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# Edge-Disjoint Circuits in Graphs on the Torus

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We give necessary and sufficient conditions for a given graph embedded on the torus, to contain edge-disjoint cycles of prescribed homotopies (under the assumption of a "parity" condition). 

## 1. INTRODUCTION

We prove a theorem on edge-disjoint cycles of prescribed homotopies in an undirected graph embedded on the torus. It forms a sharpening (integer version) of a theorem proved in [1] for general compact orientable surfaces.

Let $G=(V, E)$ be an undirected graph embedded on the torus $T$, and let $C_1, ..., C_k$ be closed curves on $T$. We are interested in conditions under which

there exist pairwise edge-disjoint cycles $\tilde{C}_1, ..., \tilde{C}_k$ in $G$ so that $\tilde{C}_i$ is freely homotopic to $C_i$, for $i=1, ..., k$. (1)

We will identify an embedded graph with its image in $T$. A *closed curve* on $T$ is a continuous function $C: S_1 \to T$, where $S_1$ is the unit circle in the complex plane.

A *cycle* in $G$ is a sequence $(v_0, e_1, v_1, ..., e_d, v_d)$ so that $e_i$ is an edge connecting $v_{i-1}$ and $v_i$ ($i=1, ..., d$), with $v_0=v_d$. (If $e_j$ is a loop, we associate an orientation with $e_j$.) In a natural way we can identify such a cycle in $G$



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with a closed curve on $T$. We call a collection of cycles *pairwise edge-disjoint* if no two cycles have an edge in common, and moreover, no cycle traverses the same edge more than once.

Two closed curves $C$ and $\tilde{C}$ on $T$ are called *freely homotopic*, in notation: $C \sim \tilde{C}$, if there exists a continuous function $\Phi: S_1 \times [0, 1] \to T$ so that $\Phi(z, 0) = C(z)$ and $\Phi(z, 1) = \tilde{C}(z)$ for each $z \in S_1$. (So there is not necessarily a point fixed.)

The following *cut condition* is a necessary condition for (1): for each closed curve $D$ on $T$, intersecting $G$ only a finite number of times and not intersecting $V$, one has

$$\mathrm{cr}(G, D) \geqslant \sum_{i=1}^{k} \min \mathrm{cr}(C_i, D). \tag{2}$$

Here we use the notation (for closed curves $C$ and $D$)

$$\begin{aligned} \mathrm{cr}(G, D) &:= |\{z \in S_1 \mid D(z) \in G\}|, \\ \mathrm{cr}(C, D) &:= |\{(y, z) \in S_1 \times S_1 \mid C(y) = D(z)\}|, \\ \min \mathrm{cr}(C, D) &:= \min\{\mathrm{cr}(\tilde{C}, \tilde{D}) \mid \tilde{C} \sim C, \tilde{D} \sim D\}. \end{aligned} \tag{3}$$

Condition (2) is not sufficient for (1), as is shown by Fig. 1, where the wriggled lines indicate closed curves $C_1$ and $C_2$ and where the torus arises by identifying the two segments $\alpha$ and identifying the two segments $\beta$ (the fact that the cut condition is satisfied can be seen by observing that a "half-integer" packing of required cycles exists). A second example arises by

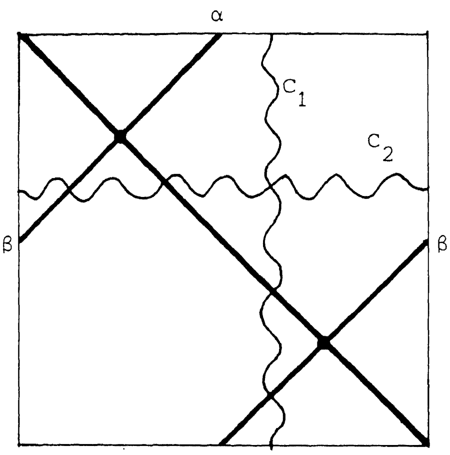


FIGURE 1

taking for $G$ a graph consisting of two vertices, each attached with a loop (pairwise disjoint and nonnullhomotopic), and for $C_1$ a closed curve going twice around one of the loops (see Fig. 2).

We show that (2) is sufficient for (1) if each $C_i$ is *simple* (i.e., is a one-to-one function), and the following *parity condition* holds:

> (parity condition) for each closed curve $D$ on $T$, not intersecting vertices of $G$, the number of crossings of $D$ with edges of $G$, plus the number of crossings with $C_1, ..., C_k$, is an even number. (4)

One easily checks that the parity condition implies that each vertex of $G$ has even degree.

THEOREM. *Let $G = (V, E)$ be a graph embedded on the torus $T$, and let $C_1, ..., C_k$ be simple closed curves on $T$, such that the parity condition holds. Then there exist pairwise edge-disjoint closed cycles $\tilde{C}_1, ..., \tilde{C}_k$ in $G$ so that $\tilde{C}_i \sim C_i$ $(i = 1, ..., k)$, if and only if the cut condition holds.*

(We do not require the $\tilde{C}_i$ to be simple—they may have self-intersections at vertices of $G$.)

Figures 1 and 2 show that we cannot delete the parity or the simple-ness condition. For general compact orientable surfaces the cut condition only implies the existence of a "fractional" solution to (1).

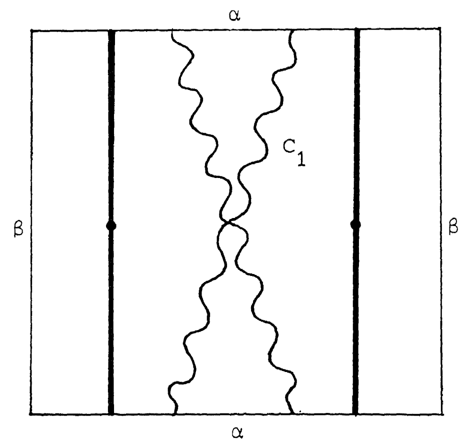


FIGURE 2

## 2. Closed Curves on the Torus and Their Crossings

Before proving the theorem (in Section 3), we show an inequality for the function min cr$(C, D)$ defined in (3). This inequality is esential in our proof and does not hold for compact orientable surfaces other than the sphere and the torus.

Let $D_1$, $D_2: S_1 \to T$ be closed curves on $T$ with $D_1(1) = D_2(1)$. Let $D_1 \cdot D_2$ denote the concatenation of $D_1$ and $D_2$. That is, $D_1 \cdot D_2: S_1 \to T$ is defined by $(D_1 \cdot D_2)(z) := D_1(z^2)$ if $\operatorname{Im} z \geqslant 0$ and $(D_1 \cdot D_2)(z) := D_2(z^2)$ if $\operatorname{Im} z < 0$. Then:

Proposition. $\min \operatorname{cr}(C, D_1 \cdot D_2) \leqslant \min \operatorname{cr}(C, D_1) + \min \operatorname{cr}(C, D_2)$.

*Proof.* Identify the torus $T$ with the product $S_1 \times S_1$ of two copies of the unit circle $S_1$ in the complex plane $\mathbb{C}$. For $m, n \in \mathbb{Z}$ we define the closed curve $C_{m,n}: S_1 \to S_1 \times S_1$ by

$$C_{m,n}(z) := (z^m, z^n) \qquad \text{for} \quad z \in S_1. \tag{5}$$

As is well known (cf. [2, Sect. 6.2.2]), the closed curves $C_{m,n}$ form a system of representatives for the free homotopy classes of closed curves on $T$. For $m, n, m', n' \in \mathbb{Z}$,

$$\min \operatorname{cr}(C_{m,n}, C_{m',n'}) = \left| \det \begin{pmatrix} m & n \\ m' & n' \end{pmatrix} \right| = |mn' - m'n|. \tag{6}$$

To see the proposition, we may assume that $D_1 = C_{m',n'}$ and $D_2 = C_{m'',n''}$ for some $m', n', m'', n'' \in \mathbb{Z}$. Then $D_1 \cdot D_2 \sim C_{m'+m'', n'+n''}$. Hence choosing $m, n$ so that $C \sim C_{m,n}$,

$$\begin{aligned} \min \operatorname{cr}(C, D_1 \cdot D_2) &= |m(n' + n'') - (m' + m'')n| \\ &\leqslant |mn' - m'n| + |mn'' - m''n| \\ &= \min \operatorname{cr}(C, D_1) + \min \operatorname{cr}(C, D_2). \quad \blacksquare \end{aligned} \tag{7}$$

## 3. Proof of the Theorem

The cut condition clearly is necessary. To see sufficiency, suppose the cut condition is satisfied, but cycles as required do not exist. We assume that we have a counterexample $G = (V, E)$ with

$$\sum_{v \in V} 2^{\deg(v)} \tag{8}$$

as small as possible. Here $\deg(v)$ denotes the degree of vertex $v$.

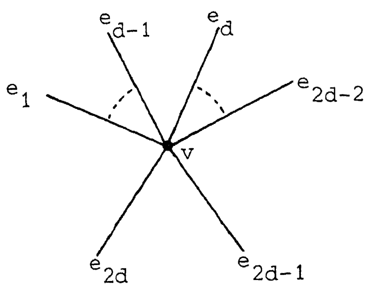


FIGURE 3

We first show:

$$\text{each vertex of } G \text{ has degree at most 4.} \tag{9}$$

Suppose to the contrary that vertex $v$ has degree $2d \geqslant 6$ (see Fig. 3). Replace Fig. 3 by Fig. 4, where there are $d-2$ parallel edges connecting $v'$ and $v''$. For the new graph $G'$ again the cut condition holds (as we may assume that the cut $D$ does not intersect the "new" edges in Fig. 4, since we can make a detour through the original edges without increasing $\mathrm{cr}(G', D)$). However, for $G'$ the sum (8) has decreased (since $2^{2d-2} + 2^{2d-2} + 2^4 < 2^{2d}$). So in $G'$ cycles as required exist. This directly gives cycles as required in the original graph $G$, contradicting our assumption. This shows (9).

We next show that in each vertex $v$ of $G$ of degree 4 the following holds. Consider a neighbourhood $N \simeq \mathbb{C}$ of $v$ not containing any vertex other than $v$ (see Fig. 5). Here $F_1, ..., F_4$ stand for the intersections of faces with $N$. We show:

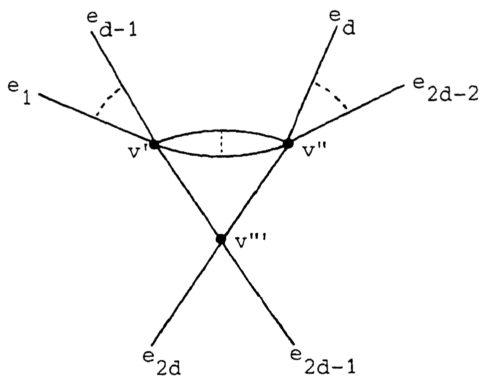


FIGURE 4

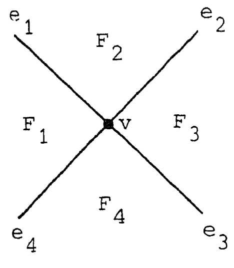


FIGURE 5

CLAIM. *There exists a closed curve* $D: S_1 \to T \setminus V$ *such that*

$$\begin{array}{ll} \text{(i)} & D \text{ contains a subcurve contained in } N \text{ connecting } F_1 \text{ and } F_3; \\ \text{(ii)} & \mathrm{cr}(G, D) = \sum_{i=1}^{k} \min \mathrm{cr}(C_i, D). \end{array} \tag{10}$$

*Proof of the Claim.* Suppose such a curve does not exist. Replace $N$ as in Fig. 5 by $N$ as in Fig. 6. Since any packing of cycles as required in the new graph $G'$ would yield a required packing in the original graph $G$, and since for $G'$ the sum (8) has decreased, the cut condition does not hold for $G'$. That is,

$$\mathrm{cr}(G', D) < \sum_{i=1}^{k} \min \mathrm{cr}(C_i, D) \tag{11}$$

for some closed curve $D$ not intersecting any vertex of $G'$. We may assume that $D$ does not traverse $v$. Let $p$ be the number of subcurves of $D$ contained in $N$ and connecting $F_1$ and $F_3$ (in one direction or the other). As $\mathrm{cr}(G', D) < \mathrm{cr}(G, D)$ we know $p \geqslant 1$. Choose $D$ so that $p$ is as small as possible. We show $p = 1$. Assume $p \geqslant 2$.

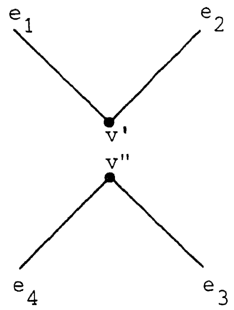


FIGURE 6

Let $P$ be any curve in $N$ from $F_1$ to $F_3$ not intersecting $v$, $v'$, or $v''$, and only crossing $e_1$ and $e_2$. Then we may assume that

$$\begin{aligned}&\text{(i)}\quad D = P \cdot D_1 \cdot P \cdot D_2, \text{ where } D_1 \text{ and } D_2 \text{ are curves from } F_3 \text{ to } F_1\text{; or}\\ &\text{(ii)}\quad D = P \cdot D_1 \cdot P^{-1} \cdot D_2, \text{ where } D_1 \text{ is a curve from } F_3 \text{ to } F_3,\\ &\qquad \text{and } D_2 \text{ is a curve from } F_1 \text{ to } F_1\end{aligned} \tag{12}$$

($P^{-1}$ denotes the curve reverse to $P$). If (12)(i) holds, then (using the proposition),

$$\begin{aligned}\operatorname{cr}(G', D) &= \operatorname{cr}(G', P \cdot D_1) + \operatorname{cr}(G', P \cdot D_2)\\ &\geqslant \sum_{i=1}^{k} \min \operatorname{cr}(C_i, P \cdot D_1) + \sum_{i=1}^{k} \min \operatorname{cr}(C_i, P \cdot D_2)\\ &\geqslant \sum_{i=1}^{k} \min \operatorname{cr}(C_i, D),\end{aligned} \tag{13}$$

since $P \cdot D_1$ and $P \cdot D_2$ are closed curves containing fewer than $p$ subcurves in $N$ connecting $F_1$ and $F_3$.

If (12)(ii) holds, then (again using the proposition),

$$\begin{aligned}\operatorname{cr}(G', D) &\geqslant \operatorname{cr}(G', D_1) + \operatorname{cr}(G', D_2)\\ &\geqslant \sum_{i=1}^{k} \min \operatorname{cr}(C_i, P \cdot D_1 \cdot P^{-1}) + \sum_{i=1}^{k} \min \operatorname{cr}(C_i, D_2)\\ &\geqslant \sum_{i=1}^{k} \min \operatorname{cr}(C_i, D),\end{aligned} \tag{14}$$

since $D_1$ and $D_2$ are closed curves containing fewer than $p$ subcurves in $N$ connecting $F_1$ and $F_3$.

Both (13) and (14) contradict (11). So $p = 1$. Hence $\operatorname{cr}(G, D) = \operatorname{cr}(G', D) + 2$. Therefore, by (11), $\operatorname{cr}(G, D) < 2 + \sum_{i=1}^{k} \min \operatorname{cr}(C_i, D)$. It follows by the parity condition that $D$ satisfies (10).

*End of proof of the Claim*

Now by the "homotopic circulation theorem" in [1], the cut condition implies the existence of a "fractional" packing of cycles. That is, there exist cycles

$$C_{1,1}, \ldots, C_{1,t_1}, C_{2,1}, \ldots, C_{2,t_2}, \ldots, C_{k,1}, \ldots, C_{k,t_k} \tag{15}$$

in $G$ and rational numbers

$$\lambda_{1,1}, \ldots, \lambda_{1,t_1}, \lambda_{2,1}, \ldots, \lambda_{2,t_2}, \ldots, \lambda_{k,1}, \ldots, \lambda_{k,t_k} > 0 \tag{16}$$

satisfying

$$\begin{aligned}
&\text{(i)} \quad C_{i,j} \sim C_i && (i=1, \ldots, k;\ j=1, \ldots, t_i),\\
&\text{(ii)} \quad \sum_{j=1}^{t_i} \lambda_{i,j} = 1 && (i=1, \ldots, k), \qquad (17)\\
&\text{(iii)} \quad \sum_{i=1}^{k} \sum_{j=1}^{t_i} \lambda_{i,j} \chi^{C_{i,j}}(e) \leqslant 1 && (e \in E).
\end{aligned}$$

Here $\chi^C(e)$ denotes the number of times the cycle $C$ traverses edge $e$.

We may assume that no $C_{i,j}$, after arriving in a vertex $v$ via an edge $e$, immediately returns over the same edge $e$ backward.

We show:

for each $i$, $j$, if $C_{i,j}$ arrives in a vertex $v$ via edge $e$, say, then it next leaves $v$ via the edge opposite to $e$. (18)

(If $e_1$, $e_2$, $e_3$, $e_4$ are the edges incident to $v$ in cyclic order, then $e_1$ and $e_3$ are called *opposite*; similarly for $e_2$ and $e_4$.) To see this, suppose that cycle $C_{1,1}$ say, contains ..., $e_1$, $v$, $e_2$, ... (where $v$, $e_1$, $e_2$, $e_3$, $e_4$ are as in Fig. 5). Let $D: S_1 \to T \setminus V$ be a closed curve satisfying (10). We may assume that $D$ crosses $e_1$ and $e_2$ successively.

However, since $C_{1,1}$ contains ..., $e_1$, $v$, $e_2$, ..., we know

$$\operatorname{cr}(C_{1,1}, D) > \min \operatorname{cr}(C_{1,1}, D). \qquad (19)$$

This gives the contradiction

$$\begin{aligned}
\operatorname{cr}(G, D) &\geqslant \sum_{i=1}^{k} \sum_{j=1}^{t_i} \lambda_{i,j} \operatorname{cr}(C_{i,j}, D)\\
&> \sum_{i=1}^{k} \sum_{j=1}^{t_i} \lambda_{i,j} \min \operatorname{cr}(C_{i,j}, D) = \sum_{i=1}^{k} \min \operatorname{cr}(C_i, D). \qquad (20)
\end{aligned}$$

This proves (18). It follows from (18) that any two of the $C_{1,1}, \ldots, C_{k,t_k}$ are pairwise edge-disjoint or are the same (up to cyclic permutation and reversal). (No $C_{i,j}$ makes more than one orbit of a cycle, as it is homotopic to a simple closed curve $C_i$.) This implies that we can select from $C_{1,1}, \ldots, C_{k,t_k}$ pairwise edge-disjoint cycles as required. ▮

## Acknowledgment

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