}{r}$ with $|e \cap f| = r-1$ and $e \in S, f \notin S$. Let $V - (e \cap f) = V'$. If T' is a spanning tree of $\binom{V'}{r}$ then $T' \cup \{f\}$ is a spanning tree of $\binom{V}{r}$, and so $S \cap (T' \cup \{f\}) \neq \varnothing$, that is, $S' \cap T' \neq \varnothing$, where $S' = S \cap \binom{V'}{r}$. Hence S' is a $(V', k-1)$-blocker, and so S' includes a spanning tree T' of $\binom{V'}{r}$, from the inductive hypothesis. Then $T' \cup \{e\} \subseteq S$ is a spanning tree of $\binom{V}{r}$, as required. ∎

We shall use (2.1)–(2.3) to prove the following, which is the main step in the proof of (1.2).

LEMMA (2.4). *Let p be prime, let $k \geqslant 1$, $r \geqslant 2$ be integers with $k \leqslant p$, let V be a set of cardinality $k(r-1)+1$, and let $w\colon \binom{V}{r} \to \mathbf{Z}_p$ be some function. Then either*

(i) *there are k spanning trees $T_1, ..., T_k$ with $w(T_1), ..., w(T_k)$ all distinct, or*

(ii) *$k \geqslant 2$ and there is a monochromatic $(V, k-1)$-blocker.*

(A subset $S \subseteq \binom{V}{r}$ is *monochromatic* if the restriction of w to S is constant.)

Proof. The result holds if $k=1$, and so we may assume that $k \geqslant 2$ and proceed by induction on k. We say that $X \subseteq V$ is *joint* if $|X| = r-1$ and $X = f_1 \cap f_2$ for some $f_1, f_2 \in \binom{V}{r}$ with $w(f_1) \neq w(f_2)$. We assume that (i) is false. We may assume that

(1) *Some set $X \subseteq V$ is joint.* For $\binom{V}{r}$ is a $(V, k-1)$-blocker since $k \geqslant 2$, and so we may assume that w is non-constant on $\binom{V}{r}$, for otherwise (ii) holds. The claim follows.

(2) *If X is joint then $k \geqslant 3$ and there exists a monochromatic $(V-X, k-2)$-blocker.* For let $X \subseteq V$ be joint. Suppose that there are $k-1$ spanning trees $T_1, ..., T_{k-1}$ of $\binom{V-X}{r}$ with $w(T_1), ..., w(T_{k-1})$ all distinct. Choose $f_1, f_2 \in \binom{V}{r}$ with $f_1 \cap f_2 = X$ and $w(f_1) \neq w(f_2)$. Now $T_i \cup \{f_1\}$ and $T_i \cup \{f_2\}$ are spanning trees of $\binom{V}{r}$ for $1 \leqslant i \leqslant k-1$, and

$$|\{w(T_i)+w(f_1) : 1 \leqslant i \leqslant k-1\} \cup \{w(T_i)+w(f_2) : 1 \leqslant i \leqslant k-1\}| \geqslant k$$

by (2.1). Hence (i) holds, a contradiction. Thus, there do not exist $k-1$ such spanning trees. From our inductive hypothesis applied to $V-X$ the claim follows.

In particular, from (1) and (2) we deduce that $k \geqslant 3$. For each joint set X, let $S(X)$ be a monochromatic $(V-X, k-2)$ blocker, and let $w(e) = q(X)$ for all $e \in S(X)$.

(3) *There exists $q \in \mathbf{Z}_p$ such that $q(X) = q$ for every joint set X.* For let X_1, X_2 be joint; we shall show that $q(X_1) = q(X_2)$. Let $X_1 \cup X_2 \subseteq Z \subseteq V$, where $|Z| = 2r-2$. Now $S(X_1)$ is a $(V-X_1, k-2)$-blocker, and so $S(X_1) \cap \binom{V-Z}{r}$ is a $(V-Z, k-2)$-blocker. By (2.3), there is a spanning tree T of $\binom{V-Z}{r}$ with $T \subseteq S(X_1)$. Similarly, $S(X_2) \cap \binom{V-Z}{r}$ is a $(V-Z, k-2)$-blocker, and so $S(X_2) \cap T \neq \varnothing$. Hence, $S(X_1) \cap S(X_2) \neq \varnothing$, and the claim follows.

Let us say a tree $T \subseteq \binom{V}{r}$ is *bad* if $|T| = k-1$ and $w(e) \neq q$ for all $e \in T$.

(4) *If* f_1 *is a leaf of a bad tree* T, *and* $f_2 \in \binom{V}{r}$ *with* $|f_2 \cap V(T-\{f_1\})| \leqslant 1$, *then* $w(f_2)=w(f_1)$. For let $V'=V(T-\{f_1\})$. If $X \subseteq V-V'$ is joint then $S(X) \cap (T-\{f_1\}) \neq \emptyset$, which is impossible by (3), since T is bad. Thus no subset of $V-V'$ is joint, and the claim follows. In particular,

(5) *If* T *is a bad tree and* T *is shiftable to* T' *then* T' *is bad.*

Now by (1), there is a joint set X. If there is a bad tree, then by $(r-1)$ applications of (2.2), it is shiftable to a tree T with $X \cap V(T)=\emptyset$; and by (5), T is bad. But then $T \cap S(X) \neq \emptyset$, a contradiction as before. We deduce that there is no bad tree, and so $\{e \in \binom{V}{r} : w(e)=q\}$ is a $(V, k-1)$-blocker. Thus (ii) holds, as required. ∎

Finally, we use (2.4) to prove (1.2).

Proof of (1.2). We proceed by induction on p. If p is prime, then $\Gamma \cong \mathbf{Z}_p$ and by (2.4) with $k=p$, either

(i) there are p spanning trees $T_1, ..., T_p$ with $w(T_1), ..., w(T_p)$ all distinct; but then one of them is zero, as required, or

(ii) for some $q \in \Gamma$ there is a $(V, p-1)$-blocker S such that $w(e)=q$ for all $e \in S$; but then S is a (V, p)-blocker and hence includes a spanning tree T, and $w(T)=\Sigma(q: e \in T)=0$ as required.

We may assume then that p is not prime, and so Γ has a proper subgroup Γ', of order p' say. Let Γ'' be the quotient group Γ/Γ', of order p'' say, where $p=p'p''$, and let $\phi: \Gamma \to \Gamma''$ be the homomorphism with kernel Γ'. For each $e \in \binom{V}{r}$, we define $w''(e)=\phi(w(e)) \in \Gamma''$. Let $r'=p''(r-1)+1$. For each $f \subseteq V$ with $|f|=r'$, we define $w'(f)$ as follows. From our inductive hypothesis applied to $\binom{f}{r}$, Γ'' and w'', there is a spanning tree $T(f)$ of $\binom{f}{r}$ such that $w''(T(f))=0$; that is, $w(T(f)) \in \Gamma'$. We define $w'(f)=w(T(f))$. From our inductive hypothesis applied to $\binom{V}{r'}$, Γ' and w', there is a spanning tree T' of $\binom{V}{r'}$ with $w'(T')=0$. Let $T=\bigcup(T(f): f \in T')$; then T is a spanning tree of $\binom{V}{r}$ and

$$w(T)=\sum_{f \in T'} \sum_{e \in T(f)} w(e)=\sum_{f \in T'} w'(f)=0$$

as required. ∎

References

1. A. Bialostocki and P. Dierker, Zero sum Ramsey theorems, *Congressus Numerantium* **70** (1990), 119–130.
2. Z. Füredi and D. J. Kleitman, On zero-trees, *J. Graph Theory*, to appear, manuscript 1989.
3. H. Halberstam and K. F. Roth, "Sequences," Vol. 1, Oxford Univ. Press, London, 1966.